Symmetries on the Contours

based on
- arXiv 2508.18346
- ongoing discussions

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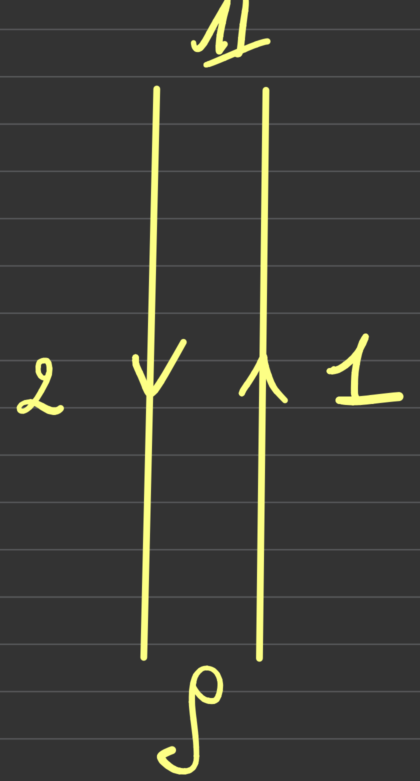
⚠ <u>Around ground state</u> : IR dynamics universally described by EFT of Goldstones for non-linearly realized symmetries

{QCD, composite H, superfluids, inflation, large Q in CFT, ...}

<u>Textbook subject</u>

⚠ <u>Around excited state</u> : IR dynamics universally described by SK-EFT for hydrodynamic modes.

<u>Novel & exciting</u>



- $G_1 \times G_2$ Ward id.
- $G_1 \times G_2 \longrightarrow G_{1+2}$ at $+\infty$ $\mathbb{1}$
- $G_1 \times G_2 \longrightarrow H$ at $-\infty$ ρ

⚠ Long range dynamics : $\dfrac{G_1 \times G_2}{H}$ coset

⚠ KMS $O_1(x) \to O_1\left(-x + \frac{i\beta}{2}\right)$ $O_2(x) \longrightarrow O_2\left(-x - \frac{i\beta}{2}\right)$

$\Longrightarrow$ field doubling necessary

▨ Generic $\dfrac{G_1 \times G_2}{H}$ is <u>not</u> a double set

Relativistic Coset construction (CCWZ - 1969)

G/H $\qquad$ $G: \Omega(x) \to g\,\Omega(x)$ $\qquad$ Left Action

$\Omega(x) \in G$ $\qquad$ $G/H: \Omega(x) \sim \Omega\, h(x)$ $\quad h \in H$ $\qquad$ Right Action

- Cartan form $\quad -i\,\Omega^{-1}\partial_\mu \Omega = \underbrace{\partial_\mu \phi}_{G-H} + \underbrace{A_\mu}_{H}$

- Matter fields in irreps of H

Coset constructions on the SK contour

- Always work with full coset $G_1 \times G_2 / \{e\}$

- $G_1 : \Omega_1 \to g_1 \Omega$ $\quad$ $G_2 : \Omega_2 \to g_2 \Omega_2$

- KMS: $\quad \Omega_1(x) \to \Omega_1(-x + i\beta/2)$

  $\quad \Omega_2(x) \to \Omega_2(-x - i\beta/2)$

- quotient by suitable Right action

A) $G_1 \times G_2 \longrightarrow H_1 \times H_2$

$$\begin{cases} \Omega_1 \sim \Omega_1\, h_1(x) \\ \Omega_2 \sim \Omega_2\, h_2(x) \end{cases}$$

KMS : $h_1(x) \longrightarrow h_1(-x + i\beta/2)$ $\quad h_2(x) \longrightarrow h_2(-x - i\beta/2)$

B) $G_1 \times G_2 \longrightarrow H_{1+2}$

$$\Omega_1 \sim \Omega_1\, h(x)$$
$$\Omega_2 \sim \Omega_2\, h(x)$$

KMS $\quad h(-x + i\beta/2) = h(-x - i\beta/2) \Rightarrow \beta^\mu \partial_\mu h = 0$

In practice $\quad h \equiv h(\vec{x})$

Non-Abelian charge diffusion: $G_1 \times G_2 \longrightarrow G_{1+2}$

- $\tilde{\Omega} = \Omega_1 \Omega_2 \equiv e^{i\phi_1 \cdot T_1} e^{i\phi_2 \cdot T_2} = e^{i\phi_a \cdot (T_1 - T_2)} e^{i\phi_r (T_1 + T_2)}$

- $\tilde{\Omega} \sim \tilde{\Omega}\, \overbrace{e^{if(x)\cdot(T_1+T_2)}}^{h^{-1}(x)} \equiv$ gauged $G_{1+2}(x) \equiv G_R$

☆ Essentially only $\dot{\phi}_r + \dots$ is physical

$\Longrightarrow$ # Hamiltonian pairs $\equiv \# \phi_a = \dim G$

$\Longrightarrow$ realizes $G_1 \times G_2 \to G_{1+2}$

Building Blocks

- $e^{i\phi_a} \xrightarrow{G_R} e^{i\phi_a}$ $\qquad$ $e^{i\phi_r} \xrightarrow{G_R} e^{i\phi_r} h^{-1}(x)$

- Cartan $\quad -i\tilde{\Omega}^{-1}\partial_\mu \tilde{\Omega} \equiv \tilde{D}_\mu \phi_a + \tilde{D}_\mu \phi_r$

	$\tilde{D}_\mu \phi_a$, $\tilde{D}_0 \varphi_r$, $\partial_0 \tilde{D}_i \varphi_r$	$\tilde{\nabla}_i = \partial_i - i[\tilde{D}_i \phi_r, \ldots]$	∂_0
G_R	adjoint	covariant	covariant

DKMS

$$\begin{pmatrix}\tilde{D}_\mu\phi'_r(x)\\ \tilde{D}_\mu\phi'_a(x)\end{pmatrix} = -\begin{pmatrix}\cosh\left(i\frac{\beta}{2}\partial_t\right) & \frac{1}{2}\sinh\left(i\frac{\beta}{2}\partial_t\right)\\ 2\sinh\left(i\frac{\beta}{2}\partial_t\right) & \cosh\left(i\frac{\beta}{2}\partial_t\right)\end{pmatrix}\begin{pmatrix}\tilde{D}_\mu\phi_r(y)\\ \tilde{D}_\mu\phi_a(y)\end{pmatrix}\Bigg|_{y=-x}$$

$+$

Unitarity

$$S[\phi_r,-\phi_a] = -S^*[\phi_r,\phi_a]$$

$$S[\phi_r,0] = 0$$

- At $O(\tilde{D}\phi)^n$ one finds 2^n-2 invariants

(see also Wang-Heinz 2002)

- $n=2$

$$\int\frac{d\omega}{2\pi}d^3\mathbf{x}\left[\chi\tilde{D}_0\phi_a(\omega)\cdot\tilde{D}_0\phi_r(-\omega)+\frac{i\sigma}{\beta}\left(-2\tanh\left(\frac{\beta\omega}{2}\right)\tilde{D}_i\phi_a(\omega)\cdot\tilde{D}_i\phi_r(-\omega)+\tilde{D}_i\phi_a(\omega)\cdot\tilde{D}_i\phi_a(-\omega)\right)\right]$$

$\Downarrow$ $O(\beta\partial_t)$

$$\int dt\, d^3\mathbf{x}\left[\chi\tilde{D}_0\phi_a\cdot\tilde{D}_0\phi_r+\sigma\left(-\tilde{D}_i\phi_a\cdot\partial_t\tilde{D}_i\phi_r+\frac{i}{\beta}\tilde{D}_i\phi_a\cdot\tilde{D}_i\phi_a\right)\right]$$

Doing away with G_R

- $\phi_1(x)$ $\quad \overset{KMS}{\Longrightarrow} \quad \phi_1(-x+i\beta/2)$ $\qquad \phi_a(x)$ $\quad \Longrightarrow$ "more involved"
  $\phi_2(x)$ $\qquad\qquad \phi_2(-x-i\beta/2)$ $\qquad \phi_r(x)$

- $[KMS, G_R] = 0 \quad \Longrightarrow$ G_R invariants mapped among themselves

- G_R is non-linearly realized:
  It must be possible to describe dynamics through local gauge invariant d.o.f's

Ex Massive gauge theory

- $A_\mu \to A_\mu - \partial_\mu \alpha \qquad \pi \to \pi + M\alpha$

- $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu \pi + M A_\mu\right)^2$

- $\mathcal{E}_\mu = A_\mu + \frac{\partial_\mu \pi}{M}$ = gauge invariant vector

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu \mathcal{E}_\nu - \partial_\nu \mathcal{E}_\mu\right)^2 + \frac{M^2}{2} \mathcal{E}_\mu \mathcal{E}^\mu$$

equivalent to gauge fix $\pi \to 0$

The Matter Field Approach

Akyuz, Goon, Penco '24

- $e^{i\phi_a} e^{i\phi_r} \Longrightarrow \underbrace{g\, e^{i\phi_a}\, \tilde{h}^{-1}_{(g,\phi_a)}}_{\frac{G_1 \times G_2}{G_{1+2}}}\, \tilde{h}_{(g,\phi_a)}\, e^{i\phi_r}\, h(x) = g\, e^{i\phi_a} e^{i\phi_r}\, h(x)$

- Can locally rewrite action using:

- $\rho_r \equiv e^{i\phi_r} \tilde{D}_0 \phi_r e^{-i\phi_r}$ $\qquad D_\mu \phi_a = e^{i\phi_r} \tilde{D}_\mu \phi_a e^{-i\phi_r}$

- $\tilde{\nabla}_i \longrightarrow \nabla_i = \partial_i + A_i(\varphi_a)$ $\qquad \partial_t \longrightarrow \mathcal{D}_t = \partial_t + i\left(A_0(\varphi_a) - \rho_r\right)$

- $F_{0i}(A) + \nabla_i \rho_r = e^{i\phi_r} \partial_t \tilde{D}_i \phi_r e^{-i\phi_r}$

KMS would be hard to guess

- KMS action on ρ_r, φ_a involves

$$W_{b,a}(t,x) = P\exp\left[-\int_a^b ds\,\big(\rho_r(t+is,x) - A_0(t+is,x)\big)\right]$$

$$e^{i\phi'_a(x)\cdot T} \equiv \Omega'_a(x) = \Omega_a^{1/2}(-x+i\beta/2)\,W_{\beta/2,-\beta/2}(-x)\,\Omega_a^{1/2}(-x-i\beta/2)$$

$$\Omega_a'^{-1/2}(x)\,\Omega_a^{1/2}(-x+i\beta/2)\,W_{\beta,0}(-x)$$

Any consequences of G_R?

⚠ Non-dissipative limit: $S = S[\Phi_1] - S[\Phi_2]$

- $G_R \xrightarrow{\text{accidentally}} G_R^1 \times G_R^2 \Rightarrow$ conserved $J_r^R(x)$

$$J_r^R(x) = \tilde{D}_0 \varphi_r = e^{-i\Phi_r} \rho_r \, e^{i\varphi_r}$$

$$\boxed{\rho_r = -i e^{i\varphi_r} \partial_t e^{-i\varphi_r}}$$

- $S = S[\dot{\phi}, \phi] \xrightarrow[\text{accident}]{\text{extra}} G_L^1(x) \times G_L^2(x) \Longrightarrow J_r^L(x)$

$$J_r^L(x) = \rho_r(x) \equiv \text{charge only diffuses through fluctuations}$$

$$\frac{d}{dt} e^{-i\varphi_r} \rho_r e^{i\varphi_r} = e^{-i\varphi_r}\left[\dot{\rho}_r + i[\rho_r, \rho_r]\right] e^{i\varphi_r} = 0$$

- degenerate instance of Kelvin's theorem
- trivial in this case because of $\exists$ of additional accidental $G_L^1(x) \times G_L^2(x)$

- Rigid body analogy

- $\rho_r = i\, e^{i\varphi_r} \partial_t e^{-i\varphi_r}$ = Angular velocity

- $L^I = I^{AB} \rho_B\, U_A{}^I(\varphi_r) = \tilde{L}^A\, U_A{}^I(\varphi_r)$

$\downarrow$ L with respect Lab frame

$\downarrow$ L with respect to body frame

- $I^{AB} = \delta^{AB} \Rightarrow SO(3)_L \times SO(3)_R$ our case

Fluids

- Haehl, Loganayagam, Rangamani 2014
- Crossley, Glorioso, Liu 2015

- $ISO(3,1)_1 \times ISO(3,1)_2 \longrightarrow ISO(3)_{1+2} \times \mathbb{R}_{1+2}$
- $ISO(3,1) \times ISO(3,1) \longrightarrow \{e\}$ $\oplus$ quotient by right action
- Left $\begin{cases} X_2^\mu(\sigma) \longrightarrow \Lambda_2{}^\mu{}_\nu X_2^\nu(\sigma) + a_2^\mu \\ X_1^\mu(\sigma) \longrightarrow \Lambda_1{}^\mu{}_\nu X_1^\nu(\sigma) + a_1^\mu \end{cases}$
- Right $\quad X_a^\mu(\sigma) \longrightarrow X_a^\mu(f^{-1}(\sigma))$
- find $f(\sigma)$'s commuting with KMS

- KMS $$\tilde{X}_1^\mu(\sigma) = -X_1^\mu\left(-\sigma^0 + \frac{i\beta}{2}, \vec{\sigma}\right) + i\frac{\beta}{2}\delta_0^\mu$$
$$\tilde{X}_2(\sigma) = -X_2\left(-\sigma^0 - \frac{i\beta}{2}, \sigma\right) - \frac{i\beta}{2}\delta_0^\mu$$

- $[KMS, G_R] = 0 \implies \partial_{\sigma_0} f^\alpha(\sigma) = 0$

$$\begin{cases} \sigma^0 \longrightarrow \sigma^0 + f^0(\sigma^i) \\ \sigma^i \longrightarrow f^i(\sigma^j) \end{cases}$$

- fluid $\implies$ maximal $G_R \equiv \vec{\sigma}$. diffs
- submaximal $G_R \longrightarrow$ solids ....

$\boxed{}$ Perfect fluid limit $\quad \beta_i^\mu = \frac{\partial x_i^\mu}{\partial \sigma^\circ}$

- $\vec{S} = \vec{S}_1 - \vec{S}_2 = \int d^4\sigma \left[\sqrt{\frac{\partial x_1}{\partial \sigma}}\, P(\beta_1) - \sqrt{\frac{\partial x_2}{\partial \sigma}}\, P(\beta_2) \right]$

- accidental $(\vec{\sigma}\text{.diff})_1 \times (\vec{\sigma}\text{.diff})_2$

$\Longrightarrow$ Kelvin's theorem

Matter Field Approach

- $X_r^\mu \equiv \frac{1}{2}(X_1+X_2)^\mu \qquad X_a^\mu \equiv (X_1 - X_2)^\mu$

$$\boxed{\begin{aligned} X_a^\mu(\sigma) &= X_a^\mu(\sigma(X_r)) = X_a^\mu(X_r) \\ \beta^\mu &\equiv \left.\frac{\partial X_r^\mu(\sigma)}{\partial \sigma^0}\right|_{\sigma(X_r)} = \beta^\mu(X_r) \end{aligned}}$$

$\vec{\sigma}$. diff invariant

- After using $X_{1,2}^\mu(\sigma)$ to deal with KMS one can happily dispose of redondencies

Ex Perfect limit

$$\mathcal{L} = T^{\mu}{}_{\nu}\, \partial_\mu X_2^\nu \qquad T^{\mu\nu} = -p\,\eta^{\mu\nu} + (p+\rho)\frac{\beta^\mu \beta^\nu}{\beta^2}$$

- consider $\delta X_2^\mu = \xi^\mu$

$$\delta_\xi \mathcal{L} = -\frac{\partial p}{\partial \beta} u_\nu [\beta, \xi]^\nu - \partial_\mu (p\, \xi^\mu)$$

$$[\beta, \xi]^\mu \equiv \beta^\nu \partial_\nu \xi^\mu - \xi^\nu \partial_\nu \beta^\mu = \frac{\partial X_r^\mu}{\partial \sigma_a}\left(\beta^b \partial_b \xi^a - \xi^b \partial_b \beta^a\right)$$

$$[\beta, \xi]^\mu = 0 \Leftrightarrow \left(\bar{\sigma}\text{-diff}\right)_{1-2}$$

Noether: $u^\mu \left[\partial_\nu (T u_\mu) - \partial_\mu (T u_\nu) \right] = 0$

Conservation of vorticity along flow

$\equiv$ Kelvin Th.

Summary

- SK contours offer a novel arena where to play the EFT game
- Goldstone Th + KMS not immediately friends
- $\vec{x}$-dependent redondancies: clever trick to tackle difficulties
- but non redondant matter field approach is fully equivalent