

A holographic prescription for dissipative hydrodynamic actions and horizon symmetries

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Based on: MB, Arpit Das and Richard Davison

[1]. Dissipative hydrodynamic actions and horizon symmetries in gravity [yesterday - 2607.00866]

[2]. Horizon symmetries of exact hydrodynamic actions [soon]

Introduction

- Hydrodynamics provides a universal way to describe the long distance, late time dynamics of conserved quantities in many-body quantum systems or quantum field theory.
- Traditionally formulated at the level of equations of motion. E.g. in a relativistic QFT

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu} + \dots \qquad J^\mu = \rho u^\mu + \dots$$

- Energy density, pressure, charge density of functions of local chemical potential and temperature $\mu(t, \vec{x}), T(t, \vec{x})$. Dots indicate expansion in derivatives of $u^\mu(t, x), \mu(t, \vec{x}), T(t, \vec{x})$
- Equations of motion are conservation laws

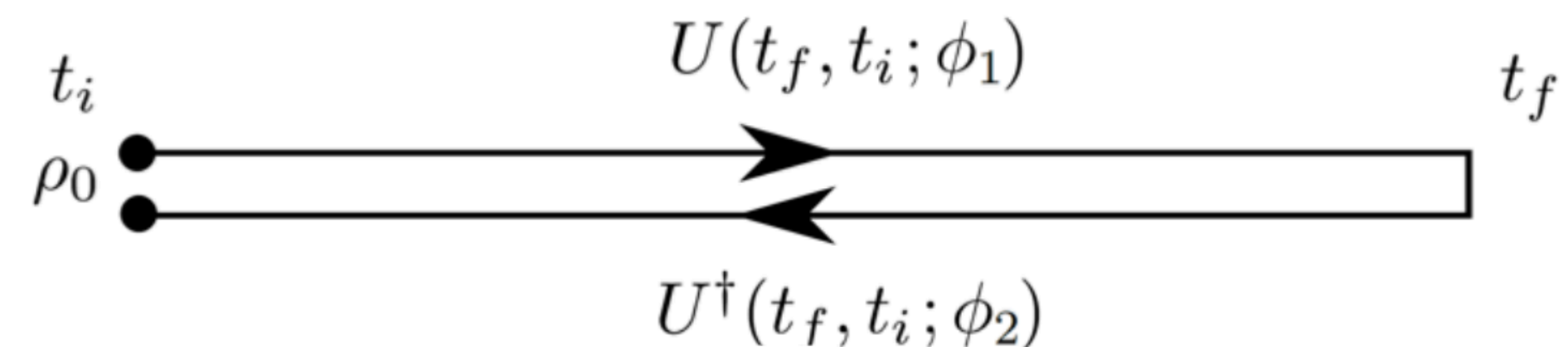
$$\nabla_\mu T^{\mu\nu} = 0 \qquad \nabla_\mu J^\mu = 0$$

- Long-standing question had been to obtain an action principle for hydrodynamics. This is non-trivial due to fact hydrodynamics is dissipative beyond ideal order and such action principle must be formulated in Schwinger-Keldysh formalism. Huge progress over last 10-15 years means this is now understood [1].
- In case of a single U(1) charge the action can be formulated in terms of fields

$$B_\mu^\sigma(t, \vec{x}) = A_\mu^{\sigma(s)}(t, \vec{x}) + \partial_\mu \varphi^\sigma(t, \vec{x})$$

- Here $\sigma = 1, 2$ is an index referring to the branch of the SK contour, $A_\mu^{\sigma(s)}$ are external sources that couple to the hydrodynamic fields φ^σ
- For energy-momentum conservation hydrodynamics fields are two copies of diffeomorphism fields X_μ^σ . Couple to external sources $g_{\mu\nu}^{\sigma(s)}$ via pull-back under diffeomorphisms.

SK contour (from Liu & Glorioso, 1805.09331)



[1]. We will be concerned with the formalism of Crossley, Glorioso & Liu (1511.0346 etc). Significant contributions/alternative approaches by others including e.g. Grozdanov & Polonyi, Haehl, Loganayagam & Rangamani.

- These hydrodynamic actions have led to significant new insights including constraints on derivative expansions from first principles and exact loop calculations in hydrodynamics.
- It has been proposed that in maximally chaotic systems such hydrodynamic actions contain exponentially growing modes that are responsible for the exponential growth of OTOCs [1]. Such modes have recently been argued to be related to symmetries of black holes horizons [2].
- The goal of our work is to systematically derive these dissipative hydrodynamic actions from gravity. There is a long history of attempting to do this, but to our knowledge no successful attempts at constructing these actions in the context of dissipative stress tensor dynamics [3].
- Armed with such actions for holographic systems, one can then directly ask whether they possess the conjectured symmetries related to chaos.

[1]. MB, Lee & Liu (1801.0010), MB & Liu (2102.11294)

[2]. Knysh, Liu & Pinzani-Fokeeva (2405.17559).

[3]. An orthogonal approach to SK generating functionals for conserved quantities is described in He, Loganayagam, Rangamani & Virrueta in 2108.03244, 2205.03415 etc.

Review - U(1) charge action

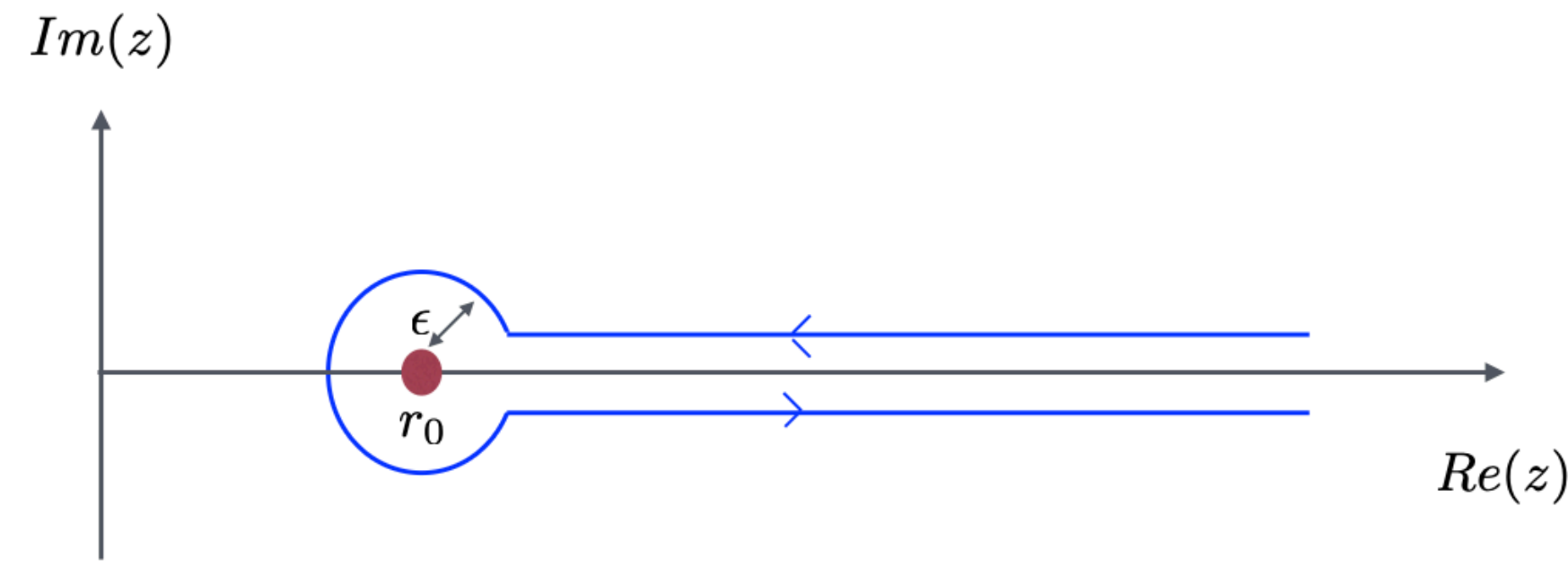
- Long history of attempts to derive hydrodynamic actions for charge and stress tensor hydrodynamics in AdS/CFT [1]. This is done by constructing a partially-on shell action - solving a subset of the bulk equations of motion. Most attempts historically are unable to include the effects of dissipation systematically.
- For a U(1) charge it has been understood how to include dissipation. There are two separate approaches in the literature [2]. We will review the approach of Crossley-Glorioso-Liu which works in terms of an analytically continued black hole geometry.
- Starting point is e.g. AdS4 Schwarzschild black hole

$$ds^2 = -f(z)dv^2 + 2dvdz + z^2(dx^2 + dy^2) \qquad f(z) = z^2 \left(1 - \frac{r_0^3}{z^3} \right)$$

1. E.g. Son & Nickel (1009.3094), Crossley, Glorioso, Liu & Wang (1504.07611), de Boer, Heller & Pinzani-Fokeeva (1504.07616)

2. Glorioso, Crossley & Liu (1812.08785), de Boer, Heller & Pinzani-Fokeeva (1812.06093)

- Consider an analytic continuation of the radial coordinate around the horizon as indicated by the contour



- We want to study fields propagating on this geometry. Conceptually easier to think of this as having two copies of fields for each branch of the contour (now with real r)

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\vec{x}^2 \qquad f(r) = r^2 \left(1 - \frac{r_0^3}{r^3} \right)$$

- For a U(1) gauge field one then considers two gauge fields $A_M^\sigma(v, r, \vec{x})$. It is convenient to decompose these in terms of a piece in radial gauge, and a bulk gauge transformation $\Lambda^\sigma(v, r, \vec{x})$

$$A_\mu^\sigma(v, r, \vec{x}) = C_\mu^\sigma(v, r, \vec{x}) + \partial_\mu \Lambda^\sigma(v, r, \vec{x}) \qquad A_r^\sigma(v, r, \vec{x}) = \partial_r \Lambda^\sigma(v, r, \vec{x})$$

$$M = v, r, x, y \qquad \mu = v, x, y$$

- One now constructs a partially on-shell action by solving a subset of the equations of motion. One solves (in a derivative expansion in ∂_v, ∂_i)

$$\nabla_N F^{N\mu,\sigma} = 0$$

- The remaining Maxwell equation $\nabla_N F^{Nr,\sigma} = 0$ is left unsolved.
- It is also necessary to impose boundary conditions. In the UV you impose standard AdS/CFT boundary conditions $A_\mu^\sigma(v, r \rightarrow \infty, \vec{x}) = A_\mu^{\sigma(s)}(v, \vec{x})$
- Near the horizon the partially on-shell solutions for A_i^σ are related in a non-trivial way by analytic continuation as implied by the CGL contour.
- One still needs to specify additional boundary conditions at the horizon for A_v^σ . Conventionally

$$A_v^\sigma(v, r_0 - \epsilon, \mathbf{x}) = 0$$

- Computing the Maxwell action $S = -\frac{1}{4} \int d^4x \sqrt{-g} F^{MN} F_{MN}$ for these partially on shell solutions, and taking into account contributions from both branches, gives the SK action

$$S = \int d^3x \left[\frac{r_0}{2} B_t^+ B_t^- + \frac{3ir_0}{4\pi} (B_x^-)^2 + \frac{3ir_0}{4\pi} (B_y^-)^2 + \dots \right]$$

$$B_\mu^\sigma(t, \vec{x}) = A_\mu^{\sigma(s)}(t, \vec{x}) + \partial_\mu \varphi^\sigma(t, \vec{x}) \quad \varphi^\sigma(v, x) = \Lambda^\sigma(v, \infty, x) - \Lambda^\sigma(v, r_0, x)$$

- The un-imposed equations of motion $\nabla_N F^{Nr, \sigma} = 0$ can be shown to be equivalent to the equations of motion of the hydrodynamic action.
- Summary: Realises a SK action, where hydrodynamic fields are relative gauge transformations between horizon and asymptotic boundaries. There is a non-trivial choice of boundary conditions at the horizon.

Stress-tensor hydrodynamic action

- We have obtained the SK action describing stress-tensor dynamics in an analogous manner. Our approach is valid to quadratic order in perturbations about the thermal state. For AdS4-Schwarzschild we have performed this at zeroth and first order in a derivative expansion [1].
- Starting point is the bulk gravitational action $S = S_{EH} + S_{GH}$ expanded to quadratic order in metric perturbations $\delta g_{MN}^\sigma(v, r, x)$

$$S_{EH} = \int d^4x \sqrt{-g} (R + 6) \qquad S_{GH} = \int d^3x \sqrt{-\gamma} (2K - 4 - R[\gamma])$$

- Metric perturbations on CGL contour $\delta g_{MN}^\sigma(v, r, x)$ decomposed in an analogous manner into pieces in radial gauge (in ingoing coordinates) $h_{\mu\nu}^\sigma(v, r, x)$ and bulk diffeomorphisms $\zeta_M^\sigma(v, r, x)$

To simplify equations I am aligning spatial dependence in the x direction WLOG

- Partially on-shell solutions means (roughly) imposing linearised Einstein equations

$$\delta E_{xx}^\sigma = \delta E_{yy}^\sigma = \delta E_{rx}^\sigma = \delta E_{rr}^\sigma = \delta E_{ry}^\sigma = \delta E_{xy}^\sigma = \delta E_{vr}^\sigma = 0$$

- These can be solved order by order in a derivative expansion in ∂_v, ∂_i . The remaining Einstein equations are left unfixed $\delta E_{vv}^\sigma, \delta E_{vx}^\sigma, \delta E_{vy}^\sigma$.
- In the UV (large r) we fix FG gauge and imposes standard boundary conditions

$$\delta g_{\mu\nu}^\sigma(v, r, x) \rightarrow r^2 \delta g_{\mu\nu}^{\sigma(s)}(v, x) + \dots$$

- Near the horizon we impose radial gauge. The solutions for $\delta g_{xy}^\sigma, \delta g_m^\sigma = \delta g_{xx}^\sigma - \delta g_{yy}^\sigma$ can be related through non-trivial analytic continuation on the CGL contour.
- One then needs 4 additional boundary conditions (per branch) to fix the remaining fields.

- In analogy with the gauge field case, it is natural to impose

$$\delta g_{vv}^\sigma(v, r_0, x) = \delta g_{vx}^\sigma(v, r_0, x) = \delta g_{vy}^\sigma(v, r_0, x) = 0$$

- To identify a final boundary condition, we consider the variational principle of the quadratic Einstein-Hilbert action. Generically, under a variation there will be a non-trivial boundary term at the horizon. This will vanish if one imposes radial gauge near the horizon and the boundary condition

$$2r_0^2 \partial_r \delta g_{vv}^\sigma(v, r_0, x) = \partial_v \delta g_{xx}^\sigma(v, r_0, x) + \partial_v \delta g_{yy}^\sigma(v, r_0, x)$$

- Note this is a mixed boundary condition [we have not added a Gibbons-Hawking term at the horizon].
- Consistency with the variational principle is important - it guarantees that the equations of motion of the hydrodynamic action (computed by going partially on-shell) are equivalent to the remaining un-imposed Einstein equations.

- These boundary conditions completely fix $h_{\mu\nu}^\sigma(v, r, x)$. Evaluating the resulting partially on-shell action gives a SK action that depends only on sources $\delta g_{\mu\nu}^{\sigma(s)}(v, x)$ and hydrodynamic fields $\xi_\mu^\sigma(v, x)$
- The hydrodynamic fields correspond to relative diffeomorphisms between the horizon and asymptotical boundaries].

$$\xi_t^\sigma(v, x) = [-\zeta^{v,\sigma}(v, r, x)]_{r_0}^\infty$$

$$\xi_x^\sigma(v, x) = [\zeta^{x,\sigma}(v, r, x) - \partial_x \zeta^{v,\sigma}(v, r, x)/r]_{r_0}^\infty$$

$$\xi_y^\sigma(v, x) = [\zeta^{y,\sigma}(v, r, x)]_{r_0}^\infty$$

- It is convenient to define the pull-backs of the external metric on each branch under these diffeomorphisms. Suppressing the branch label these are defined by

$$\mathcal{B}_{\mu\nu}(x) = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}^{(s)}(y(x)) - \eta_{\mu\nu},$$

$$y^\mu = x^\mu - \xi^\mu$$

$$g_{\mu\nu}^{(s)} = \eta_{\mu\nu} + \delta g_{\mu\nu}^{(s)}$$

- The resulting hydrodynamic action can be expressed (to quadratic order in perturbations, and first order in derivatives) as

$$S_{\text{hydro},0}[\mathcal{B}_{\mu\nu}^{\sigma}] = r_0^3 \int d^3x \left(\mathcal{B}_{tt}^- + \frac{1}{2} \mathcal{B}_p^- + \mathcal{B}_{tx}^+ \mathcal{B}_{tx}^- - \frac{1}{4} \mathcal{B}_m^+ \mathcal{B}_m^- + 2\mathcal{B}_{tt}^+ \mathcal{B}_{tt}^- + \frac{1}{2} \mathcal{B}_{tt}^+ \mathcal{B}_p^- \right. \\ \left. + \frac{1}{2} \mathcal{B}_p^+ \mathcal{B}_{tt}^- - \mathcal{B}_{xy}^+ \mathcal{B}_{xy}^- + \mathcal{B}_{ty}^+ \mathcal{B}_{ty}^- + \frac{3i}{16\pi} (\mathcal{B}_m^-)^2 + \frac{3i}{4\pi} (\mathcal{B}_{xy}^-)^2 \right).$$

$$S_{\text{hydro},1}[\mathcal{B}_{\mu\nu}^{\sigma}] = r_0^2 \int d^3x \left(\mathcal{B}_{xy}^+ \partial_v \mathcal{B}_{xy}^- + \frac{3i}{2\pi} \mathcal{B}_{ty}^- \partial_x \mathcal{B}_{xy}^- - \frac{1}{4} \mathcal{B}_m^- \partial_v \mathcal{B}_m^+ + \frac{3i}{4\pi} \mathcal{B}_{tx}^- \partial_x \mathcal{B}_m^- \right).$$

- The equations of motion of this action are precisely the un-imposed Einstein equations, and are equivalent to energy momentum conservation.
- Integrating out the hydrodynamics fields gives a generating functional for hydrodynamic response functions. These precisely match form expected from variational hydrodynamics [1].

Horizon symmetries and SK actions

- It has been argued that hydrodynamic actions for the stress-tensor should, in maximally chaotic systems, have additional symmetries [beyond those for generic hydrodynamic actions] that give rise to exponentially growing zero modes [1].
- Due to the symmetries such modes do not carry energy or momentum (i.e. the constitutive relations are invariant under them). However they do contribute to OTOCs.
- A specific proposal for the form of the such symmetries in holographic systems was proposed by Knysh et al [2], where they correspond to certain ‘horizon symmetries’. Schematically the action of symmetries on hydro fields has the form

$$\delta\xi_\mu(t, x, y) \sim \alpha_\mu(x, y) + \beta_\mu(x, y)e^{-2\pi Tt} + \gamma_\mu(x, y)e^{2\pi Tt}$$

[1]. MB, Lee & Liu (1801.0010), MB & Liu (2102.11294)

[2]. Knysh, Liu & Pinzani-Fokeeva (2405.17559).

- Hydrodynamic actions computed using our method should exhibit symmetries corresponding to infra-red (IR) diffeomorphisms that preserve the horizon boundary conditions.
- In the case of JT and AdS3 gravity we have computed the hydrodynamic actions exactly using our methods. For natural boundary conditions consistent with the VP the actions precisely realise the horizon symmetries of Knysh et al. [1]
- For AdS4 gravity we can analyse the expected symmetries arising from diffeos $\chi^\sigma(v, r, x, y)$ preserving radial gauge (near the horizon) and the boundary conditions

$$\delta g_{v\mu}^\sigma(v, r_0, x, y) = 0 \qquad 2r_0^2 \partial_r \delta g_{vv}^\sigma(v, r_0, x, y) = \partial_v \delta g_{xx}^\sigma(v, r_0, x, y) + \partial_v \delta g_{yy}^\sigma(v, r_0, x, y)$$

- Expressing such diffeomorphisms in the form

$$\chi^\sigma(v, r, x, y) = f^\sigma(v, x, y) \partial_v + Z^\sigma(v, x, y) \partial_r + \tilde{X}^\sigma(v, x, y) \partial_x + \tilde{Y}^\sigma(v, x, y) \partial_y + \mathcal{O}(r - r_0)$$

- Our boundary conditions are preserved when

$$\begin{aligned}
f^\sigma(v, x, y) &= b_1^\sigma(x, y) + b_2^\sigma(x, y)e^{-\kappa_0 v} + d_1^\sigma(x, y)e^{\kappa_0 v}, \\
Z^\sigma(v, x, y) &= \bar{b}_3^\sigma(x, y)e^{\kappa_0 v}, \\
\tilde{X}^\sigma(v, x, y) &= \bar{b}_x^\sigma(x, y) - \frac{1}{\kappa_0 r_0^2} \partial_x \bar{b}_3^\sigma(x, y)e^{\kappa_0 v}, \\
\tilde{Y}^\sigma(v, x, y) &= \bar{b}_y^\sigma(x, y) - \frac{1}{\kappa_0 r_0^2} \partial_y \bar{b}_3^\sigma(x, y)e^{\kappa_0 v},
\end{aligned}$$

$$\kappa_0 = 2\pi T = 3r_0/2$$

$$2(-\nabla^2 + 3r_0^2)b_3^\sigma + 3r_0^2(3r_0^2 + \nabla^2)d_1^\sigma = 0$$

- These are similar, but not identical, to two copies of the horizon symmetries of Knysh et al - the form of the exponentially growing mode is different. These symmetries shift the hydrodynamic fields as

$$\begin{aligned}
\delta_H \xi_t^\sigma(t, x, y) &= b_1^\sigma(x, y) + b_2^\sigma(x, y)e^{-\kappa_0 t} + d_1^\sigma(x, y)e^{\kappa_0 t}, \\
\delta_H \xi_x^\sigma(t, x, y) &= -b_x^\sigma(x, y) + \frac{1}{r_0} \partial_x b_2^\sigma(x, y)e^{-\kappa_0 t} + \frac{1}{\kappa_0 r_0^2} \partial_x b_3^\sigma(x, y)e^{\kappa_0 t}, \\
\delta_H \xi_y^\sigma(t, x, y) &= -b_y^\sigma(x, y) + \frac{1}{r_0} \partial_y b_2^\sigma(x, y)e^{-\kappa_0 t} + \frac{1}{\kappa_0 r_0^2} \partial_y b_3^\sigma(x, y)e^{\kappa_0 t}.
\end{aligned}$$

$$b_i^\sigma = \bar{b}_i^\sigma - \partial_i b_1^\sigma / r_0$$

$$b_3^\sigma = \bar{b}_3^\sigma + \kappa_0 r_0 d_1^\sigma$$

- If these were the only boundary conditions imposed we would expect two independent sets of symmetries, one on each branch of the SK contour. However we also imposed continuity of $\delta g_{xy}^\sigma, \delta g_m^\sigma$ at the horizon.
- These conditions break a subset of the off-diagonal symmetries and impose (at non-zero wave-vector)

$$b_3^- = d_3^- = \bar{b}_x^- = \bar{b}_y^- = 0$$

- With these conditions, the resulting shifts of the hydro fields should be zero modes of the action. The constitutive relations and equations of motion should also be invariant under such shifts.
- Because we only have the action in a derivative expansion, we cannot explicitly test if the exponential modes are symmetries. However we can test the time-independent symmetries, and both our action and constitutive relations are indeed non-trivially invariant under them precisely when $\bar{b}_x^- = \bar{b}_y^- = 0$

Relation to Knysh et al

- Knysh et al argued for symmetries of hydrodynamic actions arising from horizon symmetries. Their definition of horizon symmetries is equivalent, for a black brane geometry and for radial gauge near the horizon, to identifying IR diffeomorphisms that preserve [1]

$$\delta g_{v\mu}^\sigma(v, r_0, x, y) = 0$$

$$\partial_r \delta g_{vv}^\sigma(v, r_0, x, y) = 0$$

- Such diffeomorphisms are of the form we identified earlier, but with $d_1^\sigma(x, y) = 0$ and $b_3^\sigma(x, y)$ unconstrained.
- Imposing these as boundary conditions does not manifestly respect the variational principle at the horizon. Nevertheless, it is interesting to compute the hydrodynamic action with these as boundary conditions and see what changes.

- Somewhat surprisingly, to first order in derivatives the resulting action is identical to what we obtained with our original boundary conditions.
- However there is an important subtlety - because the variational principle is not respected, the equations of motion are no longer obviously equivalent to the Einstein equations.
- It turns out that order by order in derivatives, one can argue that this will not affect the solutions that are obtained by putting the hydrodynamic action fully on shell.
- However non-perturbatively, one is not guaranteed that the hydrodynamic equations of motion and Einstein equations agree. This prevents us arguing that the exponentially growing symmetries of Knysh et al are guaranteed to be symmetries of the resulting action.

Conclusion

- We derived a SK action for the stress tensor hydrodynamics dual to perturbations of the AdS4 Schwarzschild black hole, to first order in a derivative expansion. We believe this is a significant technical breakthrough, and the first time this has been done consistently. [1].
- We gave a prescription to identify symmetries of hydrodynamic actions, arising from IR diffeomorphisms that preserve the horizon boundary conditions.
- In JT and AdS3 gravity, the actions can be computed exactly, and [for the natural choice of boundary conditions respecting the VP] they realise precisely the horizon symmetries of Knysh et al. [2]
- In AdS4 gravity, the natural boundary conditions [respecting the VP] are not invariant under the horizon symmetries of Knysh et al. Such boundary conditions imply a (slightly) different form of the symmetries.

[1]. We believe there are errors in the analysis of Bu & Sun (2503.03374).

[2]. MB, Das & Davison - (to appear)