Problem Sheet 1

A. Invariant separation in (A)dS

A useful way to describe maximally symmetric spaces is as hypersurfaces in a higher-dimensional flat embedding space. The embedding-space inner product gives a simple invariant measure of the separation between two points.

1. **Warm-up: the sphere.** Consider a two-sphere S^2 of radius ℓ embedded in $\mathbb{R}^3$,
$$X \cdot X = \ell^2, \tag{1}$$
and define
$$P(x, y) \equiv \frac{X(x) \cdot X(y)}{\ell^2}\,. \tag{2}$$
Parameterize the sphere by
$$X = \ell(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{3}$$
and show that
$$P(x, y) = \cos\gamma, \tag{4}$$
where γ is the angular separation between the two points. What are the values of P for coincident and antipodal points? How is P related to the geodesic distance on the sphere?

2. **de Sitter space.** Embed dS_{d+1} in $\mathbb{R}^{1,d+1}$ as
$$-X_0^2 + \sum_{i=1}^{d+1} X_i^2 = \ell^2 \tag{5}$$
and define
$$P(x, y) \equiv \frac{\eta_{AB} X^A(x) X^B(y)}{\ell^2}\,. \tag{6}$$
Show that
$$(X(x) - X(y))^2 = 2\ell^2\big(1 - P(x, y)\big). \tag{7}$$
Deduce that $P = 1$ for coincident or null-separated points. If x_A is the antipodal point, $X(x_A) = -X(x)$, show that $P(x, x_A) = -1$.

3. In global coordinates,
$$X_0 = \ell\sinh\frac{t}{\ell}, \qquad X_i = \ell\cosh\frac{t}{\ell}\, n_i, \qquad n_i n_i = 1, \tag{8}$$
show that
$$P(x, y) = -\sinh\frac{t}{\ell}\sinh\frac{t'}{\ell} + \cosh\frac{t}{\ell}\cosh\frac{t'}{\ell}\cos\gamma, \tag{9}$$
where $n \cdot n' = \cos\gamma$. For two points at equal global time, show that
$$P = 1 - \cosh^2\frac{t}{\ell}\,(1 - \cos\gamma). \tag{10}$$
What happens to P as $t \to +\infty$ for any fixed $\gamma \neq 0$? In particular, what happens for $\gamma = \pi$? Relate this behaviour to the fact that future infinity in de Sitter space is spacelike.

4. Now use static-patch coordinates,
$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{1 - r^2/\ell^2} + r^2 d\Omega_{d-1}^2, \qquad 0 \le r < \ell. \tag{11}$$
A convenient embedding is
$$X_0 = \sqrt{\ell^2 - r^2}\sinh\frac{t}{\ell}, \qquad X_{d+1} = \sqrt{\ell^2 - r^2}\cosh\frac{t}{\ell}, \qquad X_i = r n_i. \tag{12}$$

Show that

$$P(x,y) = \sqrt{1-\frac{r^2}{\ell^2}}\sqrt{1-\frac{r'^2}{\ell^2}}\cosh\frac{t-t'}{\ell} + \frac{rr'}{\ell^2}\cos\gamma. \tag{13}$$

Show that for any two points strictly inside the same static patch,

$$P(x,y) > -1. \tag{14}$$

Compare this with the range of P found in global coordinates.

5. **Anti-de Sitter space.** Embed AdS_{d+1} in $\mathbb{R}^{2,d}$ as

$$-X_{-1}^2 - X_0^2 + \sum_{i=1}^{d} X_i^2 = -\ell^2 \tag{15}$$

and define

$$P(x,y) \equiv -\frac{X(x)\cdot X(y)}{\ell^2}, \tag{16}$$

so that again $P(x,x)=1$. In global coordinates,

$$X_{-1} = \ell\cosh\rho\cos\tau, \qquad X_0 = \ell\cosh\rho\sin\tau, \qquad X_i = \ell\sinh\rho\, n_i, \tag{17}$$

derive

$$P(x,y) = \cosh\rho\cosh\rho'\cos(\tau-\tau') - \sinh\rho\sinh\rho'\cos\gamma. \tag{18}$$

At equal times and equal large radius, $\rho=\rho'=R$, show that for $\gamma\neq 0$,

$$P \to +\infty \qquad (R\to\infty). \tag{19}$$

More generally, show that near the boundary

$$P(x,y) \sim \frac{e^{\rho+\rho'}}{4}\left[\cos(\tau-\tau') - \cos\gamma\right]. \tag{20}$$

Thus $|P|$ generically diverges as the points approach the AdS boundary, with a sign that depends on their boundary causal separation. Compare this with the large-t behaviour at future infinity in de Sitter space. How does this reflect the timelike nature of the AdS boundary versus the spacelike nature of the dS boundary?

B. Invariant two-point functions

1. Consider a free scalar field of mass m in either dS_{d+1} or AdS_{d+1}, and suppose that the state is invariant under all spacetime isometries. Explain why the two-point function

$$G(x,y) = \langle\phi(x)\phi(y)\rangle \tag{21}$$

can depend on the two points only through the invariant separation introduced above,

$$G(x,y) = G(P(x,y)). \tag{22}$$

2. Introduce

$$\epsilon = \begin{cases} +1, & \mathrm{dS}_{d+1}, \\ -1, & \mathrm{AdS}_{d+1}. \end{cases} \tag{23}$$

With the conventions for P used above, show that

$$(\nabla P)^2 = \epsilon\,\frac{1-P^2}{\ell^2}, \qquad \Box P = -\epsilon\,\frac{d+1}{\ell^2}P. \tag{24}$$

You may use the embedding-space identity

$$g^{\mu\nu}\partial_\mu X^A \partial_\nu X^B = \eta^{AB}_{\mathrm{emb}} - \epsilon\,\frac{X^A X^B}{\ell^2}, \tag{25}$$

where η^{AB}_{emb} is the inverse metric of the relevant embedding space: it has signature $(-,+,\dots,+)$ for dS and $(-,-,+,\dots,+)$ for AdS.

3. Away from coincident points, the Klein–Gordon equation implies

$$(\Box_x - m^2)G(x,y) = 0. \tag{26}$$

Using the chain rule and the identities above, show that

$$\boxed{\epsilon\left[(1-P^2)G''(P) - (d+1)P\,G'(P)\right] - m^2\ell^2 G(P) = 0}. \tag{27}$$

Write this equation explicitly for de Sitter and Anti-de Sitter space.

4. Consider $|P| \gg 1$. Use $G(P) \sim (-P)^{-\Delta}$ for the dS future-boundary limit $P \to -\infty$, and $G(P) \sim P^{-\Delta}$ for spacelike-separated points approaching the AdS boundary, $P \to +\infty$. Show that in de Sitter

$$\Delta(d-\Delta) = m^2\ell^2, \tag{28}$$

while in Anti-de Sitter

$$\Delta(\Delta - d) = m^2\ell^2. \tag{29}$$

Hence find

$$\Delta_\pm^{\rm dS} = \frac{d}{2} \pm \nu, \qquad \nu \equiv \sqrt{\frac{d^2}{4} - m^2\ell^2}, \tag{30}$$

and

$$\Delta_\pm^{\rm AdS} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2}. \tag{31}$$

5. What happens to $\Delta_\pm^{\rm dS}$ when $m\ell > d/2$? What is qualitatively different in AdS? We will rediscover these behaviours by solving the Klein–Gordon equation explicitly below.

C. Free scalar field theory in de Sitter space

Consider a free scalar field of mass m in the expanding Poincaré patch of de Sitter space,

$$ds^2 = \frac{\ell^2}{\eta^2}\left(-d\eta^2 + d\vec{x}^2\right), \qquad -\infty < \eta < 0\,. \tag{32}$$

Future infinity is at $\eta \to 0^-$. The action is

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right). \tag{33}$$

1. Fourier decompose

$$\phi(\eta,\vec{x}) = \int \frac{d^dk}{(2\pi)^d}\,\phi_k(\eta)e^{i\vec{k}\cdot\vec{x}} \tag{34}$$

and derive from $(\Box - m^2)\phi = 0$ the mode equation

$$\eta^2\phi_k'' - (d-1)\eta\phi_k' + \left(k^2\eta^2 + m^2\ell^2\right)\phi_k = 0. \tag{35}$$

2. Set $u = -k\eta$ and write $\phi_k = (-\eta)^{d/2}f(u)$. Show that f obeys Bessel's equation with index

$$\nu = \sqrt{\frac{d^2}{4} - m^2\ell^2}, \tag{36}$$

precisely the quantity found in Problem B. Hence

$$\phi_k(\eta) = (-\eta)^{d/2}\left[A_k H_\nu^{(1)}(-k\eta) + B_k H_\nu^{(2)}(-k\eta)\right]. \tag{37}$$

3. Using the small-argument behaviour of the Hankel functions, show that close to future infinity

$$\phi_k(\eta) \sim \alpha_k(-\eta)^{\Delta_-} + \beta_k(-\eta)^{\Delta_+}, \qquad \Delta_\pm = \frac{d}{2} \pm \nu. \tag{38}$$

Compare directly with the result obtained from the invariant two-point function in Problem B.

4. Distinguish the two cases

$$0 < m^2\ell^2 < \frac{d^2}{4} \qquad \text{and} \qquad m^2\ell^2 > \frac{d^2}{4}. \tag{39}$$

In the second case write $\nu = i\mu$ and, using $\eta = -\ell e^{-t/\ell}$, show that at late times

$$\phi_k(t) \sim e^{-dt/(2\ell)}e^{\mp i\mu t/\ell}. \tag{40}$$

The first range is the *complementary series*, while the heavy fields in the second range belong to the *principal series*. What is oscillating and what is decaying in the principal-series solution?

D. The Bunch–Davies state

The mode equation does not specify a quantum state. For $-k\eta \gg 1$, wavelengths are much shorter than the curvature radius, so the modes should approach Minkowski positive-frequency modes.

1. Use the large-argument behaviour of the Hankel functions to determine which linear combination in Problem C behaves as
$$\phi_k(\eta) \propto (-\eta)^{(d-1)/2} \frac{e^{-ik\eta}}{\sqrt{2k}} \qquad (-k\eta \gg 1) \tag{41}$$
up to an overall k-independent normalization and phase. This choice defines the Bunch–Davies (or Euclidean) state.

2. Why is some additional condition of this kind necessary in de Sitter space? Compare with Minkowski space, where time translations give a preferred notion of positive frequency.

3. Now specialize to $d = 3$ and a minimally coupled massless field, $m = 0$. Then $\nu = 3/2$. Using the explicit form of $H^{(1)}_{3/2}$, show that the positive-frequency mode selected by the Bunch–Davies prescription can be written, up to an irrelevant overall phase, as
$$\phi_k(\eta) = \frac{1}{\ell\sqrt{2k^3}}(1 + ik\eta)e^{-ik\eta} . \tag{42}$$
What happens to this mode after horizon crossing, $-k\eta \ll 1$?

4. Consider the equal-time variance
$$\langle \phi^2(\eta, \vec{x}) \rangle = \int \frac{d^3k}{(2\pi)^3} |\phi_k(\eta)|^2 . \tag{43}$$
Show that the late-time contribution of the massless modes contains
$$\langle \phi^2 \rangle \sim \frac{1}{4\pi^2\ell^2} \int \frac{dk}{k} . \tag{44}$$
What kind of divergence is this? Is it a short-distance or a long-distance problem?

5. For a light but non-zero mass, write $\nu = 3/2 - \epsilon$ with $0 < \epsilon \ll 1$. Using the small-argument behaviour of the Hankel function, show that the infrared part of the same integral behaves schematically as
$$\int_0 dk\, k^{-1+2\epsilon} . \tag{45}$$
Explain how the mass regulates the divergence, and why the limit $m \to 0$ is singular.

6. The range $0 < m^2\ell^2 < d^2/4$ is the *complementary series*. The massless minimally coupled scalar has an additional zero-mode subtlety (closely related to the scalar *discrete series*). What warning does the calculation above give about defining a de Sitter-invariant two-point function for this field?

E. The de Sitter temperature

The Bunch–Davies state can also be characterized by Euclidean continuation. In global coordinates,
$$ds^2 = -dt^2 + \ell^2 \cosh^2(t/\ell) d\Omega_d^2 . \tag{46}$$

1. Make the continuation
$$t = i\ell\left(\theta - \frac{\pi}{2}\right) \tag{47}$$
and show that the metric becomes that of a round $(d+1)$-sphere,
$$ds_E^2 = \ell^2 \left(d\theta^2 + \sin^2\theta\, d\Omega_d^2\right) . \tag{48}$$
This gives another way of selecting the Bunch–Davies/Euclidean state: require regularity on the Euclidean sphere.

2. Now consider the dS static patch,

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{1 - r^2/\ell^2} + r^2 d\Omega_{d-1}^2 \,. \tag{49}$$

The cosmological horizon is at $r = \ell$. Wick rotate $t = -i\tau$ and expand the Euclidean metric close to the horizon. Show that regularity requires

$$\tau \sim \tau + 2\pi\ell \,. \tag{50}$$

3. Show that the Euclidean static-patch geometry is the same round sphere obtained from the global continuation above. In particular, set

$$r = \ell \sin\theta, \qquad \varphi = \frac{\tau}{\ell}, \tag{51}$$

and show that

$$ds_E^2 = \ell^2 \left(d\theta^2 + \cos^2\theta \, d\varphi^2 + \sin^2\theta \, d\Omega_{d-1}^2\right), \tag{52}$$

with $\varphi \sim \varphi + 2\pi$. Recognize this as another coordinate system on the round S^{d+1}. Thus the Euclidean continuation of the static patch and the Euclidean continuation of global de Sitter describe the same smooth geometry.

4. Deduce the Gibbons–Hawking temperature

$$T_{\mathrm{dS}} = \frac{1}{2\pi\ell} \,. \tag{53}$$

5. The Bunch–Davies state is a pure state of the full de Sitter spacetime, but a static observer has access to only one static patch. Explain qualitatively how these two statements can be compatible with the observer seeing a thermal density matrix.

6. Compare this with a familiar flat-space example: the Minkowski vacuum restricted to a Rindler wedge. What is similar, and what is genuinely new in de Sitter space?

F. Free scalar field theory in Anti-de Sitter space

Consider a scalar field in the Poincaré patch of AdS_{d+1},

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2\right), \qquad z > 0 \,. \tag{54}$$

The conformal boundary is at $z = 0$.

1. Take

$$\phi(z, t, \vec{x}) = f(z) e^{-i\omega t + i\vec{k}\cdot\vec{x}} \,. \tag{55}$$

Derive the radial Klein–Gordon equation. Compare its form with the de Sitter equation in Problem C.

2. Without solving the full equation, assume $f(z) \sim z^\Delta$ near $z = 0$. Recover the relation already found in Problem B,

$$\Delta(\Delta - d) = +m^2\ell^2, \tag{56}$$

and hence the two AdS falloffs z^{Δ_-} and z^{Δ_+}.

3. Compare these falloffs with the de Sitter late-time weights obtained in Problems B and C. In particular:

   (a) Why do sufficiently heavy fields have complex late-time weights in dS but not in AdS?

   (b) What happens instead if $m^2 < 0$ in AdS? By demanding real Δ, recover the Breitenlohner–Freedman bound.

   (c) The relations look almost like an analytic continuation of one another. What continuation of the curvature radius would exchange them?

4. In global AdS there is a globally timelike Killing vector and one can define the vacuum using positive-frequency normal modes with respect to global time. De Sitter has no globally timelike Killing vector. Explain qualitatively why this makes the choice of state in dS conceptually different, even though the Bunch–Davies state plays a distinguished role.

G. Geodesics in (A)dS

There is a particularly simple classical comparison in static coordinates. Consider radial motion in

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2\,, \tag{57}$$

with

$$f_{\text{dS}}(r) = 1 - \frac{r^2}{\ell^2}\,, \qquad f_{\text{AdS}}(r) = 1 + \frac{r^2}{\ell^2}\,. \tag{58}$$

1. For a radial timelike geodesic parametrized by proper time τ, show that

$$E = f(r)\dot{t} \tag{59}$$

is conserved and that the normalization of the velocity implies

$$\dot{r}^2 = E^2 - f(r)\,. \tag{60}$$

2. Differentiate this equation with respect to proper time and show that

$$\ddot{r} = +\frac{r}{\ell^2} \quad \text{in dS,} \qquad \ddot{r} = -\frac{r}{\ell^2} \quad \text{in AdS.} \tag{61}$$

Solve both equations. What is the qualitative difference between radial timelike geodesics in the two spacetimes?

3. Explain how this reflects the very different causal roles of the two geometries: AdS behaves like a confining box, while in a dS static patch generic radial timelike motion is driven toward the cosmological horizon.

4. **Geodesic approximation for heavy fields.** In the semiclassical limit $m\ell \gg 1$, the Euclidean propagator is approximated by

$$G(x, y) \sim \exp[-m\,\mathcal{L}_{\text{geo}}(x, y)]\,. \tag{62}$$

When the endpoints are taken to the AdS boundary, this gives the large-dimension approximation

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim \exp\left[-m\,\mathcal{L}_{\text{geo}}^{\text{ren}}(x, y)\right]\,, \qquad m\ell \gg 1, \tag{63}$$

where the divergent part of the geodesic length has been subtracted. In Euclidean AdS_2, take

$$ds^2 = \frac{\ell^2}{z^2}(dz^2 + dx^2)\,. \tag{64}$$

Show that the geodesic connecting two regulated boundary points $(z, x) = (\varepsilon, \pm L/2)$ is, as $\varepsilon \to 0$, a semicircle, and that its regulated length is

$$\mathcal{L}_{\text{geo}} = 2\ell \log \frac{L}{\varepsilon} + \cdots\,. \tag{65}$$

Use the geodesic approximation to show that the boundary two-point function has the expected power-law dependence on L. How does the large-m result compare with $\Delta_+^{\text{AdS}} \simeq m\ell$?

5. Returning to dS, recall that for two spacelike-separated points connected by a real geodesic of length $\mathcal{L}$,

$$P = \cos\frac{\mathcal{L}}{\ell}. \tag{66}$$

A real spacelike geodesic therefore has $-1 \le P \le 1$. But in Problem A you found that in global dS one can have $P < -1$. What does this suggest about the geodesic connecting such points? By analytically continuing the relation above, show that for $P < -1$ one may write

$$\mathcal{L} = \pi\ell \pm i\ell\,\text{arccosh}(-P). \tag{67}$$

For $-P \gg 1$, use $G \sim e^{-m\mathcal{L}}$ to show that the result contains oscillatory factors $(-P)^{\mp i m\ell}$. Compare this with the principal-series exponents $\Delta_\pm = d/2 \pm i\mu$ when $m\ell \gg 1$.

Take-home message. A free scalar already sees several of the key peculiarities of de Sitter space: invariant separation behaves very differently near the dS and AdS conformal boundaries, late-time data replace the usual AdS radial data, sufficiently heavy particles have complex late-time weights and oscillate, light fields develop strong infrared sensitivity, the choice of state requires extra physical input, and the distinguished Euclidean state looks thermal to a static-patch observer.