Multipartite Entanglement

Lecture notes

§ Motivation, introduction

§ Central charges and others

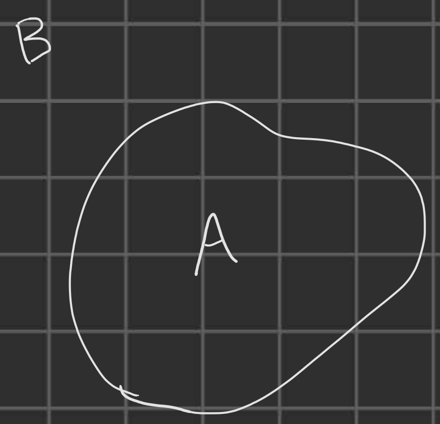
§ Multipartite entanglement quantities

§ "Caculations"

§ Motivation 1

"Entanglement probes" for phases of matter

e.g. entanglement entropy.



$|\psi\rangle$: many body wfn

$$\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$$

$$S_A = -\mathrm{Tr}\, \rho_A \ln \rho_A .$$

Different phases $\overset{?}{\leftrightarrow}$ different behaviors of S_A

§. Topological EE

e.g. $|\psi\rangle$: Topologically ordered

GS in (2+1) d.

✓ gapped GS / phase

✓ No SSB, no LRO

✓ Support anyons

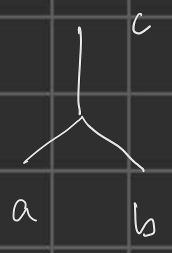
✓ Unitary modular tensor Category (UMTC)

✓ other data

anyons $a, b, c \ldots$ $\{a\}$

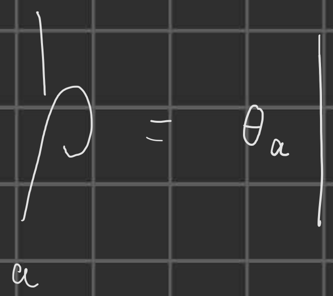
fusion rule $a \times b = \sum_c N_{ab}^c \, c$

quantum dim $\{d_a\}$



Total q-dim $D = \sqrt{\sum_a d_a^2}$

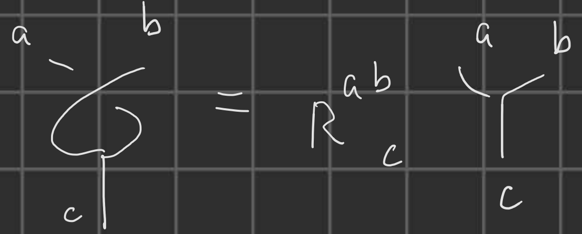
Topological spin $\{\theta_a\}$



$$\theta_a = e^{2\pi i h_a}$$

fusion category

Braiding



$$R^{ab}_c = e^{\pi i (h_c - h_a - h_b)}$$

braided fusion

modular S-matrix

$$S_{ab} = \frac{1}{D} \sum_c N_{ab}^c \frac{\theta_c}{\theta_a \theta_b} d_c$$

braided + spherical

Fusion Category

Braided fusion	+ braiding
Pivotal fusion	$X \cong X^{**}$
Spherical fusion	pivotal + $tr_L = tr_R$
premodular / ribbon	braided + spherical
modular tensor	premodular + non degenerate S
Unitary fusion	unitary

non degenerate S matrix

modular $\iff$ $\det S \neq 0$

$\longrightarrow$ $SL(2,\mathbb{Z})$ 表現の出現

unitary

braiding が量子力学の unitary evolution として実現される.

$$\rho_A = \mathrm{Tr}_{\bar{A}} |\psi\rangle\langle\psi|$$

$$S_n(A) \equiv \frac{1}{1-n} \mathrm{Tr}\, \rho_A^n \quad \text{or} \quad \ln \mathrm{Tr}\, \rho_A^n$$

$$S(A) = \underbrace{\text{const.}\, l}_{\text{Area law}} - \underbrace{\gamma}_{\text{Topo. EE}}$$

$$\gamma = \ln D = \ln \sqrt{\sum_a d_a^2} > 0$$

"Smoking gun of TO"

§ Motivation 2

- TEE does not distinguish different TO

E.g. $\mathbb{Z}_2$ spin liquid TO, Toric code

$\updownarrow$

Ising TO.

$$D = \sqrt{1+1+1+1} = 2$$

$$= \sqrt{1+\sqrt{2}^2+1} = 2$$

- Finer entanglement probe?

§. More recent results

Multipartite quantities

✓ multi-entropy ungappable

total central charge

✓ modular commutator

chiral central charge

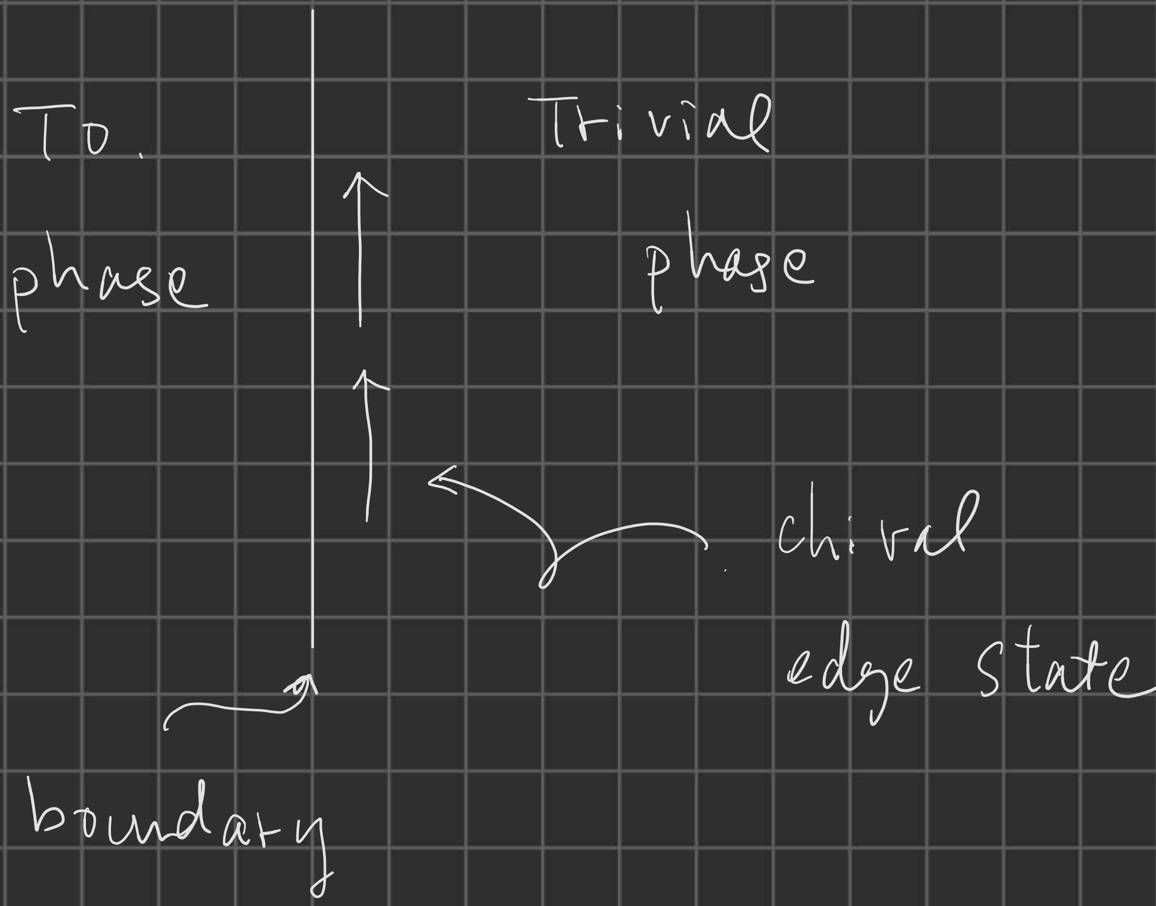
✓ lense-space multi-entropy

higher central charge

§ Central charges

① <u>chiral central charge</u>.

✓ Defined in terms of <u>edge</u>



$$\kappa / T = \frac{\pi^2 k_B^2}{3h} \times c_-$$

$6\hbar$

thermal conductance

- non chiral edge

$$c_- = c_L - c_R$$

- c_- : gravitational anomaly $\quad \partial_\mu T^{\mu\nu} \neq 0$

$c_- \neq 0$ ungappable

✓ Chiral central charge is an additional top. inv. to UMTC

✓ But related to bulk thru Gauss - Milgram (Gauss Sum)

$$\frac{1}{D} \sum_a d_a^2 \theta_a = e^{2\pi i c_-/8}$$

("def" of c_- mod 8)

✓ UMTC knows c_- only

up to 8 c.f

E_8 state $c_- = 8$ (no anyon)

can be stacked

✓ Another (?) connection to

c.f. bulk.

framing anomaly

$$Z(M) \longrightarrow Z(M)\, e^{2\pi i\, c_-/24}$$

② gappable edge

✓ non-chiral edge can be $\overbrace{\text{gapped}}^{\text{fully or partially}}$

✓ partially gapping

$$C_{Tot} = C_L + C_R \longrightarrow C'$$

$$C_- = C_L - C_R \quad \text{unchanged}$$

✓ Some non-chiral edge state cannot be gapped (although non-chiral) e.g. Abelian

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} \partial_t \phi^I \partial_x \phi^J + \cdots$$

$\exists \{ l_a \}_{a=1}^{N}$ integer vectors

$$l_a^T K^{-1} l_b = \text{integer} \quad \forall a, b$$

③ Higher central charge

$$\zeta_n \equiv \frac{\sum_a d_a^2 \theta_a^n}{\left| \sum_a d_a^2 \theta_a^n \right|}$$

$$n = 1 \rightarrow \zeta_1 = e^{\frac{2\pi i}{8} c_-}$$

✓ gappability

gappable $\Rightarrow$ $\zeta_n = 1$,

$\forall n$ with $\gcd(n, N_{FS}) = 1$

i.e. $\exists n$, $\gcd(n, N_{FS}) = 1$, $\zeta_n \neq 1$ $\Rightarrow$ ungappable

For abelian

gappable $\Leftrightarrow$ $\zeta_n = 1$,

$\forall n$ with $\gcd(n, N_{FS}) = 1$

Multi entropy "measures"

$$|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$$

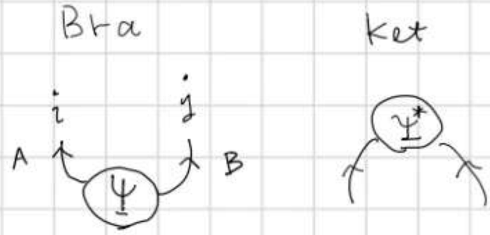
$$\langle\Psi|^{\otimes R} \; \pi_A \; \pi_B \; \pi_C \; |\Psi\rangle^{\otimes R}$$

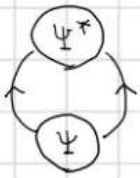
permutations

Wavefunction lego

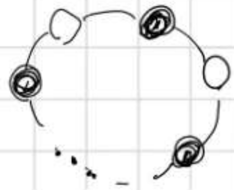
Bipartition

$$|\Psi\rangle = \sum_{ij} \Psi_{ij} |i\rangle_A |j\rangle_B$$



$\langle \Psi | \Psi \rangle =$  $=$ 

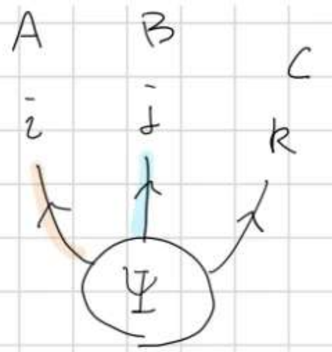
$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| =$  $=$ 

$\mathrm{Tr}_A\, \rho_A^n =$ 

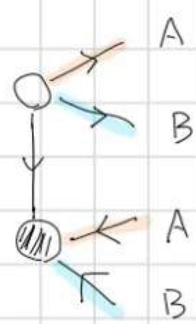
$$S_A = \lim_{n \to 1} \frac{1}{1-n} \mathrm{Tr}_A\, \rho_A^n$$

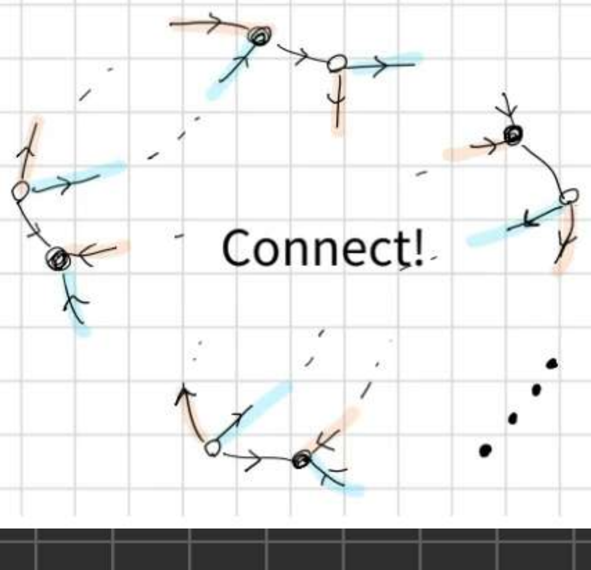
Tripartition

$$|\Psi\rangle = \sum_{ij} \Psi_{ijk} |i\rangle_A |j\rangle_B |k\rangle_C$$

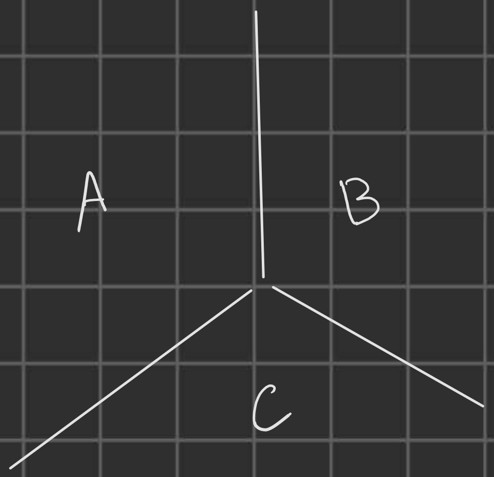


$\rho_{AB} = \mathrm{Tr}_C |\Psi\rangle\langle\Psi|$

$=$ 



Multi entropy $\quad S_n^{(q)}$



$$S_{n=2}^{(q=3)} \equiv \frac{1}{1-n} \frac{1}{n^{q-2}} \ln Z(A:B:C)$$

$$Z(A:B:C) \equiv \langle \Psi^{\otimes 4} | \Pi_A \Pi_B \Pi_C | \Psi^{\otimes 4} \rangle$$

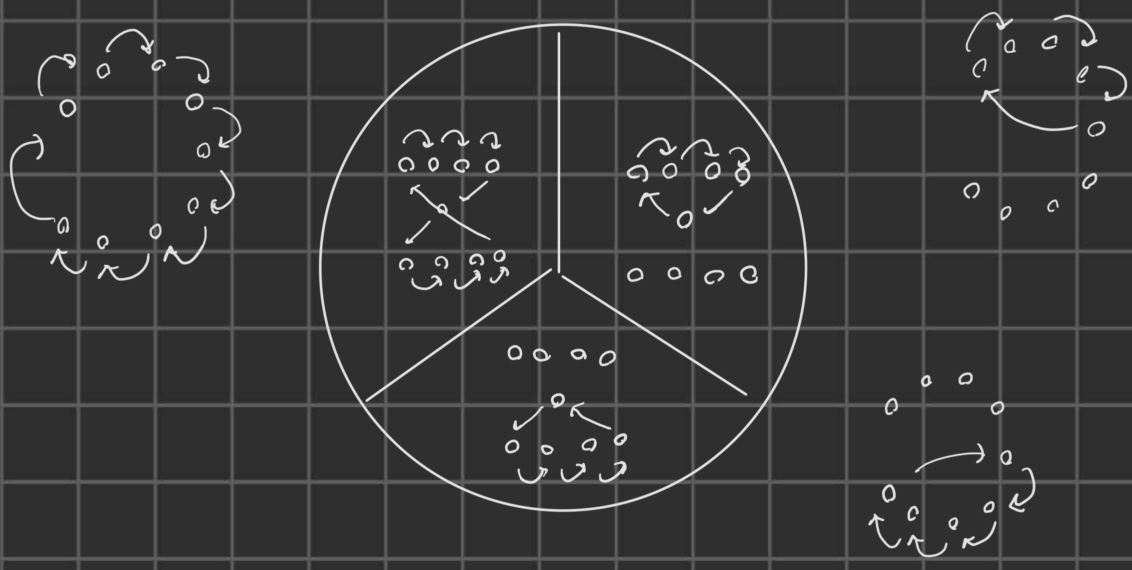
$$G(A:B:C) \equiv -\frac{1}{2} \ln Z(A:B:C)$$

$$\kappa \equiv G(A:B:C) - \frac{1}{2}\left(S_2(A) + S_2(B) + S_2(C) \right)$$

$$\kappa \geq \frac{c_L + c_R}{8} \ln 2$$

Renyi modular Commutator J_n

$R = 2n+1$



- Choose: $R = 2n+1$ and

$$\pi_A = (1, \dots, 2n+1), \quad \pi_B = (1, \dots, n+1),$$
$$\pi_C = (n+1, \dots, 2n+1)$$

- Motivated by the modular commutator [Kim-Shi-Kato-Albert(22)]

$$J(A,B,C)_\rho := i\,\mathrm{Tr}\left(\rho_{ABC}\left[K_{AB}, K_{BC}\right]\right)$$

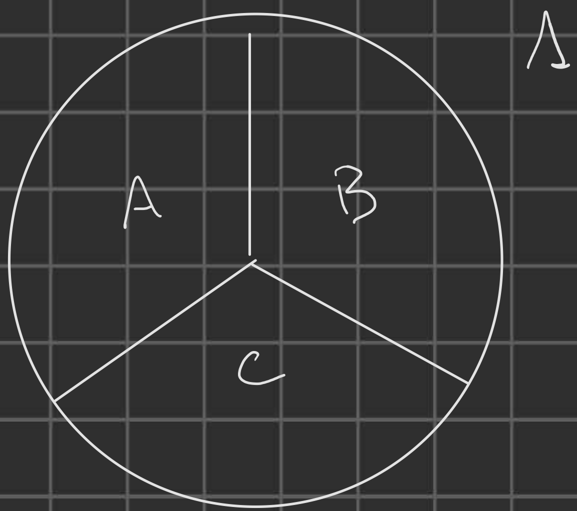
where $K_{AB} = -\ln \rho_{AB}$ is the entanglement Hamiltonian (modular operator).

- The modular commutator and chiral central charge: $J = \frac{\pi}{3} c_-$.
- J is related to J_n in the replica limit: $\frac{i}{n^2}(J_n - \overline{J_n}) \overset{n \to 0}{\rightarrow} J$

$$J_n \propto \exp\left(-\frac{2\pi i c_-}{24} \frac{2n^2}{(2n+1)(n+1)}\right)$$

Modular Commutator

[Kim - Shi - Kato - Albert 22]



$$K_A \equiv - \ln \rho_A$$ Entanglement (Modular) Hamiltonian

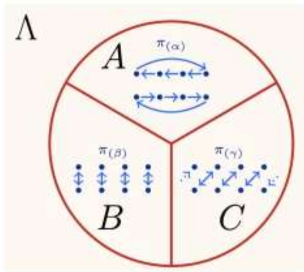
$$J \equiv i \operatorname{Tr} \rho_{ABC} [K_{AC} , K_{AB}]$$

$$J = \frac{\pi}{3} c_-$$ chiral central charge

Lens-space multi enthropy

- Choose: $R = 2r$ for $r \geq 2$, and

$$\pi_A(1,t) = (1, t-1); \quad \pi_A(2,t) = (2, t+1),$$
$$\pi_B(s,t) = (s+1, t),$$
$$\pi_C(1,t) = (2, t-1); \quad \pi_C(2,t) = (1, t+1),$$



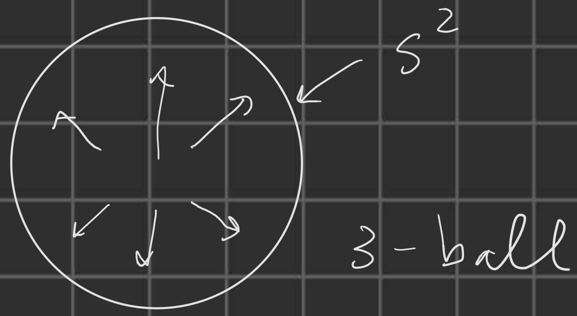
Indexing the replicas with a tuple (s,t) with $s = 1, 2$, $t = 1, ..., r$; addition is defined mod 2 for s and mod r for t.

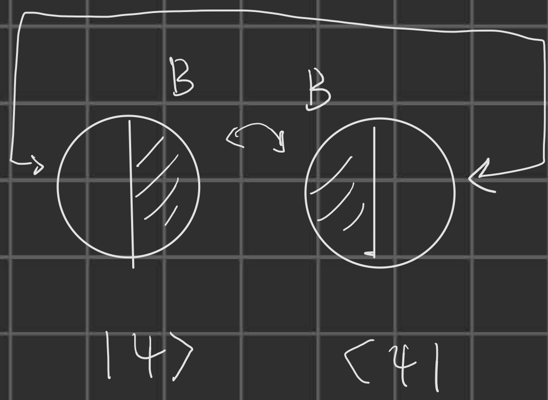
$$\Phi_r \propto e^{\frac{2\pi i c_-}{24}\left(-r - \frac{2}{r}\right)} \sum_a d_a^2 \theta_a^r$$

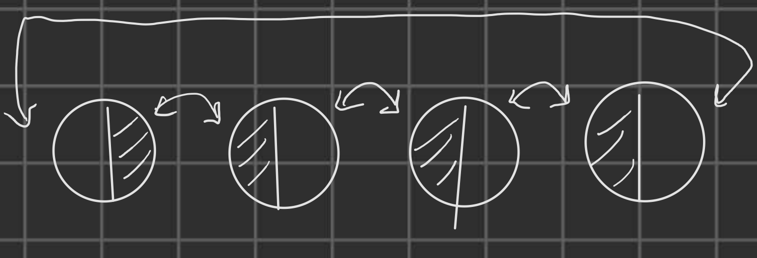
§ "Calculations"

"Derivations"

§ warm up: bipartition

$|\psi\rangle =$ 

$\mathrm{Tr}\,\rho_A =$ 

$\mathrm{Tr}\,\rho_A^2 =$ 

$$\frac{\mathrm{Tr}\,\rho_A^n}{(\mathrm{Tr}\,\rho_A)^n} = \frac{Z(S^3)}{[Z(S^3)]^n} = [Z(S^3)]^{1-n} = [S_0{}^0]^{1-n}$$

UV divergent term is missing

§ Warmup : bipartition

$A \quad \uparrow \Big| \downarrow \quad B$

$$\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$$

Bulk - boundary Correspondence.

Ansatz

$$\rho_A \propto e^{-\varepsilon' H^A_{CFT}}$$

supported only near ∂A

$\varepsilon' \sim$ (bulk gap)

$$S_n(A) = \frac{1}{1-n} \log \mathrm{Tr}\, \rho_A^n$$

In fact, we can approx GS itself

$$|\Psi\rangle \underset{\substack{\text{near} \\ \partial A}}{\sim} e^{-\varepsilon H^{AB}_{CFT}} \underbrace{|\Omega\rangle}_{\text{bdry states}}$$

$$H^{AB}_{CFT} = H^{A}_{CFT} + H^{B}_{CFT}$$

$$|\Omega\rangle = \text{"Maximally Entangled state"}$$

$$= \sum_n |E_n\rangle_A |E_n^*\rangle_B$$

Can check:

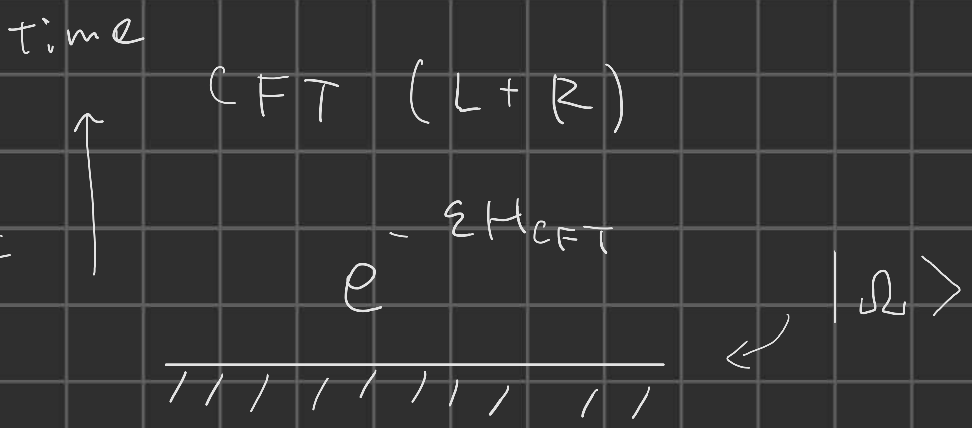
$$|\Psi\rangle = \sum_n e^{-2\varepsilon E_n} |E_n\rangle |E_n^*\rangle$$

$$\mathrm{Tr}_B |\Psi\rangle\langle\Psi| = e^{-4\varepsilon H^{A}_{CFT}}$$

* If. Topological degeneracy

$$|\Omega\rangle_a = \sum_n |E_n^a\rangle_A \, |E_n^{a\,*}\rangle_B$$

$|\psi\rangle =$



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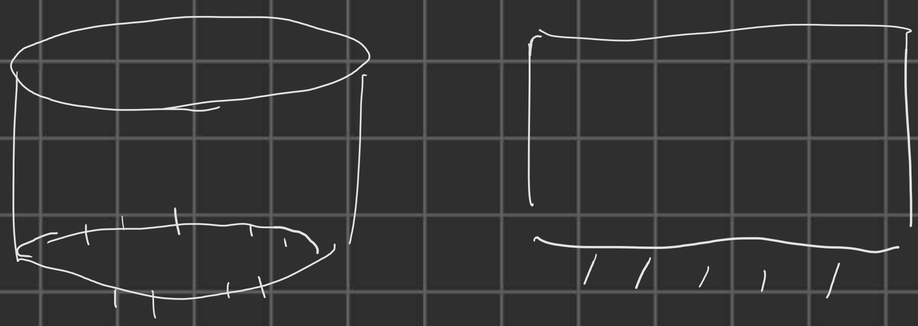
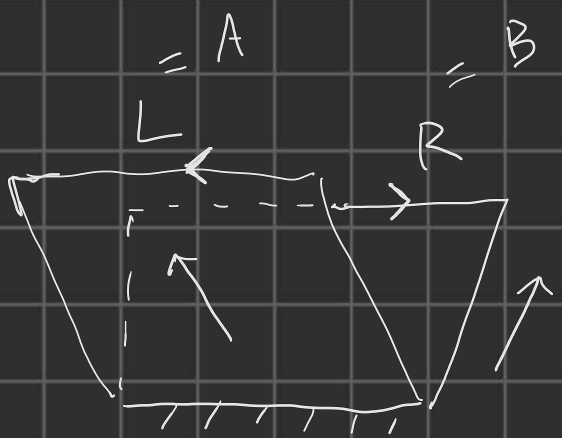
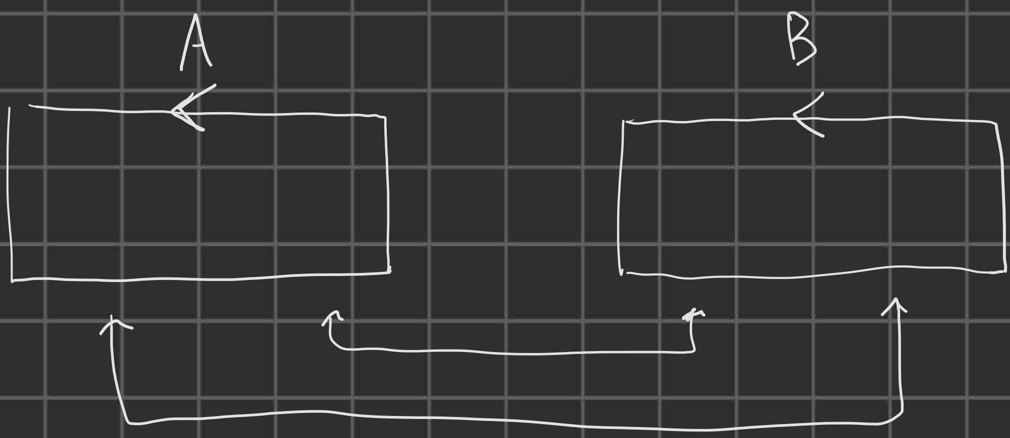



= A
= B
L
R


A
B

- Trace of the reduced density matrix:

$$\mathrm{Tr}\,(\rho_A)^n = \frac{1}{\mathfrak{n}^n}\mathrm{Tr}\left[e^{-n\epsilon H^A_{edge}}\right] = \frac{1}{\mathfrak{n}^n}\mathrm{Tr}\left[e^{-n\epsilon\frac{8\pi}{l}(L_0-c/24)}\right]$$
$$= \frac{1}{\mathfrak{n}^n}\chi\left(e^{-\frac{8\pi n\epsilon}{l}}\right) = \frac{\chi\left(e^{-\frac{8\pi n\epsilon}{l}}\right)}{\chi\left(e^{-\frac{8\pi\epsilon}{l}}\right)^n}$$

- 熱力学極限 $l/\epsilon\to\infty$ をとりたい。そこで、S モジュラー変換を行う：

$$\chi\left(e^{-\frac{8\pi n\epsilon}{l}}\right) = \sum_{a'}\mathcal{S}_{0a'}\chi_{a'}\left(e^{-\frac{\pi l}{2n\epsilon}}\right) \to \mathcal{S}_{00}\times e^{\frac{\pi cl}{48n\epsilon}} \quad (l/\epsilon\to\infty),$$

i.e., only the identity field I, labeled by "0" here, survives the limit.

- Hence, in the thermodynamic limit $l/\epsilon\to\infty$:

$$\mathrm{Tr}\,(\rho_A)^n = \frac{\sum_{a'}\mathcal{S}_{0a'}\chi_{a'}\left(e^{-\frac{\pi l}{2n\epsilon}}\right)}{\left[\sum_{a'}\mathcal{S}_{0a'}\chi_{a'}\left(e^{-\frac{\pi l}{2\epsilon}}\right)\right]^n} \to e^{\frac{\pi cl}{48\epsilon}\left(\frac{1}{n}-n\right)}(\mathcal{S}_{00})^{1-n},$$

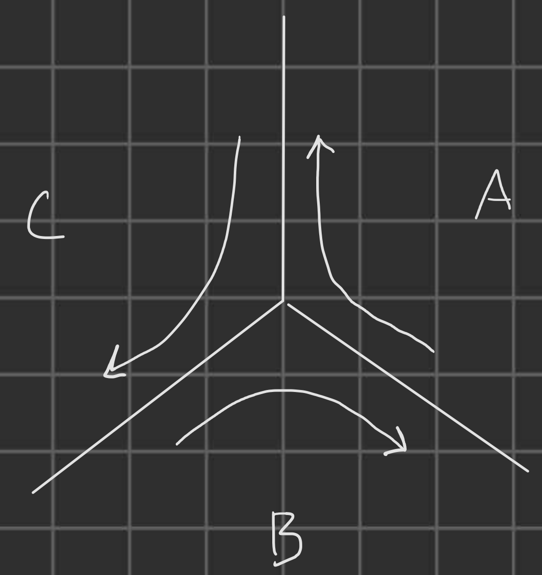
- Final result:

$$S_A^{(n)} = \frac{1+n}{n}\cdot\frac{\pi c}{48}\cdot\frac{l}{\epsilon} - \ln\mathcal{D} + \frac{1}{1-n}\ln d_0^{1-n}$$
$$S_A^{\mathrm{vN}} = \frac{\pi c}{24}\cdot\frac{l}{\epsilon} - \ln\mathcal{D} + \ln d_0$$

where the quantum dimension is related to the S matrix via

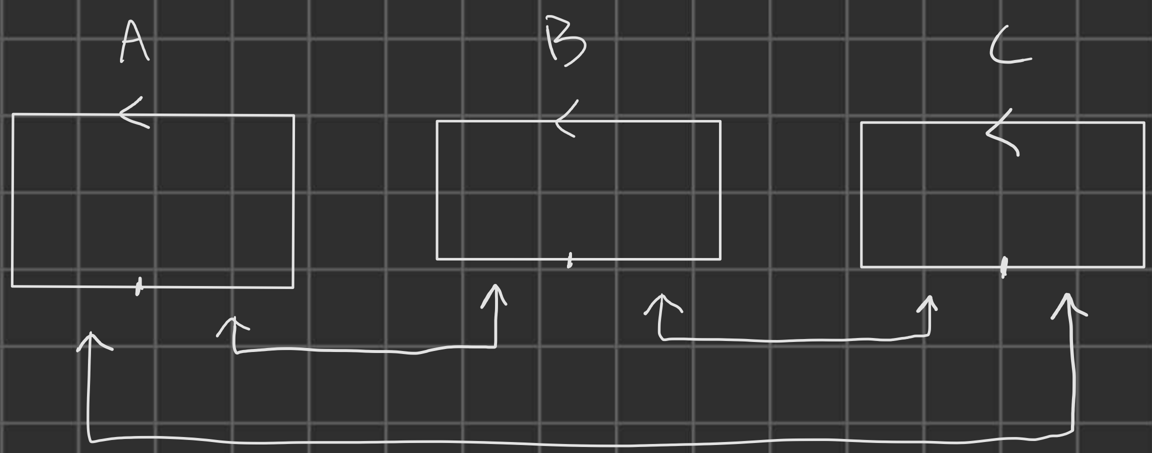
$$\mathcal{S}_{a0} = d_a/\mathcal{D}$$

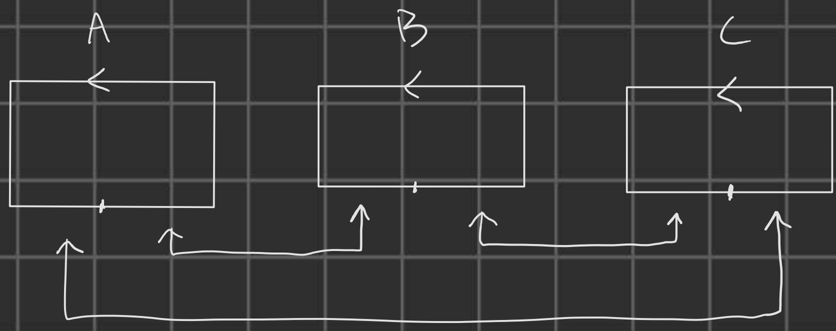
tri partition

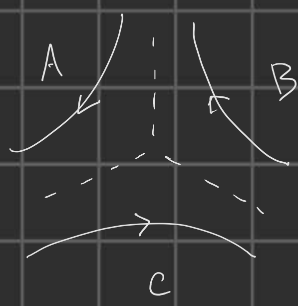


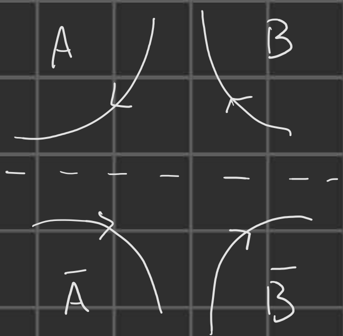
tri partition ansatz

$$|\psi\rangle \underset{\text{near bdry.}}{\sim} e^{-\beta H^{ABC}_{CFT}} |V\rangle$$

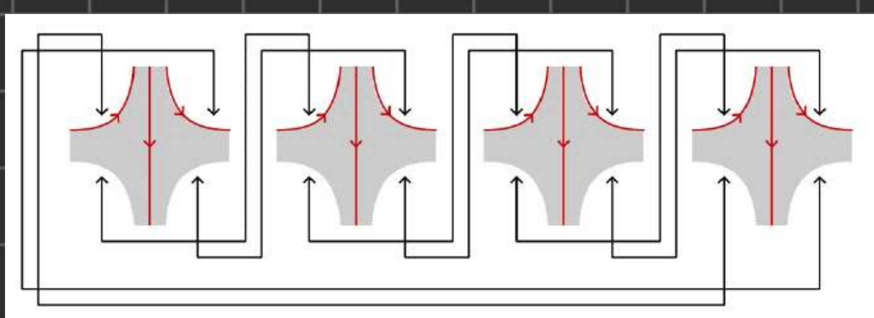


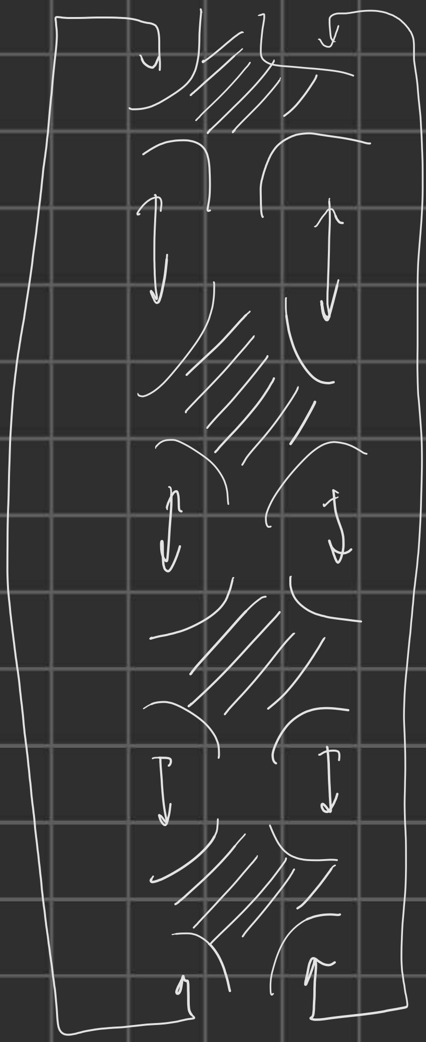
$|\psi\rangle =$ 

$=$ 

$\rho_{AB} =$ 

$$Z(A:B:C) = \mathrm{Tr}\left[\left\{\left(\rho_{AB}^{R}\right)^{\dagger} \rho_{AB}^{R}\right\}^{2}\right]$$

$=$ 



2β

2β

2β

2β

cylinder with circumference $2\beta \times 4 = 8\beta$

→ Riemann surface

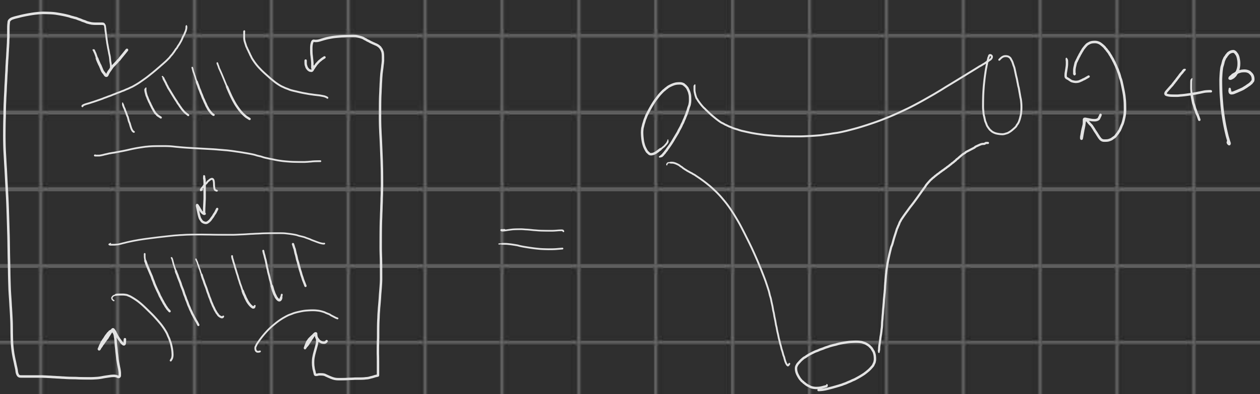
PBC の時 $g = 5$ 2つの球面 + 6本のcylinder

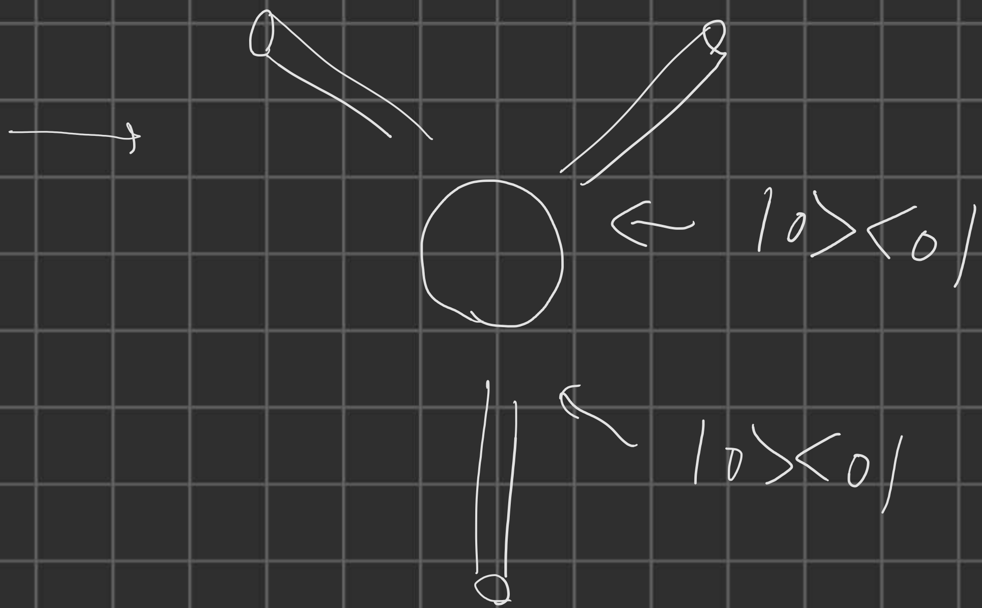
OBC なら $g = 0$ 球面 + 6本の cylinder

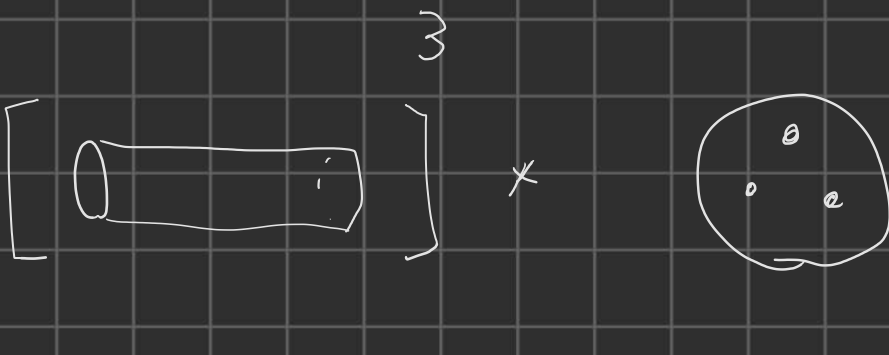
"folded" picture

最小レプリカ数の時,

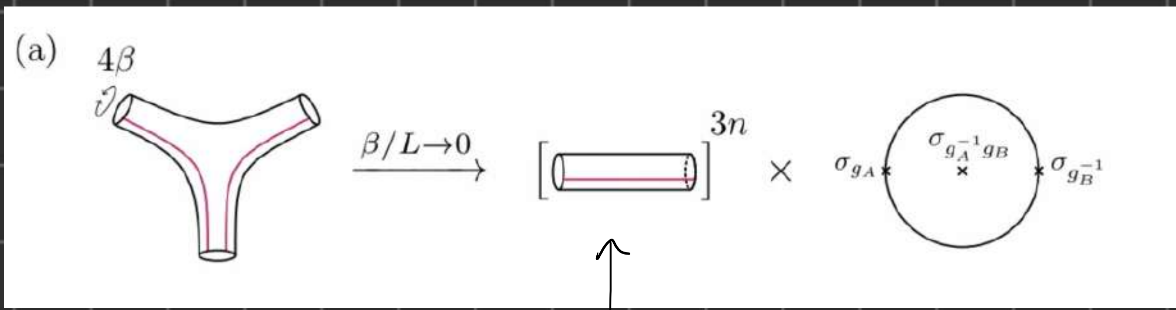
$$\langle \psi|^{\otimes R}\, \Pi_A \Pi_B \Pi_C\, |\psi\rangle^{\otimes R} \xrightarrow[R=1]{} \langle \psi|\psi\rangle$$





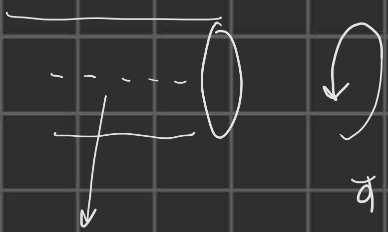
$=$ 

一般のレプリカ n



folded picture

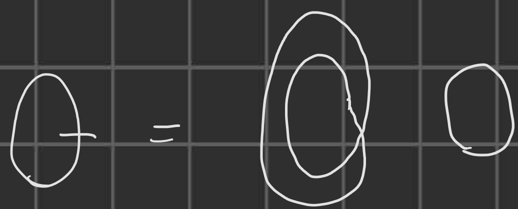
unfolded cylinder



branch

すべてのレプリカを訪れる

folded picture



folded picture

unfolded picture (2 cylinders)

$n = 3 = 2 + 1$

Cylinderの数 = permutation

$g_A,\ g_A^{-1} g_B,\ g_B^{-1}$

のcycleの数

cylinder から来る (e.g. n=2)

$$H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)$$

part. fun $\chi(8\beta, L) = \mathrm{Tr}\, e^{-8\beta H}$

$$\sim e^{\frac{\pi c L}{6 \times 8\beta}} \quad (c = c_L + c_R)$$

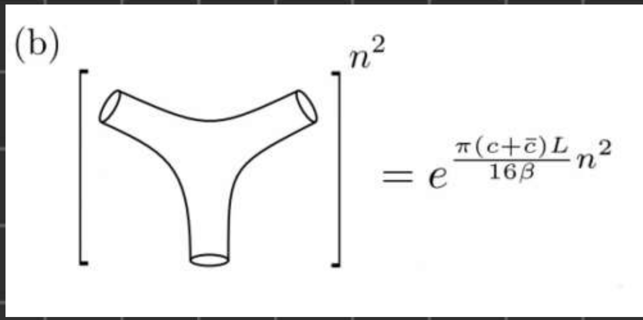
six cylinders

$$\chi^6 \sim e^{\frac{\pi c L}{6 \times 8\beta} \times 6}$$

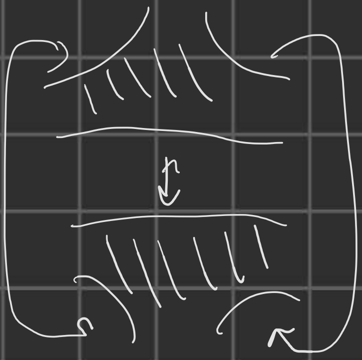
$$\frac{\chi^6}{\langle \psi | \psi \rangle^4} = \frac{e^{\pi c L / 8\beta}}{e^{\frac{\pi c L}{8\beta} \times 4}}$$

normalization

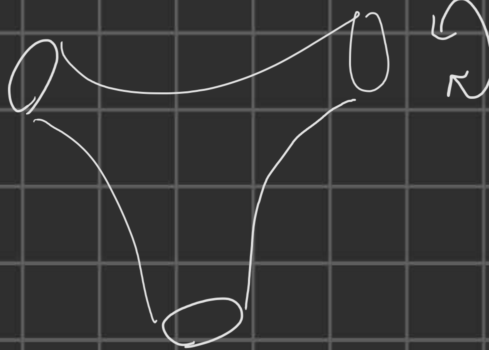
$$\langle 4|4\rangle^{4=n^2} =$$



$$\left[\quad\right]^{n^2} = e^{\frac{\pi(c+\bar{c})L}{16\beta}n^2}$$



$$=$$



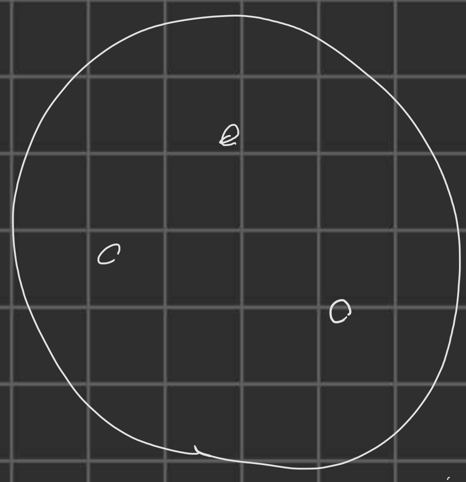
$$\sim \chi(4\beta, L)^{\times 3}$$

$$= \left(e^{\pi c L / 6\times 4\beta}\right)^{\times 3} = e^{\frac{\pi c L}{8\beta}}$$

$$\ln Z = -\frac{3\pi c L}{8\beta}$$

$$G = \frac{-1}{2}\ln Z = \frac{-1}{2}\left(-\frac{3\pi c L}{8\beta}\right) = \frac{3\pi c L}{16\beta}$$

Sphere からのキヨ



【値】Sphere 上の3つの twist operator の **3点関数**：

$$Z_{\text{sph}} = \langle \sigma_1\,\sigma_2\,\sigma_3 \rangle_{\text{sphere}} = \left(\sin\frac{\pi}{3}\right)^{\frac{c+\bar{c}}{24}(n^2-1)\times 3} \times C_n$$

- C_n：OPE coefficient（CFT data）
- $\sin(\pi/3)$：trijunction の opening angle $2\pi/3$ から来る **幾何学的因子**

【補足】$SL(2,\mathbb{C})$ で 3 点を $(0,1,\infty)$ に標準化して評価（CFTの標準的手法）。詳細は **Appendix D**。

球面上の3つの twist operator の3点関数：

$$Z_{\text{sph}} = \langle \sigma_1(z_1)\,\sigma_2(z_2)\,\sigma_3(z_3) \rangle_{\text{sphere}}$$

CFT の一般論により、 $SL(2,\mathbb{C})$ で 3 点を $(0,1,\infty)$ に標準化可能：

$$\langle \sigma_1(0)\,\sigma_2(1)\,\sigma_3(\infty) \rangle = C_{123} \quad \text{(OPE coefficient)}$$

座標因子は元の3点位置の比から決まり、**trijunction の対称な配置**（$2\pi/3$ 等分）から $\sin(\pi/3)$ が現れる。

$$S_n^{(3)}(A:B:C) = \underbrace{\left(1+\tfrac{1}{n}\right)\frac{\pi L(c+\bar{c})}{16\beta}}_{\text{Step 1: area law}} + \underbrace{\left(1+\tfrac{1}{n}\right)\frac{c+\bar{c}}{8}\ln\sin\tfrac{\pi}{3}}_{\text{Step 2: corner}} + \underbrace{\frac{1}{1-n}\frac{1}{n}\ln C_n}_{\text{Step 2: OPE}}$$

$$S_n(A) = \left(1+\tfrac{1}{n}\right)\frac{\pi L(c+\bar{c})}{24\beta} + \left(1+\tfrac{1}{n}\right)\frac{c+\bar{c}}{12}\ln\sin\tfrac{\pi}{3}$$

$$\boxed{\kappa_n = \frac{1}{1-n}\frac{1}{n}\ln C_n}$$

$$\boxed{\kappa = \frac{c+\bar{c}}{8}\ln 2}$$

§ Calculation (quad partition)

各 region 境界の neighborhood を「太い管」で正則化（cutting map i_ϵ）。
結果として得られる重要な分解：

$$\boxed{\mathcal{M} \propto \frac{1}{N^R} C_{\{0\}}\, Z_{\mathrm{topo}}(M)\, Z_{\mathrm{CFT}}(\Sigma)}$$

- M：closed 3-manifold → **TQFT 情報**（$\mathcal{D}, \zeta_n$ 等）
- $\Sigma = \partial W$：高 genus 2次元面 → **chiral CFT** → c_-

・Focus on $Z_{\mathrm{CFT}}(\Sigma) \longrightarrow c_-$

14.1 鍵となる事実：CFT vacuum の回転位相

Chiral CFT の vacuum を「角度 τ だけ回転」させると、Casimir momentum から位相が出る：

$$\boxed{\langle 0|e^{i\tau P}|0\rangle = e^{-i\tau c_-/24}}$$

【物理】Modular parameter / partition function の話と同じ起源。
$\tau \neq 0$ で **chiral central charge が exposed** される。

14.2 高 genus surface への一般化

Surface Σ は thin tubes で繋がった3つ穴の球面（**pair-of-pants, POP**）の集合。
極限 $\epsilon/L \to 0$ では tubes が無限に細く → **vacuum のみ伝播**。

各 tube に "twist" τ_i が associated：

$$\boxed{Z(\Sigma) \propto \exp\left[-\frac{ic_-}{24}\sum_i \tau_i\right]}$$

→ **全ての非自明性は $\sum_i \tau_i$ に集約！**

14.3 τ_i の組合せ的決定

τ_i は permutation π_A, π_B, π_C の構造から組合せ的に決まる。

① Construct covering surf Σ_{V_i}
(= pop)

② Glue $\Sigma_{V_i} \longrightarrow \Sigma$

[looks like $g = 3$ surface, w/ 6 tubes
But actually a higher genus surf.

③ Calculate $\sum_i \tau_i$

$\uparrow$ $i = 1 \cdots 6$

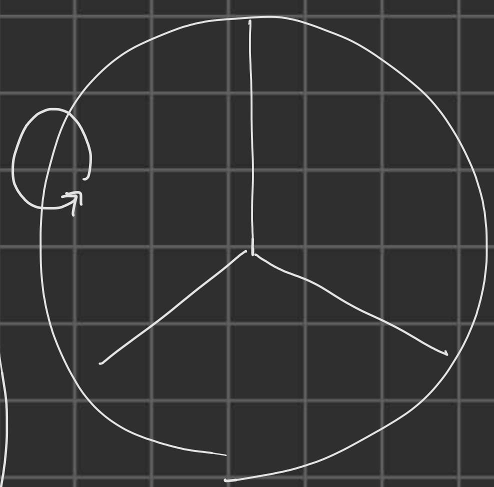
example. $\bar{J}_1$

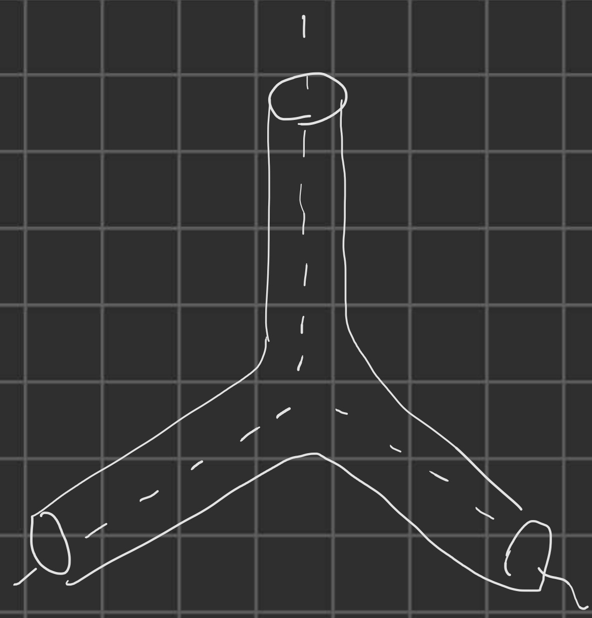
$\bar{J}_{n=1}$, $2n+1 = 3$ copies

$\pi_A = (1, 2, 3)$

$\pi_B = (2, 3)$

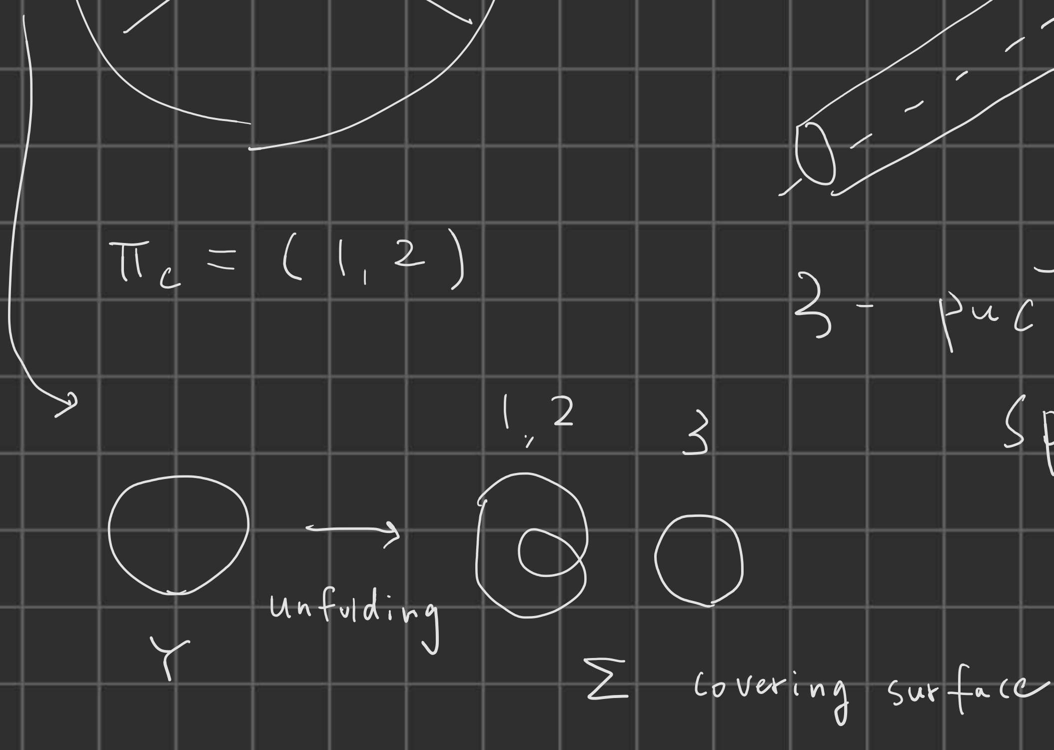
$\pi_C = (1, 2)$

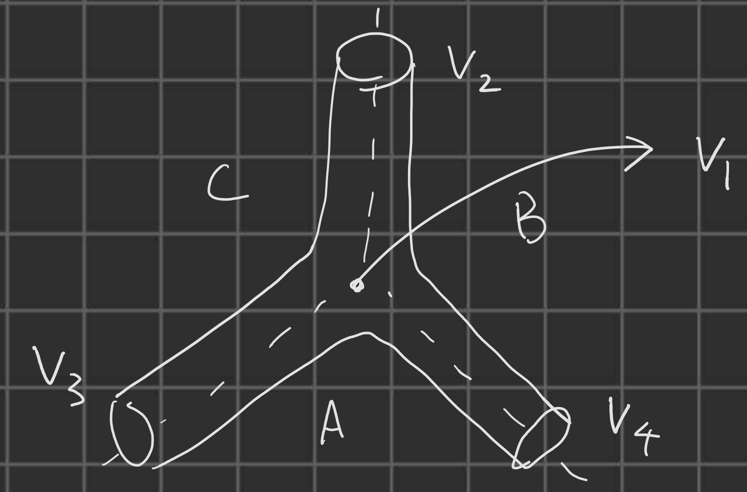




$\pi_c = (1, 2)$

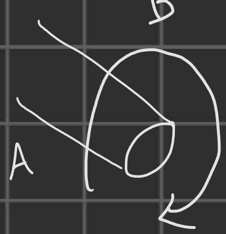
3- puctured sphere





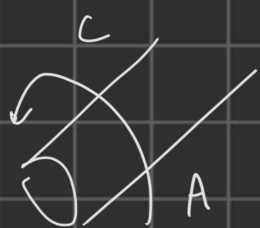
If we "go around" V_4, V_3, V_2

V_4



$$3 \xrightarrow{A} 1^* \xrightarrow{B} 1 \xrightarrow{A} 2^* \xrightarrow{B} 3$$

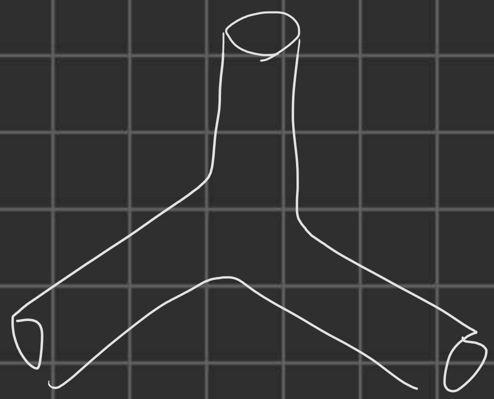
V_3



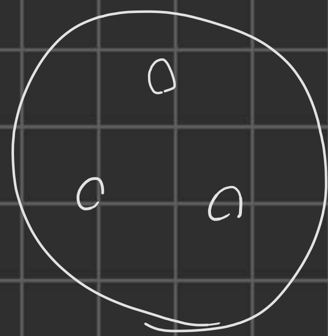
$$3 \xrightarrow{A} 1^* \xrightarrow{C} 2 \xrightarrow{A} 3^* \xrightarrow{C} 3$$

V_2

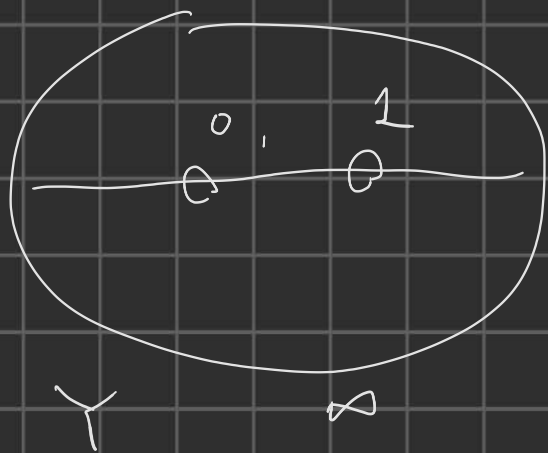
$$3 \xrightarrow{C} 3^* \xrightarrow{B} 2 \xrightarrow{C} 1^* \xrightarrow{B} 1 \xrightarrow{C} 2^* \xrightarrow{B} 3$$



$\simeq$



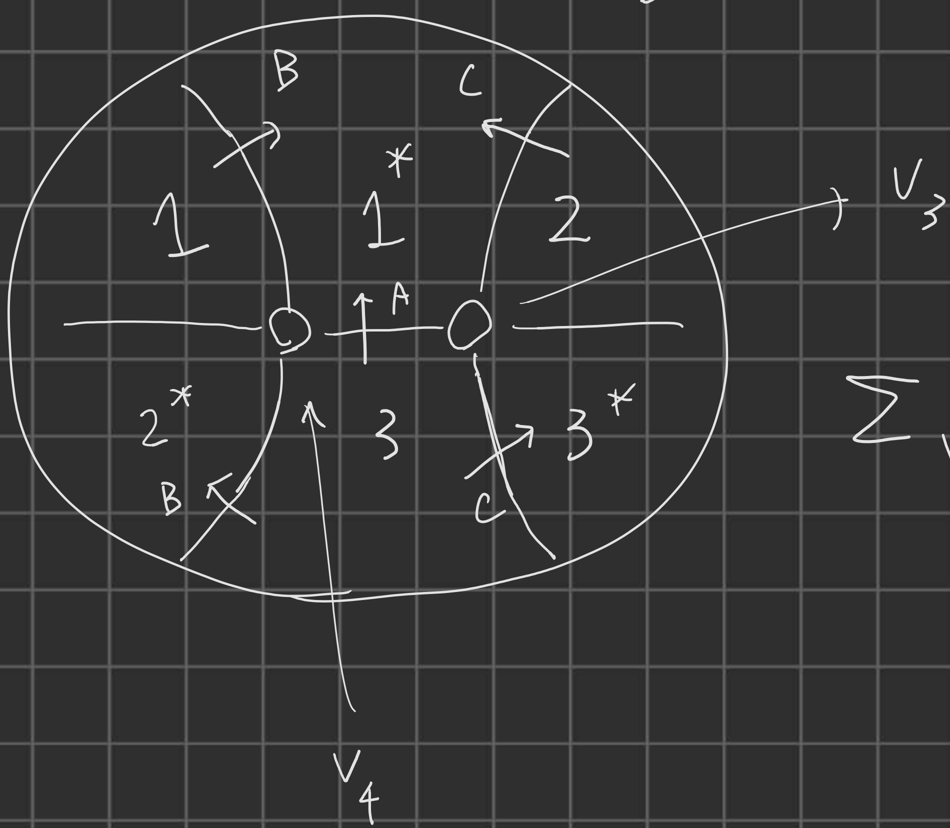
$\simeq$



"redrawing of previous page"

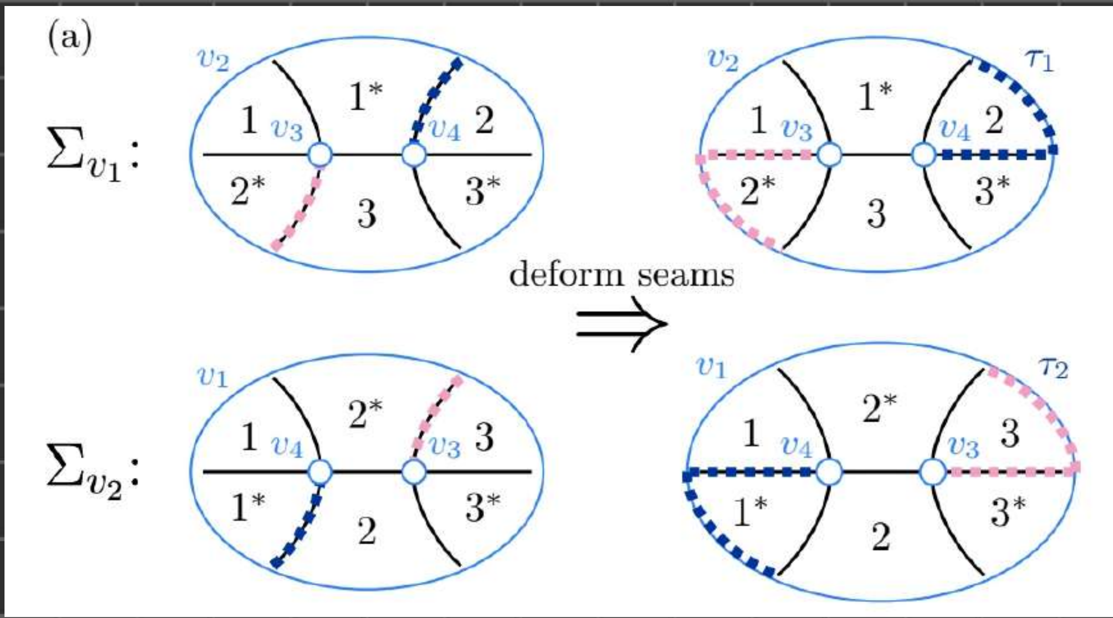
$\Downarrow$

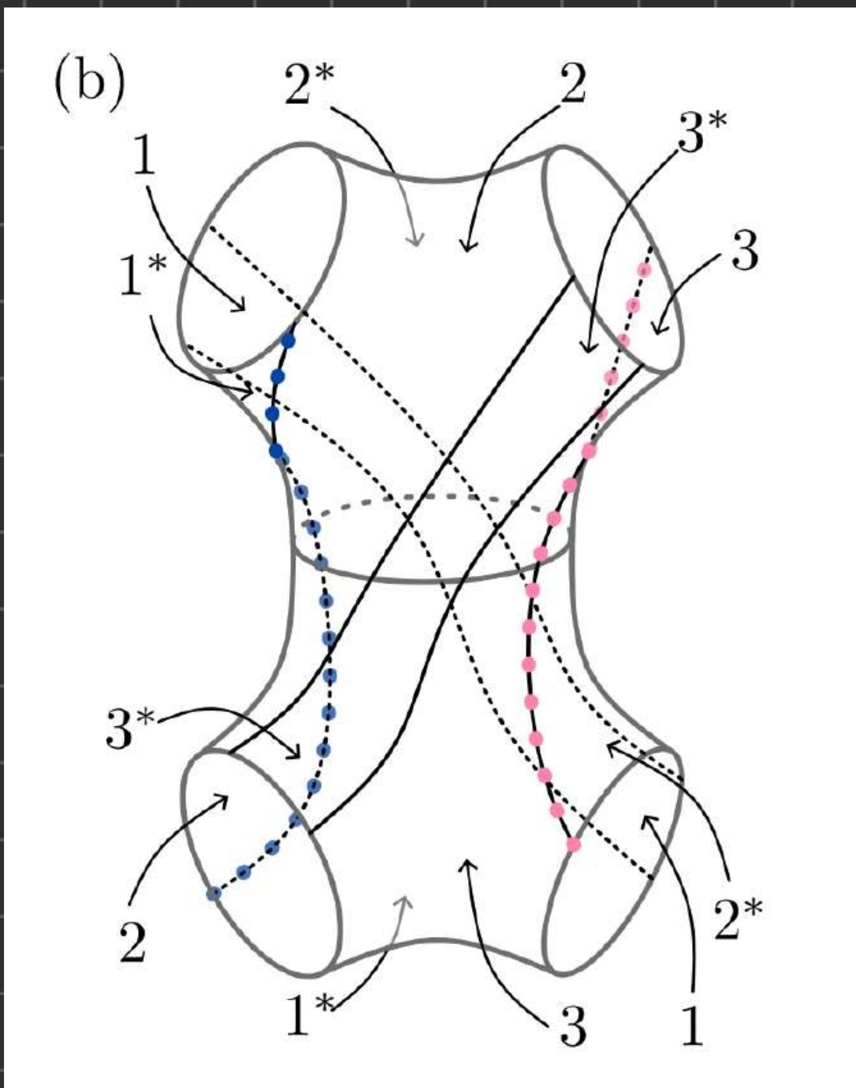
unfolding



Σ_{V_1}

glueing Σ's (covering surf, POP)





Σ_{v_2}

Σ_{v_1}

Gappable bdry

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} \partial_t \phi^I \partial_x \phi^J$$

$$\phi^I \equiv \phi^I + 2\pi$$

$$e^{i n^T \cdot \phi} , \quad n \in \mathbb{Z}^{2N}$$ quasi particles

$$\mathcal{A} \equiv \mathbb{Z}^{2N} / K \mathbb{Z}^{2N}$$ anyon group

Levin

edgeでcondenseするエニオン
の集合Mを考え，これが
Lagrangian subgroup
になっていればgappable

① $M \subset \mathcal{A} = \mathbb{Z}^{2N}/K\mathbb{Z}^{2N}$
が存在して，

$m^T K m' \in \mathbb{Z} \quad \forall m, m' \in M$

「Mの元はすべて互いにボゾンで
互いに局所」

(規約によっては $m^T K m' = 0$
と書くのが適当か)

③ $l \in M$ なら，ある $m \in M$ で

$$m^T K^{-1} l \notin \mathbb{Z}.$$

「M が最大」

$$M^{\perp} = \left\{ \begin{array}{c} M\text{の全ての元と} \\ \text{相互統計が trivial} \\ \text{なエニオンの集合} \end{array} \right\}$$

$$= \{ a \in A \mid a^T K^{-1} m \in \mathbb{Z} \quad \forall m \in M \}$$

$M \subset M^{\perp}$: isotropic 条件

$\iff$ ①

$M^{\perp} \subset M$: M は maximal

$\iff$ ②

$M^{\perp} = M$: Lagrangian 条件

$\iff$ ① + ②

isotropic をこえて

これ以上大きく

できない

① + ② $\Longrightarrow$

$$|M|^2 = |A| = |\det K|$$

エニオン $a \in M$ が境界で

condnce $\langle a \rangle \neq 0$

新しい真空の一部となる

相互統計が trivial な b

$\wedge$

a と

は $\langle a \rangle$ を気にせず通過

できる

〃 non trivial な b

$\langle a \rangle$ の中を移動すると

位相をひろいつづける

$\longrightarrow$ confinement.

② の maximality 条件より,

M の外にあるエニオンが全て

confine される

Non abelian

$$A = \bigoplus_a n_a a \longleftrightarrow M \subset A$$

⓪ $n_1 = 1$ $\quad 1 \in M$

① $n_a > 0 \Rightarrow \theta_a = 1$

② for a, b in A $\quad M_{ab} = \dfrac{S_{ab} S_{11}}{S_{a1} S_{1b}} = 1$

③ $a, b \in A \rightarrow a \times b \in A$

④ $\dim A = \sum_a n_a d_a = D$

② $\longleftrightarrow$ A is commutative alg.

$$\mu \circ c_{A,A} = \mu$$

③ $\longleftrightarrow$ $\mu : A \times A \to A$

結合的乗法の存在

abelian では automatic

④ $\dim A^2 = D^2 = \sum_a d_a^2$

$\longleftrightarrow$ $|M|^2 = |A|$

for abelian

③ "algebra structure" が存在

1 $\mu : A \times A \longrightarrow A$

2 $\eta : 1 \longrightarrow A$

3 $\delta : A \to A \times A$

4 $\mu \circ c_{A,A} = \mu$