Schwinger-Keldysh for Gauge Theories from Top Down in the BRST formalism

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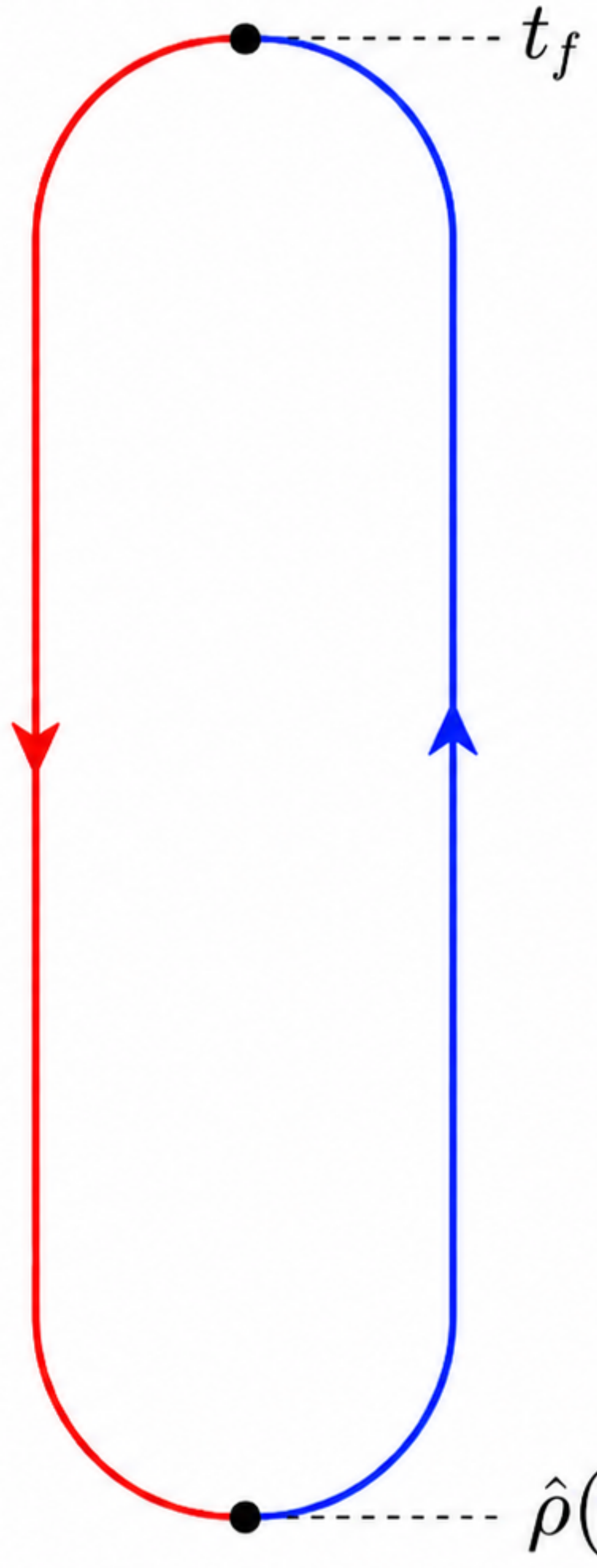


Work done with

Greg Kaplanek (Syracuse)

Maria Mylova (IPMU)

2512.17089 Gauging Open EFTs from the top down

2604.26941 Schwinger-Keldysh Path Integral for Gauge Theories

Schwinger 1960 (Brandeis lectures)

Mahanthappa, Bakshi (1962/1963)

Kadanoff+Baym (1962)

Robert Mills (1962) (Birmingham)

Konstantinov and Perel (1960/1961),
Dzyaloshinskil (1962)

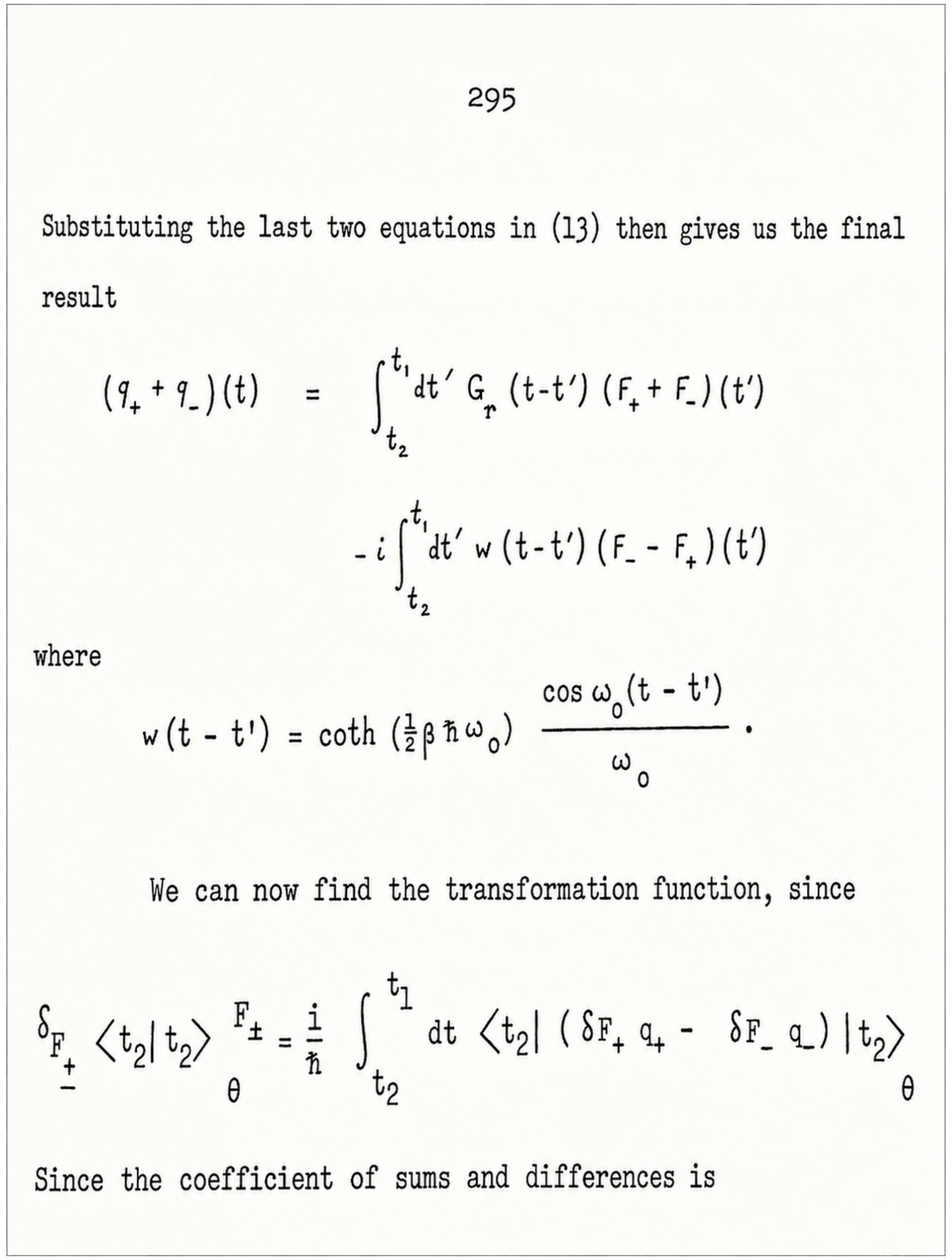
Feynman-Vernon (1963)

Keldysh (1964)

FIELD THEORY METHODS
IN
NON-FIELD-THEORY CONTEXTS

J. Schwinger
Harvard University

Notes by A.C. Manoharan



Substituting the last two equations in (13) then gives us the final result

$$(q_+ + q_-)(t) = \int_{t_2}^{t_1} dt'\, G_r(t-t')\,(F_+ + F_-)(t') - i\int_{t_2}^{t_1} dt'\, w(t-t')\,(F_- - F_+)(t')$$

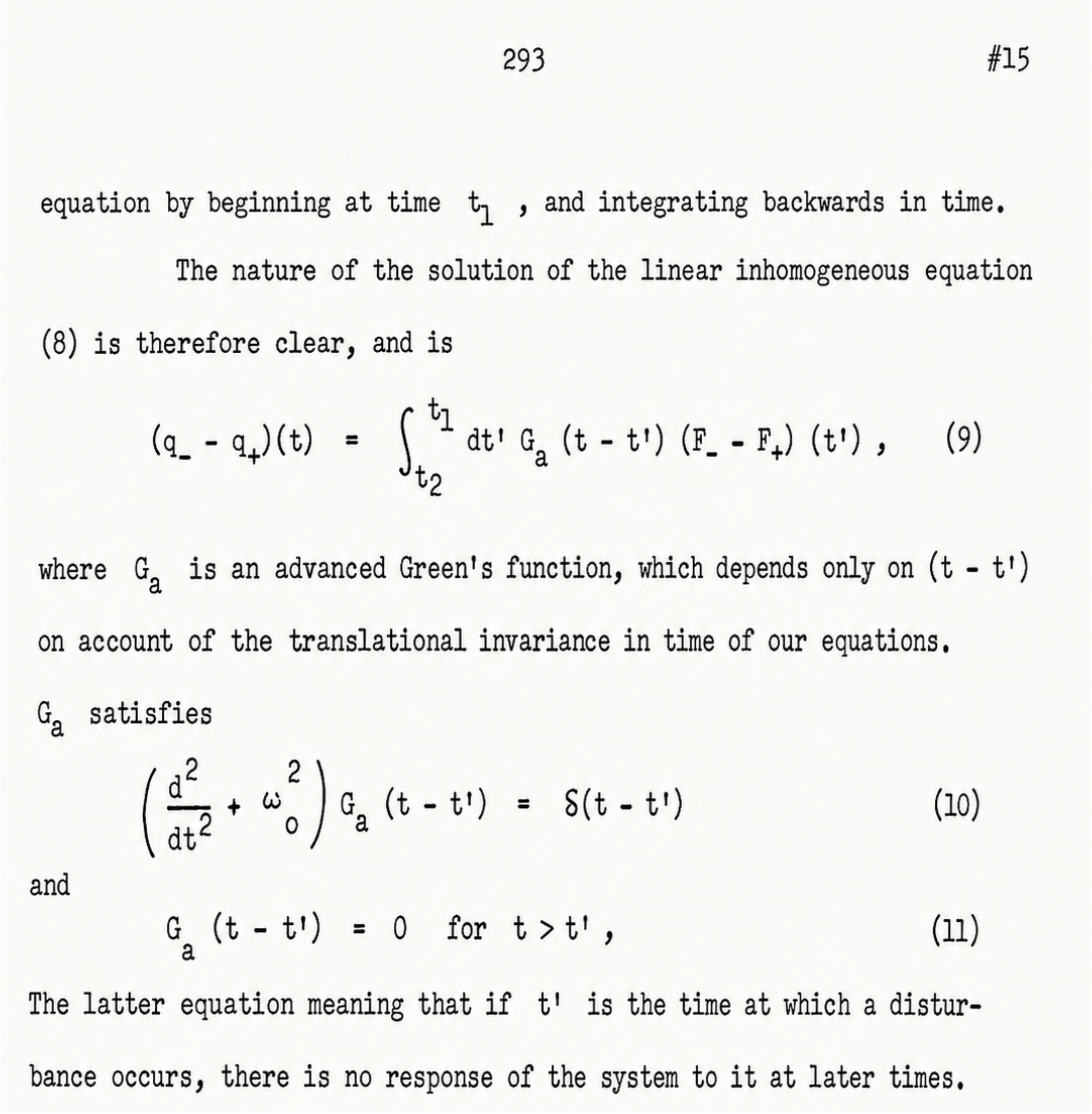
where

$$w(t-t') = \coth\left(\tfrac{1}{2}\beta\hbar\omega_0\right)\frac{\cos\omega_0(t-t')}{\omega_0}\,.$$

We can now find the transformation function, since

$$\delta_{F_\pm}\langle t_2|t_2\rangle^{F_\pm}_{\theta} = \frac{i}{\hbar}\int_{t_2}^{t_1} dt\,\langle t_2|\,(\delta F_+ q_+ - \delta F_- q_-)\,|t_2\rangle_{\theta}$$

Since the coefficient of sums and differences is



equation by beginning at time t_1, and integrating backwards in time.

The nature of the solution of the linear inhomogeneous equation (8) is therefore clear, and is

$$(q_- - q_+)(t) = \int_{t_2}^{t_1} dt'\, G_a(t-t')\,(F_- - F_+)(t')\,, \qquad (9)$$

where G_a is an advanced Green's function, which depends only on $(t - t')$ on account of the translational invariance in time of our equations. G_a satisfies

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right) G_a(t-t') = \delta(t-t') \qquad (10)$$

and

$$G_a(t-t') = 0 \quad \text{for } t > t'\,, \qquad (11)$$

The latter equation meaning that if t' is the time at which a disturbance occurs, there is no response of the system to it at later times.

SK Path Integral (No Symmetry)

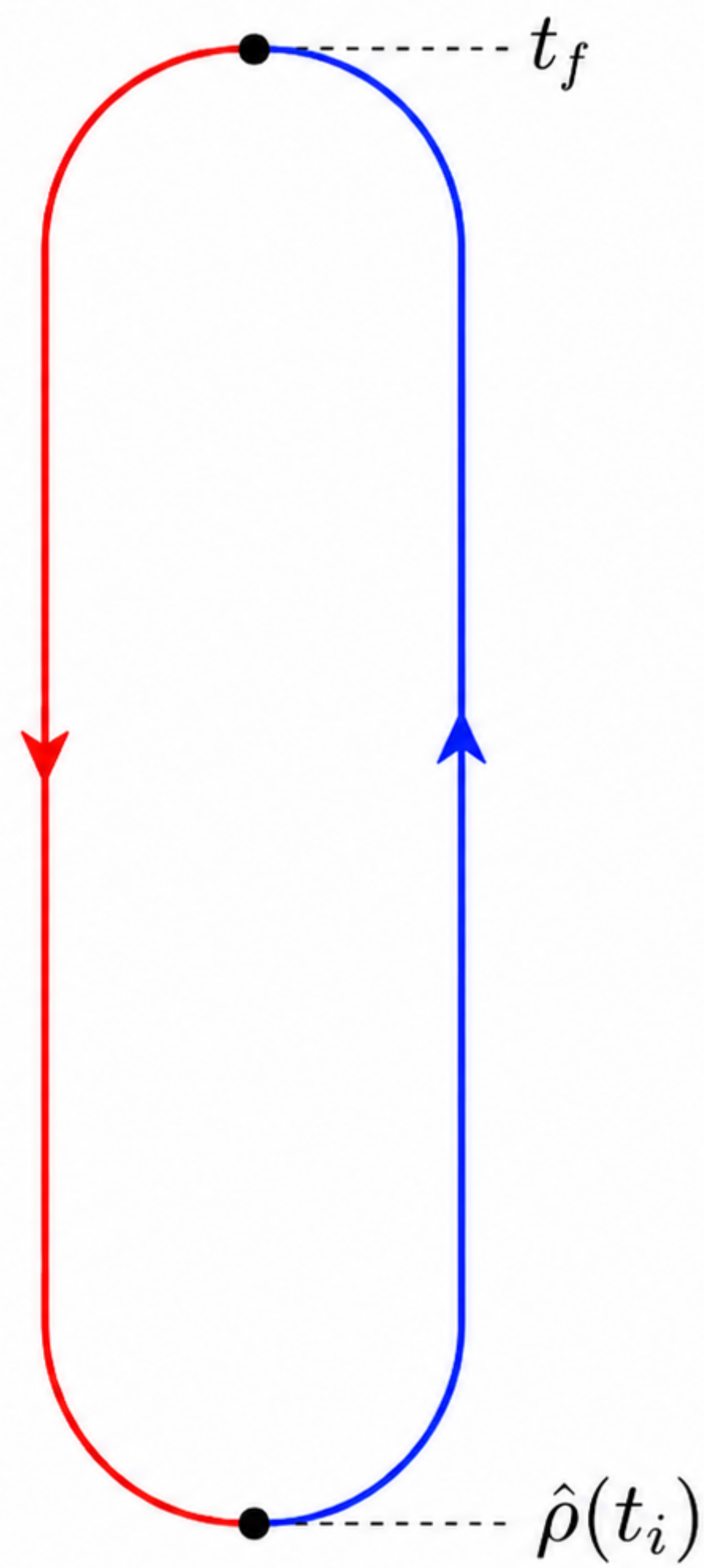
$$Z[J_+, J_-] = \int^{\phi_+(t_f,\mathbf{x})=\phi_-(t_f,\mathbf{x})} \mathcal{D}\phi_+\, \mathcal{D}\pi_+\, \mathcal{D}\phi_-\, \mathcal{D}\pi_- \times \langle \phi_+(t_i)|\, \hat{\rho}_0\, |\phi_-(t_i)\rangle\; e^{iS_{\text{SK}}[\phi_\pm,\pi_\pm;J_\pm]}.$$

$$S_{\text{SK}} = \int d^4x \Big[\pi_+\, \partial_t\phi_+ - \mathcal{H}(\phi_+,\pi_+) + J_+\phi_+ - \pi_-\, \partial_t\phi_- + \mathcal{H}(\phi_-,\pi_-) - J_-\phi_-\Big]$$

OR

$$Z[J_+, J_-] = \int^{\pi_+(t_f,\mathbf{x})=\pi_-(t_f,\mathbf{x})} \mathcal{D}\phi_+\, \mathcal{D}\pi_+\, \mathcal{D}\phi_-\, \mathcal{D}\pi_- \times \langle \pi_+(t_i)|\, \hat{\rho}_0\, |\pi_-(t_i)\rangle\; e^{iS^{(\pi)}_{\text{SK}}[\phi_\pm,\pi_\pm;J_\pm]}.$$

$$S^{(\pi)}_{\text{SK}} = \int d^4x \Big[-\phi_+\, \partial_t\pi_+ - \mathcal{H}(\phi_+,\pi_+) + J_+\phi_+ + \phi_-\, \partial_t\pi_- + \mathcal{H}(\phi_-,\pi_-) - J_-\phi_-\Big]$$



SK Path Integral (Global Symmetry)

E.g. complex scalar field U(1)

$$Z[J_+, J_-] = \int^{\substack{\phi_+(t_f,\mathbf{x})=\phi_-(t_f,\mathbf{x})\\ \phi_+^*(t_f,\mathbf{x})=\phi_-^*(t_f,\mathbf{x})}} \mathcal{D}\phi_+\,\mathcal{D}\phi_+^*\,\mathcal{D}\pi_+\,\mathcal{D}\pi_+^*\,\mathcal{D}\phi_-\,\mathcal{D}\phi_-^*\,\mathcal{D}\pi_-\,\mathcal{D}\pi_-^*$$

$$\times \left\langle \phi_+(t_i), \phi_+^*(t_i) \right| \hat{\rho}_0 \left| \phi_-(t_i), \phi_-^*(t_i) \right\rangle \, e^{iS_{\rm SK}[\phi_\pm, \pi_\pm; J_\pm]}$$

with doubled action

$$S_{\rm SK} = \int d^4x \Big[\pi_+\,\partial_t\phi_+ + \pi_+^*\,\partial_t\phi_+^* - \mathcal{H}(\phi_+, \phi_+^*, \pi_+, \pi_+^*) + J_+^*\phi_+ + J_+\phi_+^*$$

$$- \pi_-\,\partial_t\phi_- - \pi_-^*\,\partial_t\phi_-^* + \mathcal{H}(\phi_-, \phi_-^*, \pi_-, \pi_-^*) - J_-^*\phi_- - J_-\phi_-^* \Big]$$

Global Symmetry Breaking

Strong Symmetry

$$\left\langle e^{iq\theta_0}\phi_+, e^{-iq\theta_0}\phi_+^* \right| \hat{\rho}_0 \left| \phi_-, \phi_-^* \right\rangle = e^{iQ\theta_0} \left\langle \phi_+, \phi_+^* \right| \hat{\rho}_0 \left| \phi_-, \phi_-^* \right\rangle$$

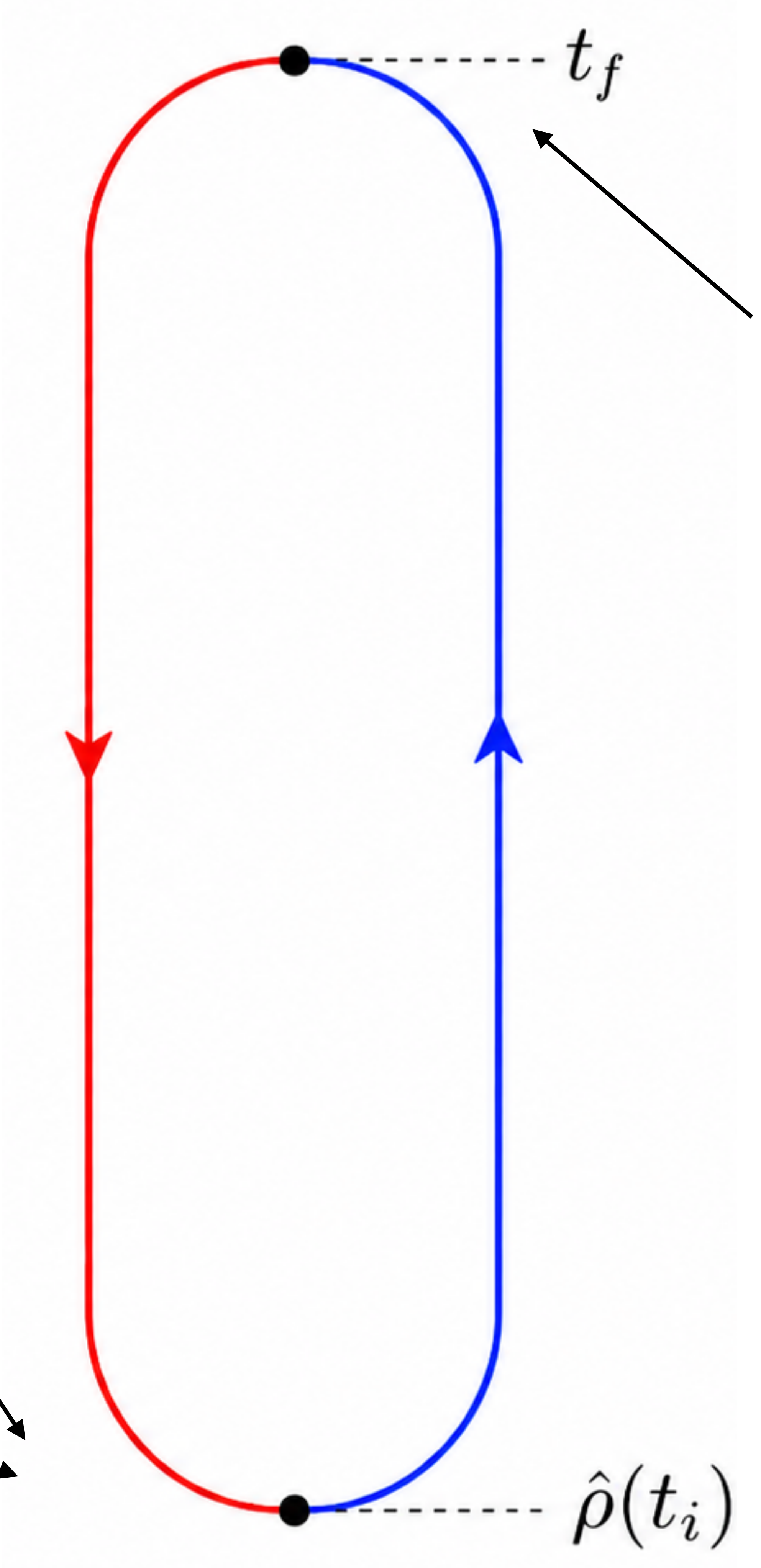
Spontaneous breaking of Strong Symmetry

$$\left\langle e^{iq\theta_0}\phi_+, e^{-iq\theta_0}\phi_+^* \right| \hat{\rho}_0 \left| \phi_-, \phi_-^* \right\rangle \neq e^{iQ\theta_0} \left\langle \phi_+, \phi_+^* \right| \hat{\rho}_0 \left| \phi_-, \phi_-^* \right\rangle$$

$$\left\langle e^{iq\theta_0}\phi_+, e^{-iq\theta_0}\phi_+^* \right| \hat{\rho}_0 \left| e^{iq\theta_0}\phi_-, e^{-iq\theta_0}\phi_-^* \right\rangle = \left\langle \phi_+, \phi_+^* \right| \hat{\rho}_0 \left| \phi_-, \phi_-^* \right\rangle$$

Spontaneous breaking of weak symmetry

$$\left\langle e^{iq\theta_0}\phi_+, e^{-iq\theta_0}\phi_+^* \right| \hat{\rho}_0 \left| e^{iq\theta_0}\phi_-, e^{-iq\theta_0}\phi_-^* \right\rangle \neq \left\langle \phi_+, \phi_+^* \right| \hat{\rho}_0 \left| \phi_-, \phi_-^* \right\rangle$$



$$\phi_+(t_f) = \phi_-(t_f)$$

Explicit Breaking of Advanced Global Symmetry (no Goldstone bosons)

Typical NR EFT for Global Case

Sieberer, Buchhold, Diehl 2015

$$S = \int d^4x \Big\{ \phi_a^* \left(i\partial_t + K_c\nabla^2 - r_c + ir_d\right)\phi_r + \text{c.c.}$$

$$- \left[\left(u_c - iu_d\right)\left(\phi_a^*\phi_r^*\phi_r^2 + \phi_a^*\phi_r^*\phi_a^2\right) + \text{c.c.}\right]$$

$$+ i2\left(\gamma + 2u_d\phi_r^*\phi_r\right)\phi_a^*\phi_a \Big\}.$$

Manifest Weak (Retarded) Global Symmetry

Explicitly Broken (advanced) Global Symmetry

EFT for Global Case

$$S = S_{\rm diag} + S_{A\text{-break}},$$

Strong Symmetry

$$S_{\rm diag} = \int d^4x \Big[\phi_+^* \left(i\partial_t + K_c \nabla^2 - r_c \right) \phi_+ - \phi_-^* \left(i\partial_t + K_c \nabla^2 - r_c \right) \phi_- \\ - u_c \left(|\phi_+|^4 - |\phi_-|^4 \right) + i\gamma \left(|\phi_+|^2 + |\phi_-|^2 \right) + i u_d \left(|\phi_+|^4 + |\phi_-|^4 \right) \Big]$$

Explicit breaking of Strong, Unbroken Weak Symmetry

$$S_{A\text{-break}} = \int d^4x \left[-i\gamma_p \, \phi_+^* \phi_- - i\gamma_l \, \phi_-^* \phi_+ - 2iu_d \, (\phi_-^*)^2 \phi_+^2 \right].$$

Breaking of Adv. Symmetry = Ret. Current not conserved

Sieberer, Buchhold, Diehl 2015

$$S_{A\text{-break}} = \int d^4x \left[-i\gamma_p\, \phi_+^* \phi_- - i\gamma_l\, \phi_-^* \phi_+ - 2iu_d\, (\phi_-^*)^2 \phi_+^2 \right].$$

$$\partial_\mu J^\mu_{r,\text{Noether}} = \gamma_p\, \phi_+^* \phi_- - \gamma_l\, \phi_-^* \phi_+ - 4u_d\, (\phi_-^*)^2 \phi_+^2 \neq 0$$

$$\Big\downarrow$$

On identifying contours

$$\boxed{\partial_\mu J^\mu_{r,\text{Noether}} = (\gamma_p - \gamma_l)|\phi_r|^2 - 4u_d\, |\phi_r|^4 \neq 0}$$

e.g. particle number not conserved

Toy Example (Charged Caldeira-Leggett)

Akjuz, Penco 2025

Closed system

$$S[\Phi, \phi_I] = \int d^4x \left[-\partial_\mu \Phi^* \, \partial^\mu \Phi - m^2 |\Phi|^2 + \sum_I \left(\partial_t \phi_I^* \, \partial_t \phi_I - \Gamma_I^2 |\phi_I|^2 + g_I \left(\phi_I^* \Phi + \Phi^* \phi_I \right) \right) \right]$$

On integrating out the environment (eg at finite temperature)

$$S_{\text{eff}}[\Phi_+, \Phi_-] = S_\Phi[\Phi_+] - S_\Phi[\Phi_-] + S_{\text{IF}}[\Phi_+, \Phi_-].$$

$$S_\Phi[\Phi] = \int d^4x \left[-\partial_\mu \Phi^* \, \partial^\mu \Phi - m^2 |\Phi|^2 \right]$$

$$S_{\text{IF}}^{\text{tree}} = i \sum_I g_I^2 \int d^4x \, d^4y \, \begin{pmatrix} \Phi_+^*(x) & -\Phi_-^*(x) \end{pmatrix} \mathbf{G}_I(x, y) \begin{pmatrix} \Phi_+(y) \\ -\Phi_-(y) \end{pmatrix}$$

Toy Example (Charged Caldeira-Leggett)

$$S_{\rm IF}^{\rm tree} = i \sum_I g_I^2 \int d^4x\, d^4y\, \left(\Phi_+^*(x) \quad -\Phi_-^*(x)\right) \mathbf{G}_I(x,y) \begin{pmatrix} \Phi_+(y) \\ -\Phi_-(y) \end{pmatrix}$$

$$\mathbf{G}_I(x,y) = \begin{pmatrix} G_I^{++}(x,y) & G_I^{+-}(x,y) \\ G_I^{-+}(x,y) & G_I^{--}(x,y) \end{pmatrix} = \begin{pmatrix} \left\langle T\, \hat\phi_I(x)\hat\phi_I^\dagger(y) \right\rangle_{\rm env} & \left\langle \hat\phi_I^\dagger(y)\hat\phi_I(x) \right\rangle_{\rm env} \\ \left\langle \hat\phi_I(x)\hat\phi_I^\dagger(y) \right\rangle_{\rm env} & \left\langle \bar T\, \hat\phi_I(x)\hat\phi_I^\dagger(y) \right\rangle_{\rm env} \end{pmatrix} \qquad = \begin{pmatrix} \left\langle \hat\phi_I^+(x)\, \hat\phi_I^{+\dagger}(y) \right\rangle_{\rm env} & \left\langle \hat\phi_I^+(x)\, \hat\phi_I^{-\dagger}(y) \right\rangle_{\rm env} \\ \left\langle \hat\phi_I^-(x)\, \hat\phi_I^{+\dagger}(y) \right\rangle_{\rm env} & \left\langle \hat\phi_I^-(x)\, \hat\phi_I^{-\dagger}(y) \right\rangle_{\rm env} \end{pmatrix}.$$

$$\phi_I^\pm(x) \to e^{\pm iq\theta_\pm} \phi_I^\pm(x) \qquad\qquad \phi_I{}^{\pm\dagger}(x) \to e^{\mp iq\theta_\pm} \phi_I^{\pm\dagger}(x)$$

Explicitly breaking advanced global symmetry

$$S_{\rm IF} = \int d^4x \left\{ \frac{\gamma}{2} \Big[(\Phi_+^* - \Phi_-^*)\partial_t(\Phi_+ + \Phi_-) + (\Phi_+ - \Phi_-)\partial_t(\Phi_+^* + \Phi_-^*) \Big] + i\, 2\gamma T\, (\Phi_+^* - \Phi_-^*)(\Phi_+ - \Phi_-) \right\}$$

Global to Local?

Global Symmetries are Symmetries

Local Symmetries are Redundancies

$$A^{\pm}_{\mu} = A^{r}_{\mu} \pm \frac{1}{2} A^{a}_{\mu}$$

Global Symmetries can be broken

Local Symmetries are never actually broken

Conclusion:

EFT must preserve two copies of gauge symmetry, even when global symmetry is broken

The Gauge Charge Problem

Key problem:

In a gauge theory, there are no local gauge-invariant operators carrying gauge charge.

$$\psi(x) \to e^{iq\theta(x)}\psi(x) \qquad A_\mu(x) \to A_\mu(x) + \partial_\mu\theta(x)$$

Naive single particle states are not gauge invariant

$$\psi(x)^\dagger|0\rangle \to e^{iq\theta(x)}\psi^\dagger(x)|0\rangle$$

Violate Gauss law/BRST cohomology

The relativistic concept of a charged particle does not exist.

Kulish and Faddeev, 1970

The Importance of Dressing Well

Dirac's Solution, dress the states

$$\tilde{\psi}(x) = e^{-iq\frac{1}{\nabla^2}(\partial_i A_i(x))}\psi(x)$$

Dressed state is invariant under small gauge transformations (satisfies Gauss law constraint/BRST cohomology)

Dressed state transforms under global transformations!

$$\tilde{\psi}(x) \to e^{iq\theta_0}\tilde{\psi}(x)$$

Thus creating gauge invariant charged states!!!

There are lots of ways to dress

e.g. Lavelle–McMullan dressing (v=0)

$$\tilde{\psi}(x) = \exp\left[iq \int_{-\infty}^{x^0} ds\, \frac{1}{\nabla^2} \partial_i F^{i0}(s, \mathbf{x}) \right] \exp\left[-iq\, \frac{1}{\nabla^2} \partial_i A^i(x) \right] \psi(x).$$

Equivalent to Wilson line + Dirac Dressing at initial time

$$\tilde{\psi}(x) = \exp\left[-iq \int_{t_{\text{in}}}^{x^0} ds\, A_0(s, \mathbf{x}) \right] \exp\left[-iq\, \frac{1}{\nabla^2} \partial_i A^i(t_{\text{in}}, \mathbf{x}) \right] \psi(x).$$

Other examples: Fadeev-Kulish dressing (specific for S-matrix asymptotic states)

+ LOTs of Modern Literature dealing with IR divergences, soft theorems etc

Dirac Dressing suitable for NR systems

$$S_{\text{diag}}[A_+, A_-] = \int d^4x \Big[\phi_+^* \left(iD[A_+]_t + K_c D[A_+]_i^2 - r_c \right) \phi_+ - \phi_-^* \left(iD[A_-]_t + K_c D[A_-]_i^2 - r_c \right) \phi_- \\ - u_c \left(|\phi_+|^4 - |\phi_-|^4 \right) + i\gamma \left(|\phi_+|^2 + |\phi_-|^2 \right) + i u_d \left(|\phi_+|^4 + |\phi_-|^4 \right) \Big]$$

$$S_{a\text{-break}} = \int d^4x \Big[- i\gamma_p \, e^{+iq\Theta_a[A_a](x)} \, \phi_+^* \phi_- - i\gamma_l \, e^{-iq\Theta_a[A_a](x)} \, \phi_-^* \phi_+ - 2iu_d \, e^{-2iq\Theta_a[A_a](x)} \, (\phi_-^*)^2 \phi_+^2 \Big]$$

Only need dressing with advanced field

$$\Theta_a[A_a](x) = \nabla^{-2} \partial_i A_{a,i}(x).$$

EFT Manifestly Invariant under two copies of gauge symmetry, but breaks advanced global symmetry

Maxwell $\neq$ Noether

EFT Manifestly invariant under two copies of gauge symmetry, but breaks advanced global symmetry

Noether (Advanced Global) current is not conserved

Maxwell (Advanced Local) current is conserved

Curvature couplings

$$J_\mu^{\text{Maxwell}} = J_\mu^{\text{Noether}} - P_\mu{}^\nu\,\partial_\nu \frac{1}{\nabla^2}\left(\partial^\rho J_\rho^{\text{Noether}}\right) + \partial^\nu B_{\nu\mu}$$

$B_{\nu\mu} = -B_{\mu\nu}$

$P_0{}^\nu = 0, \qquad P_i{}^j = \delta_i{}^j$

$$\partial^\nu F^r_{\nu\mu} = -J^{\text{Maxwell}}_{r\mu} + \text{noise}$$

Dress like Schwinger

Dressing should preserve SK **final time boundary condition**

Dirac dressing does this, but disguises relativistic causality

Necessarily requires a different treatment of retarded and advanced gauge symmetry - required to preserve causality

Dressing is naturally incorporated into boundary conditions on solutions of propagator equations (Schwinger-Dyson equations)

Schwinger-Wilson Dressing

EFT covariant under retarded gauge transformations (if weak/retarded global symmetry is unbroken)

$$\widetilde{\phi}_{\pm}(x) = \exp\left[-iq\int_{t_{\rm in}}^{x^0} ds\, A_{r,0}(s,\mathbf{x})\right] \exp\left[-iq\,\frac{1}{\nabla^2}\partial_i A_r^i(t_{\rm in},\mathbf{x})\right] \times \exp\left[\pm\frac{iq}{2}\int_{x^0}^{t_f} ds\, A_{a,0}(s,\mathbf{x})\right]\phi_{\pm}(x)$$

No need to Dirac dress at final time due to final time boundary conditions!!!!

Dressed fields invariant under both retarded and advanced gauge transformations

Toy Example (Gauged Caldeira-Leggett)

Kaplanek, Mylova, AJT 2025

Closed system

$$S_{\rm SK} = S[A_+, \Phi_+, \phi_{I+}] - S[A_-, \Phi_-, \phi_{I-}].$$

$$S[A_\pm, \Phi_\pm, \phi_{I\pm}] = \int d^4x \Bigg[- \left(D_\mu[A_\pm]\Phi_\pm\right)^* D^\mu[A_\pm]\Phi_\pm - m^2|\Phi_\pm|^2 + \sum_I \Big(\left(D_t[A_\pm]\phi_{I\pm}\right)^* D_t[A_\pm]\phi_{I\pm} - \Gamma_I^2|\phi_{I\pm}|^2 + g_I \left(\phi^*_{I\pm}\Phi_\pm + \Phi^*_\pm\phi_{I\pm}\right)\Big)\Bigg]$$

On integrating out the environment (eg at finite temperature)

$$S_{\rm eff}[A_+, A_-; \Phi_+, \Phi_-] = S_\Phi[A_+, \Phi_+] - S_\Phi[A_-, \Phi_-] + S_{\rm IF}[A_+, A_-; \Phi_+, \Phi_-]$$

Toy Example (Gauged Caldeira-Leggett)

Kaplanek, Mylova, AJT 2025

On integrating out the environment (eg at finite temperature)

$$S_{\rm eff}[A_+, A_-; \Phi_+, \Phi_-] = S_\Phi[A_+, \Phi_+] - S_\Phi[A_-, \Phi_-] + S_{\rm IF}[A_+, A_-; \Phi_+, \Phi_-]$$

With

$$S_\Phi[A_\pm, \Phi_\pm] = \int d^4x \left[- \left(D_\mu[A_\pm]\Phi_\pm \right)^* D^\mu[A_\pm]\Phi_\pm - m^2 |\Phi_\pm|^2 \right]$$

And

$$S_{\rm IF}^{\rm tree} = i \sum_I g_I^2 \int d^4x\, d^4y\, \begin{pmatrix} \Phi_+^*(x) & -\Phi_-^*(x) \end{pmatrix} \mathbf{G}_I[A_+, A_-](x, y) \begin{pmatrix} \Phi_+(y) \\ -\Phi_-(y) \end{pmatrix}$$

Greens function equation

The Greens function now satisfies

$$\begin{pmatrix} D_0[A_+]^2+\Gamma_I^2 & 0 \\ 0 & D_0[A_-]^2+\Gamma_I^2 \end{pmatrix} \mathbf{G}_I[A_+,A_-](x,y) = -i\,\delta^{(4)}(x-y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Solution is

$$S_{\rm IF}^{\rm tree} = i\sum_I g_I^2 \int d^4x\,d^4y\,\left(\tilde{\Phi}_+^*(x) \quad -\tilde{\Phi}_-^*(x)\right)\mathbf{G}_I[A_r,A_r](x,y)\begin{pmatrix} \tilde{\Phi}_+(y) \\ -\tilde{\Phi}_-(y) \end{pmatrix}$$

With dressing

$$\widetilde{\Phi}_\pm(x) = \exp\left[\pm\frac{iq}{2}\int_{x^0}^{t_f} ds\,A_0^a(s,\mathbf{x})\right]\Phi_\pm(x).$$

So in local limit:

$$S_{\rm IF} = \int d^4x\left\{\frac{\gamma}{2}\left[(\widetilde{\Phi}_+^* - \widetilde{\Phi}_-^*)D_t[A_r](\widetilde{\Phi}_+ + \widetilde{\Phi}_-) + (\widetilde{\Phi}_+ - \widetilde{\Phi}_-)D_t^*[A_r](\widetilde{\Phi}_+^* + \widetilde{\Phi}_-^*)\right] + i\,2\gamma T\,(\widetilde{\Phi}_+^* - \widetilde{\Phi}_-^*)(\widetilde{\Phi}_+ - \widetilde{\Phi}_-)\right\}$$

Maxwell $\neq$ Noether

$$S_{\mathrm{IF}} = \int d^4x \left\{ \frac{\gamma}{2} \left[(\widetilde{\Phi}^*_+ - \widetilde{\Phi}^*_-) D_t[A_r] (\widetilde{\Phi}_+ + \widetilde{\Phi}_-) + (\widetilde{\Phi}_+ - \widetilde{\Phi}_-) D^*_t[A_r] (\widetilde{\Phi}^*_+ + \widetilde{\Phi}^*_-) \right] + i\, 2\gamma T\, (\widetilde{\Phi}^*_+ - \widetilde{\Phi}^*_-)(\widetilde{\Phi}_+ - \widetilde{\Phi}_-) \right\}$$

Once again EFT Manifestly invariant under two copies of gauge symmetry, but breaks advanced global symmetry

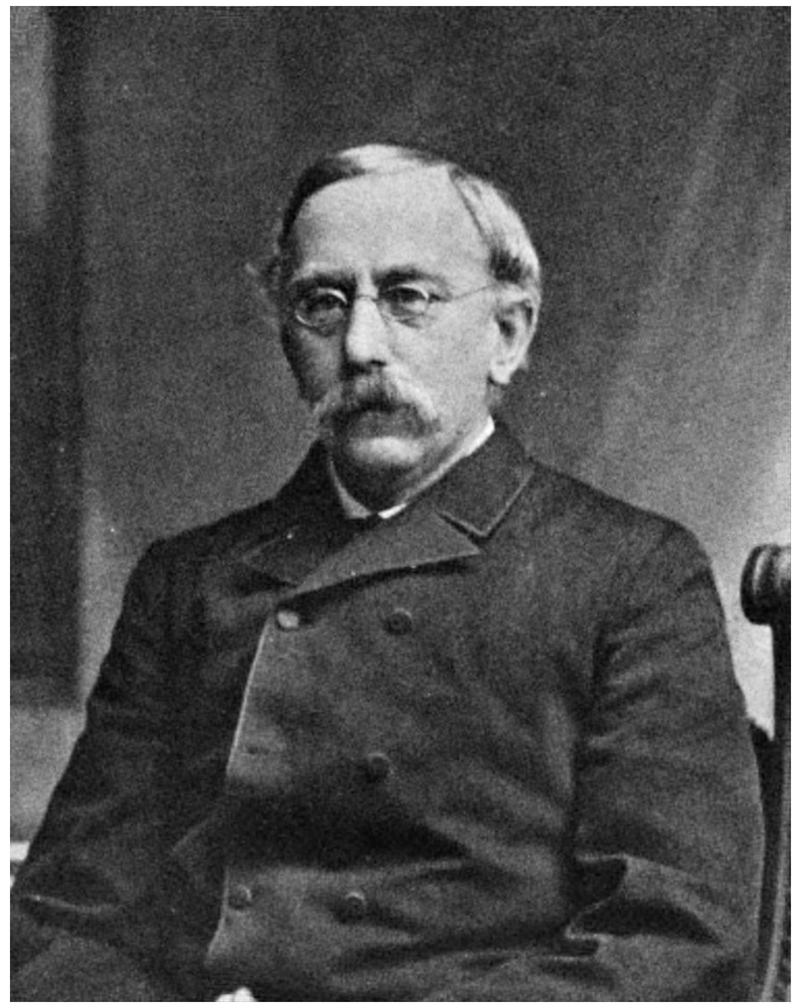
$$J^\mu_{\mathrm{Maxwell}}(x) = J^\mu_{\mathrm{Noether}}(x) - n^\mu \int_{t_{\mathrm{in}}}^{x^0} ds\, \partial_\nu J^\nu_{\mathrm{Noether}}(s, \mathbf{x}) + \partial^\nu B_{\nu\mu}, \qquad n^\mu = (1, \mathbf{0})$$

$$B_{\nu\mu} = -B_{\mu\nu}$$

$$\partial^\nu F^r_{\nu\mu} = -J^{\mathrm{Maxwell}}_{r\mu} + \mathrm{noise}$$

Lorenz-Lorentz Dressing

In a relativistic theory, its easier to see relativistic causality in Lorenz gauge



Focussing on advanced gauge fields, natural Lorentz invariant dressing is

$$\tilde{\phi}_{\pm}(x) = e^{\pm \frac{i}{2} q \int d^4 y G_a(x,y)(\partial^{\mu} A^a_{\mu}(y))} \phi_{\pm}(x)$$

$$\Box G_a(x,y) = \delta^4(x,y)$$

Advanced Greens function

Lorenz-Lorentz Dressing

$$\tilde{\phi}_\pm(x) = e^{\pm\frac{i}{2}q\int d^4y G_a(x,y)(\partial^\mu A^a_\mu(y))}\phi_\pm(x)$$

Curvature couplings

$$J_\mu^{\text{Maxwell}}(x) = J_\mu^{\text{Noether}}(x) - \partial_\mu^x \int d^4y\, G_r(x,y)\partial^\nu J_\nu^{\text{Noether}}(y) + \partial^\nu B_{\mu\nu}(x)$$

$$B_{\nu\mu} = -B_{\mu\nu}$$

Retarded Greens function

Local version

Dirac dressing

$$S_{SK} = \int d^4x\, \partial_i \kappa_r(x)(\partial_i \Theta_A(x) - A_i) + \mathcal{S}_{\text{local}}[A_+, A_-, \phi_+, \phi_-, \phi_+^*, \phi_-^*, \Theta_A]$$

Schwinger-Wilson dressing

$$S_{SK} = \int d^4x\, \partial^0 \kappa_r(x)(\partial_0 \Theta_A(x) - A_0) + \mathcal{S}_{\text{local}}[A_+, A_-, \phi_+, \phi_-, \phi_+^*, \phi_-^*, \Theta_A]$$

Lorenz-Lorentz dressing

$$S_{SK} = \int d^4x\, \partial^\mu \kappa_r(x)(\partial_\mu \Theta_A(x) - A_\mu) + \mathcal{S}_{\text{local}}[A_+, A_-, \phi_+, \phi_-, \phi_+^*, \phi_-^*, \Theta_A]$$

Massless open EM = Massless Limit of Massive open EM

No vDVZ discontinuity

Dirac dressing

$$S_{SK} = \int d^4x \, \frac{1}{2} m^2 (-A_i^{+\,2} + A_i^{-\,2}) + \mathcal{S}_{\text{local}}^{\text{unitary gauge}}[A_+, A_-, \phi_+, \phi_-, \phi_+^*, \phi_-^*]$$

Schwinger-Wilson dressing

$$S_{SK} = \int d^4x \, \frac{1}{2} m^2 (A_0^{+\,2} - A_0^{-\,2}) + \mathcal{S}_{\text{local}}^{\text{unitary gauge}}[A_+, A_-, \phi_+, \phi_-, \phi_+^*, \phi_-^*]$$

Lorenz-Lorentz dressing

$$S_{SK} = \int d^4x \, \frac{1}{2} m^2 (-A_\mu^+ A^{+\mu} + A_\mu^- A^{-\mu}) + \mathcal{S}_{\text{local}}^{\text{unitary gauge}}[A_+, A_-, \phi_+, \phi_-, \phi_+^*, \phi_-^*]$$

SK for Gauge Theories a la BRST

Central benefits of BRST formalism:

Replace LOCAL symmetry with GLOBAL Symmetry

Make relativistic locality manifest but control unphysical states

SK rules for Global Symmetries apply - Final time boundary conditions explicitly break naive doubled BRST down to single diagonal/retarded copy

BRST cohomology = Physical States respect **strong BRST symmetry** (no spontaneous breaking!!) although more on this later

BRST Symmetry

Nilpotent transformations

$$\delta = \eta\hat{s}$$

$$\begin{aligned} \hat{s}A^a_\mu &= D_\mu c^a = \partial_\mu c^a + g f^{abc} A^b_\mu c^c, \\ \hat{s}c^a &= -\frac{1}{2} g f^{abc} c^b c^c, \\ \hat{s}\bar{c}^a &= B^a, \\ \hat{s}B^a &= 0 \end{aligned}$$

Yang-Mills in covariant gauge

$$S = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}^4x \left[-\frac{1}{4} {F^a_{\mu\nu}}^2 + B^a(\partial_\mu A^\mu_a) + \frac{\xi}{2} B^2_a + \partial_\mu \bar{c}^a D^\mu c^a \right]$$

Schrodinger Wavefunctionals

In Lorenz, covariant gauges typically used in particle physics, natural choice of fields are

$$A_\mu^a, c^a, \bar{c}^a$$

Two subtleties arise

1. Naively non normalisable wave functions of A_0
2. Ghosts are Grassman odd, how do we make sense?

Indefinite Hilbert space Norm

In covariant gauges, A_0 has negative norm states. Naive Lorentz invariant vacuum wavefunctional is **not normalisable**

$$\psi[A_\mu] \propto e^{-\int d^3x \frac{1}{2} A_\mu \sqrt{-\nabla^2} A^\mu} \qquad \int DA_\mu |\psi[A_\mu]|^2 = \infty$$

Solution due to Pauli: in an indefinite Hilbert space, hermitian operators can have imaginary eigenvalues)

$$A_0 = iA_4$$

$$\langle \psi_1 | \psi_2 \rangle = \int D[A_i] \int D[A_4] \, \psi_1^*[A_i, -A_4] \psi_2[A_i, A_4]$$

Odd in A_0 wavefunctionals have negative norm, consistent with canonical quantisation

Vacuum is normalisable with positive norm $\psi[A_\mu] \propto e^{-\int d^3x \frac{1}{2} A_i \sqrt{-\nabla^2} A^i} e^{-\int d^3x \frac{1}{2} A_4 \sqrt{-\nabla^2} A_4}$

Indefinite Hilbert space Norm II

In covariant gauges, FPDW ghosts have second order equations of motion

$$\Box c = \Box \bar{c} = 0$$

Unlike physical fermions, we **do not need to use** coherent states!!!

Meaningful notation of Grassmann odd wavefunctional $\psi[c, \bar{c}]$

$$\langle \psi_1 | \psi_2 \rangle = \int Dc \int D\bar{c}\, \psi_1^\dagger[c, \bar{c}] \psi_2[c, \bar{c}]$$

$$Tr[\hat{O}] = \int Dc \int D\bar{c}\, \langle -c, -\bar{c} | \hat{O} | c, \bar{c} \rangle$$

Fermionic property

Free theory Fock States

Vacuum wavefunctional is now **manifestly normalisable**

$$\psi[A, c, \bar{c}] \propto e^{-\int d^3x \frac{1}{2} A_i \sqrt{-\nabla^2} A^i - \int d^3x \frac{1}{2} A_4 \sqrt{-\nabla^2} A^4} e^{\int d^3x \bar{c} \sqrt{-\nabla^2} c}$$

Generic normalisable Fock states can be constructed by applying Schrodinger picture creation operators

$$\hat{a}_i^\dagger(\mathbf{k}) \equiv \int \mathrm{d}^3\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}} \left(\sqrt{-\nabla^2}\, A_i(\mathbf{x}) - \frac{\delta}{\delta A_i(\mathbf{x})}\right) \qquad \hat{a}_0^\dagger(\mathbf{k}) \equiv i \int \mathrm{d}^3\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}} \left(\sqrt{-\nabla^2}\, A_4(\mathbf{x}) - \frac{\delta}{\delta A_4(\mathbf{x})}\right)$$

Physical States

Generic physical states satisfy

$$\hat{Q}|\psi_{\text{phys}}\rangle = 0 \qquad \hat{Q}\hat{\rho}_{\text{phys}} = \hat{\rho}_{\text{phys}}\hat{Q} = 0$$

Problem, with $\psi[A_\mu, c, \bar{c}]$ is $\hat{s}A_0^a = D_0[A]c^a = \text{conjugate momenta}$

Condition $\langle A_\mu, c, \bar{c}|\hat{Q}|\psi\rangle = 0$ is then a functional differential equation

Nakanishi-Lautrup Representation

A_0 ⬌ B Canonically conjugate

$$S_{\mathrm{NL}} = \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} \mathrm{d}^4x \left[-\frac{1}{4} F^a_{\mu\nu}{}^2 - \partial_\mu B^a A^\mu_a + \frac{1}{2} B_a^2 + \partial_\mu \bar{c}^a D^\mu c^a \right]$$

Work with $\psi[A_i, B, c, \bar{c}]$

$$\begin{aligned} \hat{s}A_i^a &= D_i c^a = \partial_i c^a + g f^{abc} A_i^b c^c, \\ \hat{s}c^a &= -\frac{1}{2} g f^{abc} c^b c^c, \\ \hat{s}\bar{c}^a &= B^a, \\ \hat{s}B^a &= 0 \end{aligned}$$

$\langle A_i, B, c, \bar{c} | \hat{Q} | \psi_{\mathrm{phys}} \rangle = 0$ ➡ $\hat{s}\psi_{\mathrm{phys}}[A_i, B, c, \bar{c}] = 0$

Batalin-Marnelius Construction

This allows us to give a general solution of the BRST cohomology

$$\psi_{\text{phys}}[A_i, B, c, \bar{c}] = e^{\hat{s}G[A_i, B, c, \bar{c}]}\,\Psi[A_i]$$

BRST exact

Gauge invariant under 3D gauge transformations

E.g. Vacuum Wavefunctionals

$$\psi_0[A_i, B, c, \bar{c}] \propto e^{\hat{s}G} e^{-\frac{1}{4}\int d^3\mathbf{x}\; F_{ij}\frac{1}{\sqrt{-\nabla^2}}F^{ij}}$$

BRST exact

Gauge invariant under 3D gauge transformations

$$G = \int d^3\mathbf{x}\left[\frac{1}{2}\bar{c}\frac{1}{\sqrt{-\nabla^2}}B + \bar{c}\frac{1}{\sqrt{-\nabla^2}}\partial_i A_i\right]$$

BRST Strong to Weak

Physical States are **BRST Strong**

$$\hat{Q}\hat{\rho}_{\rm phys} = \hat{\rho}_{\rm phys}\hat{Q} = 0$$

$$\langle A_i^+ + \eta\hat{s}A_i^+, B^+ + \eta\hat{s}B^+, c^+ + \eta\hat{s}c^+, \bar{c}^+ + \eta\hat{s}\bar{c}^+ |\hat{\rho}_{\rm phys}| A_i^-, B^-, c^-, \bar{c}^- \rangle = 0$$

$$\left\langle A_i^+, B^+, c^+, \bar{c}^+ \right| \hat{\rho}_{\rm phys} \left| A_i^- + \eta\hat{s}A_i^-, B^- + \eta\hat{s}B^-, c^- + \eta\hat{s}c^-, \bar{c}^- + \eta\hat{s}\bar{c}^- \right\rangle = 0$$

However Hata-Kugo showed that

$$\mathrm{Tr}[\hat{\rho}_{\rm phys}\hat{O}] = \mathrm{Tr}[e^{\pi\hat{Q}_G}\hat{\rho}\hat{O}]$$

Provided

$$[\hat{Q}, \hat{\rho}] = 0$$
$$[\hat{Q}, \hat{O}] = 0$$

Hata-Kugo means we only need weak BRST symmetry

$\hat{\rho}_{\mathrm{phys}}$ (Strong BRST) → $\hat{\rho}$ (Weak BRST)

$$[\hat{Q}, \hat{\rho}] = 0$$

$$\mathrm{Tr}[\hat{\rho}_{\mathrm{phys}}\hat{O}] = \mathrm{Tr}[e^{\pi \hat{Q}_G}\hat{\rho}\hat{O}]$$

$$\left\langle A_i^+ + \eta\hat{s}A_i^+, B^+ + \eta\hat{s}B^+, c^+ + \eta\hat{s}c^+, \bar{c}^+ + \eta\hat{s}\bar{c}^+ \middle| \hat{\rho} \middle| A_i^- + \eta\hat{s}A_i^-, B^- + \eta\hat{s}B^-, c^- + \eta\hat{s}c^-, \bar{c}^- + \eta\hat{s}\bar{c}^- \right\rangle = 0$$

$$\hat{s}_{\mathrm{diagonal}} \left\langle A_i^+, B^+, c^+, \bar{c}^+ \middle| \hat{\rho} \middle| A_i^-, B^-, c^-, \bar{c}^- \right\rangle = 0.$$

Historically this is why different Thermal b.c allowed

Bernard-Hata-Kugo

$$\hat{\rho} = Z^{-1} e^{-\beta \hat{H}_{BRST}}$$

Landshoff-Rehban

$$\hat{\rho} = Z^{-1} e^{-\beta \hat{H}_{\text{Transverse polarizations}}}$$

Weak BRST state different, Strong state the same!

Ghost boundary conditions

$$\mathrm{Tr}[\hat{\rho}_{\mathrm{phys}}\hat{O}] = \mathrm{Tr}[e^{\pi \hat{Q}_G}\hat{\rho}\hat{O}]$$

$$e^{\pi \hat{Q}_G}|c, \bar{c}\rangle = |-c, -\bar{c}\rangle$$

$$Tr[\hat{O}] = \int Dc \int D\bar{c}\, \langle -c, -\bar{c}|\hat{O}|c, \bar{c}\rangle \quad \Rightarrow \quad Tr[e^{\pi \hat{Q}_G}\hat{O}] = \int Dc \int D\bar{c}\, \langle c, \bar{c}|\hat{O}|c, \bar{c}\rangle$$

BRST path integral recipe

1. Work in Nakanishi-Lautrup representation
2. Double fields, double action, CTP contour
3. Take an initial density state that respects WEAK (diagonal) BRST symmetry only
4. Perform path integral with final time boundary conditions

$$B^a_+(t_f) = B^a_-(t_f)\,, \quad A_i^{+a}(t_f) = A_i^{-a}(t_f)\,, \quad c^a_+(t_f) = c^a_-(t_f)\,, \quad \bar{c}^a_+(t_f) = \bar{c}^a_-(t_f)$$

5. At initial time include - sign for physical fermions but NOT for ghosts (Hata-Kugo)

MANIFEST DIAGONAL BRST SYMMETRY

BRST Path Integral

$$e^{iW[\boldsymbol{J}^+,\boldsymbol{J}^-;\boldsymbol{K}^+,\boldsymbol{K}^-]} = \int^{\boldsymbol{f}^+(t_\mathrm{f})=\boldsymbol{f}^-(t_\mathrm{f})} \mathcal{D}[\boldsymbol{f}_+,\boldsymbol{f}_-]\,\mu[\boldsymbol{f}_+,\boldsymbol{f}_-]\,\langle(-1)^{\hat{N}_\psi}\mathbf{f}_\mathrm{i}^+|\hat{\rho}(t_\mathrm{i})|\mathbf{f}_\mathrm{i}^-\rangle$$
$$\times\; e^{iS_\mathrm{NL}[\boldsymbol{f}^+]+i\int_{t_\mathrm{i}}^{t_\mathrm{f}}\mathrm{d}^4x\,(\boldsymbol{J}^+\cdot\boldsymbol{f}^+ + \boldsymbol{K}^+\cdot\hat{s}\boldsymbol{f}^+)-iS_\mathrm{NL}[\boldsymbol{f}^-]-i\int_{t_\mathrm{i}}^{t_\mathrm{f}}\mathrm{d}^4x\,(\boldsymbol{J}^-\cdot\boldsymbol{f}^- + \boldsymbol{K}^-\cdot\hat{s}\boldsymbol{f}^-)}$$

Sources

$$\boldsymbol{J}\cdot\boldsymbol{f}+\boldsymbol{K}\cdot\hat{s}\boldsymbol{f} = J_a^\mu A_\mu^a + J_B^a B^a + \bar{J}_c^a c^a + \bar{c}^a J_c^a + J_\Phi^\dagger\Phi + \Phi^\dagger J_\Phi + \bar{\eta}\psi + \bar{\psi}\eta$$
$$+K_a^\mu D_\mu c^a + K_a\left(-\frac{g}{2}f^{abc}c^b c^c\right) + ig\kappa_\Phi^\dagger c^a T_\Phi^a\Phi - ig\Phi^\dagger c^a T_\Phi^a\kappa_\Phi + ig\kappa_\psi^\dagger c^a T_\psi^a\psi - ig\bar{\psi}c^a T_\psi^a\kappa_\psi$$

Final time b.c.

$$A_i^+(t_\mathrm{f}) = A_i^-(t_\mathrm{f})\,,\quad B^+(t_\mathrm{f}) = B^-(t_\mathrm{f})\,,\quad c^+(t_\mathrm{f}) = c^-(t_\mathrm{f})\,,\quad \bar{c}^+(t_\mathrm{f}) = \bar{c}^-(t_\mathrm{f})$$

Zinn-Justin equation for Weak BRST

All of the Ward-Takahashi-Slavnov-Taylor identifies are encoded in the CTP Zinn-Justin equation that follows from diagonal BRST

$$\mathcal{S}(\Gamma) \equiv \mathcal{S}_+(\Gamma) - \mathcal{S}_-(\Gamma) = 0$$

$$\mathcal{S}_\pm(\Gamma) \equiv \int_{t_{\rm i}}^{t_{\rm f}} \mathrm{d}^4x \left[\frac{\delta\Gamma}{\delta A^a_{\pm\mu}} \frac{\delta\Gamma}{\delta K^\mu_{\pm a}} + \frac{\delta\Gamma}{\delta c^a_\pm} \frac{\delta\Gamma}{\delta K_{\pm a}} + B^a_\pm \frac{\delta\Gamma}{\delta \bar{c}^a_\pm} \right.$$
$$\left. + \frac{\delta\Gamma}{\delta\Phi_\pm} \frac{\delta\Gamma}{\delta K^\dagger_{\pm\Phi}} + \frac{\delta\Gamma}{\delta\Phi^\dagger_\pm} \frac{\delta\Gamma}{\delta K_{\pm\Phi}} + \frac{\delta\Gamma}{\delta\psi_\pm} \frac{\delta\Gamma}{\delta K^\dagger_{\pm\psi}} + \frac{\delta\Gamma}{\delta\bar{\psi}_\pm} \frac{\delta\Gamma}{\delta K_{\pm\psi}} \right]$$

At `tree’ level

$$\hat{s}\left(S_{\rm NL}[\mathbf{f}_+] - S_{\rm NL}[\mathbf{f}_-] - i \ln\left(\langle (-1)^{\hat{N}_\psi} \mathbf{f}_{\rm i}^+ | \hat{\rho}(t_{\rm i}) | \mathbf{f}_{\rm i}^- \rangle \right) \right) = 0$$

Keldysh contraction of diagonal BRST symmetry

In Keldysh variables, diagonal BRST symmetry is

$$
\begin{aligned}
\hat{s}A^a_{\mathbf{r}\mu} &= D_\mu[A_\mathbf{r}]c^a_\mathbf{r} + \frac{1}{4}gf^{abc}A^b_{\mathbf{a}\mu}c^c_\mathbf{a}\,, & \hat{s}c^a_\mathbf{r} &= -\frac{1}{2}gf^{abc}c^b_\mathbf{r}c^c_\mathbf{r} - \frac{1}{8}gf^{abc}c^b_\mathbf{a}c^c_\mathbf{a}\,,\\
\hat{s}\bar{c}^a_\mathbf{r} &= B^a_\mathbf{r}\,, & \hat{s}B^a_\mathbf{r} &= 0\,,\\
\hat{s}A^a_{\mathbf{a}\mu} &= D_\mu[A_\mathbf{r}]c^a_\mathbf{a} + gf^{abc}A^b_{\mathbf{a}\mu}c^c_\mathbf{r}\,, & \hat{s}c^a_\mathbf{a} &= -gf^{abc}c^b_\mathbf{r}c^c_\mathbf{a}\,,\\
\hat{s}\bar{c}^a_\mathbf{a} &= B^a_\mathbf{a}\,, & \hat{s}B^a_\mathbf{a} &= 0\,,
\end{aligned}
$$

In `classical' limit

$$\boldsymbol{f}^\mathbf{a} \to \epsilon \boldsymbol{f}^\mathbf{a}$$

$$
\begin{aligned}
\hat{s}_K A^a_{\mathbf{r}\mu} &= D_\mu[A_\mathbf{r}]c^a_\mathbf{r}\,, & \hat{s}_K c^a_\mathbf{r} &= -\frac{1}{2}gf^{abc}c^b_\mathbf{r}c^c_\mathbf{r}\,,\\
\hat{s}_K \bar{c}^a_\mathbf{r} &= B^a_\mathbf{r}\,, & \hat{s}_K B^a_\mathbf{r} &= 0\,,\\
\hat{s}_K A^a_{\mathbf{a}\mu} &= D_\mu[A_\mathbf{r}]c^a_\mathbf{a} + gf^{abc}A^b_{\mathbf{a}\mu}c^c_\mathbf{r}\,, & \hat{s}_K c^a_\mathbf{a} &= -gf^{abc}c^b_\mathbf{r}c^c_\mathbf{a}\,,\\
\hat{s}_K \bar{c}^a_\mathbf{a} &= B^a_\mathbf{a}\,, & \hat{s}_K B^a_\mathbf{a} &= 0\,.
\end{aligned}
$$

Keldysh BRST

Keldysh BRST symmetry is an EXACT symmetry of the SK action truncated at quadratic order in the Keldysh expansion

$$S_K[\boldsymbol{f}^{\mathbf{r}}, \boldsymbol{f}^{\mathbf{a}}] = \int \mathrm{d}^4x \, \frac{\delta S_{\mathrm{EFT}}}{\delta \boldsymbol{f}^{\mathbf{a}}(x)}\bigg|_{\boldsymbol{f}^{\mathbf{a}}=0} \cdot \boldsymbol{f}^{\mathbf{a}}(x) + \frac{1}{2} \int \mathrm{d}^4x \int \mathrm{d}^4y \, \boldsymbol{f}^{\mathbf{a}}(x) \cdot \frac{\delta^2 S_{\mathrm{EFT}}}{\delta \boldsymbol{f}^{\mathbf{a}}(x) \delta \boldsymbol{f}^{\mathbf{a}}(y)}\bigg|_{\boldsymbol{f}^{\mathbf{a}}=0} \cdot \boldsymbol{f}^{\mathbf{a}}(y)$$

$$\boxed{\hat{s}_K S_K[\boldsymbol{f}^{\mathbf{r}}, \boldsymbol{f}^{\mathbf{a}}] = 0}$$

Retarded Gauge Invariance + Covariant Abelian gauge invariance

$$\hat{s}_K = \hat{s}_{\mathbf{r}} + \hat{s}_{\mathbf{a}}$$

Pure retarded BRST

$$\begin{aligned}
&\hat{s}_{\mathbf{r}} A^a_{\mathbf{r}\mu} = D_\mu[A_{\mathbf{r}}] c^a_{\mathbf{r}}\,, &\qquad& \hat{s}_{\mathbf{r}} c^a_{\mathbf{r}} = -\frac{1}{2} g f^{abc} c^b_{\mathbf{r}} c^c_{\mathbf{r}}\,,\\
&\hat{s}_{\mathbf{r}} \bar{c}^a_{\mathbf{r}} = B^a_{\mathbf{r}}\,, && \hat{s}_{\mathbf{r}} B^a_{\mathbf{r}} = 0\,,\\
&\hat{s}_{\mathbf{r}} A^a_{\mathbf{a}\mu} = g f^{abc} A^b_{\mathbf{a}\mu} c^c_{\mathbf{r}}\,, && \hat{s}_{\mathbf{r}} c^a_{\mathbf{a}} = -g f^{abc} c^b_{\mathbf{r}} c^c_{\mathbf{a}}\,,\\
&\hat{s}_{\mathbf{r}} \bar{c}^a_{\mathbf{a}} = 0\,, && \hat{s}_{\mathbf{r}} B^a_{\mathbf{a}} = 0\,,
\end{aligned}$$

Abelian BRST (covariant under retarded)

$$\begin{aligned}
&\hat{s}_{\mathbf{a}} A^a_{\mathbf{r}\mu} = 0\,, &\qquad& \hat{s}_{\mathbf{a}} c^a_{\mathbf{r}} = 0\,,\\
&\hat{s}_{\mathbf{a}} \bar{c}^a_{\mathbf{r}} = 0\,, && \hat{s}_{\mathbf{a}} B^a_{\mathbf{r}} = 0\,,\\
&\hat{s}_{\mathbf{a}} A^a_{\mathbf{a}\mu} = D_\mu[A_{\mathbf{r}}] c^a_{\mathbf{a}}\,, && \hat{s}_{\mathbf{a}} c^a_{\mathbf{a}} = 0\,,\\
&\hat{s}_{\mathbf{a}} \bar{c}^a_{\mathbf{a}} = B^a_{\mathbf{a}}\,, && \hat{s}_{\mathbf{a}} B^a_{\mathbf{a}} = 0\,.
\end{aligned}$$

General structure of BRST EFT

$$S^{SK}_{BRST} = S^{\text{gauge invariant}}_{SK}[A_+, A_-] + \hat{s}_{\text{diagonal}} F[A_i^{\pm}, B^{\pm}, c^{\pm}, \bar{c}^{\pm}]$$

Or to quadratic order in Keldysh expansion

$$S^{SK}_{BRST,K} = S^{\text{non-Abelian ret. + abelian adv.}}_{SK}[A_+, A_-] + \hat{s}_K F[A_i^{\pm}, B^{\pm}, c^{\pm}, \bar{c}^{\pm}]$$

$k \sim gT \quad q \sim T$

Hard Thermal Loop EFT

Caron-Huot 2007

$$E_i[A_{\mathbf{r}}] = F_{i0}[A_{\mathbf{r}}]$$

$$S^{\mathrm{SK}}_{\mathrm{HTL}}[A_{\mathbf{r}}, A_{\mathbf{a}}] = m_D^2 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \int \mathrm{d}^4x\, v^\mu A^a_{\mu\mathbf{a}}(x) \left[\frac{1}{v\cdot D[A_{\mathbf{r}}]_{\mathrm{ret}}}\, v^i E_i[A_{\mathbf{r}}]\right]^a (x) \; + \frac{iTm_D^2}{2} \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \int \mathrm{d}^4x\, v^\mu A^a_{\mu\mathbf{a}}(x) \left[\left(\frac{1}{v\cdot D[A_{\mathbf{r}}]_{\mathrm{ret}}} - \frac{1}{v\cdot D[A_{\mathbf{r}}]_{\mathrm{adv}}}\right) v^\nu A_{\nu\mathbf{a}}\right]^a (x)$$

Quadratic order in Keldysh expansion. Manifestly invariant under retarded gauge transformations

$$\delta_{\mathbf{a}} A^a_{\mu\mathbf{a}} = \left(D_\mu[A_{\mathbf{r}}]\lambda_{\mathbf{a}}\right)^a, \qquad \delta_{\mathbf{a}} A^a_{\mu\mathbf{r}} = 0,$$

$$\delta_a S^{SK}_{HTL} = 0$$

➡

$$\hat{s}_K S^{\mathrm{SK}}_{\mathrm{HTL}} = 0$$

Satisfies Dynamical KMS (Abe, Nishii 2026)

Kaplanek, Mylova, AJT 2026

Open EFT for Yang Mills with SSB

Unitary gauge

$$\mathcal{A}_{\mu\mathbf{r}} = \frac{1}{2}\left(\mathcal{A}_{\mu+} + \mathcal{A}_{\mu-}\right) = U(\pi_{\mathbf{r}}) A_{\mu\mathbf{r}} U^{-1}(\pi_{\mathbf{r}}) + \frac{i}{g} U(\pi_{\mathbf{r}}) \partial_\mu U^{-1}(\pi_{\mathbf{r}})$$

$$\mathcal{A}_{\mu\mathbf{a}} = \mathcal{A}_{\mu+} - \mathcal{A}_{\mu-} = U(\pi_{\mathbf{r}}) \left(A_{\mu\mathbf{a}} + D_\mu[A_{\mathbf{r}}]\pi_{\mathbf{a}}\right) U^{-1}(\pi_{\mathbf{r}})\,.$$

Stuckelberg form

$$\tilde{D}_\mu[A_{\mathbf{r}}]\pi_{\mathbf{r}} \equiv A_{\mu\mathbf{r}} + \frac{i}{g}\left(\partial_\mu U^{-1}(\pi_{\mathbf{r}})\right) U(\pi_{\mathbf{r}})$$

$$\begin{aligned}
S_{K,EFT} = & \int \mathrm{d}^d x \, \left(A^a_{\mathbf{a}\mu}(x) + D_\mu[A_{\mathbf{r}}]\pi^a_{\mathbf{a}}(x)\right) E^{a\mu}(x; A_{\mathbf{r}}, \tilde{D}[A_{\mathbf{r}}]\pi_{\mathbf{r}}) \\
& + \frac{i}{2}\int \mathrm{d}^d x \int \mathrm{d}^d y \, \left(A^a_{\mathbf{a}\mu}(x) + D_\mu[A_{\mathbf{r}}]\pi^a_{\mathbf{a}}(x)\right) N^{ab\,\mu\nu}\left(x, y; A_{\mathbf{r}}, \tilde{D}[A_{\mathbf{r}}]\pi_{\mathbf{r}}\right) \\
& \qquad \times \left(A^b_{\mathbf{a}\nu}(y) + D_\nu[A_{\mathbf{r}}]\pi^b_{\mathbf{a}}(y)\right) \\
& + \hat{s}_K G_K[A_{\mathbf{r}}, A_{\mathbf{a}}, \pi_{\mathbf{r}}, \pi_{\mathbf{a}}, c_{\mathbf{r}}, c_{\mathbf{a}}, \bar{c}_{\mathbf{r}}, \bar{c}_{\mathbf{a}}, B_{\mathbf{r}}, B_{\mathbf{a}}]\,.
\end{aligned}$$

Causality

$$\frac{\delta E^{a\mu}(x)}{\delta A^{\nu b}_{\mathbf{r}}(y)} = 0\,, \quad \text{unless } x^0 \geq y^0\,,\ (x-y)^2 \leq 0\,.$$

$$\frac{\delta E^{a\mu}(x)}{\delta \pi^b_{\mathbf{r}}(y)} = 0\,, \quad \text{unless } x^0 \geq y^0\,,\ (x-y)^2 \leq 0\,.$$

Unitarity

$$\int \mathrm{d}^d x \int \mathrm{d}^d y \, f^a_\mu(x) N^{ab\,\mu\nu}(x, y; \mathcal{A}_{\mathbf{r}}) f^b_\nu(y) \geq 0$$

$$N^{ab\,\mu\nu}(x,y) = N^{ba\,\nu\mu}(y,x)$$

Conclusions

- Gauge charge in open systems is subtle: Open EFTs may break advanced/global charge conservation but must preserve two copies of gauge symmetry

- In the absence of SSB, gauge invariance is naturally maintained by Wilson-line dressings of advanced fields ending at the SK final-time surface.

- Open EFTs for gauge theories can be constructed systematically with weak/diagonal BRST symmetry. SK final-time boundary condition breaks the naive doubled BRST symmetry down to weak/diagonal BRST.

- Applications: HTL effective theory becomes a natural example where causality, unitarity, and dynamical KMS are manifest.

- Spontaneously broken non-Abelian open gauge theories can be treated similarly.