Alternative renormalization schemes/prescriptions

1. RI/SMOM scheme
2. Coordinate-space/Gauge-invariant scheme (GIRS)
3. Wilson/gradient flow

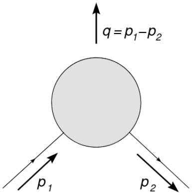
Extended Twisted Mass Collaboration

ETMC Meeting, Cyprus, 26 February 2026

RI/SMOM

- RI/MOM vs RI/SMOM

$$\Lambda_\Gamma(p_1,p_2) = S_u^{-1}(p_1)\langle u(p_1)\mathcal{O}_\Gamma(x)\bar{d}(p_2)\rangle S_d^{-1}(p_2)$$



RI/MOM: $p_1 = p_2 = p,\ q = p_1 - p_2 = 0,\ p^2 = \mu^2$

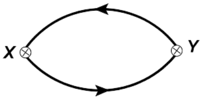
RI/SMOM: $p_1^2 = p_2^2 = q^2 = \mu^2,\ q = p_1 - p_2$

- Advantages of RI/SMOM:
  - Infrared effects are expected to be more suppressed.
  - Conversion factors to the $\overline{\text{MS}}$ scheme converge faster than in the case of RI/MOM
- Disadvantages of RI/SMOM:
  - Conversion factors known only to 2 loops
  - Additional mixing with total derivative operators (if applicable)

Coordinate-space/Gauge-invariant scheme

- Green's functions:

$$\Pi_{\mathcal{O}_\Gamma,\mathcal{O}_\Gamma}(t) = \frac{1}{L^3}\sum_{\vec{x},\vec{y}} \langle \mathcal{O}_\Gamma(\vec{x},x_4)\mathcal{O}_\Gamma^\dagger(\vec{y},y_4)\rangle|_{x_4-y_4=t}, \qquad t \neq 0$$



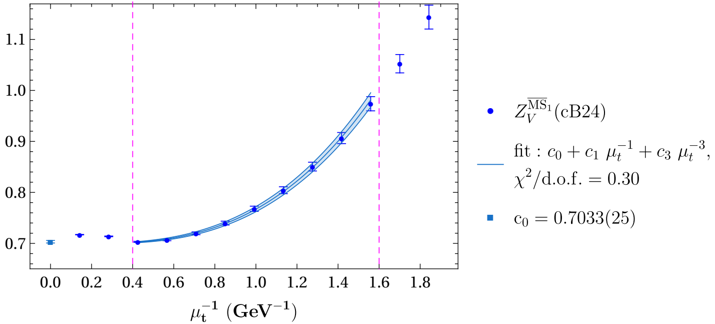
- Renormalization conditions:

$$Z^2_{\mathcal{O}_\Gamma}\ \Pi_{\mathcal{O}_\Gamma,\mathcal{O}_\Gamma}(t)\Big|_{|t|\sim 1/\mu_t} = \Pi^{\text{free}}_{\mathcal{O}_\Gamma,\mathcal{O}_\Gamma}(t)\Big|_{|t|\sim 1/\mu_t},$$

  where $\Lambda_{\text{QCD}} \ll \mu_t \ll a^{-1}$ (renormalization window).
- Good features:
  No need for gauge-fixing, No contact terms, No need to consider mixing with gauge-non-invariant operators, Consider on-shell Green's functions
- Disadvantages:
  Strong non-perturbative effects (as predicted by OPE)

Example Z_V

Work by Cyprus group (C. Alexandrou, G. Spanoudes, S. Yamamoto), $\mu_t \sim 1/t$



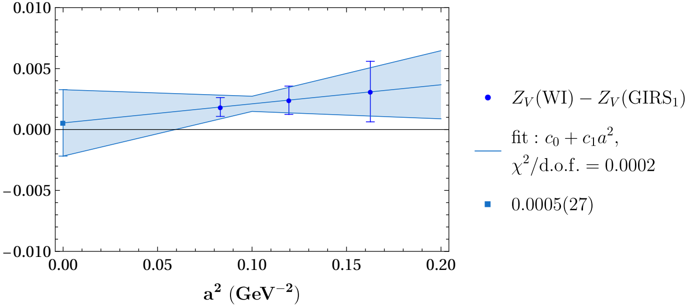
$$\Pi_{VV}^{\mathrm{OPE}}(t) = \Pi_{VV}^{\mathrm{pert}}(t) + \sum_{\mathcal{O}} c_{\mathcal{O}}^{V}(t)\ t^{\dim\mathcal{O}-3}\ \langle\mathcal{O}\rangle,$$

$$\mathcal{O} \in \{m_q\bar{q}q, a_s G^2, m_q g_s \bar{q}Gq, a_s(\bar{q}q)^2, \cdots\}$$

In general $c_{\mathcal{O}}^{V}$ depends on t logarithmically as $\ln(t^2\mu^2)$ and on the quark mass $m_q^2 t^2$.

Example Z_V

Work by Cyprus group (C. Alexandrou, G. Spanoudes, S. Yamamoto)



Four-quark operators

Work by Cyprus+Temple group (M. Constantinou, M. Costa, H. Panagopoulos, G. Spanoudes)

$$\mathcal{O}_{\Gamma\Gamma'}(x) = \bar{\psi}_{f_1}(x)\Gamma\psi_{f_2}(x)\bar{\psi}_{f_3}(x)\Gamma'\psi_{f_4}(x)$$

- Operator mixing in groups

$$\begin{cases} Q_1^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VV} \pm \mathcal{O}^F_{VV}\right] + \frac{1}{2}\left[\mathcal{O}_{AA} \pm \mathcal{O}^F_{AA}\right], \\ Q_2^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VV} \pm \mathcal{O}^F_{VV}\right] - \frac{1}{2}\left[\mathcal{O}_{AA} \pm \mathcal{O}^F_{AA}\right], \\ Q_3^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{SS} \pm \mathcal{O}^F_{SS}\right] - \frac{1}{2}\left[\mathcal{O}_{PP} \pm \mathcal{O}^F_{PP}\right], \\ Q_4^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{SS} \pm \mathcal{O}^F_{SS}\right] + \frac{1}{2}\left[\mathcal{O}_{PP} \pm \mathcal{O}^F_{PP}\right], \\ Q_5^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{TT} \pm \mathcal{O}^F_{TT}\right], \end{cases}$$

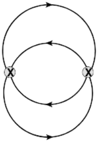
$$\left\{\mathcal{Q}_1^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VA} \pm \mathcal{O}^F_{VA}\right] + \frac{1}{2}\left[\mathcal{O}_{AV} \pm \mathcal{O}^F_{AV}\right],\right.$$

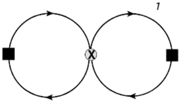
$$\begin{cases} \mathcal{Q}_2^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VA} \pm \mathcal{O}^F_{VA}\right] - \frac{1}{2}\left[\mathcal{O}_{AV} \pm \mathcal{O}^F_{AV}\right], \\ \mathcal{Q}_3^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{PS} \pm \mathcal{O}^F_{PS}\right] - \frac{1}{2}\left[\mathcal{O}_{SP} \pm \mathcal{O}^F_{SP}\right], \end{cases}$$

$$\begin{cases} \mathcal{Q}_4^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{PS} \pm \mathcal{O}^F_{PS}\right] + \frac{1}{2}\left[\mathcal{O}_{SP} \pm \mathcal{O}^F_{SP}\right], \\ \mathcal{Q}_5^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{T\tilde{T}} \pm \mathcal{O}^F_{T\tilde{T}}\right]. \end{cases}$$

- GIRS renormalization conditions involve two-point and three-point functions [M. Constantinou et al., PRD 110 (2024) 074506, G. Spanoudes et al., PRD 113 (2026) 034504]

$$\int d^3\vec{z}\ \left\langle Q_i^{\pm}(x+z)\left(Q_j^{\pm}(x)\right)^{\dagger}\right\rangle, \qquad \int d^3\vec{z}\int d^3\vec{z}'\ \left\langle \mathcal{O}_{\Gamma}(x+z)\ Q_i^{\pm}(x)\ \mathcal{O}_{\Gamma}(x-z')\right\rangle$$





Wilson/gradient flow I

Slide from talk of Nico Battelli, Stefan Sint at Lattice 2021

Renormalization of composite operators

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Given such composite operators

$$\mathcal{O}_\Gamma^{ij}(x) = \overline{\psi}_i(x)\Gamma\psi_j(x), \quad (i,j\text{: flavour indices})$$
$$\mathcal{Q}_\Gamma^{ij}(t,x) = \overline{\chi}_i(t,x)\Gamma\chi_j(t,x), \quad (\Gamma\text{: some gamma matrix})$$

the renormalization for the flowed one is obtained by counting the number of fermionic insertions ($Z_\chi^{-1/2}$ for each fermion field) while the non-flowed renormalizes with the appropriate factor

$$\left(\mathcal{O}_\Gamma^{ij}(x)\right)_R = Z_{\mathcal{O}_\Gamma}\mathcal{O}_\Gamma^{ij}(x) \qquad \left(\mathcal{Q}_\Gamma^{ij}(t,x)\right)_R = Z_\chi \mathcal{Q}_\Gamma^{ij}(t,x).$$

Then we can define the renormalization constant $Z_{\mathcal{O}_\Gamma}$ for every non-flowed operator in the following ways (**wave function renormalization cancellation**)

$$i) \qquad Z_{\mathcal{O}_\Gamma}\frac{t^{3/2}\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_1}^{ij}(t,x)\mathcal{O}_{\Gamma_1}^{ji}(y)\rangle}{\sqrt{t^{3/2}\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_2}^{ij}(t,x)\mathcal{Q}_{\Gamma_2}^{ji}(t,y)\rangle}}\Bigg|_{\mathbf{y}=\mathbf{0}} \equiv \frac{t^{3/2}\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_1}^{ij}(t,x)\mathcal{O}_{\Gamma_1}^{ji}(y)\rangle}{\sqrt{t^{3/2}\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_2}^{ij}(t,x)\mathcal{Q}_{\Gamma_2}^{ji}(t,y)\rangle}}\Bigg|_{\mathbf{y}=\mathbf{0},\ \text{tree level}},$$

$$ii) \qquad Z_{\mathcal{O}_\Gamma}\frac{\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_1}^{ij}(t,x)\mathcal{O}_{\Gamma_1}^{ji}(y)\rangle}{\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_2}^{ij}(t,x)\mathcal{Q}_{\Gamma_2}^{ji}(0,y)\rangle}\Bigg|_{\mathbf{y}=\mathbf{0}} \equiv \frac{\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_1}^{ij}(t,x)\mathcal{O}_{\Gamma_1}^{ji}(y)\rangle}{\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_2}^{ij}(t,x)\mathcal{Q}_{\Gamma_2}^{ji}(0,y)\rangle}\Bigg|_{\mathbf{y}=\mathbf{0},\ \text{tree level}},$$

- $Z_{\mathcal{O}_\Gamma}$ is equal to 1 at tree level in the continuum limit,
- Because of the Gaussian damping factor $\sim e^{-tp^2}$ in the propagator due to the flow, we have no finiteness problem at $x = y$,
- Fixed values of t, x_0 and y_0 correspond to different renormalization conditions.

Wilson/gradient flow II

- **Main idea** [M. Lüscher, JHEP 08 (2010) 071, JHEP 04 (2013) 123]

  • Introduce a fictitious "flow time" t (units of length^2)

  • Evolve gauge and fermion fields as a function of t, according to gradient flow equations

  • The flow time acts as a UV regulator and smooths out the fields over region $\sim \sqrt{8t}$

  • Flowed gauge field does not need renormalization [M. Lüscher, P. Weisz, JHEP 02 (2011) 051]

  • Flowed quark field needs a multiplicative renormalization [M. Lüscher, P. Weisz, JHEP 02 (2011) 051]

  • Flowed gluon operators are finite. Flowed quark bilinear operators have identical renormalization factors, regardless of their Dirac structure or the inclusion of covariant derivatives.

  • No further renormalization? (Finite regularization-dependent mixing effects allowed by the symmetries of lattice actions?)

  • We can take the continuum limit ($a \to 0$) at fixed t (after renormalizing flowed quark fields). Divergences in t can be removed by continuum perturbation theory.

Wilson/gradient flow II

- **Procedure** E.g. twist-2 quark bilinear operators [A. Shindler, PRD 110 (2024) L051503]
  1. Calculate matrix elements of flowed operators in lattice QCD

$$\langle N|\bar{\chi}(x,t)\Gamma\overleftrightarrow{D}\cdots\overleftrightarrow{D}\chi(x,t)|N\rangle$$

  2. Renormalize the flowed quark fields of the operator by using “ringed” fields. [H. Makino, H. Suzuki, PTEP 2014 (2014) 063B02]

$$\left\langle \mathring{\bar{\chi}}_r(x,t)\overleftrightarrow{\not{D}}\mathring{\chi}_r(x,t)\right\rangle = -\frac{N_c}{(4\pi)^2t^2}\,.$$

  Alternatively: Take ratios of the matrix elements of two different quark bilinear operators in order to cancel the renormalization factor[1].
  3. Take the continuum limit $a \to 0$ at fixed t.
  4. Match to the $\overline{\rm MS}$ scheme at a reference scale μ and zero flow time $t = 0$ using the short-flow time expansion (SFTX) [M. Lüscher, PoS LATTICE2013 (2014) 016].
  5. Extrapolate to $t \to 0$ to remove any residual dependence on t.

[1] In this case, we need to use as an input the one of the two matrix elements renormalized in $\overline{\rm MS}$ using the conventional lattice QCD

THANK YOU FOR YOUR ATTENTION!

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