Exercises for La Ricotta 2026 lectures on Asymptotic Observables and Analyticity

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1. Consider the "entangled squeezed state" between two harmonic oscillators:

$$|c\rangle = \mathcal{N} \exp\left(b_1^\dagger b_2^\dagger c\right)|0\rangle \tag{1}$$

where b_i are ladder operators ($[b_i, b_j^\dagger] = \delta_{ij}$), c is a constant with $|c| < 1$ and $\mathcal{N}$ a normalization.

   (a) Show that tracing out the b_2 oscillator yields a thermal density matrix for b_1. How is the temperature related to c?

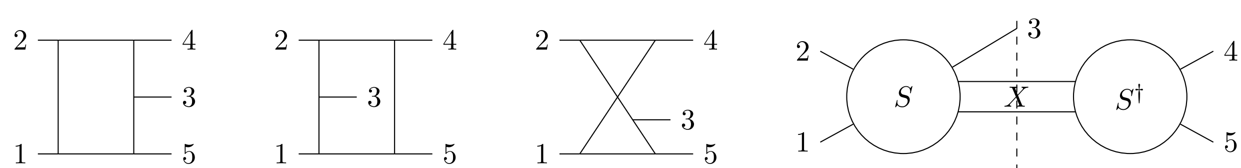
   (b) Show that this state is equivalent to the vacuum $a_i|0\rangle_a = 0$ via the Bogoliubov transform (*Hint:* which combinations of b and $b^\dagger$'s annihilate $|0\rangle_a$?):

$$b_1 = \frac{a_1 + c a_2^\dagger}{\sqrt{1-|c|^2}}, \quad b_2 = \frac{a_2 + c a_1^\dagger}{\sqrt{1-|c|^2}}\,. \tag{2}$$

   (c) Verify that ${}_a\langle 0|b_i^\dagger b_i|0\rangle_a = \sum_j |\beta_{ij}|^2$ as derived for a general transformation $b_i = \alpha_{ij}a_j + \beta_{ij}a_j^\dagger$.

   (d) What does this model tell us about the state of the radiation of a black hole, if each exterior mode b_1 is entangled with some interior mode b_2 as in this model?

2. Consider the following diagram topologies and blob pattern for the in-in expectation value ${}_{\text{in}}\langle 45|b_3|12\rangle_{\text{in}}$ or "waveform":



(In the third diagram imagine that the lines do not touch at the center.)

Throughout, assume that external particles have positive energies and are kinematically stable: no cut can isolate a single 3-particle vertex.

(a) Enumerate all cuts of the diagrams that match the blob pattern, ie. contribute to the waveform.

Recall that each S or $S^\dagger$ blob can contain disconnected spectators, including an identity term where no scattering occurs. You should find rather few nonvanishing options (maybe only a single nontrivial cut)!

(b) According to Schwinger-Keldysh LSZ formulas (`https://arxiv.org/pdf/2308.02125`), the in-in waveform can be computed by summing over ways of assigning a SK branch '1' or '2' to each vertex in each diagram, except for the vertex to which 3 attached, which is always '1';[1] the cut then separates 1's from 2's. Discarding vanishing cuts, check if this reproduces your result from (a).

(c) Explain how each cut in (a) can be computed by modifying propagators in the usual time-ordered expression (ie. which $i\epsilon$'s get reversed and which propagators become δ-functions).

3. The Schwarzschild metric has the form:

$$ds^2 = -dt^2(1 - 2GM/r) + \frac{dr^2}{1 - 2GM/r} + r^2 d\Omega_2^2 \,. \tag{3}$$

(a) Find a change of coordinates $\tilde{r}(r)$ and time rescaling $\eta = \kappa t$ near $r = 2GM$ such that the metric becomes proportional to the Rindler form $d\tilde{r}^2 - \tilde{r}^2 d\eta^2$ (close to $r \approx 2GM$). Use this to relate the Killing vector ∂_t to the near-horizon boost ∂_η.

(b) The Euclidean version of the Schwarzschild metric is obtained by Wick rotating $\eta \to i\theta$. The required (why?) identification $\theta \simeq \theta + 2\pi$ implies a periodicity in Euclidean time: what is the corresponding temperature?

(c) The Euclidean Schwarzschild geometry is sometimes described as a "cigar"; try different ways of visualizing it.

(d) A finite-temperature Schwinger-Keldysh contour (for quantum fields in this geometry) can be defined by gluing a Euclidean arc of angle $2\pi - \epsilon$ (ie. almost the complete circle) with a real back-and-forth in real η.

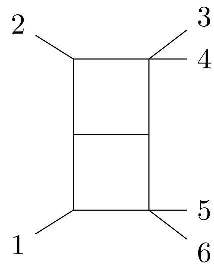
Try different ways of visualizing this geometry, and hopefully connecting the Euclidean cigar with the Penrose diagram of the exterior of the black hole.

4. The largest time equation states that multiplying forward and backward time evolution with the same sources is trivial: $U^\dagger[J]U[J] = \mathbb{I}$. In Schwinger-Keldysh perturbation theory, this implies that summing over all 1 vs 2 timefold assignments for *every* vertex in any diagram produces zero,

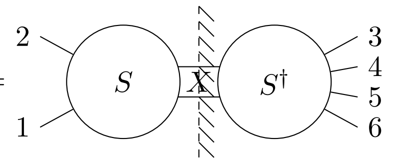
[1] Can you explain why this recipe is natural, given what you learned from Enrico about ϕ_r/ϕ_a sources?

which is an interesting identity related to the Cutkosky rules:

$$\sum_{\substack{\text{cuts that}\\ \text{match pattern}}}$$



$$= 0, \qquad \text{pattern}=$$

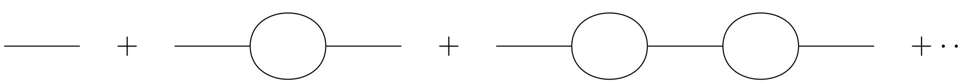


(4)

Use either the vertex-coloring method or the pattern-matching method to make explicit this identity for the shown diagram. For the pattern matching method, recall that each S blob can contain any number of passive spectators and connected components.

Assume that all external particles are on-shell and stable, and that $p_{1,2}$ are incoming and $p_{3,4,5,6}$ outgoing, and try to clean up your answer by removing cut diagrams that vanish for kinematic reasons. (Hint: besides the uncut diagram with all T-ordered propagators, and its complex conjugate, I believe there are exactly 6 cut diagrams.)

5. Consider the self-energy chain for a scalar field with $g\phi^3$ self-interaction:



(5)

   (a) In ordinary QFT, such chains give geometric series that are easy to resum. Show (eg. combinatorially by assigning r's and a's to each field at each vertex) that the same is true for the retarded correlator in the r/a formalism: $\langle \phi_r(P)\phi_a\rangle$ resums to $G_{\text{ret}}^{\text{resummed}}(P) = -i/(P^2 - M^2 + \Pi_{\text{ret}}(P))$ where Π_{ret} is the one-loop self-energy we computed in class.

   (b) For the $\langle \phi_r\phi_r\rangle$ two-point function, show that you get a sum of geometric series involving $G_{\text{ret}}^{\text{resummed}}(P)$ and multiplying two extra ingredients: either Π_{aa} that we computed or $G_{rr}^{(0)}$.

6. Consider a massless scalar field coupled to a worldline theory ("black hole") with effective action:

$$\begin{aligned} S = &-\int d^4x\, \partial_\mu\phi_r \partial^\mu \phi_a + \\ &+ \int dt\, \left[\phi_a\left(\chi_0 + \chi_1\partial_t + \ldots\right)\phi_r(t, \vec{x}=0) + ik\phi_a^2 + (\text{higher derivatives})\right]. \end{aligned} \tag{6}$$

   (a) Working to linear order in χ and k, show that the off-shell correlator ${}_{\text{in}}\langle \text{BH}|\phi_r(p')\phi_a(-p)|\text{BH}\rangle_{\text{in}}$ extracts χ but not k.

   (b) Determine if/how k relates to the zero-frequency fluctuations at large distances:

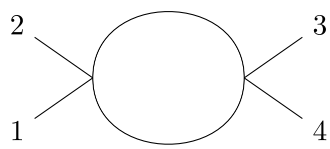
$$\int_{-\infty}^{+\infty} dt\ {}_{\text{in}}\langle \text{BH}|\phi_r(\vec{x_1}, t)\phi_r(\vec{x}_2, 0)|\text{BH}\rangle_{\text{in}} = k \times (?). \tag{7}$$

(c) Taking ϕ on-shell, a natural (why?) guess is that LSZ amputation of (a) gives a commutator of asymptotic fields: ${}_{\rm in}\langle \mathrm{BH}|[b_{p'}, a_p^\dagger]|\mathrm{BH}\rangle_{\rm in}$ (see 2308.02125).

Show that the LSZ amputation of the time-ordered correlator gives you instead a more complicated combination of both χ and k, even in the zero-frequency limit. Is it possible to interpret the discrepancy by stringing together words like "spontaneous emission", "absorption" and "stimulated emission"?

7. Consider an amplitude that is the sum of two one-loop bubble diagrams:

$$\mathcal{M}(s,t) =$$



$$+ (2 \leftrightarrow 3) \propto \mathrm{Bub}(s) + \mathrm{Bub}(u)\,,$$

$$\mathrm{Bub}(s) \equiv \Gamma(\epsilon) \int_0^1 \frac{da\, \mu^{2\epsilon}}{(-a(1-a)s + m^2 - i0)^\epsilon} \xrightarrow[\epsilon, m \to 0]{} -\log(\frac{-s - i0}{\mu^2}) + \text{constant}\,.$$

Here $s = -(p_1+p_2)^2$ and $u = 4m^2 - s - t$ are Mandelstam invariants ($s > 0$ $p_1 + p_2$ is timelike), and we have given both an exact formula for the diagram (using Feynman parameter a and dimensional regularization to $d = 4 - 2\epsilon$) and an approximate formula for the massless limit. Throughout, take $t < 0$ to be real and fixed.

Note that $\log(-s - i0)$ is real when $s < 0$ and has the imaginary part $-i\pi$ when $s > 0$; the notation is supposed to make this intuitive, make sure you see why!

(a) Starting with s positive, analytically continue the massless limit of $\mathcal{M}(s,t)$ through the upper-half s-plane to an endpoint where s and u are swapped.

Show that the result does *not* coincide with simply swapping s and u in the formula (eg. doing nothing), but rather it gives $\mathcal{M}^*$.

(b) Do the same conclusions hold for the full bubble function with $m \neq 0$? Try to continue $\mathrm{Bub}(s)$ without actually computing it.

(c) Is there another continuation path which would give $\mathcal{M}$ in this example? Are there restrictions on t and m for your path to exist?

8. This problem is based on the analysis of the single-scalar EFT without EFT loops in 2011.02957, adapted to the more modern crossing-symmetric symmetric formalism. At low energies:

$$\mathcal{M}_{\rm low}(s,t) = g^2 \left(\frac{1}{m_\phi^2 - s} + \frac{1}{m_\phi^2 - t} + \frac{1}{m_\phi^2 - u} \right) - g_0 + g_2(s^2+t^2+u^2) + g_3(stu)$$
$$+ g_4(s^2+t^2+u^2)^2 + g_5(s^2+t^2+u^2)(stu) + g_6(s^2+t^2+u^2)^3 + g_6'(stu)^2 + \ldots$$

(a) This low-energy theory admits an action

$$S_{\text{low}} = \int d^d x \left(-\frac{1}{2}(\partial\phi^2) - \frac{1}{2}m_\phi^2\phi^2 - \frac{g}{6}\phi^3 - \frac{g_0}{4!}\phi^4 + g'(\partial\phi)^2\phi + \# g_2(\partial\phi)^4 + \dots \right)$$

Verify that the g' term can be removed by a field redefinition, order by order in the small coupling expansion (at the cost of shifting the value of other couplings): there is only one genuine 3-point coupling, g.

(b) From now, set $m_\phi = 0$. The first two crossing-symmetric dispersive sum rules (with $k = 2, 4$ subtractions) give (see 2501.18465 or 2506.09884 and references therein if you want to rederive this):

$$\begin{aligned} k=2: \qquad & 2g_2 + p^2 g_3 = \left\langle \frac{2m^2+3p^2}{m^6}\mathcal{P}_J\left(\sqrt{\frac{m^2-3p^2}{m^2+p^2}}\right)\right\rangle \\ k=4: \qquad & 4g_4 + 2p^2 g_5 + p^4 g_6' = \left\langle \frac{(2m^2+3p^2)(m^2+p^2)}{m^{12}}\mathcal{P}_J\left(\sqrt{\frac{m^2-3p^2)}{m^2+p^2}}\right)\right\rangle \end{aligned} \tag{8}$$

where $\langle\cdots\rangle$ is a sum over heavy states with mass $m \geq M$ and even spin $J \geq 0$ and *positive* unknown coefficients; $\mathcal{P}_J(x) = {}_2F_1(-J, J+d-3, \frac{1}{2}(d-2), \frac{1}{2}(1-x))$ is a Legendre/Gegenbauer polynomial, and $p^2 < M^2/3$.

Assuming (untrue with EFT loops!!!) that a regular Taylor series as $p^2 \to 0$ exists, deduce the sum rules (in $d = 4$):

$$g_2 = \left\langle \frac{1}{m^4}\right\rangle, \quad g_3 = \left\langle \frac{3-2J(J+1)}{m^6}\right\rangle, \quad g_4 = \left\langle \frac{1}{2m^8}\right\rangle, \quad 0 = \left\langle \frac{J(J+1)(J(J+1)-8)}{m^8}\right\rangle.$$

Use positivity of $\langle\cdots\rangle$ to show that $g_4 \leq \frac{g_2}{2M^4}$ and $g_3 \leq \frac{3g_2}{M^2}$ (easy).

(c) Prove the lower bound $g_3 \geq -10.61249\frac{g_2}{M^2}$ using only the g_2, g_3 and "0" sum rules above. This can be done analytically (not so easy; hint: key constraints come from $J = 2, 4$) or numerically with a computer; numerical hints below.

Note: The bound can be improved since using more *null constraints* $0 = \langle\cdots\rangle$ on unknown heavy couplings. Finding the optimal bound with this method maps to a *semidefinite linear problem* that can be solved by powerful algorithms; the optimal bound found in this way is $-10.346 \leq \frac{g_3 M^2}{g_2} \leq 3$, see [?].

(d) * Show numerically (or argue generally) that the same bounds can be obtained by integrating (8) against wavepackets $\psi_k(p)$ with $0 < p < M/\sqrt{3}$. (Wavepackets, as opposed to forward limits, are more stable and practically essential to deal with either gravity or EFT loops.)

On solving linear optimization programs

Given a matrix `mat` and vectors `obj` and `norm`, suppose we want the vector v which maximizes the "objective" `obj.v`, subject to the constraints `mat.v`$\geq$0 and normalization `norm.v==1`. Mathematica's `LinearOptimization[-obj,{mat,0*mat[[;;,1]]},{{norm},{1}}]` will return just that.

Concretely, for question 4(c) with the three sum rules $\{g_2, g_3, 0\}$, v is a three vector, `norm=`$\{0, 1, 0\}$ and `obj=`$\{-1, 0, 0\}$ (explain why!); `mat` is a $N \times 3$ matrix whose rows represent N discrete values of J and m. Since this method imposes positivity at only discrete values, it is important to plot the outcome and verify positivity for all $m \geq 1$, refining your sampling if necessary!

For larger problems, or problems involving semi-definite matrices, the powerful solver SDPB is generally more stable and efficient; SDPB also handles polynomials in a variable $x \geq 0$.