A Quantum Algorithm for Measuring Entanglement Asymmetry and Quantum Mpemba Effect

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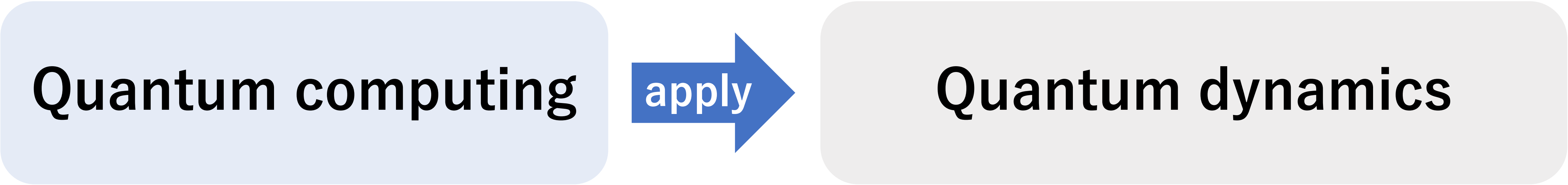
2026/5/19, 25min talk

Based on ongoing work with
Keisuke Fujii, Masazumi Honda and Duc Truyen Le.

My HP

1. Introduction

The main topic of this talk:



1. Introduction

The main topic of this talk:

Quantum computing → apply → **Symmetry restoration dynamics**

Take home message :

Quantum computing enables the study of out-of-equilibrium symmetry restoration dynamics, such as Quantum Mpemba Effect.

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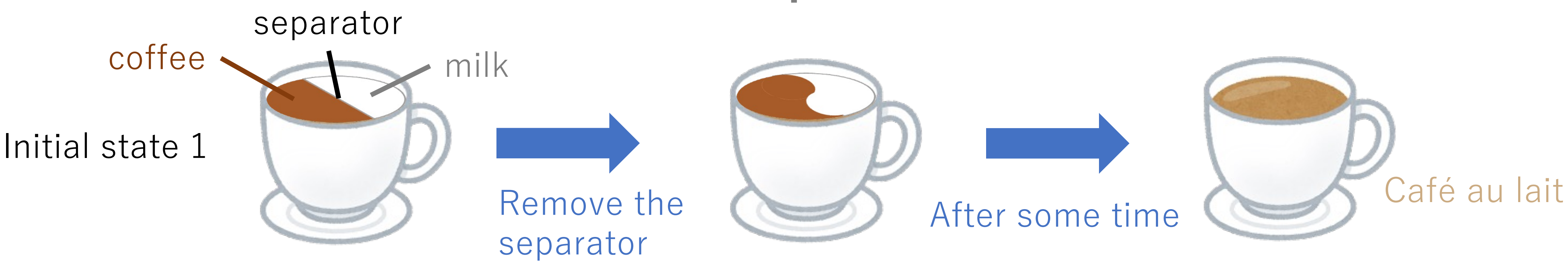
↑

What is this ?

1. Introduction

Mpemba effect is an anomalous out-of-equilibrium relaxation.
[E B Mpemba and D G Osborn, 1969]

Metaphor



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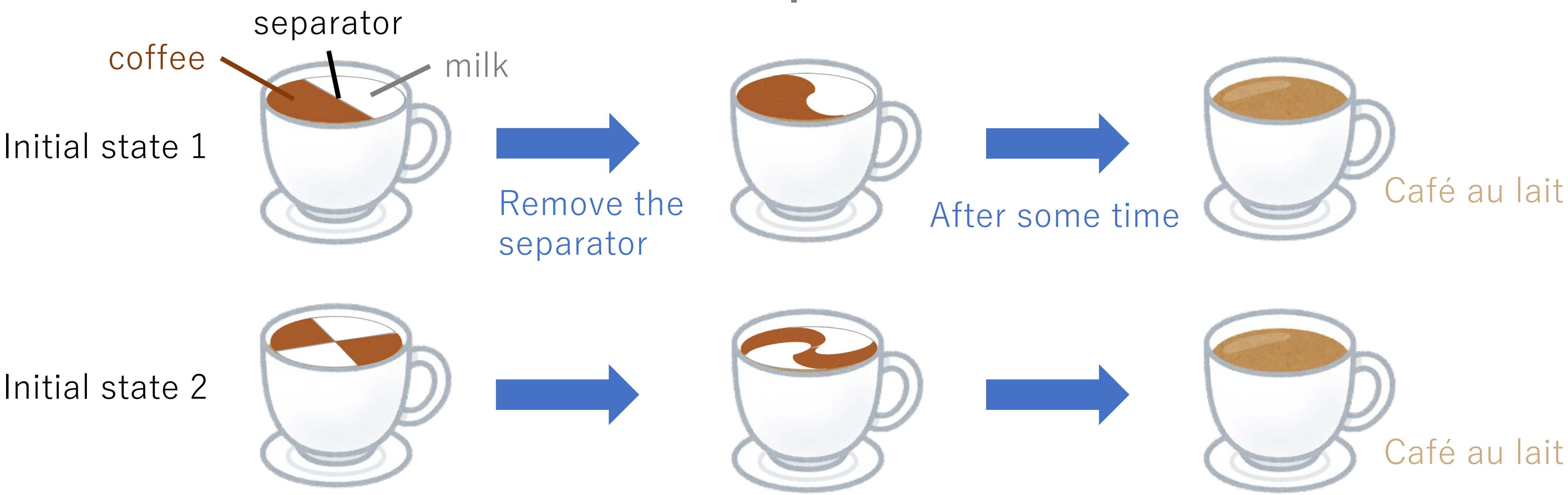

Intuitively, one expects the initial state 2 to relax faster than the initial state 1.

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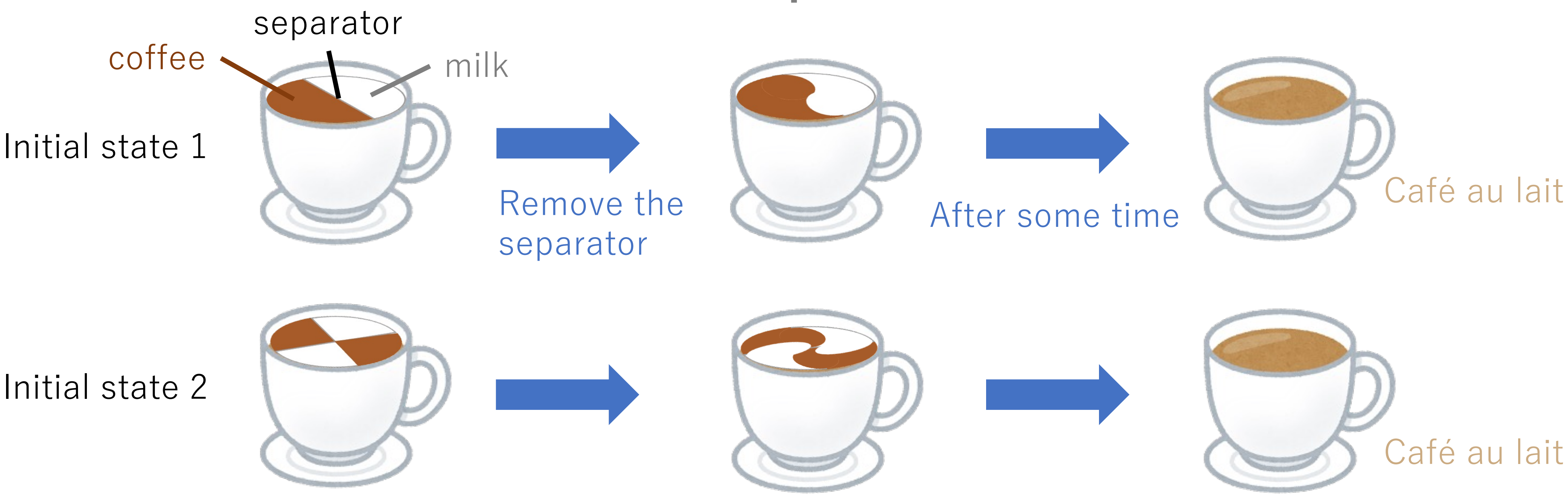

Intuitively, one expects the initial state 2 to relax faster than the initial state 1.

However, under certain conditions, the opposite behavior is observed.

➡ **Counterintuitive phenomenon!**

1. Introduction

Mpemba effect is an anomalous out-of-equilibrium relaxation.
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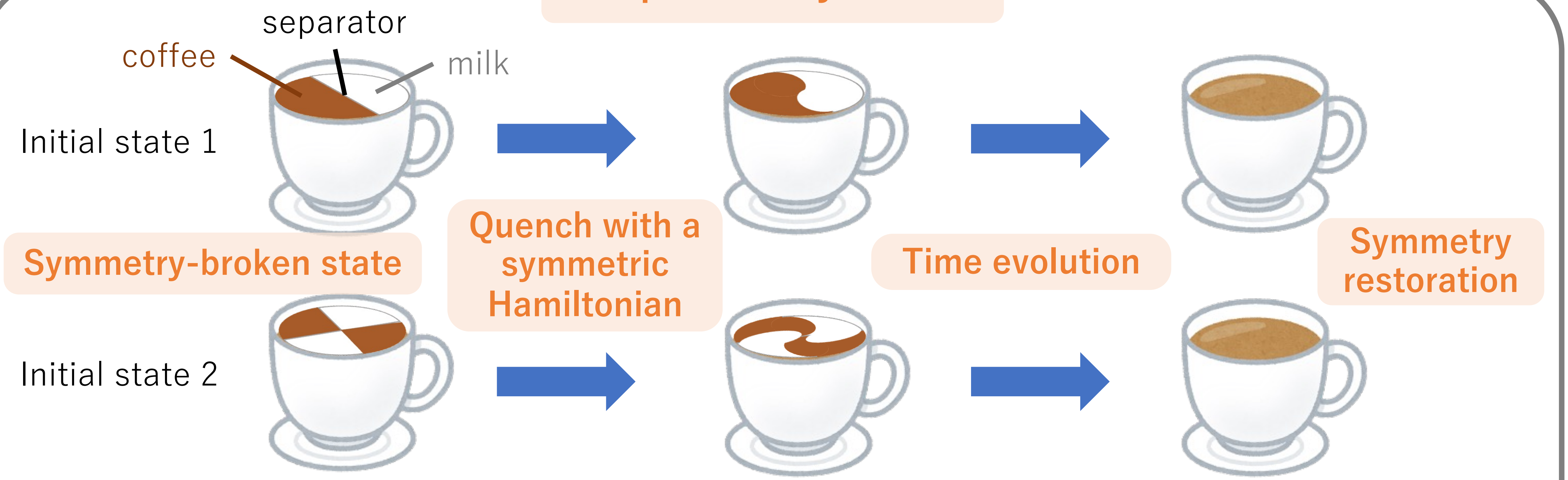
In quantum system…

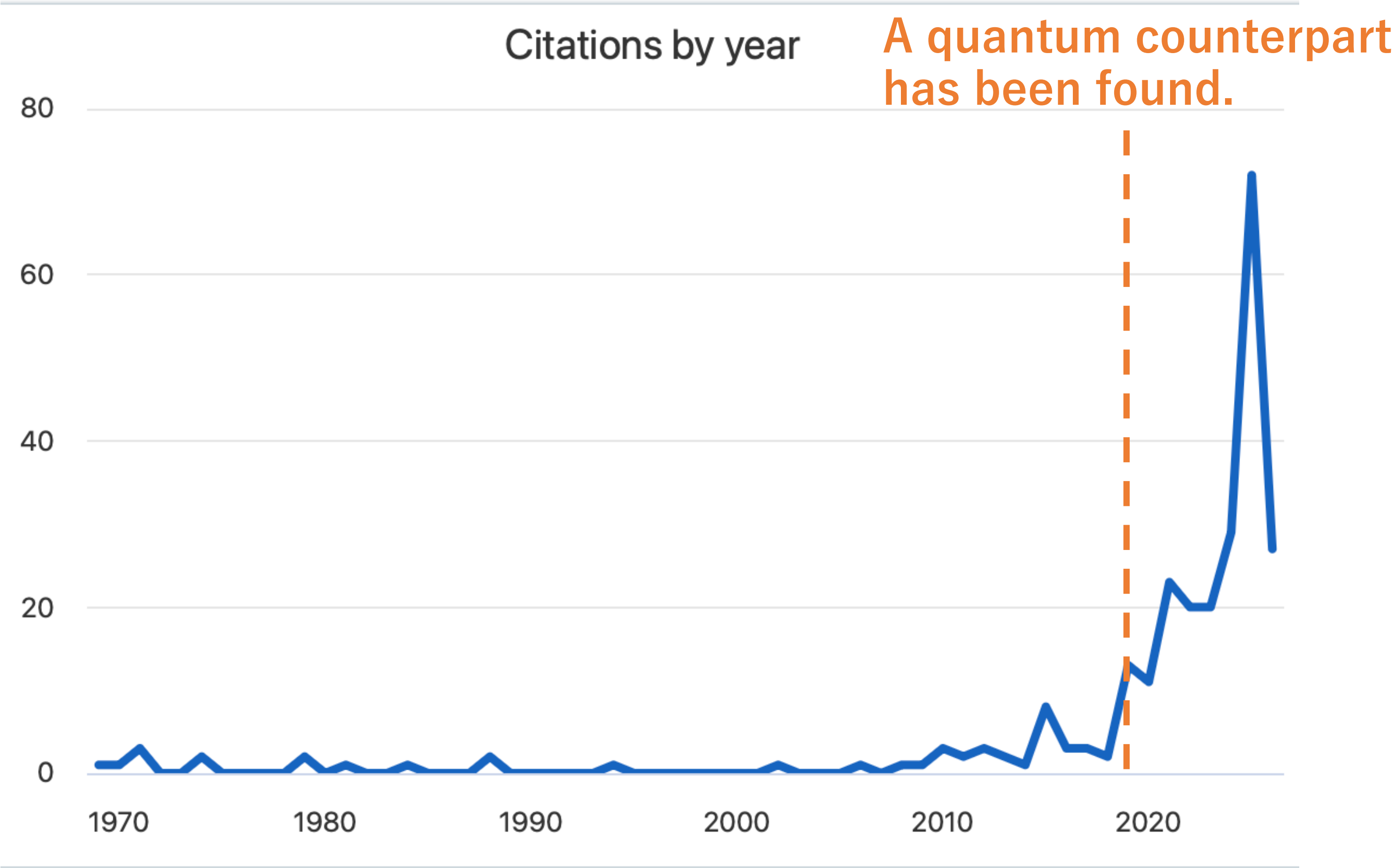


Quantum Mpemba effect

The more symmetry is initially broken, the faster it is restored.

1. Introduction

The Mpemba effect was first observed in classical systems, but a quantum counterpart has been found.
[E B Mpemba and D G Osborn, 1969]



Research on the quantum Mpemba effect has rapidly expanded in recent years!

1. Introduction

Question: How can we study the quantum Mpemba effect?

Conventionally, an order parameter $\langle \mathcal{O} \rangle$ is used to characterize symmetry breaking.

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However, to study the quantum Mpemba effect, we need a **new "quantifier"** satisfying the following requirements.

Requirements for a quantifier of the quantum Mpemba effect

1. Quantifies the degree of symmetry breaking
2. Well-defined for out-of-equilibrium dynamics
3. Computable

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A new quantifier: Entanglement Asymmetry

[Ares-Murciano-Calabrese, 2022]

1. Introduction

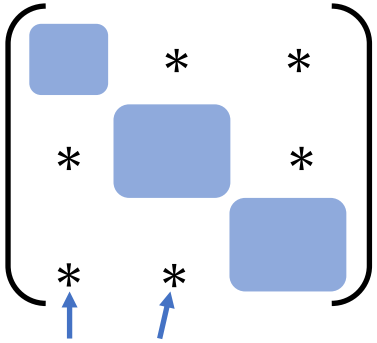
We assume that the theory has a symmetry with charge

$$Q = Q_A + Q_B$$

A = Subsystem of interest, B = Environment.

Symmetry-broken state

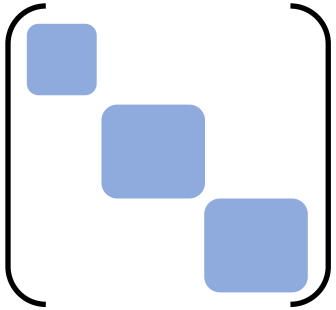
$\rho_A =$



Off-diagonal elements in eigen basis of Q_A

Symmetric state

$\rho_{A,S} =$



Block diagonal

Entanglement Asymmetry: a quantifier of symmetry breaking

[Ares-Murciano-Calabrese, 2022]

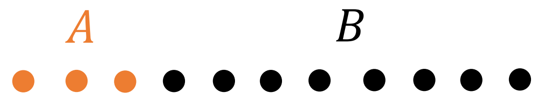
$$\Delta S_A \equiv \Delta S(\rho_A | \rho_{A,S}) = \mathrm{Tr}_A[\rho_A(\log \rho_A - \log \rho_{A,S})]$$

Relative entropy

1. Introduction

Typical protocol to investigate the quantum Mpemba effect.

Quench dynamics with a symmetric Hamiltonian



Spin chain or QFT

Total system= $A \cup B$

A : subsystem of interest

STEP1: Prepare an initial state $|\psi_{AB}(0)\rangle$ that **explicitly** breaks the symmetry.

STEP2: Perform the time evolution.

$$|\psi_{AB}(t)\rangle = e^{-iHt}|\psi_{AB}(0)\rangle,$$ where H is a symmetric Hamiltonian.

STEP3: Compute the entanglement asymmetry at time t.

$$\Delta S_A(t)$$

By this protocol, people has been extensively studied the quantum Mpemba effect in various fields, including condensed matter physics, high-energy physics, and quantum information.

1. Introduction

However, several challenges remain⋯

Challenges

- Computing the entanglement asymmetry beyond CFTs or integrable systems.
- Numerical approaches by classical computer : $\mathcal{O}(2^{N_B} + 2^{2N_A})$ → **exponentially costly**

  $N_{A/B}$: the size of subsystem A/B
- Monte Carlo methods suffer from the sign problem in real-time dynamics.
- Tensor-network approaches become inefficient due to the rapid growth of entanglement.

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- Tensor-network approaches become inefficient due to the rapid growth of entanglement.

Large systems, including quantum field theories, are therefore difficult to access…

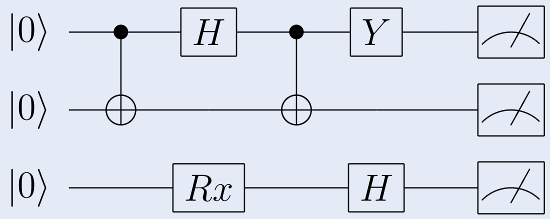
⬇

We need a new efficient method to simulate Entanglement Asymmetry.

1. Introduction

Our approach:

Quantum computing



Simulation

Entanglement Asymmetry

$\Delta S_A(t)$

Quantum Mpemba effect

Short summary of our work

- We propose a scalable quantum algorithm that efficiently compute the entanglement asymmetry.
- As an application, we study Schwinger model and demonstrate that our quantum algorithm can be used to investigate the quantum Mpemba effect in quantum field theories.
- Finally, we estimate the resources required to implement the quantum algorithm and show its scalability.

Outline

1. Introduction

2. Quantum Algorithm for Entanglement Asymmetry

3. Quantum Mpemba Effect in Schwinger Model

4. Summary

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2. Quantum Algorithm for Entanglement Asymmetry

To simplify the analysis, we focus on the Rényi entanglement asymmetry:

$$\Delta S_A^{(n)} \equiv \frac{1}{n-1}\left(\log \mathrm{Tr}_A[\rho_A^n] - \log \mathrm{Tr}_A\left[\rho_{A,S}^n\right]\right), \quad \lim_{n\to 1} \Delta S_A^{(n)} = \Delta S_A$$

For concreteness, we consider $n = 2$.

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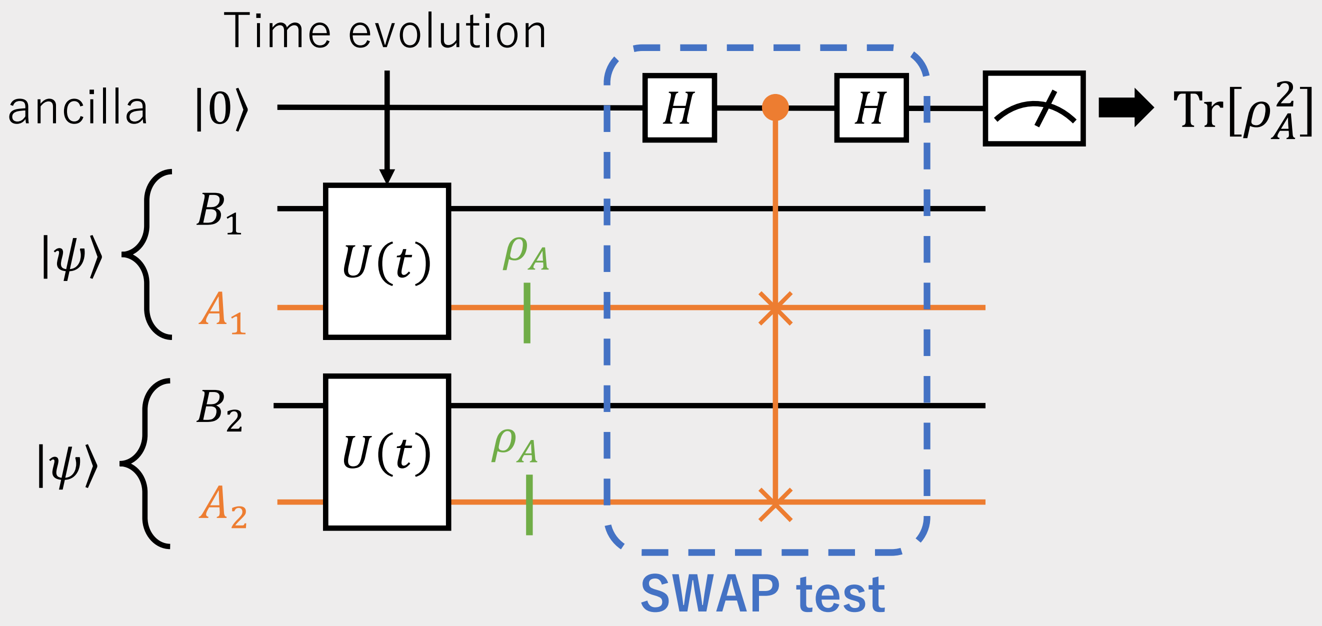
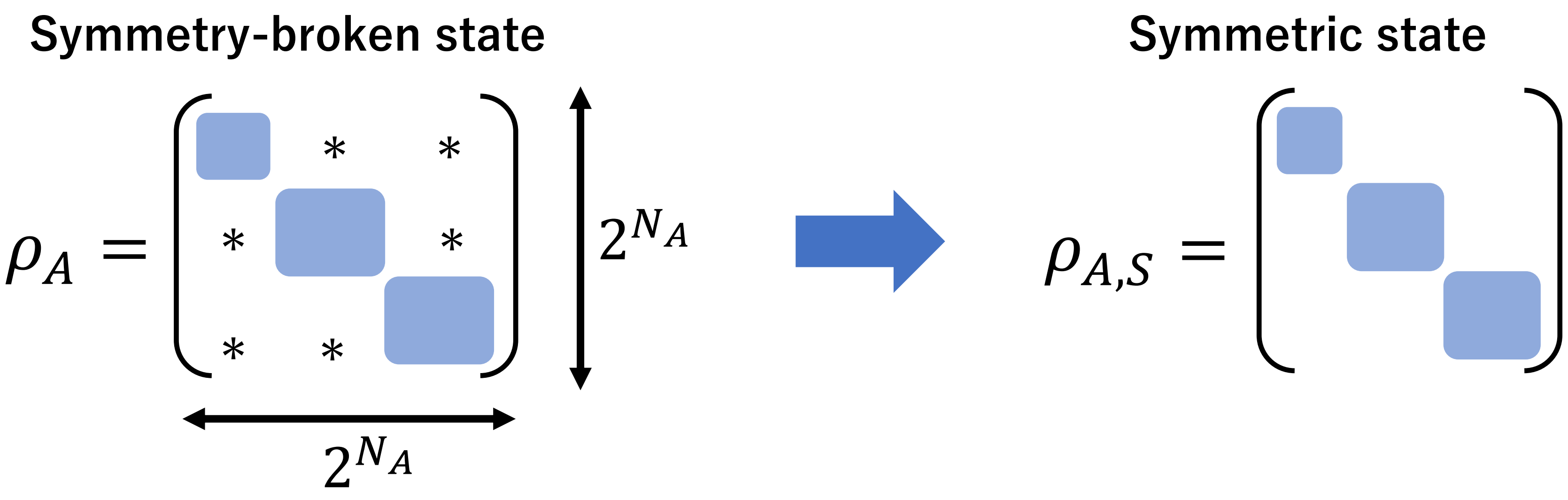
Quantum circuit for computing $\mathrm{Tr}_A[\rho_A^2]$



How can we compute $\mathrm{Tr}_A[\rho_{A,S}^2]$?

(Our work)

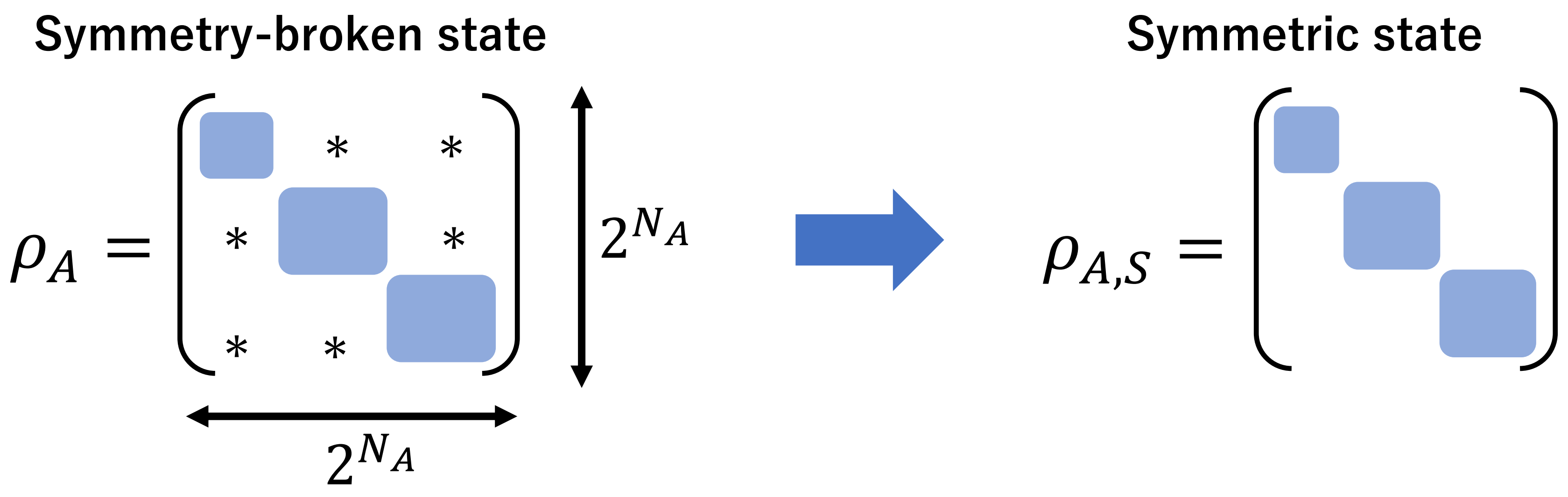
2. Quantum Algorithm for Entanglement Asymmetry



Naively, one need to remove $\mathcal{O}(2^{2N_A})$ off-diagonal elements.

Question: How can we realize this efficiently in a quantum circuit?

2. Quantum Algorithm for Entanglement Asymmetry



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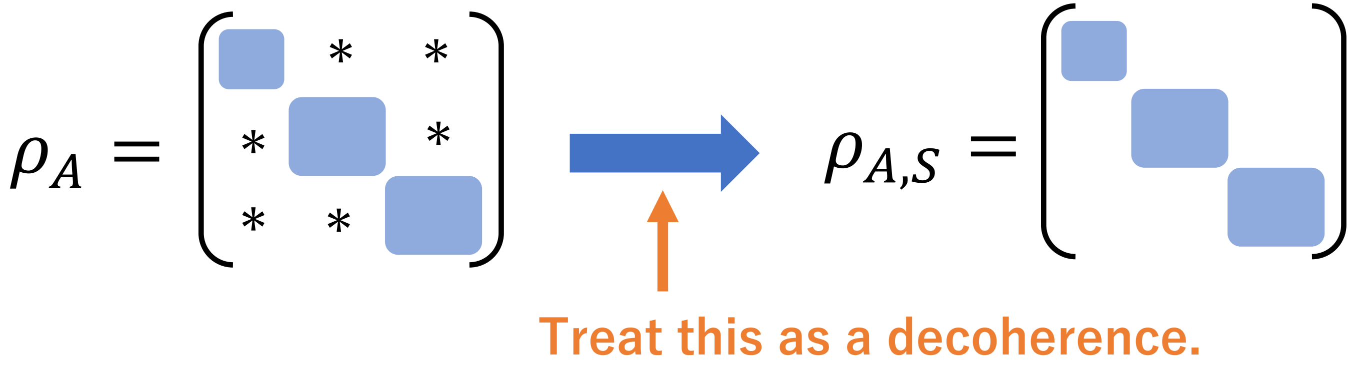
Our idea: Reformulating it as "decoherence"

2. Quantum Algorithm for Entanglement Asymmetry

Decoherence: the loss of quantum coherence due to interactions with environment.



, in eigen basis of Q_A

2. Quantum Algorithm for Entanglement Asymmetry

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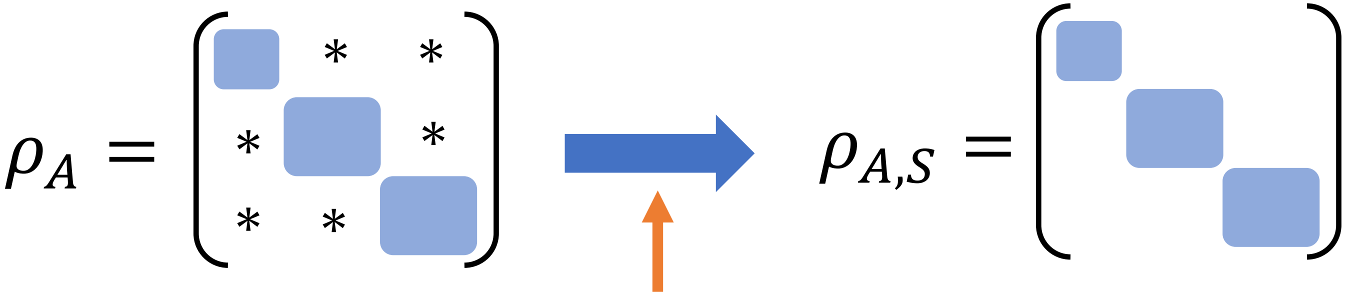
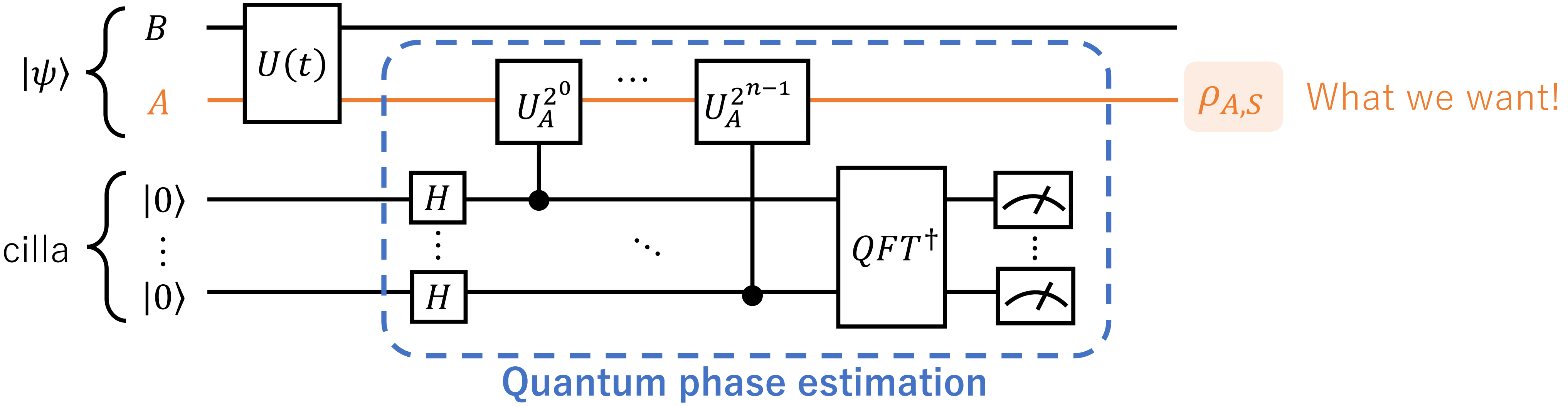


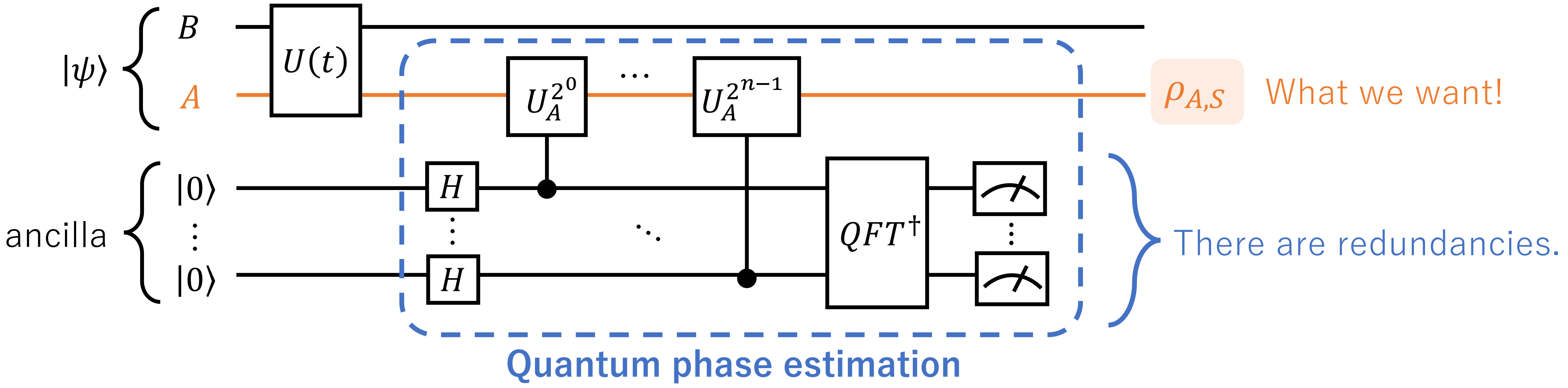
, in eigen basis of Q_A

We implement this decoherence by employing a **quantum phase estimation** of $U_A = e^{2\pi i Q_A/2^n}$, $n > \log_2 N_A$



$\rho_{A,S}$ can be prepared in a quantum circuit!

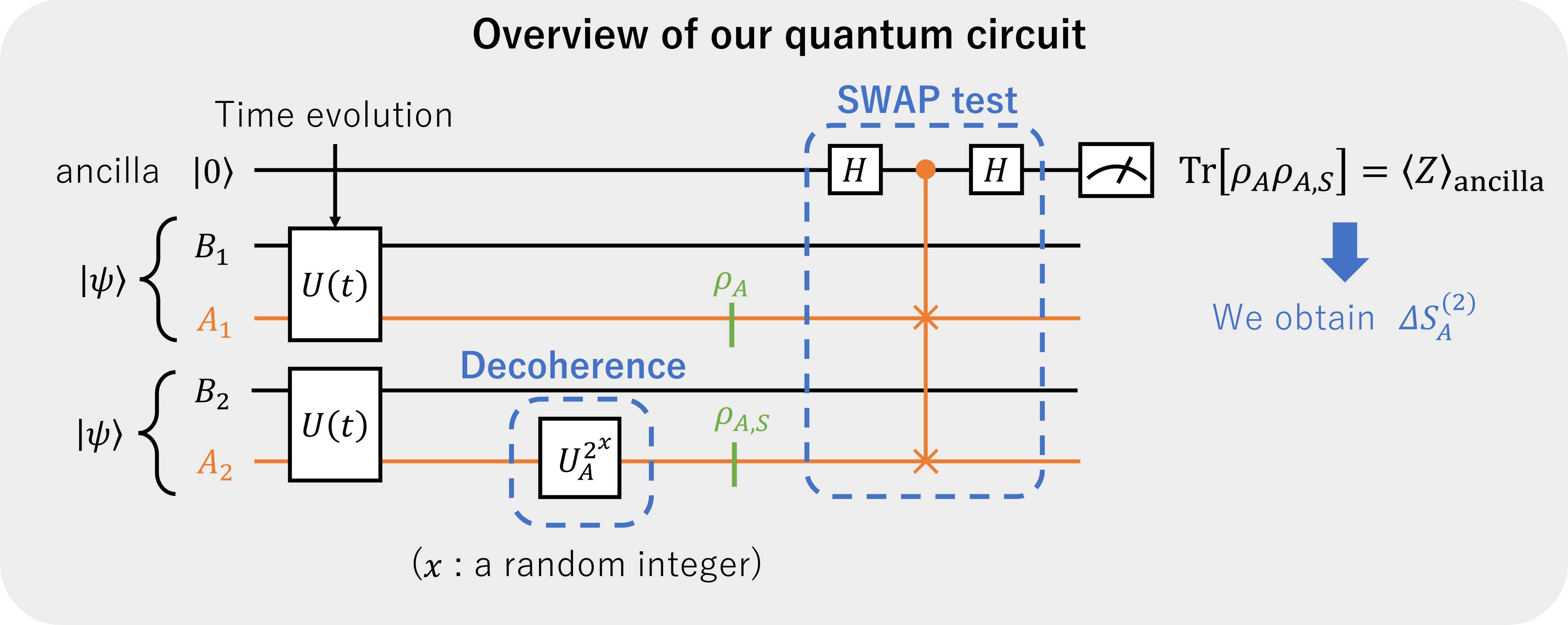
2. Quantum Algorithm for Entanglement Asymmetry



simplify

We can drastically simplify the quantum circuit.

2. Quantum Algorithm for Entanglement Asymmetry



By combining the idea of quantum phase estimation with SWAP test, we can estimate the Rényi Entanglement Asymmetry.

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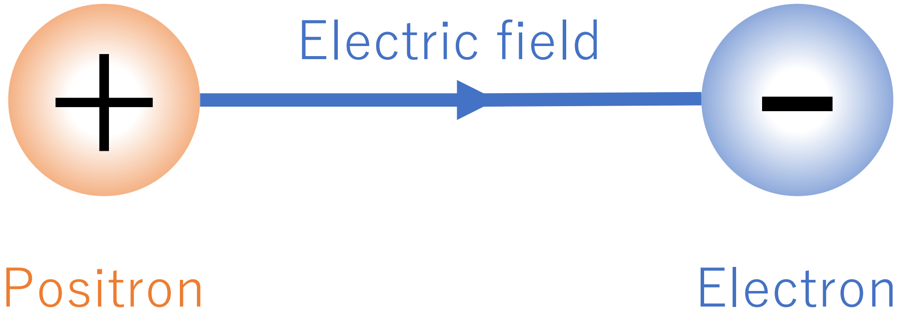
3. Quantum Mpemba Effect in Schwinger Model

Schwinger model in the continuum [Schwinger, 1962]

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}e^{i\theta\gamma^5}\psi$$

||

The simplest model of quantum electrodynamics in (1+1) dimensions



3. Quantum Mpemba Effect in Schwinger Model

Schwinger model in the continuum [Schwinger, 1962]

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + \mathrm{i}gA_{\mu})\psi - m\bar{\psi}e^{\mathrm{i}\theta\gamma^5}\psi$$

Temporal gauge $A_0 = 0$

Kogut-Susskind formalism

Open boundary condition

Gauss law constraint

Jordan-Wigner transformation

Schwinger model as a spin model [Masazumi Honda et al, 2022]

$$H = H_{ZZ} + H_{\pm} + H_Z$$

$$H_{ZZ} \sim \sum_n \sum_{1\le k<\ell\le n} Z_k Z_\ell \ , \quad H_{\pm} \sim \sum_n (X_n X_{n+1} + Y_n Y_{n+1}) \ , \quad H_Z = \sum_n Z_n$$

X_n, Y_n, Z_n : Pauli matrices on the n-th site.

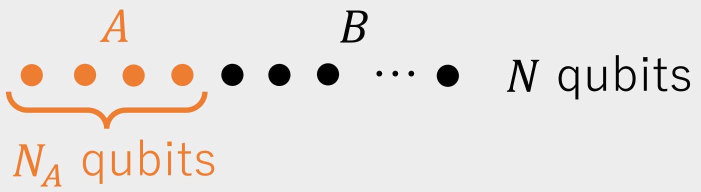
3. Quantum Mpemba Effect in Schwinger Model

Our setup:

Symmetry

$U(1)$ symmetry, $Q = \frac{1}{2}\sum_n Z_n$

Configuration of subsystem



The procedure of our simulation:

STEP1: Prepare the following initial state.

$$|\phi\rangle = e^{-\mathrm{i}\frac{\phi}{2}\sum_n Y_n Y_{n+1}}|\uparrow\uparrow\cdots\uparrow\rangle$$, ϕ : Parameter of initial symmetry breaking.

STEP2: Perform the time evolution by second order Trotterization

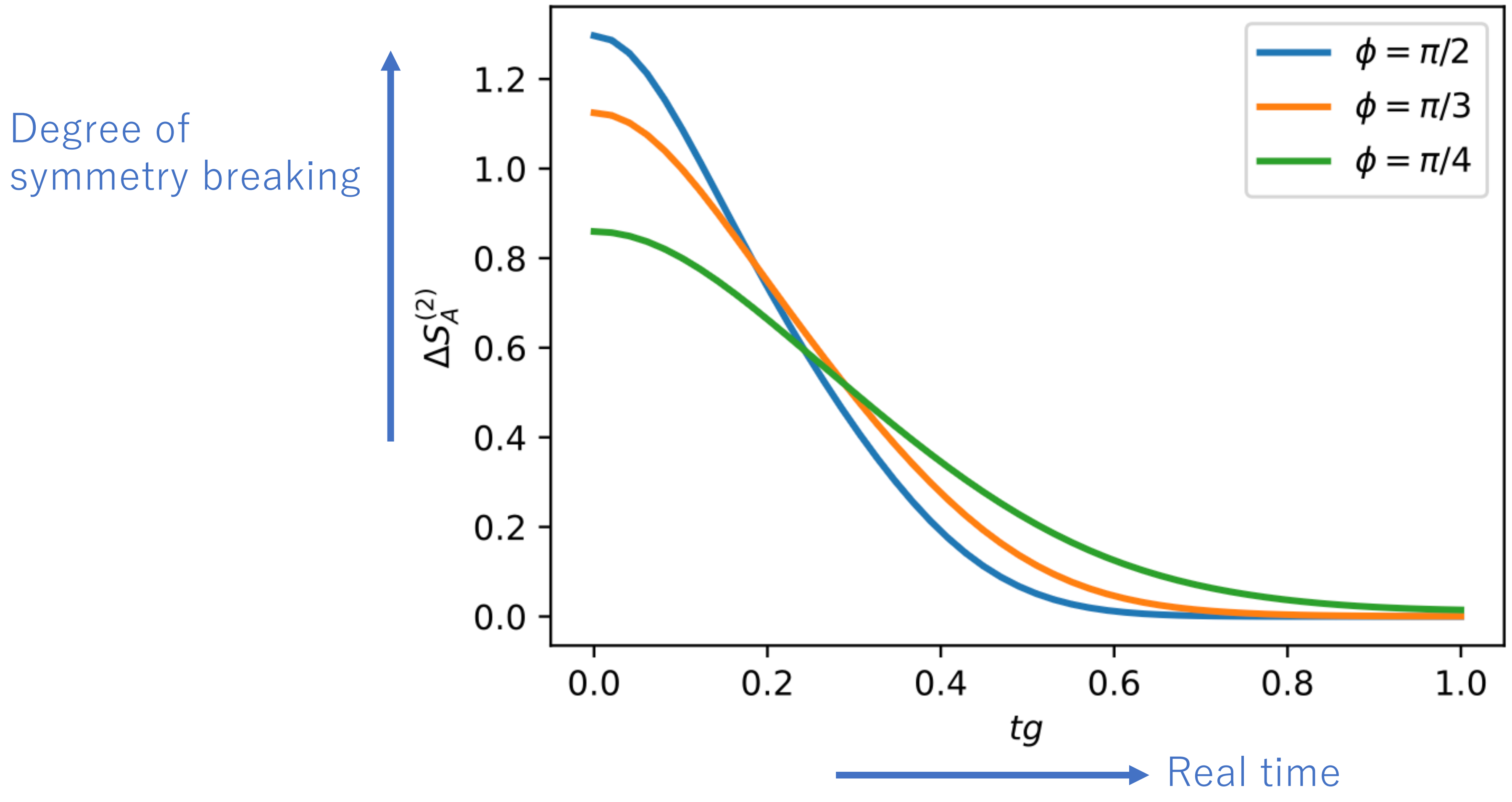
$$H = H_{ZZ} + H_{XY} + H_Z, \qquad H_{XY} = H_{XY,\mathrm{even}} + H_{XY,\mathrm{odd}}$$

STEP3: Compute $\Delta S_A^{(2)}(t)$ which is obtained by our algorithm.

We performed above quantum simulations using Qulacs, a Python library for classical simulation.

3. Quantum Mpemba Effect in Schwinger Model

An example of our result:



parameters

$N = 18, N_A = 4$
$m/g = 4, \theta = \pi/3$
$ag = 0.25, \delta t = 0.03$
(# of Trotter step) $= 50$

3. Quantum Mpemba Effect in Schwinger Model

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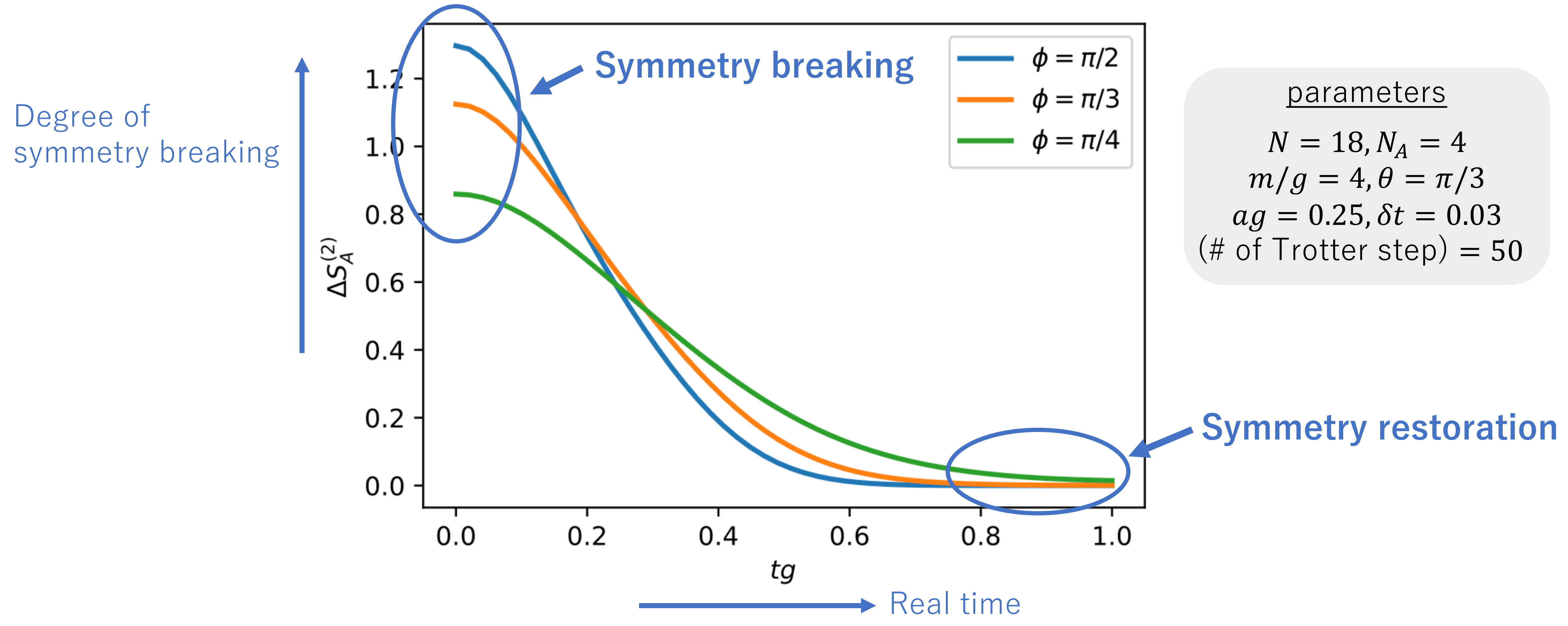

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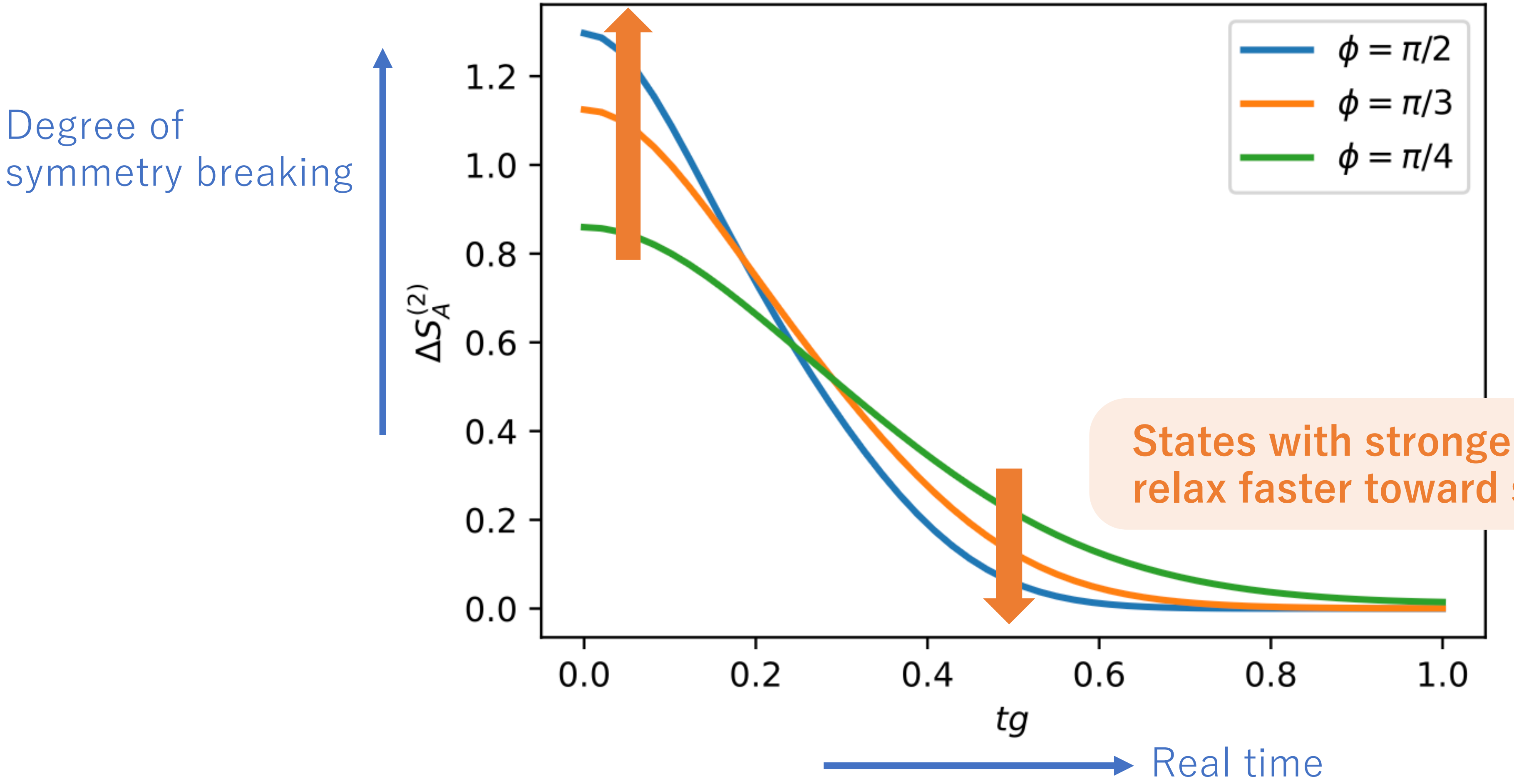

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$N = 18, N_A = 4$
$m/g = 4, \theta = \pi/3$
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$(\# \text{ of Trotter step}) = 50$

States with stronger symmetry breaking relax faster toward symmetry restoration.

Using our quantum algorithm, we demonstrate the quantum Mpemba effect in the Schwinger model.

3. Quantum Mpemba Effect in Schwinger Model

Resource estimation of our algorithm

There are two errors:

Statistical error of $\langle Z \rangle_{\text{ancilla}}$ in SWAP test

Required number of measurements N_{shot}:

$N_{\text{shot}} = \mathcal{O}(\delta^{-2})$, where δ is standard deviation

Does not depend on system size!

Systematic error of Trotterization

There is an error $\epsilon = \left\| e^{iHt} - U^M(t/M) \right\|$, where $U(t/M)$ is one step of Trotterization.

Required number of gates to achieve an error tolerance ϵ:

[A. M. Childs et al, 2021]

(gate complexity) $\leq \mathcal{O}\left(N^{4+1/p} t^{1+1/p} \epsilon^{-1/p}\right)$ Theoretical upper bound

p: order of Trotterization

No exponential cost!

Our quantum algorithm is scalable and applicable to large systems.

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Summary

- In this work, we propose an efficient quantum algorithm for computing the entanglement asymmetry.
- We demonstrate the quantum Mpemba effect in the Schwinger model.
- The required number of measurements is independent of the system size, and the gate complexity scales polynomially ⇒ Our algorithm is suitable for large systems.
- **The quantum Mpemba effect is one of the promising application of quantum computing!**

Future directions

- Estimating feasibility in early FTQC era ⇒ the viewpoint of the STAR architecture(Ongoing).
- Implementation on current Noisy Intermediate-Scale Quantum(NISQ) devices(Ongoing).
- Application to other models(our algorithm works for general quantum theories)

Appendix

An example of decoherence

Decoherence: the loss of quantum coherence due to interactions with environment.

Simple example of decoherence

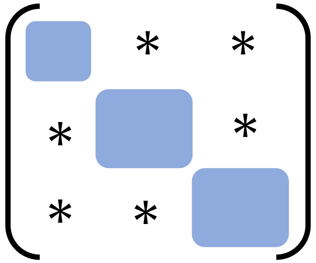
State: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ → **Measurement of Z** → State: $\begin{cases} |0\rangle \text{ with probability } |\alpha|^2 \\ |1\rangle \text{ with probability } |\beta|^2 \end{cases}$

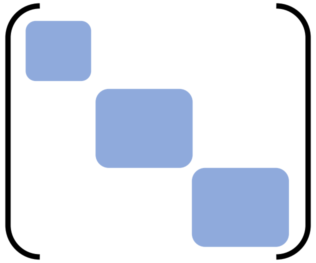
$$\rho = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \quad \Rightarrow \quad \rho = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$$

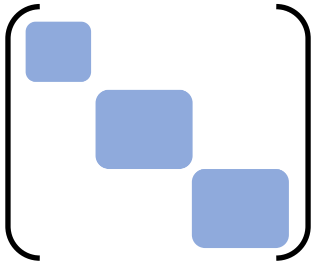
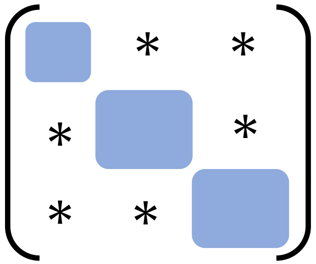
in eigen basis of Z

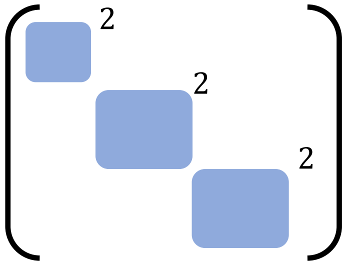
Measurement (or entanglement with the environment) removes the off-diagonal element.

Technical detail

$\rho_A =$ 

$\rho_{A,S} =$  , in eigenbasis of Q_A

$\mathrm{Tr}[\rho_A \rho_{A,S}] =$ Tr  

$=$ Tr 

$= \mathrm{Tr}[\rho_{A,S}^2]$

In general, the following identity holds

$$\mathrm{Tr}[\rho_A \rho_{A,S}^{n-1}] = \mathrm{Tr}[\rho_{A,S}^n]$$

Detail of Hamiltonian of Schwinger model

Schwinger model as a spin model

$$H = H_{ZZ} + H_{\pm} + H_Z$$

$$H_{ZZ} = \frac{J}{2}\sum_{n=2}^{N-1}\sum_{1\le k<\ell\le n} Z_k Z_\ell \ , \quad H_{\pm} = \frac{1}{2}\sum_{n=1}^{N-1}\left(w - (-1)^n \frac{m}{2}\sin\theta\right)(X_n X_{n+1} + Y_n Y_{n+1})$$

$$H_Z = \frac{m\cos\theta}{2}\sum_{n=1}^{N}(-1)^n Z_n - \frac{J}{2}\sum_{n=1}^{N-1}(n \bmod 2)\sum_{\ell}^{n} Z_\ell \quad , \quad J = \frac{g^2 a}{2}, w = \frac{1}{2a}$$

a : lattice spacing

Initial state and EA

Symmetry

$U(1)$ symmetry, $Q = \frac{1}{2}\sum_n Z_n$

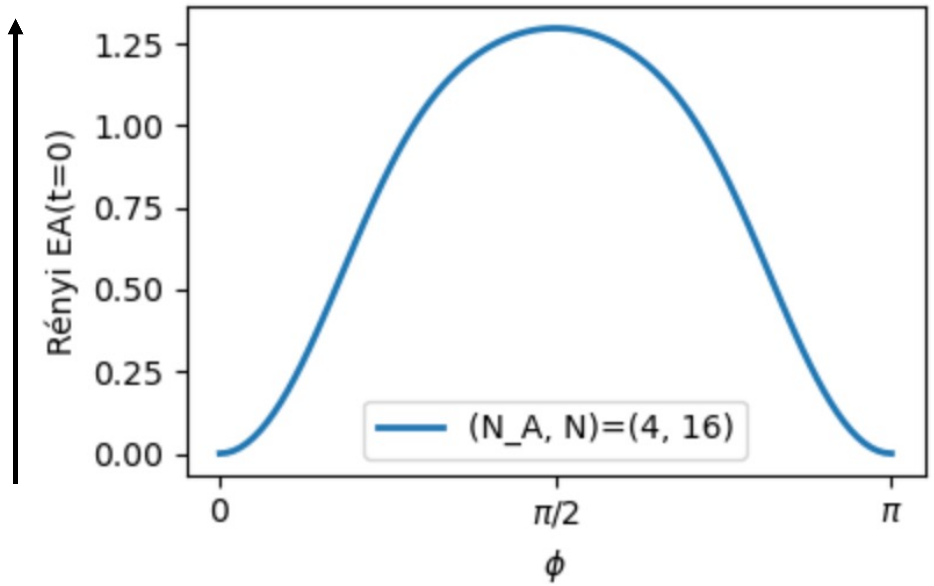
: Rotational symmetry around Z axis

Initial state

$$|\phi\rangle = e^{-\mathrm{i}\frac{\phi}{2}\sum_n Y_n Y_{n+1}} |\uparrow\uparrow\cdots\uparrow\rangle$$

ϕ : Parameter of initial symmetry breaking.

Degree of symmetry breaking at $t = 0$



Initial angle ϕ

$\Delta S_A^{(2)}(t)$ takes maximal value at $\phi = \pi/2$ and increasing function for $\phi \in \left[0, \frac{\pi}{2}\right]$