Five-lecture course on tensor networks for highly entangled states

This short course provides an introduction to tensor-network methods for the description of many-body quantum systems. Tensor networks offer a compact language to represent large tensors, such as many-body wavefunctions, probability distributions, partition functions, and time-evolution operators, by decomposing them into networks of smaller building blocks. The course starts from the basic definitions of tensors, graphical notation, tensor contractions, and tensor decompositions, and gradually develops the matrix product state formalism as the central example of a one-dimensional tensor network.

A key theme of the course is the relation between tensor networks and entanglement. Matrix product states provide efficient descriptions of one-dimensional ground states with limited entanglement, such as those of gapped local Hamiltonians. However, the course will also emphasize that tensor networks are not restricted to weakly entangled states. They can be used to study highly entangled many-body states by adapting the geometry of the network, by exploiting renormalization-group ideas, by using finite-entanglement scaling, or by contracting tensor networks in alternative directions. In this way, tensor-network methods can provide controlled access to critical systems, higher-dimensional systems, thermal states, imaginary-time path integrals, and out-of-equilibrium dynamics, even in regimes where the entanglement structure is too complex to be captured exactly by a finite-bond-dimension ansatz.

The course will connect several complementary viewpoints. Tensor networks will first be introduced as algebraic decompositions of large tensors and as graphical tools for organizing contractions. They will then be related to entanglement through the singular value decomposition and the Schmidt decomposition. This will lead naturally to matrix product states, their canonical forms, and their use in computing norms, expectation values, correlation functions, and reduced density matrices. The course will then discuss how the scaling of entanglement explains the success and limitations of the density matrix renormalization group, and how DMRG can be understood as a variational optimization over matrix product states.

In the later lectures, the course will broaden the perspective from variational wavefunctions to tensor networks representing imaginary-time evolution, ground-state projectors, and renormalization-group transformations. The two-dimensional tensor networks arising from Trotter decompositions of (e^{-\beta H}) will be connected to boundary matrix product states and transfer matrices. Finally, the course will discuss tensor-network renormalization, finite-entanglement scaling, and the challenges posed by real-time dynamics, where entanglement typically grows rapidly. Particular attention will be given to strategies that go beyond the standard spatial MPS picture, including transverse contractions of spatio-temporal tensor networks and wavefunction branching.

By the end of the course, students should understand tensor networks both as practical numerical tools and as conceptual frameworks for organizing many-body quantum information. They should be able to recognize when tensor networks provide efficient representations, why entanglement controls their computational cost, and how tensor-network ideas can still be useful in highly entangled regimes when combined with suitable geometries, renormalization concepts, and scaling theories.

Key references

General introductions and tutorials

- **Tensors.net**, basic tutorials and code snippets for tensor-network methods, including the `ncon` tensor-contraction notation and implementation resources:
  `www.tensors.net`
- J. C. Bridgeman and C. T. Chubb, “Hand-waving and Interpretive Dance: An Introductory Course on Tensor Networks,” arXiv:1603.03039.
  Introductory lecture notes covering tensor-network notation, matrix product states, tensor-network algorithms, and applications to quantum many-body systems.
- S.-J. Ran, E. Tirrito, C. Peng, X. Chen, L. Tagliacozzo, G. Su, and M. Lewenstein, *Tensor Network Contractions: Methods and Applications to Quantum Many-Body Systems*, Lecture Notes in Physics **964**, Springer,(2020). DOI: 10.1007/978-3-030-34489-4.

Tensor-network libraries

- **TeNPy — Tensor Network Python**
  J. Hauschild, J. Unfried, S. Anand, B. Andrews, M. Bintz, U. Borla, S. Divic, M. Drescher, J. Geiger, M. Hefel, K. Hémery, W. Kadow, J. Kemp, N. Kirchner, V. S. Liu, G. Möller, D. Parker, M. Rader, A. Romen, S. Scalet, L. Schoonderwoerd, M. Schulz, T. Soejima, P. Thoma, Y. Wu, P. Zechmann, L. Zweng, R. S. K. Mong, M. P. Zaletel, and F. Pollmann, “Tensor Network Python (TeNPy) version 1,” *SciPost Physics Codebases* **41** (2024). DOI: 10.21468/SciPostPhysCodeb.41.
- **ITensor**
  M. Fishman, S. R. White, and E. M. Stoudenmire, “The ITensor Software Library for Tensor Network Calculations,” *SciPost Physics Codebases* **4** (2022). DOI: 10.21468/SciPostPhysCodeb.4.
- **quimb**
  J. Gray, “quimb: A python package for quantum information and many-body calculations,” *Journal of Open Source Software* **3**, 819 (2018). DOI: 10.21105/joss.00819.
- **ITransverse.jl**
  S. Carignano, “The ITransverse.jl library for transverse tensor network contractions,” *SciPost Physics Codebases* **63** (2026). DOI: 10.21468/SciPostPhysCodeb.63.
- **YASTN — Yet Another Symmetric Tensor Network**
  M. M. Rams, G. Wójtowicz, A. Sinha, and J. Hasik, “YASTN: Yet another symmetric tensor networks; A Python library for Abelian symmetric tensor network calculations,” *SciPost Physics Codebases* **52** (2025). DOI: 10.21468/SciPostPhysCodeb.52.
- **SeeMPS**
  P. García-Molina, J. J. Rodríguez-Aldavero, J. Gidi, and J. J. García-Ripoll,

“SeeMPS: A Python-based Matrix Product State and Tensor Train Library,” arXiv:2601.16734.
Python library for tensor-network algorithms based on Matrix Product States and Quantized Tensor Trains.

Entanglement, area laws, and matrix product states

- J. Eisert, M. Cramer, and M. B. Plenio, “Area laws for the entanglement entropy,” *Reviews of Modern Physics* **82**, 277–306 (2010).
  Review of entanglement scaling in quantum many-body systems and its connection to efficient tensor-network descriptions.
- L. Tagliacozzo, T. R. de Oliveira, S. Iblisdir, and J. I. Latorre, “Scaling of entanglement support for matrix product states,” *Physical Review B* **78**, 024410 (2008).
  Foundational work on finite-entanglement scaling in critical matrix product states.
- F. Pollmann, S. Mukerjee, A. M. Turner, and J. E. Moore, “Theory of finite-entanglement scaling at one-dimensional quantum critical points,” *Physical Review Letters* **102**, 255701 (2009).
  Development of the finite-entanglement scaling theory for critical systems represented by finite-bond-dimension MPS.
- L. Tagliacozzo et al., “Self-congruent point in critical matrix product states: An effective field theory for finite-entanglement scaling,” arXiv:2411.03954.
  Recent perspective on finite-entanglement scaling, emphasizing its interpretation as an RG flow induced by finite bond dimension.

Renormalization group, MERA, and tensor-network renormalization

- G. Vidal, “Entanglement Renormalization: an introduction,” arXiv:0912.1651.
  Introductory reference on entanglement renormalization, real-space RG, and MERA.
- G. Evenbly and G. Vidal, “Tensor Network Renormalization,” arXiv:1412.0732.
  Introduction of tensor network renormalization as a coarse-graining method for classical partition functions and Euclidean path integrals.
- G. Evenbly and G. Vidal, “Tensor network renormalization yields the multi-scale entanglement renormalization ansatz,” arXiv:1502.05385.
  Explains how applying TNR to the Euclidean path integral or ground-state projector produces a MERA representation of the ground state.
- G. Evenbly G. Vidal , “Tensor network states and geometry,” arXiv:1106.1082.
  Discussion of tensor-network states from the perspective of emergent geometry, including the relation between MPS, PEPS, MERA, and holographic geometries.
- V. Vanthilt, A. Naravane, C. Meng, A. Ueda “A Practical Introduction to Tensor Network Renormalization with TNRKit.jl,” arXiv:2604.06922. Recent practical introduction to tensor-network renormalization methods using the Julia package TNRKit.jl.

Time evolution and out-of-equilibrium dynamics

- M. Paeckel et al., “Time-evolution methods for matrix-product states,” arXiv:1901.05824.

Review of MPS time-evolution methods, including TEBD, TDVP, Krylov approaches, and MPO-based methods.

- “Spatio-temporal tensor-network approaches to out-of-equilibrium dynamics bridging open and closed systems,” arXiv:2502.20214.
  Review of approaches based on contracting full spatio-temporal tensor networks, including influence functionals, process tensors, and transverse contraction strategies.
- “Overcoming the entanglement barrier with sampled tensor networks,” arXiv:2505.09714.
  Variational Monte Carlo and sampled tensor-network approach to overcoming the entanglement barrier in real-time dynamics.
- A. J. A. James, M. C. Bañuls, and L. Tagliacozzo, “Converting long-range entanglement into mixture: tensor-network approach to local equilibration,” arXiv:2308.04291.
  Tensor-network approach to local equilibration based on transforming long-range entanglement into mixture.
- J. K. Taylor and I. P. McCulloch, “Wavefunction branching: when you can’t tell pure states from mixed states,” *Quantum* **9**, 1670 (2025).
  Introduction of wavefunction branching as a way to characterize situations in which pure-state superpositions are operationally indistinguishable from mixed states.

1 Tensors, contractions, decompositions, and constrained matrix product states

- Definition of tensors as multidimensional arrays.
- Examples of tensors:
  - vectors,
  - matrices,
  - higher-rank tensors,
  - many-body wavefunction coefficients.
- Graphical notation for tensors:
  - tensors as nodes,
  - indices as legs,
- Definition of tensor contraction.
  - contractions as connected legs,
  - open indices as physical or boundary indices.
- Examples of simple contractions:
  - scalar product,
  - matrix-vector multiplication,
  - matrix multiplication,
  - contraction of higher-rank tensors.
- Tensor decompositions:
  - eigenvalue decomposition,

    - singular value decomposition,
    - low-rank approximations.
- Introduction to matrix product states as decompositions of many-body coefficient tensors.
- Example: counting bit strings without two consecutive ones.
- Construction of an MPS whose coefficients in the computational basis are:
    - one for allowed strings,
    - zero for forbidden strings
- General construction of MPS representations for constrained strings adding different weights to zeros and ones without changing the bond dimension.

2 Entanglement, tensor-network contractions, and basic properties of MPS

- Definition of bipartite entanglement for pure states.
- Reduced density matrix and partial trace.
- Von Neumann entanglement entropy.
- Rényi entanglement entropies.
- Entanglement as a property determined only by the spectrum of the reduced density matrix.
- Relation between the reduced density matrix and the SVD of the coefficient tensor.
- Schmidt decomposition from the SVD.
- Simple examples:
    - product states and zero entanglement,
    - Bell states,
    - maximally entangled states,
    - states with minimal and maximal entropy.
- Entanglement upperbound in tensor networks
- Entanglement bound imposed by the MPS bond dimension.
- Cost of contracting a tensor network.
- Efficient contraction of one-dimensional tensor networks.
- Matrix product states:
    - definition,
    - physical indices,
    - virtual indices,
    - bond dimension.
    - Calculation of the norm of an MPS.
    - Transfer matrix associated with an MPS.
    - Calculation of expectation values of local operators.
    - Calculation of two-point correlation functions.
    - Correlation length from the spectrum of the transfer matrix.
    - Calculation of reduced density matrices from MPS.

3 Entanglement structure, real-space RG, DMRG, and canonical MPS

- Patterns of entanglement in many-body quantum systems.
- Entanglement in random states.
- Entanglement in physically relevant states.
  - Ground states of local gapped one-dimensional systems: area law,
  - Ground states of critical one-dimensional systems:logarithmic violation of the area law,
  - Ground states in higher dimensions: gapped and critical
- Introduction to real-space renormalization group ideas:
  - change of length scale,
  - coarse graining,
  - flow of Hamiltonians,
  - fixed points,
  - relevant and irrelevant perturbations,
  - preservation of low-energy or ground-state expectation values.
- Real-space RG as motivation for DMRG.
- DMRG prescription:
  - truncation of the Hilbert space,
  - keeping the most relevant states,
  - use of the reduced density matrix,
  - relation between truncation and entanglement.
- Cost of DMRG from entanglement scaling:
  - random states,
  - gapped one-dimensional ground states,
  - critical one-dimensional ground states,
  - higher-dimensional systems.
- Gauge freedom in matrix product states.
- Canonical forms of MPS.
  - Left-canonical and right-canonical tensors.
  - Mixed canonical form.
  - Orthogonality center.
- Connection between DMRG and MPS in canonical form.

4 Variational DMRG, imaginary-time tensor networks, and boundary MPS

- Variational formulation of DMRG.
  - MPS as a variational manifold.
  - Mapping the original Hamiltonian to an effective Hamiltonian on a coarse-grained Hilbert space.
  - Local optimization of one tensor or two tensors at a time.
  - Iterative sweeping procedure.
  - Interpretation of DMRG as variational energy minimization over MPS.
- The ground-state projector
  - Representation of the imaginary-time path integral as a two-dimensional tensor network.
  - Cost of exact contraction of 2D TN

- Tension between two descriptions:
  - DMRG gives a finite-bond-dimension MPS approximation to gapped one-dimensional ground states,
  - imaginary-time evolution gives an infinite two-dimensional tensor network.
- Solution to the tension
  - The two-dimensional tensor network as an MPS with infinite bond dimension.
  - Transfer matrix of the two-dimensional tensor network and its gap
  - Boundary contraction of two-dimensional tensor networks.
  - Boundary MPS approximation.

5 Tensor-network renormalization, finite-entanglement scaling, and dynamics

- Renormalization-group perspective on tensor networks.
  - Fixed points of tensor-network transformations.
  - Scaling operators.
  - Relevant and irrelevant perturbations.
  - Tensor-network renormalization as a real-space RG transformation.
- Use of TNR transformations to justify the emergence of finite-bond-dimension MPS descriptions.
- Ground states of gapless Hamiltonians with infinite TEBD.
  - Infinite MPS and translational invariance the special case of the critical point
  - Finite bond dimension as a relevant perturbation.
  - Finite bond dimension induces a finite correlation length.
- Finite-entanglement scaling:
  - scaling of the correlation length with bond dimension,
  - scaling of the entanglement entropy with bond dimension,
  - extraction of universal data.
  - RG interpretation of finite-entanglement scaling.
- Tensor networks beyond exact entanglement scaling:
  - using finite-(\chi) tensor networks together with scaling theory,
  - extracting universal properties even when the ansatz does not reproduce the true asymptotic entanglement.
- Out-of-equilibrium dynamics.
  - Growth of entanglement after quantum quenches.
  - Limitations of standard MPS time evolution.
- Possible strategies around entanglement growth:
  - transverse contraction of spatio-temporal tensor networks,
  - influence-functional and temporal-MPS viewpoints,
  - wavefunction branching,
  - alternative tensor-network geometries for dynamics.
- Concluding perspective:
  - tensor networks as compression tools,
  - tensor networks as variational manifolds,
  - tensor networks as RG transformations,

- tensor networks as controlled approximations when combined with scaling theory.