Problem sheet La Ricotta Summer School 2026

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Exercises on density matrices, quantum maps, and Lindblad evolution

Unless stated otherwise, assume finite-dimensional Hilbert spaces.

Ex. 1: The qubit and the Bloch ball (introductory)

Any Hermitian unit-trace 2×2 matrix can be written as

$$\rho = \frac{1}{2}\left(\mathbb{1} + \mathbf{b}\cdot\boldsymbol{\sigma}\right), \qquad \mathbf{b} \in \mathbb{R}^3.$$

(a) Show that

$$(\mathbf{b}\cdot\boldsymbol{\sigma})^2 = |\mathbf{b}|^2\mathbb{1}.$$

(b) Find the eigenvalues of ρ and show that

$$\rho \geq 0 \qquad \Longleftrightarrow \qquad |\mathbf{b}| \leq 1.$$

(c) Compute

$$\mathrm{Tr}\left(\rho^2\right) = \frac{1}{2}(1 + |\mathbf{b}|^2).$$

(d) Show that ρ is pure iff $|\mathbf{b}| = 1$, and maximally mixed iff $\mathbf{b} = 0$.

(e) Express the von Neumann entropy $S(\rho)$ in terms of $|\mathbf{b}|$.

Ex. 2: From unitary evolution to Kraus operators

Let the initial system–environment state be

$$\rho_{SE}(0) = \rho \otimes \rho_E, \qquad \rho_E = \sum_\beta p_\beta |\beta\rangle\langle\beta|,$$

and let the full system evolve with a unitary operator U.

Choose an orthonormal environment basis $\{|\alpha\rangle\}$ and define

$$M_{\alpha\beta} := \sqrt{p_\beta}\,\langle\alpha|U|\beta\rangle.$$

(a) Starting from

$$\Phi[\rho] = \mathrm{Tr}_E\left[U(\rho\otimes\rho_E)U^\dagger\right],$$

derive

$$\Phi[\rho] = \sum_{\alpha,\beta} M_{\alpha\beta}\rho M_{\alpha\beta}^\dagger.$$

(b) Use $U^\dagger U = \mathbb{1}$ to prove

$$\sum_{\alpha,\beta} M_{\alpha\beta}^\dagger M_{\alpha\beta} = \mathbb{1}_S.$$

(c) Simplify the construction when the environment begins in a pure state,

$$\rho_E = |\chi\rangle\langle\chi|.$$

Ex. 3: Properties of a Kraus map

Consider

$$\Phi[X] = \sum_\mu M_\mu X M_\mu^\dagger, \qquad \sum_\mu M_\mu^\dagger M_\mu = \mathbb{1}.$$

Prove directly that Φ is:

(a) linear;

(b) Hermiticity preserving,

$$X = X^\dagger \quad \Longrightarrow \quad \Phi[X] = \Phi[X]^\dagger;$$

(c) positive,

$$X \geq 0 \quad \Longrightarrow \quad \Phi[X] \geq 0;$$

(d) trace preserving,

$$\operatorname{Tr}\Phi[X] = \operatorname{Tr}X;$$

(e) completely positive: for any auxiliary Hilbert space $\mathcal{H}_A$, prove that

$$X \geq 0 \quad \Longrightarrow \quad (\mathbb{1}_A \otimes \Phi)[X] \geq 0.$$

For the last part, use

$$(\mathbb{1}_A \otimes \Phi)[X] = \sum_\mu (\mathbb{1}_A \otimes M_\mu) X (\mathbb{1}_A \otimes M_\mu^\dagger).$$

Finally, show that if a Kraus map is trace preserving for every X, then

$$\sum_\mu M_\mu^\dagger M_\mu = \mathbb{1}.$$

Ex. 4: A positive map that is not completely positive

In a fixed basis, define the transpose map

$$T[X] = X^T.$$

(a) Show that T is linear, trace preserving, and positive.

(b) Consider the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).$$

Write its density matrix explicitly.

(c) Compute

$$(\mathbb{1} \otimes T)\left[|\Phi^+\rangle\langle\Phi^+|\right].$$

(d) Show that its eigenvalues are

$$\frac{1}{2}, \qquad \frac{1}{2}, \qquad \frac{1}{2}, \qquad -\frac{1}{2}.$$

(e) Conclude that the transpose map is positive but not completely positive.

Ex. 5: Hermitian jump operators and dephasing

Consider one Hermitian jump operator $L = L^\dagger$:

$$\dot{\rho} = -i[H, \rho] + \gamma \left(L\rho L - \frac{1}{2}\{L^2, \rho\} \right).$$

(a) Show that

$$L\rho L - \frac{1}{2}\{L^2, \rho\} = -\frac{1}{2}[L, [L, \rho]].$$

(b) Assume that H and L commute and choose a common eigenbasis,

$$H|n\rangle = E_n|n\rangle, \qquad L|n\rangle = \ell_n|n\rangle.$$

Show that

$$\dot{\rho}_{nm} = \left[-i(E_n - E_m) - \frac{\gamma}{2}(\ell_n - \ell_m)^2\right] \rho_{nm}.$$

(c) Solve for $\rho_{nm}(t)$.

(d) Show that the populations remain constant while coherences between states with $\ell_n \neq \ell_m$ decay.

(e) Which coherences survive if two different states have the same eigenvalue of L?

Ex. 6: From Lindblad jumps to the Pauli master equation

Let $\{|n\rangle\}$ be energy eigenstates and introduce

$$L_{nm} = \sqrt{W_{nm}}\,|n\rangle\langle m|, \qquad n \neq m.$$

(a) Interpret W_{nm} as the rate for which transition?

(b) Starting from the Lindblad equation, derive

$$\dot{p}_n = \sum_m \left(W_{nm}p_m - W_{mn}p_n\right), \qquad p_n := \rho_{nn}.$$

(c) Show explicitly that

$$\frac{d}{dt}\sum_n p_n = 0.$$

(d) Let

$$p_n^{(\beta)} = \frac{e^{-\beta E_n}}{Z}.$$

Show that this distribution is stationary if

$$\frac{W_{nm}}{W_{mn}} = e^{-\beta(E_n - E_m)}.$$

(e) Explain why this condition suppresses upward transitions at low temperature.

Ex. 7: The open qubit

Consider

$$H = \frac{\omega_0}{2}\sigma_z, \qquad L_\uparrow = \sqrt{\gamma_\uparrow}\,\sigma_+, \qquad L_\downarrow = \sqrt{\gamma_\downarrow}\,\sigma_-.$$

(a) Write the full Lindblad equation.

(b) By explicit matrix multiplication, derive

$$\dot{\rho}_{\uparrow\uparrow} = -\gamma_{\downarrow}\rho_{\uparrow\uparrow} + \gamma_{\uparrow}\rho_{\downarrow\downarrow}, \tag{1}$$

$$\dot{\rho}_{\uparrow\downarrow} = -\left[i\omega_0 + \frac{\gamma_{\uparrow} + \gamma_{\downarrow}}{2}\right]\rho_{\uparrow\downarrow}. \tag{2}$$

(c) Defining

$$\Gamma := \gamma_{\uparrow} + \gamma_{\downarrow},$$

solve for the population and coherence:

$$\rho_{\uparrow\uparrow}(t) = \rho_{\uparrow\uparrow}^{(*)} + \left[\rho_{\uparrow\uparrow}(0) - \rho_{\uparrow\uparrow}^{(*)}\right] e^{-\Gamma t}, \tag{3}$$

$$\rho_{\uparrow\downarrow}(t) = e^{-i\omega_0 t} e^{-\Gamma t/2} \rho_{\uparrow\downarrow}(0). \tag{4}$$

Determine $\rho_{\uparrow\uparrow}^{(*)}$.

(d) Find the full stationary density matrix.

(e) Show that it is thermal at inverse temperature β when

$$\frac{\gamma_{\uparrow}}{\gamma_{\downarrow}} = e^{-\beta\omega_0}.$$

(f) Study the limits of zero and infinite temperature.

(g) Rewrite the evolution in terms of the Bloch vector

$$\rho = \frac{1}{2}\left(\mathbb{1} + b_x\sigma_x + b_y\sigma_y + b_z\sigma_z\right)$$

and show that

$$\dot{b}_x = -\frac{\Gamma}{2} b_x - \omega_0 b_y, \tag{5}$$

$$\dot{b}_y = -\frac{\Gamma}{2} b_y + \omega_0 b_x, \tag{6}$$

$$\dot{b}_z = -\Gamma b_z + \gamma_{\uparrow} - \gamma_{\downarrow}. \tag{7}$$

Describe the resulting trajectory inside the Bloch ball.

Ex. 8: Perturbative evolution and secular resummation (short)

Consider

$$\dot{p}(t) = -\epsilon\gamma p(t), \qquad p(0) = p_0, \qquad \epsilon \ll 1.$$

(a) Solve the equation exactly.

(b) Expand the solution through order ϵ^3.

(c) Show that the fixed-order approximation fails when

$$\epsilon\gamma t \sim 1.$$

(d) Explain why the first-order approximation eventually predicts a negative value of $p(t)$.

(e) Show that solving the leading-order generator exactly resums all terms of the form

$$\frac{(-\epsilon\gamma t)^n}{n!}.$$

(f) Now suppose

$$\mathcal{L} = \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2 + \cdots .$$

Expand $e^{t\mathcal{L}}$ through order ϵ^2 and identify which terms are included in

$$e^{\epsilon t\mathcal{L}_1}$$

and which are omitted.

Ex. 9: Hermitian jump operators and unital evolution (short)

Consider the Lindblad equation

$$\dot{\rho} = \mathcal{L}[\rho] = -i[H,\rho] + \sum_\mu \left(L_\mu \rho L_\mu^\dagger - \frac{1}{2}\{L_\mu^\dagger L_\mu, \rho\} \right),$$

where every jump operator is Hermitian:

$$L_\mu = L_\mu^\dagger.$$

(a) Evaluate $\mathcal{L}[\mathbb{1}]$ and show that

$$\mathcal{L}[\mathbb{1}] = 0.$$

Conclude that the dynamical map is unital:

$$\Phi_t[\mathbb{1}] = \mathbb{1}.$$

Ex. 10: Deriving the thermal density matrix (optional)

Find the density matrix that maximizes

$$S(\rho) = -\operatorname{Tr}(\rho \ln \rho)$$

subject to

$$\operatorname{Tr}\rho = 1, \qquad \operatorname{Tr}(\rho H) = U.$$

(a) Introduce Lagrange multipliers λ and β and extremize

$$\mathcal{F}[\rho] = S(\rho) - \lambda\big(\operatorname{Tr}\rho - 1\big) - \beta\big(\operatorname{Tr}(\rho H) - U\big).$$

(b) Using

$$\delta S = -\operatorname{Tr}\left[\delta\rho\,(\ln\rho + 1)\right],$$

show that $\delta\mathcal{F} = 0$ implies

$$\rho = C\, e^{-\beta H}.$$

(c) Determine C from normalization and obtain

$$\rho_\beta = \frac{e^{-\beta H}}{Z(\beta)}, \qquad Z(\beta) = \operatorname{Tr}\Big(e^{-\beta H}\Big).$$

(d) In the energy basis, $H|n\rangle = E_n|n\rangle$, find the probabilities p_n.

(e) Show that

$$U = -\frac{\partial}{\partial\beta}\ln Z, \qquad S(\rho_\beta) = \beta U + \ln Z.$$

Exercises on Schwinger-Keldysh formalism and dynamical map

Ex. 11: Propagators in the r/a basis

Starting from

$$\langle\!\langle x_+(t)x_+(t')\rangle\!\rangle = \langle T\, x(t)x(t')\rangle, \tag{8}$$
$$\langle\!\langle x_-(t)x_-(t')\rangle\!\rangle = \langle \bar{T}\, x(t)x(t')\rangle, \tag{9}$$
$$\langle\!\langle x_+(t)x_-(t')\rangle\!\rangle = \langle x(t')x(t)\rangle, \tag{10}$$
$$\langle\!\langle x_-(t)x_+(t')\rangle\!\rangle = \langle x(t)x(t')\rangle, \tag{11}$$

introduce

$$x_r = \frac{x_+ + x_-}{2}, \qquad x_a = x_+ - x_-.$$

Write out the four r/a correlators explicitly. Using

$$T\, x(t)x(t') = \theta(t-t')x(t)x(t') + \theta(t'-t)x(t')x(t), \tag{12}$$
$$\bar{T}\, x(t)x(t') = \theta(t'-t)x(t)x(t') + \theta(t-t')x(t')x(t), \tag{13}$$

show that

$$\langle\!\langle x_r(t)x_r(t')\rangle\!\rangle = \frac{1}{2}\langle\{x(t), x(t')\}\rangle, \tag{14}$$
$$\langle\!\langle x_r(t)x_a(t')\rangle\!\rangle = \theta(t-t')\langle[x(t), x(t')]\rangle, \tag{15}$$
$$\langle\!\langle x_a(t)x_r(t')\rangle\!\rangle = \theta(t'-t)\langle[x(t'), x(t)]\rangle, \tag{16}$$
$$\langle\!\langle x_a(t)x_a(t')\rangle\!\rangle = 0. \tag{17}$$

Ex. 12: Constraints on the Schwinger–Keldysh effective action

Assume an initially factorized state with a pure environment state

$$\rho_q^{(0)} = |\Omega_q\rangle\langle\Omega_q|.$$

The influence functional is

$$e^{iS_{\mathrm{IF}}[x_+,x_-]} = \int dq\, dq_i\, dq_i' \int_{q_i}^{q} \mathcal{D}q_+ \int_{q_i'}^{q} \mathcal{D}q_- \times e^{iS_{\mathrm{tot}}[x_+,q_+] - iS_{\mathrm{tot}}[x_-,q_-]} \rho_q^{(0)}(q_i, q_i'). \tag{18}$$

Treat x_+ and x_- as prescribed external sources for the environment and introduce the corresponding evolution operator

$$U_q[x] := U_q(t, t_0; \{x\}).$$

(a) Use the path-integral representation of transition amplitudes,

$$\int_{q_i}^{q} \mathcal{D}q_+\, e^{iS_{\mathrm{tot}}[x_+,q_+]} = \langle q|U_q[x_+]|q_i\rangle, \tag{19}$$
$$\int_{q_i'}^{q} \mathcal{D}q_-\, e^{-iS_{\mathrm{tot}}[x_-,q_-]} = \langle q_i'|U_q^{\dagger}[x_-]|q\rangle. \tag{20}$$

(b) Write

$$\rho_q^{(0)}(q_i, q_i') = \langle q_i|\Omega_q\rangle\langle\Omega_q|q_i'\rangle$$

and perform the integrals over q_i and q_i'.

(c) Insert

$$\int dq\,|q\rangle\langle q| = \mathbb{1}_q$$

and show that

$$e^{iS_{\rm IF}[x_+,x_-]} = \langle\Omega_q|U_q^\dagger[x_-]U_q[x_+]|\Omega_q\rangle.$$

Equivalently, defining

$$|\Omega_q^{\{x_\pm\}}(t)\rangle := U_q[x_\pm]|\Omega_q\rangle,$$

show that

$$e^{iS_{\rm IF}[x_+,x_-]} = \langle\Omega_q^{\{x_-\}}(t)|\Omega_q^{\{x_+\}}(t)\rangle.$$

Use normalization, complex conjugation, and the Cauchy–Schwarz inequality to derive

$$S_{\rm IF}[x_+,x_+] = 0, \tag{21}$$

$$S_{\rm IF}[x_+,x_-] = -S^*_{\rm IF}[x_-,x_+], \tag{22}$$

$$\mathrm{Im}\,S_{\rm IF}[x_+,x_-] \ge 0. \tag{23}$$

Finally, translate these conditions to the r/a basis:

$$S_{\rm eff}[x_r,x_a=0] = 0, \tag{24}$$

$$S_{\rm eff}[x_r,x_a] = -S^*_{\rm eff}[x_r,-x_a], \tag{25}$$

$$\mathrm{Im}\,S_{\rm eff}[x_r,x_a] \ge 0. \tag{26}$$

Ex. 13: Master equation for a noisy harmonic oscillator

Consider the phase-space Schwinger–Keldysh action

$$S_{\rm SK} = \int dt\left[p_a\dot{x}_r + p_r\dot{x}_a - p_rp_a - \Omega^2(t)x_rx_a + \frac{i}{2}N_x(t)x_a^2 + \frac{i}{2}N_p(t)p_a^2\right], \tag{27}$$

where we have set $\kappa = N_{xp} = 0$.

Use

$$x_r = \frac{x_+ + x_-}{2}, \qquad x_a = x_+ - x_-, \qquad p_r = \frac{p_+ + p_-}{2}, \qquad p_a = p_+ - p_-.$$

(a) Show that the real part of the action can be written as

$$S_H[x_+,p_+] - S_H[x_-,p_-],$$

where

$$S_H[x,p] = \int dt\ (p\dot{x} - H(t)), \qquad H(t) = \frac{p^2}{2} + \frac{\Omega^2(t)}{2}x^2.$$

Conclude that these terms generate

$$-i[H(t),\rho].$$

(b) Over an infinitesimal interval, use

$$\mathcal{K}(t+\delta t,t) = \mathbb{1} + \delta t\,\mathcal{L}_t + \mathcal{O}(\delta t^2), \qquad e^{iS_{\rm SK}} = 1 + iS_{\rm SK} + \mathcal{O}(\delta t^2).$$

Verify the operator dictionary

$$x_a \longleftrightarrow [x,\rho], \qquad p_a \longleftrightarrow [p,\rho],$$

by interpreting x_+, p_+ as left actions on ρ and x_-, p_- as right actions.

(c) Use this dictionary to show that

$$\frac{i}{2} N_x x_a^2 \longrightarrow -\frac{N_x}{2}[x,[x,\rho]], \tag{28}$$

$$\frac{i}{2} N_p p_a^2 \longrightarrow -\frac{N_p}{2}[p,[p,\rho]]. \tag{29}$$

(d) Combine the Hamiltonian and noise terms and derive

$$\dot{\rho} = -i[H(t),\rho] - \frac{N_x(t)}{2}[x,[x,\rho]] - \frac{N_p(t)}{2}[p,[p,\rho]].$$

Rewrite the result in GKSL form and identify the two Hermitian jump operators. What conditions must $N_x(t)$ and $N_p(t)$ satisfy for the evolution to be CP divisible?

(e) Optional: using the dictionary above check that there is no operator ordering issue in $x_a p_a$ and $x_r p_r$, but there is one in $x_r p_a$ and $x_a p_r$.

Ex. 14: Retarded propagator and memory in a dissipative scalar field

For each momentum mode, the retarded propagator satisfies

$$\left(\partial_t^2 + \gamma \partial_t + \omega_{\mathbf{k}}^2\right) G_R(t,\mathbf{k}) = \delta(t), \qquad G_R(t,\mathbf{k}) = 0 \quad \text{for } t < 0,$$

where

$$\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m^2, \qquad \gamma \geq 0.$$

(a) Using

$$G_R(t,\mathbf{k}) = \int \frac{d\omega}{2\pi} e^{-i\omega t} G_R(\omega,\mathbf{k}),$$

find $G_R(\omega,\mathbf{k})$. Derive the dispersion relation from its poles and show that the two solutions are

$$\omega_\pm(\mathbf{k}) = -\frac{i\gamma}{2} \pm \sqrt{\omega_{\mathbf{k}}^2 - \frac{\gamma^2}{4}}.$$

Classify the poles as underdamped, critically damped, or overdamped.

(b) In the overdamped regime,

$$\gamma > 2\omega_{\mathbf{k}}, \qquad \Lambda_{\mathbf{k}} := \sqrt{\frac{\gamma^2}{4} - \omega_{\mathbf{k}}^2},$$

perform the inverse Fourier transform by closing the contour in the appropriate half-plane. Show that

$$G_R(t,\mathbf{k}) = \theta(t) \frac{e^{-(\gamma/2-\Lambda_{\mathbf{k}})t} - e^{-(\gamma/2+\Lambda_{\mathbf{k}})t}}{2\Lambda_{\mathbf{k}}}.$$

Take the limit $\Lambda_{\mathbf{k}} \to 0$ and find the propagator at critical damping.

(c) Discuss how long the system retains memory of a perturbation in the following cases:

i. $\gamma < 2\omega_{\mathbf{k}}$;
ii. $\gamma = 2\omega_{\mathbf{k}}$;
iii. $\gamma > 2\omega_{\mathbf{k}}$;
iv. $\mathbf{k}^2 + m^2 \ll \gamma^2$;

v. $m = 0$ and $\mathbf{k} = 0$;

vi. $\gamma = 0$.

In the overdamped low-frequency regime, show that the slow decay rate is

$$\Gamma_{\text{slow}} = \frac{\gamma}{2} - \Lambda_{\mathbf{k}} \simeq \frac{\mathbf{k}^2 + m^2}{\gamma}.$$

Explain when the memory time is finite, when it becomes parametrically large, and when the propagator does not decay at late times.

Ex. 15: Tree-level bispectrum from a $\dot{\phi}^3$ interaction

Consider a massless scalar field in de Sitter,

$$a(\tau) = -\frac{1}{H\tau}, \qquad u_k(\tau) = \frac{H}{\sqrt{2k^3}}(1 + ik\tau)e^{-ik\tau},$$

with cubic interaction

$$S_{\text{int}} = \frac{\lambda}{3!}\int dt\, d^3x\, a^3 \dot{\phi}^3.$$

To leading order in λ, the interaction Hamiltonian in conformal time is

$$H_{\text{int}}(\tau) = -\frac{\lambda}{3!}\int d^3x\, a(\tau)\phi'(\tau, \mathbf{x})^3.$$

(a) Using the $+/-$ Feynman rules, draw the two tree-level contact diagrams contributing to

$$\langle \phi_{\mathbf{k}_1}(0)\phi_{\mathbf{k}_2}(0)\phi_{\mathbf{k}_3}(0)\rangle\,.$$

Show that the $+$ and $-$ diagrams are complex conjugates and write their sum as

$$\langle \phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_3}\rangle' = 2\,\text{Re}\left[-i\int_{-\infty}^{0} d\tau\, \langle \phi_{\mathbf{k}_1}(0)\phi_{\mathbf{k}_2}(0)\phi_{\mathbf{k}_3}(0)H_{\text{int}}(\tau)\rangle\right]. \tag{30}$$

The prime denotes removal of

$$(2\pi)^3\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3).$$

(b) Defining

$$k_t := k_1 + k_2 + k_3,$$

reduce the calculation to an integral proportional to

$$\int_{-\infty}^{0} d\tau\, \tau^2 e^{ik_t\tau},$$

with the usual $i\epsilon$ prescription, and evaluate it.

(c) Show that the reduced tree-level bispectrum is

$$\langle \phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_3}\rangle' = \frac{\lambda H^5}{2\,k_1k_2k_3\,k_t^3}.$$

Verify that the result is symmetric under permutations of the three momenta and finite as $\tau \to 0^-$.

Ex. 16: Covariant building blocks for the Goldstone of time translations

Let

$$U(x) = t + \pi(x).$$

Under a Poincaré transformation

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + \alpha^\mu,$$

the Goldstone field transforms non-linearly as

$$\pi(x) \longrightarrow \pi'(x) = \pi(x') + \alpha^0 + \Lambda^0{}_\mu x^\mu - t.$$

(a) Show that

$$t + \pi'(x) = t' + \pi(x'), \qquad t' = \Lambda^0{}_\mu x^\mu + \alpha^0.$$

Conclude that $t + \pi$ transforms covariantly as the scalar field $U(x)$:

$$U'(x) = U(x').$$

(b) Explicitly compute the transformation of

$$-2\dot{\pi} + \partial_\mu \pi \, \partial^\mu \pi.$$

and conclude that it is a covariant scalar building block.

References