Problem Sheet 2

A. The wave equation as a well-posed initial value problem

Consider the wave equation on two-dimensional Minkowski space,

$$-\partial_t^2\phi + \partial_x^2\phi = 0\,, \tag{1}$$

with initial data

$$\phi(0,x) = f(x)\,, \qquad \partial_t\phi(0,x) = g(x)\,. \tag{2}$$

1. Introduce null coordinates

   $$u = x - t\,, \qquad v = x + t\,. \tag{3}$$

   Show that the wave equation becomes

   $$\partial_u\partial_v\phi = 0\,. \tag{4}$$

   Deduce that the most general solution can be written as

   $$\phi(t,x) = F(x-t) + G(x+t) \tag{5}$$

   for arbitrary functions F and G.

   Hint: If $\partial_u\partial_v\phi = 0$, what can $\partial_v\phi$ depend on?

2. Impose the initial data (2) on (5). Show that

   $$F(x) + G(x) = f(x)\,, \qquad -F'(x) + G'(x) = g(x)\,. \tag{6}$$

   Use these equations to solve separately for F' and G'.

   Hint: differentiate the first equation in (6) with respect to x and combine the result with the second one.

3. Integrate the resulting equations and derive d'Alembert's formula,

   $$\phi(t,x) = \frac{f(x-t) + f(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \mathrm{d}s\, g(s)\,. \tag{7}$$

   Hint: you do not need to determine the two integration constants separately: only the combination $F(x-t) + G(x+t)$ enters the solution, and the constants cancel once $F + G = f$ is imposed.

4. Use (7) to establish *existence* and *uniqueness* for arbitrary smooth initial data (f, g).

   Hint for uniqueness: if ϕ_1 and ϕ_2 have the same initial data, apply d'Alembert's formula to $\delta\phi = \phi_1 - \phi_2$.

5. Which portion of the initial surface can influence the value of $\phi(t,x)$? Draw this region in the (t,x) plane and relate it to the causal structure of the wave equation.

B. Failure of existence and uniqueness

Consider again the wave equation on the half-space $t \geq 0$, $x \geq 0$,

$$-\partial_t^2\phi + \partial_x^2\phi = 0\,, \tag{8}$$

with initial data

$$\phi(0,x) = f(x)\,, \qquad \partial_t\phi(0,x) = g(x)\,, \qquad x \geq 0\,. \tag{9}$$

Instead of the Dirichlet boundary condition discussed in the lecture, impose at $x = 0$

$$(\partial_t + \partial_x)\phi(t,0) = \widetilde{h}(t)\,, \qquad t \geq 0\,. \tag{10}$$

Recall that the general solution of the wave equation is

$$\phi(t,x) = \phi_-(x-t) + \phi_+(x+t)\,. \tag{11}$$

1. Substitute (11) into the initial data and show that

$$f'(x) + g(x) = 2\phi_+'(x)\,. \tag{12}$$

   What does the boundary condition (10) determine about ϕ_+?

2. Show that the initial and boundary data must satisfy

$$f'(x) + g(x) = \widetilde{h}(x)\,, \qquad x \geq 0\,. \tag{13}$$

   The functions f, g and $\widetilde{h}$ were supposed to be freely specifiable data. What does (13) imply about *existence* for generic data?

3. Now suppose that the data obey (13) and that $\phi(t,x)$ is one solution. Let $a(u)$ be any smooth function with compact support on the negative real axis. Show that

$$\phi(t,x) \longrightarrow \phi(t,x) + a(x-t) \tag{14}$$

   leaves both the initial data (9) and the boundary data (10) unchanged. What does this imply about *uniqueness*?

4. Which of the two characteristic modes $\phi_-(x-t)$ and $\phi_+(x+t)$ is not controlled by the boundary condition? Give a simple physical explanation for why this is a bad boundary condition for the region $x \geq 0$.

C. Loss of continuous dependence

A problem may have a solution, and that solution may be unique, but still fail to be well posed because the solution does not depend continuously on the data. Consider the Laplace equation

$$\partial_t^2\phi + \partial_x^2\phi = 0 \tag{15}$$

treated, deliberately, as an evolution equation in t. For $k > 0$, take the Cauchy data

$$\phi_k(0,x) = 0\,, \qquad \partial_t\phi_k(0,x) = e^{-\sqrt{k}}\sin(kx)\,. \tag{16}$$

You may assume that for these analytic data the solution exists and is unique.

1. Verify that the corresponding solution is

$$\phi_k(t,x) = \frac{e^{-\sqrt{k}}}{k}\sinh(kt)\sin(kx)\,. \tag{17}$$

2. Show that as $k \to \infty$ the initial data tend uniformly to zero. More strongly, for every fixed integer N, show that all x-derivatives up to order N also tend uniformly to zero. You may use

$$k^N e^{-\sqrt{k}} \longrightarrow 0\,. \tag{18}$$

3. Fix any $t > 0$. Show that

$$\sup_x |\phi_k(t,x)| = \frac{e^{-\sqrt{k}}}{k}\sinh(kt) \sim \frac{1}{2k}e^{kt-\sqrt{k}} \longrightarrow \infty\,. \tag{19}$$

   Explain why arbitrarily small changes of the initial data can therefore produce arbitrarily large changes of the solution after any non-zero evolution time.

4. Which part of the data is responsible for the instability? Explain why this is a high-frequency instability and why it signals failure of continuous dependence in any norm involving only finitely many derivatives.

D. Boundary terms for the Einstein–Hilbert action

Consider Einstein gravity in four spacetime dimensions on a manifold with a non-null boundary,

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_N} \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\,(R - 2\Lambda)\,. \tag{20}$$

Denote by h_{ij} the induced metric on $\partial\mathcal{M}$, by n^μ the unit outward normal, and by

$$K_{ij} = h_i{}^\mu h_j{}^\nu \nabla_\mu n_\nu\,, \qquad K = h^{ij} K_{ij} \tag{21}$$

the extrinsic curvature and its trace. Overall signs may change with conventions for the normal; the important point in this problem is the relative coefficient of the boundary terms.

1. Recall that the variation of the Einstein–Hilbert action contains normal derivatives of $\delta g_{\mu\nu}$ at the boundary. Show that adding the Gibbons–Hawking–York term

$$S_{\mathrm{GHY}} = \frac{1}{8\pi G_N} \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|}\, K \tag{22}$$

gives, on shell,

$$\delta(S_{\mathrm{EH}} + S_{\mathrm{GHY}}) = \frac{1}{16\pi G_N} \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|}\,(K_{ij} - K h_{ij})\,\delta h^{ij}\,. \tag{23}$$

Why does this give a well-defined Dirichlet variational principle?

Hint: a useful starting point is

$$\delta(\sqrt{-g}\,R) = \sqrt{-g}\,G_{\mu\nu}\,\delta g^{\mu\nu} + \sqrt{-g}\,\nabla_\mu(\nabla_\nu \delta g^{\mu\nu} - \nabla^\mu \delta g)\,, \qquad \delta g \equiv g_{\mu\nu}\delta g^{\mu\nu}\,. \tag{24}$$

Use Stokes' theorem to identify the boundary contribution. You may also use $\delta\sqrt{|h|} = -\frac{1}{2}\sqrt{|h|}\,h_{ij}\delta h^{ij}$. The terms containing normal derivatives of δh_{ij} are precisely cancelled by the variation of K in S_{GHY}.

2. Equivalently, writing the Dirichlet variation in terms of δh_{ij}, define the canonical momentum

$$\pi^{ij} = -\frac{\sqrt{|h|}}{16\pi G_N}(K^{ij} - K h^{ij})\,, \qquad \delta(S_{\mathrm{EH}} + S_{\mathrm{GHY}}) = \int_{\partial\mathcal{M}} \pi^{ij}\delta h_{ij}\,. \tag{25}$$

(The overall sign changes if the opposite convention for n^μ is chosen.) The action appropriate to Neumann boundary conditions can be obtained by the boundary Legendre transform

$$S_{\mathrm{N}} = S_{\mathrm{D}} - \int_{\partial\mathcal{M}} \pi^{ij} h_{ij}\,. \tag{26}$$

Show that in D spacetime dimensions it takes the form

$$S_{\mathrm{N}} = S_{\mathrm{EH}} + \frac{4-D}{16\pi G_N} \int_{\partial\mathcal{M}} \mathrm{d}^{D-1}x \sqrt{|h|}\, K\,. \tag{27}$$

What is special about $D = 4$? What boundary data are naturally held fixed if the Einstein–Hilbert action is used with no additional boundary term?

3. We now want to impose *conformal boundary conditions*: we fix the conformal class of the induced metric together with the trace of the extrinsic curvature,

$$[h_{ij}]_{\mathrm{conf}} = \text{fixed}\,, \qquad K = \text{fixed}\,. \tag{28}$$

Consider the one-parameter family of actions

$$S_\alpha = S_{\mathrm{EH}} + \alpha S_{\mathrm{GHY}}\,. \tag{29}$$

Starting from (23), show that its on-shell variation can be written as

$$\delta S_\alpha = \frac{1}{16\pi G_N} \int_{\partial\mathcal{M}} \sqrt{|h|}\,\left[(K_{ij} - \alpha K h_{ij})\delta h^{ij} + 2(\alpha - 1)\delta K\right]. \tag{30}$$

4. If the conformal class is fixed, an allowed variation is purely Weyl,

$$\delta h^{ij} = \varphi\, h^{ij} \tag{31}$$

for some arbitrary function φ on the boundary. Using also $\delta K = 0$, determine the value of α for which (30) vanishes for arbitrary allowed variations. Show that in four dimensions

$$S_{\text{CBC}} = S_{\text{EH}} + \frac{1}{3} S_{\text{GHY}} = S_{\text{EH}} + \frac{1}{24\pi G_N} \int_{\partial\mathcal{M}} \sqrt{|h|}\, K\,. \tag{32}$$

Extra. What are the boundary conditions compatible with an arbitrary $\alpha > 0$?

Problems for Friday and beyond...

E. Linear uniqueness of conformal boundary conditions

Take Minkowski space with coordinates (t, x, y, z) and a timelike boundary at $x = 0$, with the spacetime occupying $x \geq 0$. Let $m, n \in \{t, y, z\}$ denote directions tangent to the boundary. Write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{33}$$

and impose de Donder gauge,

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0\,. \tag{34}$$

The linearised equations are then wave equations, so consider individual plane-wave modes

$$h_{\mu\nu} = \gamma_{\mu\nu}(k) e^{i k_\rho x^\rho}\,, \qquad k^\mu k_\mu = 0\,. \tag{35}$$

1. Impose that the conformal class of the induced metric at $x = 0$ is fixed. Show that for a plane-wave mode this implies

$$\gamma_{mn} = \Gamma(k)\eta_{mn} \tag{36}$$

for some scalar amplitude $\Gamma(k)$.

2. Linearise the trace of the extrinsic curvature of the boundary and show that fixing K gives

$$\frac{1}{2}\partial_x h_m{}^m - \partial^m h_{mx} = 0 \qquad \text{at } x = 0\,. \tag{37}$$

Hint: work about the planar boundary of Minkowski space, for which the background extrinsic curvature vanishes.

3. Substitute (35) and (36) into the de Donder condition and (37). Show that the independent conditions can be arranged as

$$k^m\gamma_{mx} - \frac{3}{2}k_x\Gamma = 0\,, \qquad k^m\gamma_{mx} + \frac{1}{2}k_x\gamma_{xx} - \frac{3}{2}k_x\Gamma = 0\,, \qquad k_x\gamma_{mx} - \frac{1}{2}k_m(\gamma_{xx} + \Gamma) = 0\,.$$

4. First assume $k_x \neq 0$. Use the three conditions above to show that

$$\Gamma = 0\,, \qquad \gamma_{mx} = 0\,, \qquad \gamma_{xx} = 0\,. \tag{38}$$

Thus there are no non-trivial plane-wave modes of this type compatible with homogeneous conformal boundary data.

5. The case $k_x = 0$ is slightly more subtle. Show that the boundary and gauge conditions reduce to

$$k^m\gamma_{mx} = 0\,, \qquad \gamma_{xx} = -\Gamma\,. \tag{39}$$

Why has uniqueness not yet been established?

6. Impose homogeneous initial data on the spacelike surface $t = 0$,

$$h_{\mu\nu} = 0\,, \qquad \partial_t h_{\mu\nu} = 0\,, \qquad x \geq 0\,. \tag{40}$$

For example, the most general $k_x = 0$ contribution to h_{xx} is

$$h_{xx} = \int \frac{dk_y dk_z}{(2\pi)^2} \gamma_{xx}(k_y, k_z) e^{-i\sqrt{k_y^2 + k_z^2}\, t + i k_y y + i k_z z} + \text{h.c.} \tag{41}$$

Show that (40) forces $\gamma_{xx} = 0$, and hence $\Gamma = 0$. Argue similarly that $\gamma_{mx} = 0$.

7. Compare this result with the non-uniqueness of the Dirichlet problem discussed in the lecture. Which boundary degree of freedom responsible for the Dirichlet non-uniqueness has been eliminated by fixing $([h_{mn}]_{\text{conf}}, K)$?

F. Euclidean conformal thermodynamics in three dimensions

In three dimensions the Euclidean action appropriate to conformal boundary conditions is

$$I_E = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} \mathrm{d}^3x\sqrt{g}\,(R-2\Lambda) - \frac{1}{16\pi G_N}\int_{\partial\mathcal{M}} \mathrm{d}^2x\sqrt{h}\,K\,. \tag{42}$$

The boundary has topology $S^1 \times S^1$. Since only its conformal class is fixed, the natural inverse temperature is the dimensionless quantity

$$\widetilde{\beta} \equiv \frac{\beta_0\sqrt{f(r_c)}}{r_c}\,, \tag{43}$$

where β_0 is the period of the Euclidean time coordinate and $r = r_c$ is the location of the wall.

F.1 BTZ

Consider Euclidean non-rotating BTZ,

$$\mathrm{d}s^2 = f(r)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\phi^2\,, \qquad f(r) = \frac{r^2 - r_+^2}{\ell^2}\,, \qquad \Lambda = -\frac{1}{\ell^2}\,. \tag{44}$$

The region of interest is $r_+ \leq r \leq r_c$. Smoothness at the horizon gives

$$\beta_0 = \frac{2\pi\ell^2}{r_+}\,. \tag{45}$$

1. Verify that the trace of the extrinsic curvature at the wall is

$$K\ell = \frac{r_c}{\sqrt{r_c^2 - r_+^2}} + \frac{\sqrt{r_c^2 - r_+^2}}{r_c}\,. \tag{46}$$

   Conclude that $K\ell \geq 2$.

2. Let $k \equiv K\ell$. Using (43), (45), and (46), show that

$$r_+ = \frac{\pi\ell}{\widetilde{\beta}}\left(k - \sqrt{k^2-4}\right)\,. \tag{47}$$

3. Evaluating (42) on the BTZ solution gives

$$I_E^{\mathrm{BTZ}} = -\frac{\pi r_+}{4G_N}\,. \tag{48}$$

   Verify this result directly if you wish. Define

$$c_{\mathrm{BTZ}}(K) = \frac{3\ell}{4G_N}\left(k - \sqrt{k^2-4}\right) \tag{49}$$

   and show that

$$I_E^{\mathrm{BTZ}} = -\frac{\pi^2 c_{\mathrm{BTZ}}}{3\widetilde{\beta}}\,. \tag{50}$$

4. Define the conformal energy, entropy and heat capacity by

$$E_{\mathrm{conf}} = \left.\frac{\partial I_E}{\partial\widetilde{\beta}}\right|_K\,, \qquad S = \widetilde{\beta}E_{\mathrm{conf}} - I_E\,, \qquad C_K = -\widetilde{\beta}^2\left.\frac{\partial E_{\mathrm{conf}}}{\partial\widetilde{\beta}}\right|_K\,. \tag{51}$$

   Show that

$$E_{\mathrm{conf}} = \frac{\pi^2 c_{\mathrm{BTZ}}}{3\widetilde{\beta}^2}\,, \qquad S = C_K = \frac{2\pi^2 c_{\mathrm{BTZ}}}{3\widetilde{\beta}} = \frac{2\pi r_+}{4G_N}\,. \tag{52}$$

   Thus the entropy is precisely the Bekenstein–Hawking entropy.

5. Take the asymptotic-boundary limit. Show that $K\ell \to 2$ and

$$c_{\mathrm{BTZ}} \longrightarrow \frac{3\ell}{2G_N}\,. \tag{53}$$

   Recognise this as the Brown–Henneaux central charge.

F.2 de Sitter

Now consider the Euclidean continuation of a dS_3 cosmic patch,

$$\mathrm{d}s^2 = f(r)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\phi^2\,, \qquad f(r) = \frac{r_h^2 - r^2}{\ell^2}\,, \qquad \Lambda = +\frac{1}{\ell^2}\,. \tag{54}$$

Here the region is $r_c \le r \le r_h$, and the wall is an *inner* boundary of the Euclidean manifold. Since the region $r < r_c$ is excised, r_h need not equal ℓ. Smoothness at the cosmological horizon gives

$$\beta_0 = \frac{2\pi\ell^2}{r_h}\,. \tag{55}$$

The outward-pointing normal at the wall points toward decreasing r.

1. Verify that
$$K\ell = \frac{r_c}{\sqrt{r_h^2 - r_c^2}} - \frac{\sqrt{r_h^2 - r_c^2}}{r_c}\,. \tag{56}$$
In contrast with BTZ, show that $K\ell$ can take any real value.

2. Writing again $k = K\ell$, show that the horizon radius is
$$r_h = \frac{\pi\ell}{\widetilde{\beta}}\left(\sqrt{k^2+4} - k\right)\,. \tag{57}$$

3. The on-shell action is
$$I_E^{\mathrm{dS}} = -\frac{\pi r_h}{4G_N}\,. \tag{58}$$
Again, you may verify this explicitly using (42). Define
$$c_{\mathrm{dS}}(K) = \frac{3\ell}{4G_N}\left(\sqrt{k^2+4} - k\right)\,. \tag{59}$$
Show that
$$I_E^{\mathrm{dS}} = -\frac{\pi^2 c_{\mathrm{dS}}}{3\widetilde{\beta}}\,. \tag{60}$$

4. Compute E_{conf}, S, and C_K using (51). Show that
$$E_{\mathrm{conf}} = \frac{\pi^2 c_{\mathrm{dS}}}{3\widetilde{\beta}^2}\,, \qquad S = C_K = \frac{2\pi^2 c_{\mathrm{dS}}}{3\widetilde{\beta}} = \frac{2\pi r_h}{4G_N}\,. \tag{61}$$
What does the sign of C_K tell you about the thermal stability of the dS cosmic patch in the conformal canonical ensemble?

   Comment. The expressions above have precisely the Cardy form of the thermal energy and entropy of a two-dimensional CFT with effective central charge $c_{\mathrm{dS}}(K)$, for *all* values of $K\ell$.

5. What do the limits $K\ell \to -\infty$ and $K\ell \to +\infty$ correspond to in de Sitter space? Use (56) to determine in which limit the wall approaches the observer at $r_c = 0$, and in which limit it approaches the cosmological horizon at $r_c = r_h$.

6. Finally, consider the common large-positive-$K\ell$ regime, $K\ell \gg 1$. Show that in this limit the BTZ and dS_3 effective central charges agree at leading order,
$$c_{\mathrm{BTZ}}(K) \simeq c_{\mathrm{dS}}(K) \simeq c_{\mathrm{flat}}(K) \equiv \frac{3}{2G_N K}\,. \tag{62}$$
There are two useful ways of thinking about this limit. At fixed ℓ, use (46) and (56) to show that $K\ell \to +\infty$ corresponds to bringing the wall close to the corresponding horizon. Alternatively, at fixed K, the same limit can be obtained by taking $\ell \to \infty$, for which $|\Lambda| \to 0$. In this sense the common answer above points toward a local flat-space (Rindler) regime in which the distinction between positive and negative cosmological constant disappears.

   Show also that the entropy becomes $S \simeq 2\pi^2 c_{\mathrm{flat}}/(3\widetilde{\beta}) = \pi^2/(G_N K\widetilde{\beta})$. Why can c_{flat}, and therefore the entropy at fixed $\widetilde{\beta}$, remain parametrically large even after the curvature scale ℓ has dropped out of the answer? Identify the dimensionless parameter which must be small for this to happen. Interpret the result semiclassically.