

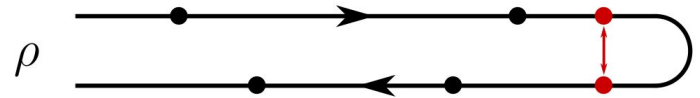
# Schwinger-Keldysh EFTs for Precision Physics and Discovery

Luca Delacrétaz | U Chicago

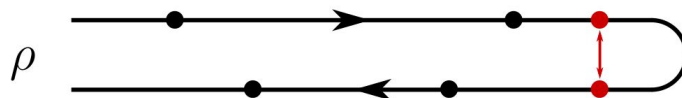
*Contours 2026 in Cambridge*  
*30 June 2026*

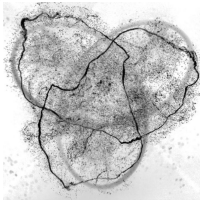


Schwinger-Keldysh EFTs have allowed for unprecedented control in otherwise strongly coupled quantum many-body systems

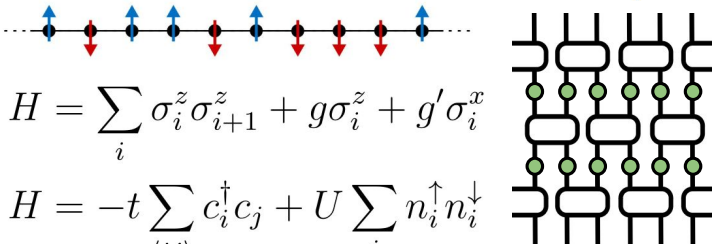


Schwinger-Keldysh EFTs have allowed for unprecedented control in otherwise strongly coupled quantum many-body systems





$$S = \int \bar{\psi} i \not{D} \psi + \frac{1}{4g^2} \text{Tr}(F^2)$$

$$S = S_{\text{CFT}} + \lambda \int \mathcal{O}$$


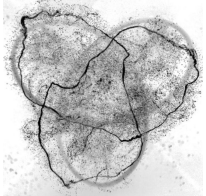
$$H = \sum_i \sigma_i^z \sigma_{i+1}^z + g \sigma_i^z + g' \sigma_i^x$$

$$H = -t \sum_{\langle ij \rangle} c_i^\dagger c_j + U \sum_i n_i^\uparrow n_i^\downarrow$$

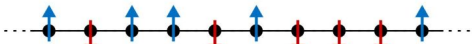
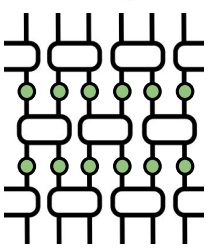
Fluctuating hydrodynamics is arguably the most universal EFT in physics

Locality + symmetry  $\Rightarrow$  hydrodynamics

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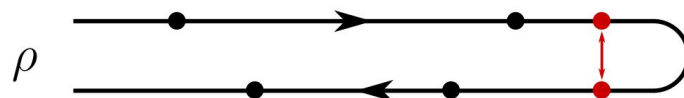
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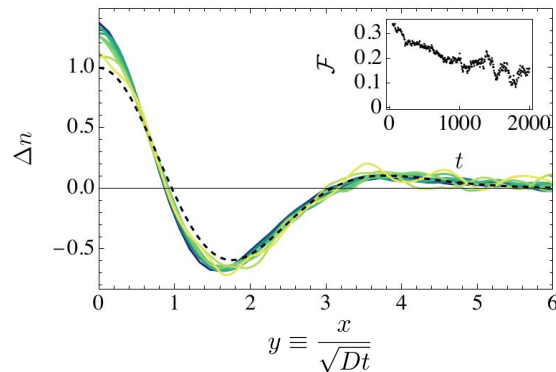
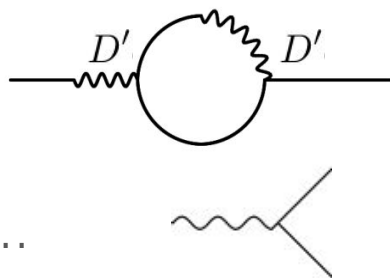
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Inspired by  $T = 0$  EFTs, it is possible to measure loop effects, nonlinear effects, ...



Fluctuating hydrodynamics is arguably the most universal EFT in physics

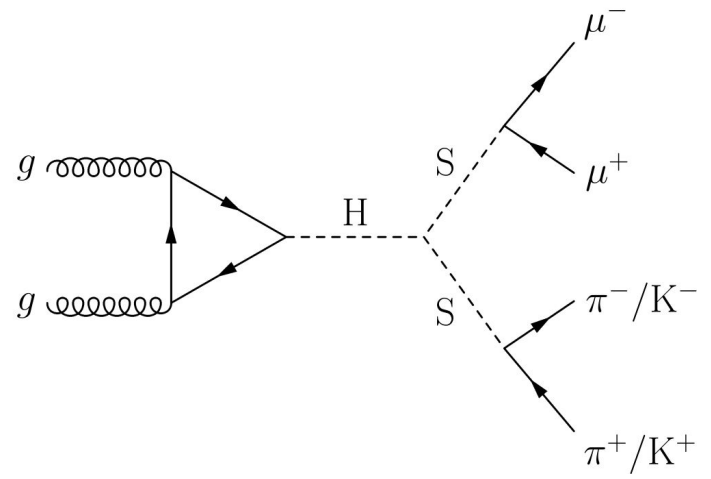
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What is this control good for?..

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In SMEFT: find new (BSM) physics



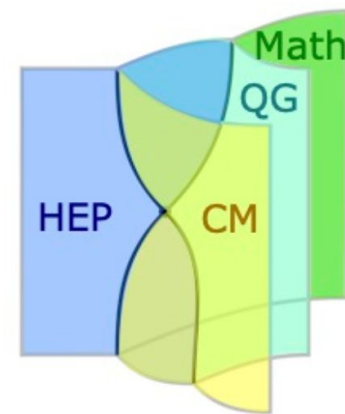
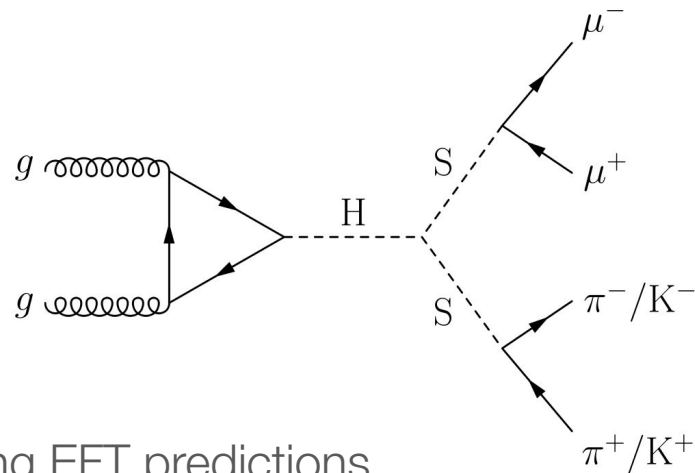
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In SMEFT: find new (BSM) physics

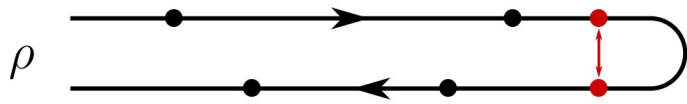
Search for many-body systems (numerics) violating EFT predictions

Because [Hydrodynamics  $\Leftrightarrow$  Symmetry], finding systems exhibiting “beyond hydro” behavior  $\approx$  generalizing notions of symmetry

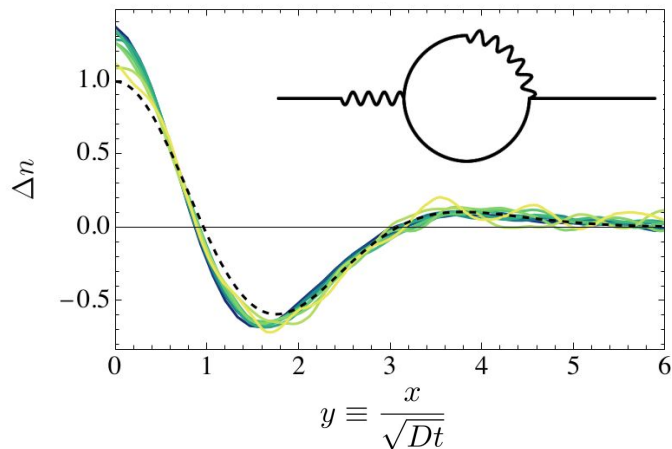
Connect to formal effort to generalize notion of symmetry.



## Tools: Precision physics in thermalizing systems



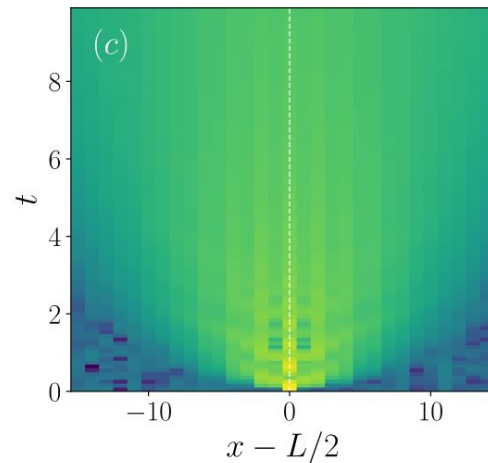
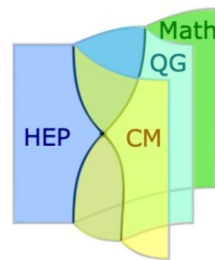
$$Z = \int D\phi_a D\mu_r e^{iS_{\text{eff}}[\phi_a, \mu_r]}$$



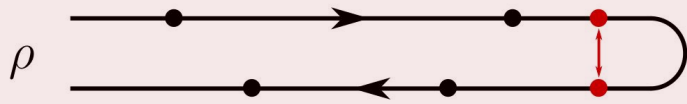
## Discovery: Hydrodynamics beyond conventional symmetry

$$S^\pm = \sum_i \dots \otimes q^{\sigma_z} \otimes q^{\sigma_z} \otimes \sigma_i^\pm \otimes \mathbf{1} \otimes \mathbf{1} \otimes \dots$$

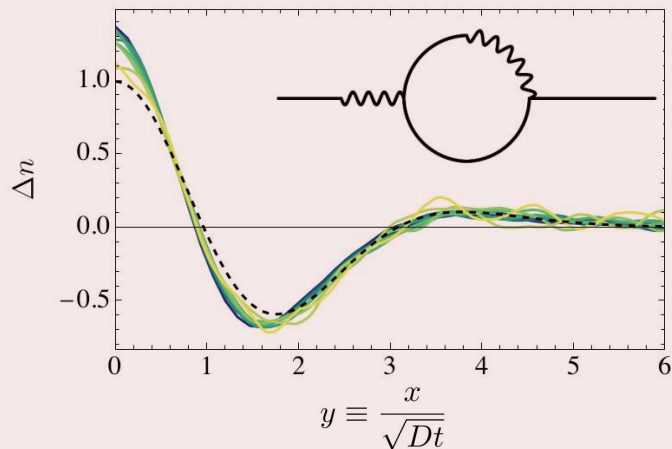
$z < 2$   
universality



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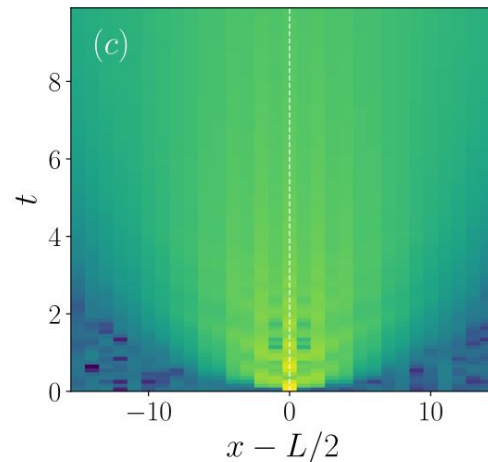
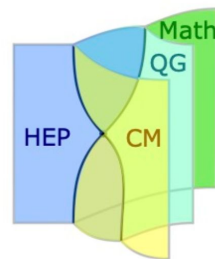


## Discovery: Hydrodynamics beyond conventional symmetry

$$S^\pm = \sum_i \dots \otimes q^{\sigma_z} \otimes q^{\sigma_z} \otimes \sigma_i^\pm \otimes \mathbf{1} \otimes \mathbf{1} \otimes \dots$$

A diagram showing a wavy line with a black dot at its end, representing a vertex in the operator expansion.

$z < 2$   
universality



# Schwinger-Keldysh EFT for Fluctuating Hydrodynamics

Martin Siggia '73 De Dominicis '76 Janssen '76 ...

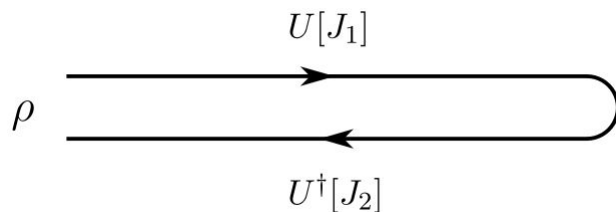
Hydrodynamics as an EFT on a Keldysh contour

Grozdanov Polonyi '14, Haehl Loganayagam Rangamani '15, Crossley Glorioso Liu '15, Jensen Pinzani-Fokeeva Yarom '17 ...

Hydro = Nambu-Goldstone modes of strong to weak SSB

Ogunnaike Feldmeier Lee '23, Akyuz Goon Penco '23, Gu Wang Wang '24, Huang Qi Zhang Lucas '24, Firat Gomes Nardi Penco Rattazzi '25 ...

# Symmetries of mixed state evolution



Natural notion of doubled symmetries on mixed states

Buca Prosen '12

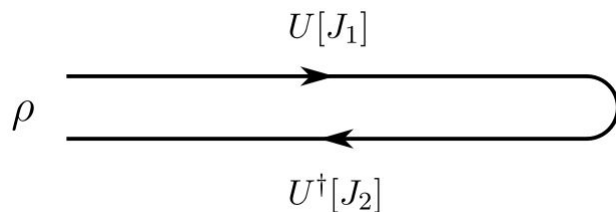
$$\rho \rightarrow e^{-i\alpha_1 Q} \rho e^{i\alpha_2 Q} \quad U(1) \times U(1)$$

Useful in the context of *open* dynamics, where Lindblad evolution can break either symmetry

$$\partial_t \rho = \mathcal{L}\rho \equiv -i[H, \rho] + \sum \left( 2L_i \rho L_i^\dagger - L_i^\dagger L_i \rho - \rho L_i^\dagger L_i \right)$$

If  $L_i$  is charged, dynamics only has “weak” symmetry  $U(1)_{\text{diag}}$

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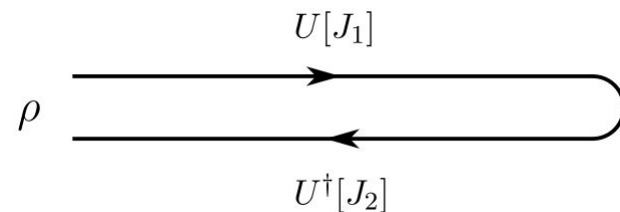
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Why care for closed systems? Thermal states have “strong to weak” SSB

$$\text{Tr} \left( \rho \psi^\dagger \psi \right) = \langle \psi_2^\dagger \psi_1 \rangle \neq 0 \quad U(1) \times U(1) \rightarrow U(1)_{\text{diag}}$$

# Symmetries of mixed state evolution



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## Hydrodynamics

Useful in the  
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“Goldstone modes for strong to weak SSB”

ak either

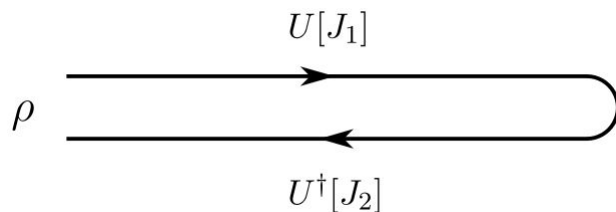
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ak either

Unifying description for open systems [review: Sieberer Buchhold Diehl '16]

If  $L_i$  is char

Builds on previous SK EFTs Haehl Loganayagam Rangamani '15, Crossley

Why care fo

Glorioso Liu '15, Jensen Marjeh Pinzani-Fokeeva Yarom '17

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# Degrees of freedom

Algorithm: double global symmetry group  $G$  and SSB to diagonal

$$G \times G \rightarrow G$$

Resulting EFT = Fluctuating hydrodynamics of conserved densities

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Works for symmetries that are non-abelian, spacetime, higher-form, anomalous, ...

MHD, ...

Navier-Stokes, ...

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Works for symmetries that are non-abelian, spacetime, higher-form, anomalous, ...

Simplest example:  $G = U(1)$

MHD, ...

Navier-Stokes, ...

$\phi_a$ ,  $\mu_r$

Goldstone  
(historically: “noise field”)

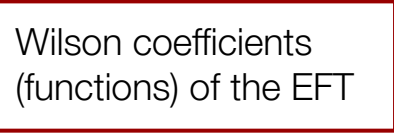
$\mu_r \sim \dot{\phi}_r$

KMS multiplet:

$$\begin{cases} \phi_a \rightarrow -(\phi_a + i\beta\mu_r + \dots) \\ \mu_r \rightarrow \mu_r + \frac{1}{4}i\beta\ddot{\phi}_a + \dots \end{cases}$$

# EFT of diffusion

$$S[\phi_a, \mu_r] = \int \dot{\phi}_a n(\mu_r) + iT\sigma(\mu_r)\nabla\phi_a (\nabla\phi_a + i\beta\nabla\mu_r) + \dots$$



Wilson coefficients  
(functions) of the EFT

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Higher derivative,  
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Identify  $U(1)$  density:  $\delta S/\delta\dot{\phi}_a \simeq n(\mu_r) = n + \chi\delta\mu_r + \frac{1}{2}\chi'(\delta\mu_r)^2 + \dots$

(equation of state)

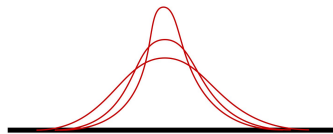
Linearized action predicts diffusion

$$S = \chi \int \dot{\phi}_a \mu_r + iTD\partial_i\phi_a (\partial_i\phi_a + i\beta\partial_i\mu_r)$$

$$G_{nn}^R(\omega, k) = \frac{\chi D k^2}{Dk^2 - i\omega} + \dots$$

$$D = \sigma/\chi$$

$$\langle n(t, x)n \rangle = \frac{\chi T}{(4\pi Dt)^{d/2}} e^{-x^2/4Dt} + \dots$$



# EFT of diffusion

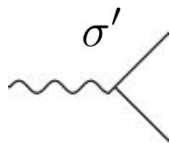
$$S[\phi_a, \mu_r] = \int \dot{\phi}_a n(\mu_r) + iT\sigma(\mu_r)\nabla\phi_a (\nabla\phi_a + i\beta\nabla\mu_r) + \dots$$

Leading nonlinearities come from  $n(\mu)$  and  $\sigma(\mu)$  (scaling argument)

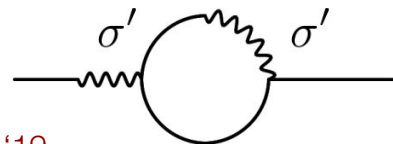
⇒ Nonlinear response tied to linear response!

$$S^{(3)} \supset \int \sigma' \mu_r \nabla\mu_r \cdot \nabla\phi_a$$

... as are corrections to transport!



LVD Mishra '23



Chen-Lin LVD Hartnoll '19  
Michailidis Abanin LVD '23

# EFT systematics

$$y \equiv \frac{x}{\sqrt{Dt}}$$

$$\begin{aligned} \langle n(t, x)n \rangle = & \frac{\chi T}{(4\pi Dt)^{d/2}} \left[ F_{0,0}(y) + \frac{1}{t} F_{0,1}(y) + \frac{1}{t^2} F_{0,2}(y) + \dots \right. \\ & + \frac{1}{t^{d/2}} \left( F_{1,0}(y) + \frac{1}{t} F_{1,1}(y) + \frac{1}{t^2} F_{1,2}(y) + \dots \right) \\ & \left. + \frac{1}{t^d} \left( F_{2,0}(y) + \frac{1}{t} F_{2,1}(y) + \frac{1}{t^2} F_{2,2}(y) + \dots \right) + \dots \right] \end{aligned}$$

# EFT systematics

Derivative expansion

$$y \equiv \frac{x}{\sqrt{Dt}}$$

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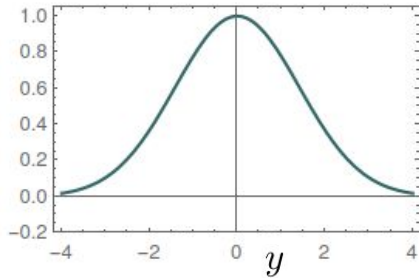
Loop expansion

Up to a handful of Wilson coefficients, EFT predicts entire *scaling functions*  $F_{\ell,n}$

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$$+ \frac{1}{t^{d/2}} \left( F_{1,0}(y) + \frac{1}{t} F_{1,1}(y) + \frac{1}{t^2} F_{1,2}(y) + \dots \right) \\ + \frac{1}{t^d} \left( F_{2,0}(y) + \frac{1}{t} F_{2,1}(y) + \frac{1}{t^2} F_{2,2}(y) + \dots \right) + \dots \left. \right]$$

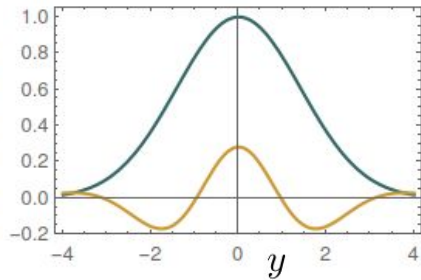
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$$F_{0,0}(y) = e^{-y^2/4}$$

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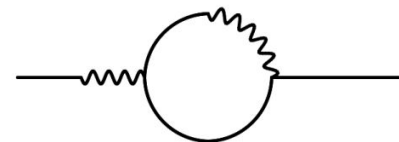


Up to a handful of Wilson coefficients, EFT predicts entire *scaling functions*  $F_{\ell,n}$

$$F_{0,0}(y) = e^{-y^2/4}$$

$$F_{1,0}(y) = \frac{\chi D'^2}{D^{5/2}} \left[ \frac{4 + y^2}{8\sqrt{\pi}} e^{-y^2/2} + \frac{y(y^2 - 10)}{16} e^{-y^2/4} \text{Erf}(y/2) \right]$$

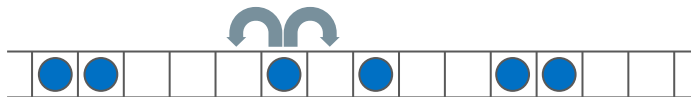
(d=1)



# Precision tests of thermalization

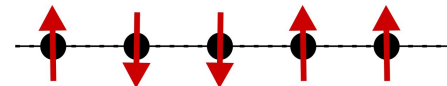
Can test these predictions numerically in many systems:

1. Classical lattice gas

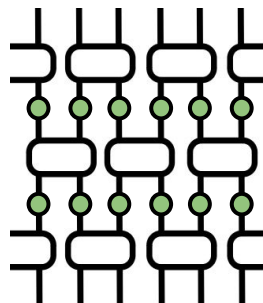


2. Hamiltonian spin chains

$$H = \sum_i \sigma_i^z \sigma_{i+1}^z + g \sigma_i^z + g' \sigma_i^x$$



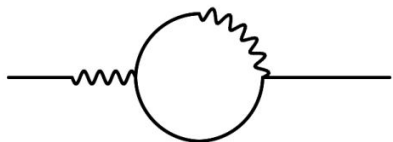
3. Floquet quantum systems



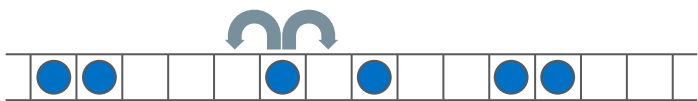
Precision test of thermalization:

Universal 1-loop correction to diffusion

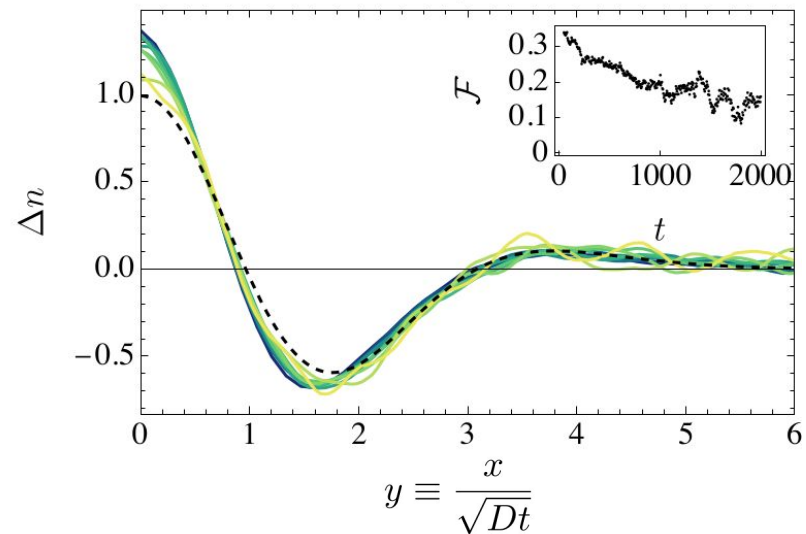
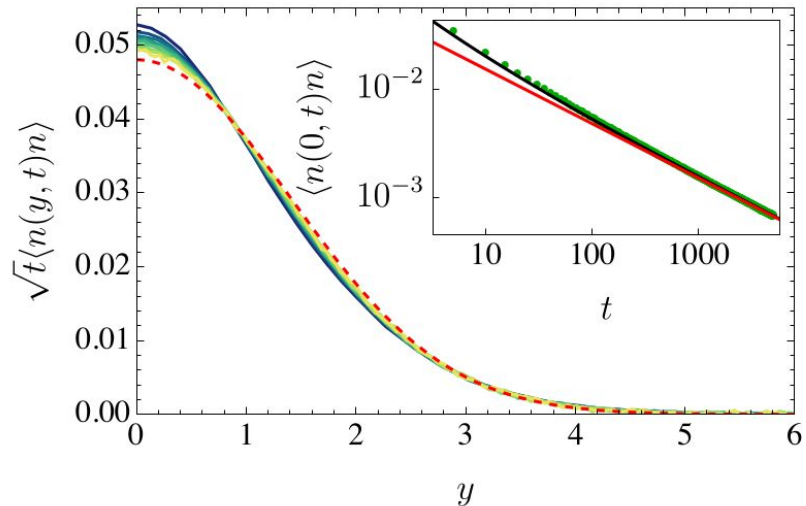
$$\langle n(x, t)n \rangle = \frac{1}{\sqrt{t}} \left[ e^{-x^2/4Dt} + \frac{1}{\sqrt{t}} F_{1,0} \left( \frac{x^2}{Dt} \right) + \dots \right]$$



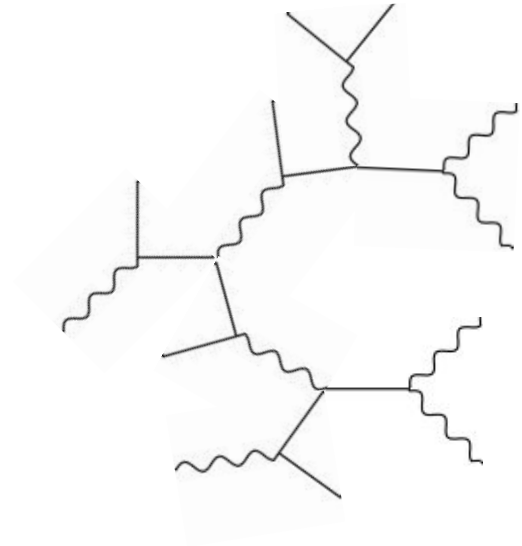
Test in a classical lattice gas [Spohn '12]



( $D(n)$  known analytically  $\rightarrow$  no fitting parameter!)



# Non-linear response in diffusive systems



$$\langle n(x_N, t_N) \cdots n(x_2, t_2) n(x_1, t_1) \rangle$$

All leading order higher point functions fixed by  $n(\mu)$  and  $\sigma(\mu)$

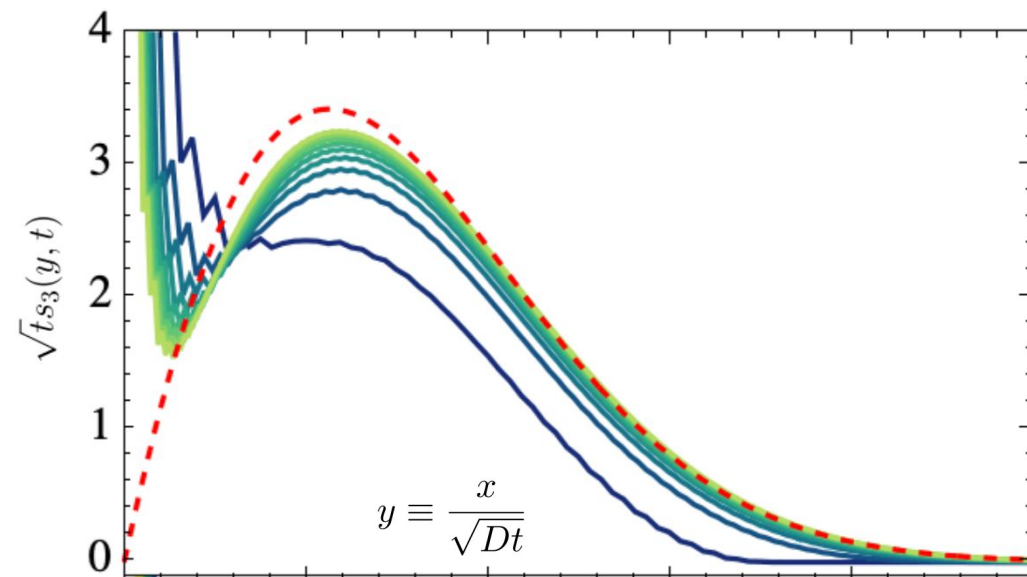
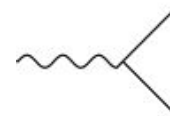
# Non-linear response in diffusive systems

LVD Mishra '23

Michailidis Abanin LVD '23

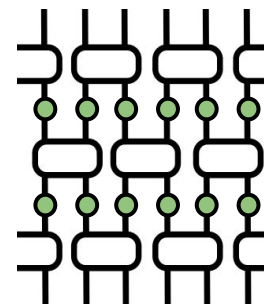
Higher point functions fixed by the same vertex:

$$\langle n(t, x)n(t, 0)n(0, 0) \rangle = \frac{1}{\sqrt{t}} \frac{D'\chi}{8\sqrt{D^3\pi}} \left( e^{-y^2/4} + \text{Erf}(y/2) - 1 \right)$$



Numerics:

Floquet quantum spin chain



# More exotic observables

EFTs can be extended to capture OTOCs

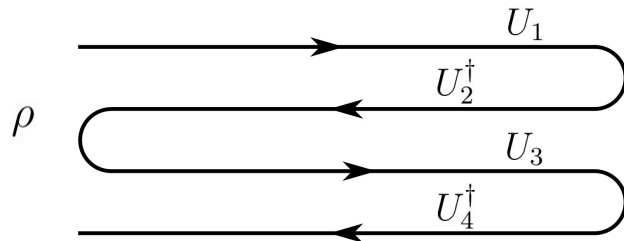
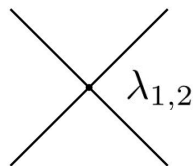
$$G \times G \times G \times G \rightarrow G_{\text{diag}}$$

3 Goldstones, one conjugate field:  $\phi, \phi, \phi, \mu$

New EFT contains all the information from previous EFT  $n(\mu), \sigma(\mu), \dots$

Also contains two new parameters at leading order – “OTO-Transport”

$$S_{(4)} = \int dt d^d x \lambda_1 \mathcal{L}_1 + \lambda_2 \mathcal{L}_2$$



# Testing OTOC predictions

Trace cyclicity  $\Rightarrow$  3 independent 4pt functions

$$g_0 = \text{Tr}(\rho O(t_4)O(t_3)O(t_2)O(t_1))$$

$$g_1 = \text{Tr}(\rho O(t_3)O(t_4)O(t_2)O(t_1))$$

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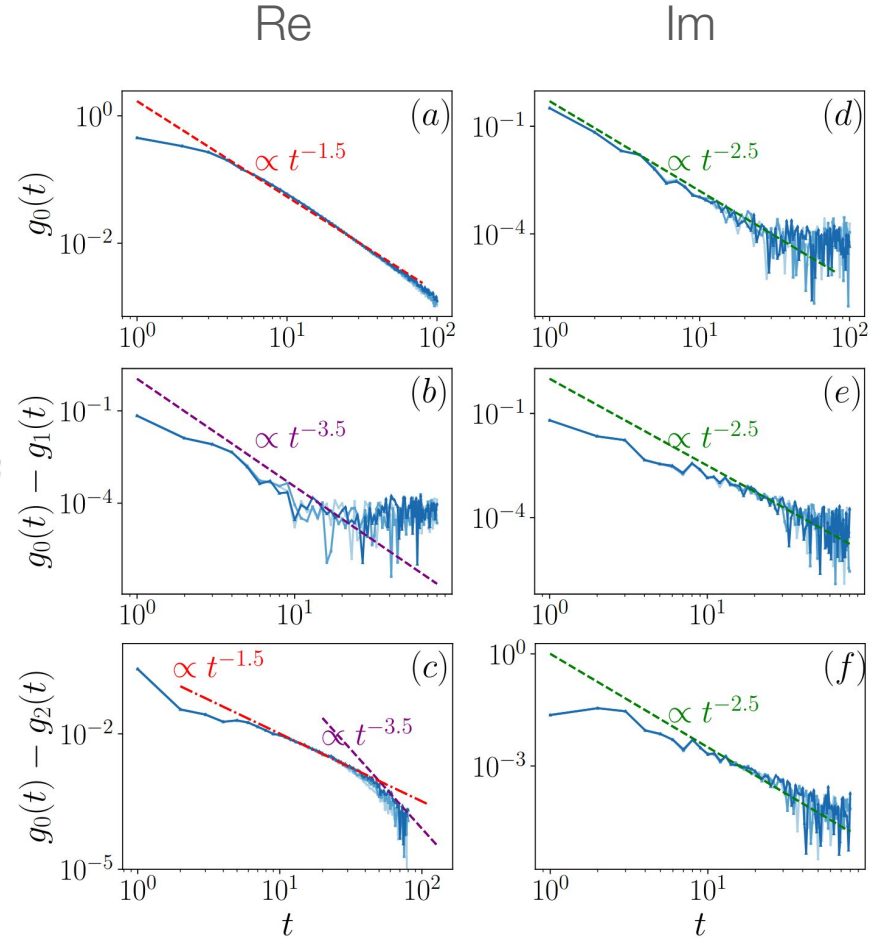
Wang Heinz '98, Haehl  
Loganayagam Narayan  
Nizami Rangamani '17

Scaling in the EFT  $\mu^R \sim \phi^A \sim \dot{\phi}^- \sim \phi^+ \sim q^{d/2}$   
implies:

$$g_0 \sim \frac{1}{t^{3/2}} + \frac{i}{t^{5/2}}$$

$$g_1 - g_0 \sim \frac{1}{t^{7/2}} + \frac{i}{t^{5/2}}$$

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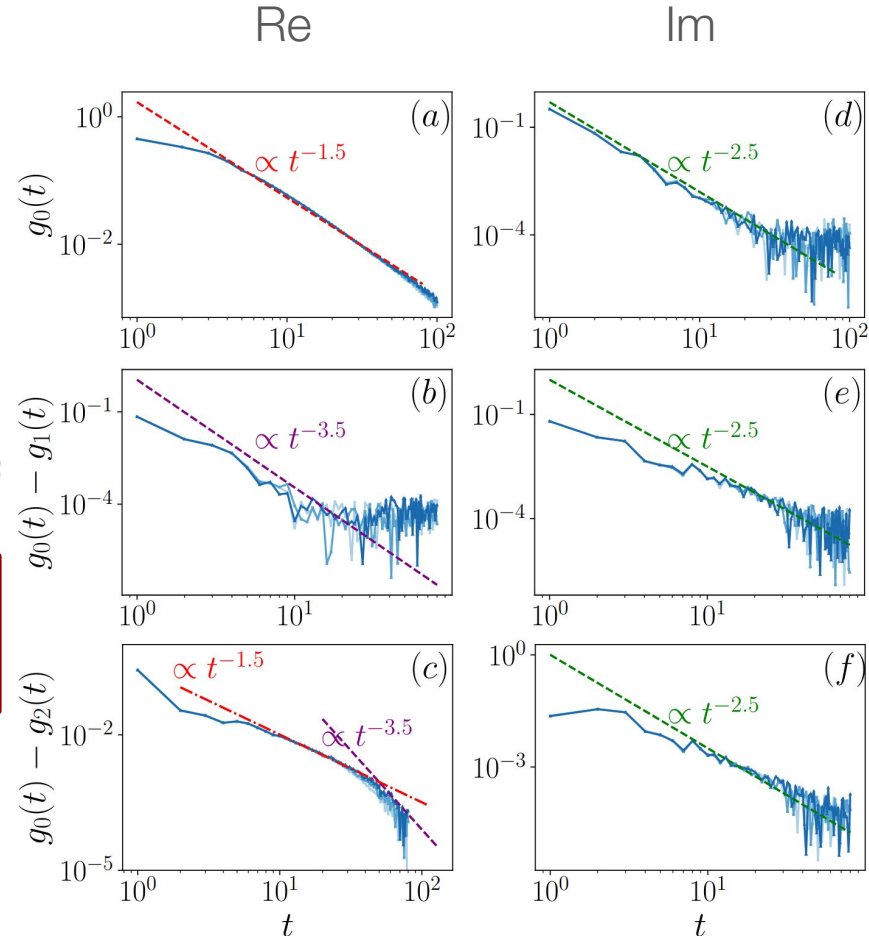
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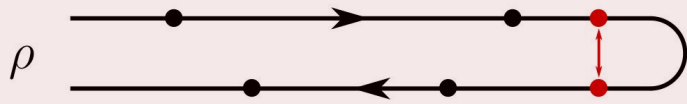
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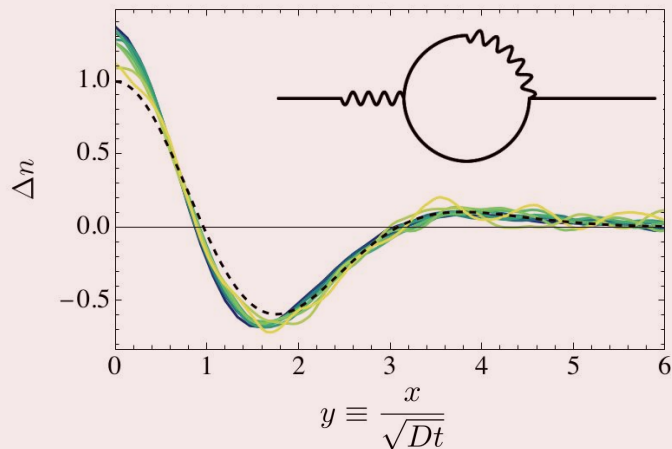
$\lambda_1, \lambda_2$



## Tools: Precision physics in thermalizing systems



$$Z = \int D\phi_a D\mu_r e^{iS_{\text{eff}}[\phi_a, \mu_r]}$$

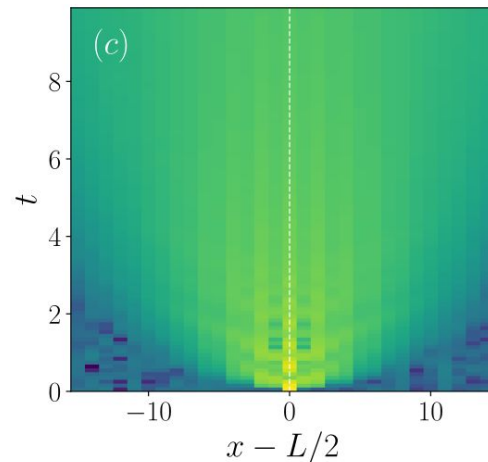
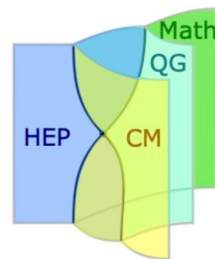


## Discovery: Hydrodynamics beyond conventional symmetry

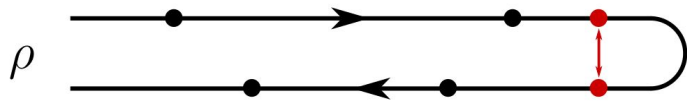
$$S^\pm = \sum_i \dots \otimes q^{\sigma_z} \otimes q^{\sigma_z} \otimes \sigma_i^\pm \otimes \mathbf{1} \otimes \mathbf{1} \otimes \dots$$

A wavy line with a black dot at its end, representing a vertex in the operator expansion.

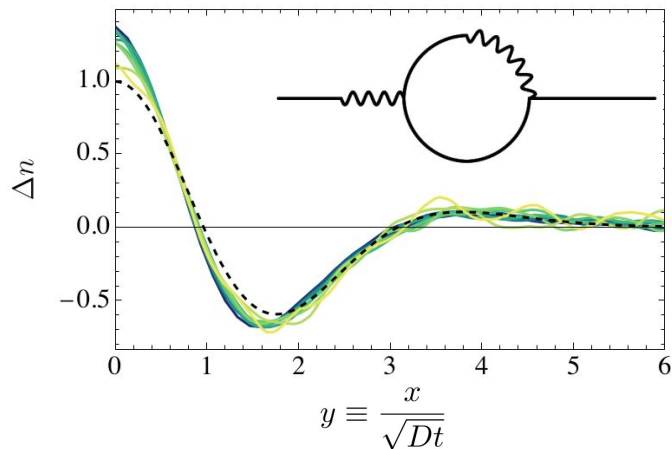
$z < 2$   
universality



## Tools: Precision physics in thermalizing systems



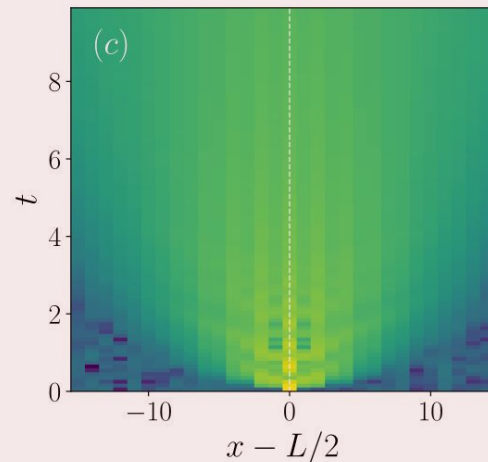
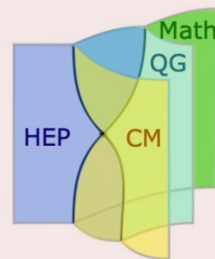
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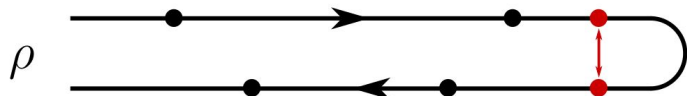
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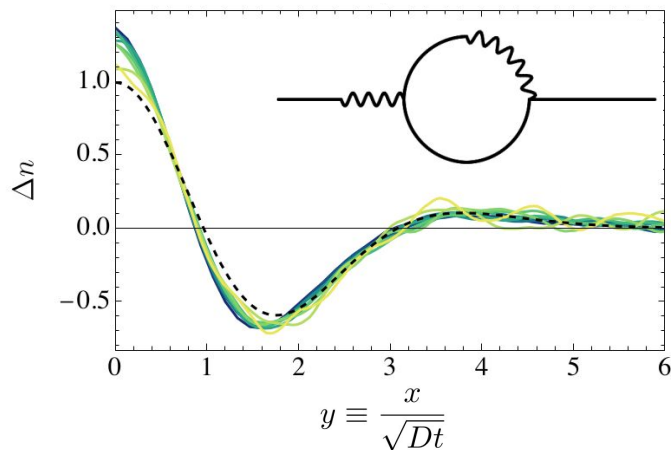
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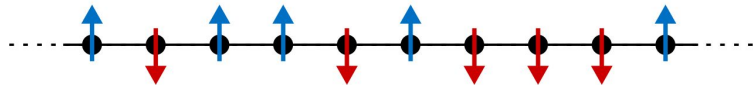
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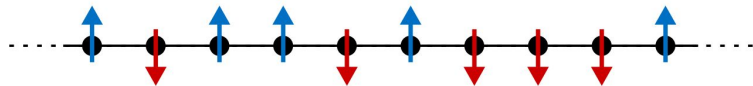


XXZ model



$$H = \sum_i \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z$$

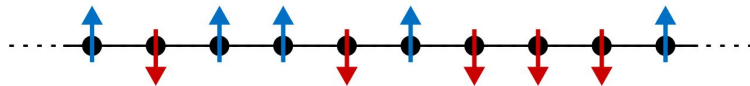
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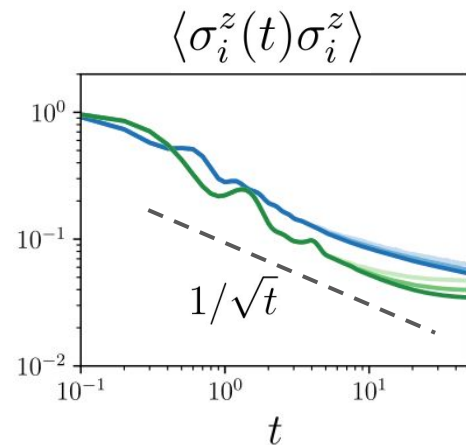


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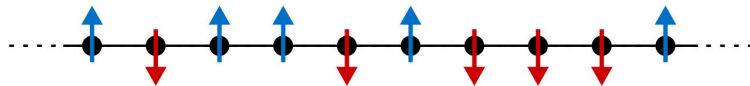
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Expect two diffusive modes:  $\sigma_i^z$  and  $h_i$



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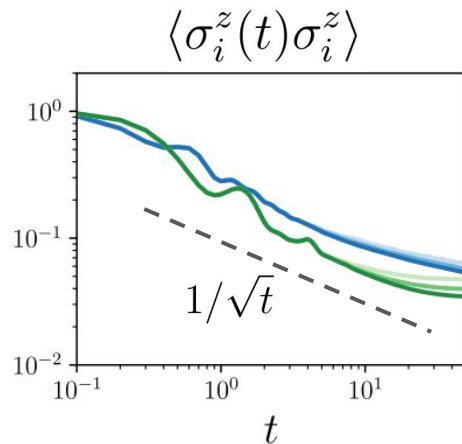
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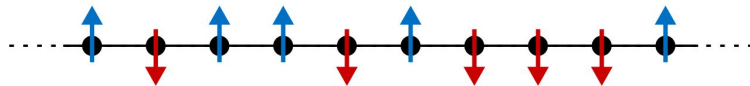
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# XXZ model



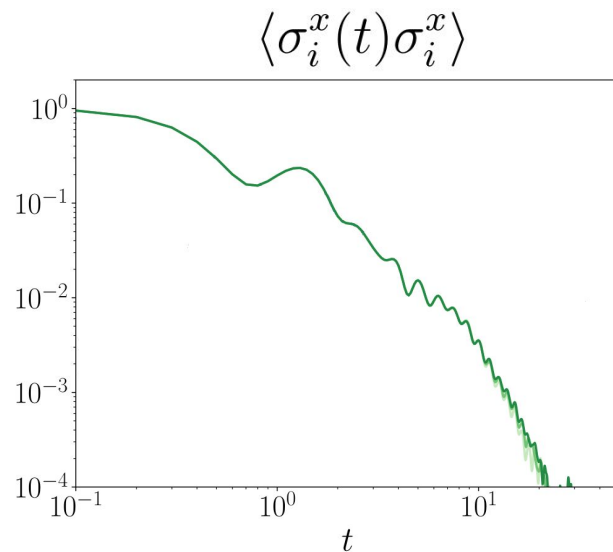
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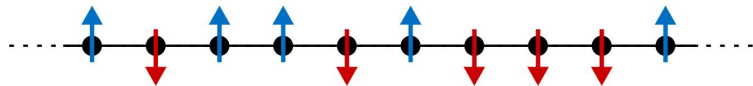
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# XXZ-like model



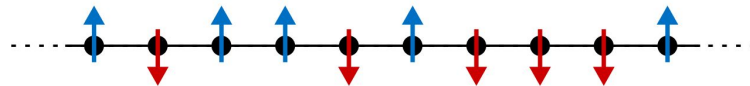
$$\tilde{H} = \sum_i \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + V_i$$

with  $V_i = V_i^{\text{nnn}} + A_i + B_i$

$$A_i = \frac{1}{4}(q - q^{-1})^2 (\sigma_i^z \sigma_{i+1}^z + \sigma_{i+1}^z \sigma_{i+2}^z)$$

$$B_i = \frac{1}{2}(q - q^{-1}) (\sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^z + \sigma_i^y \sigma_{i+1}^y \sigma_{i+2}^z - \sigma_i^z \sigma_{i+1}^x \sigma_{i+2}^x - \sigma_i^z \sigma_{i+1}^y \sigma_{i+2}^y)$$

# XXZ-like model



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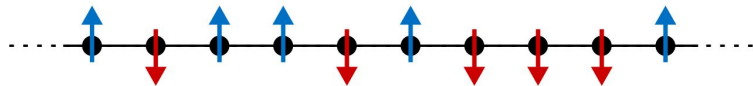
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entirely different transverse spin dynamics!

# XXZ-like model



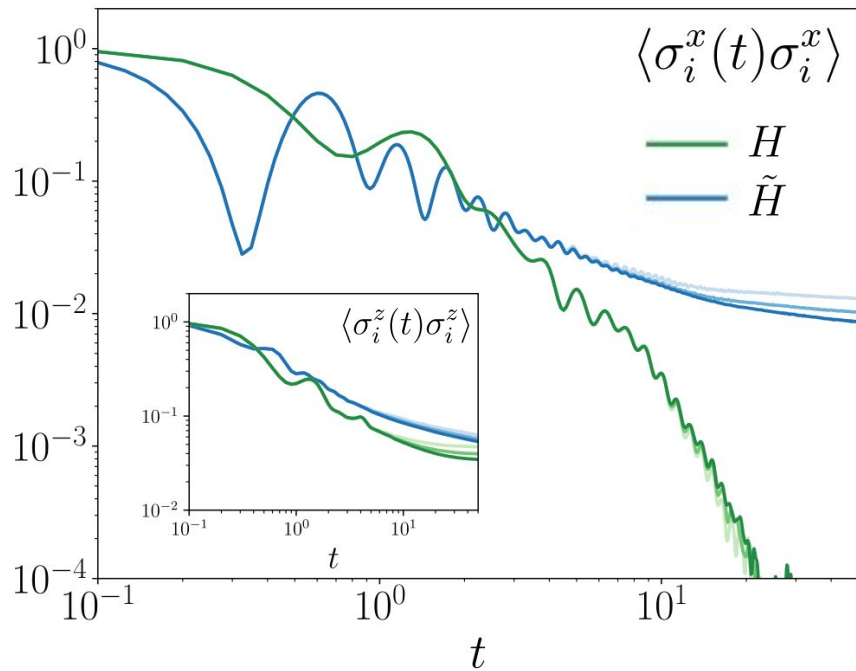
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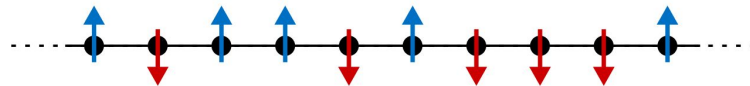
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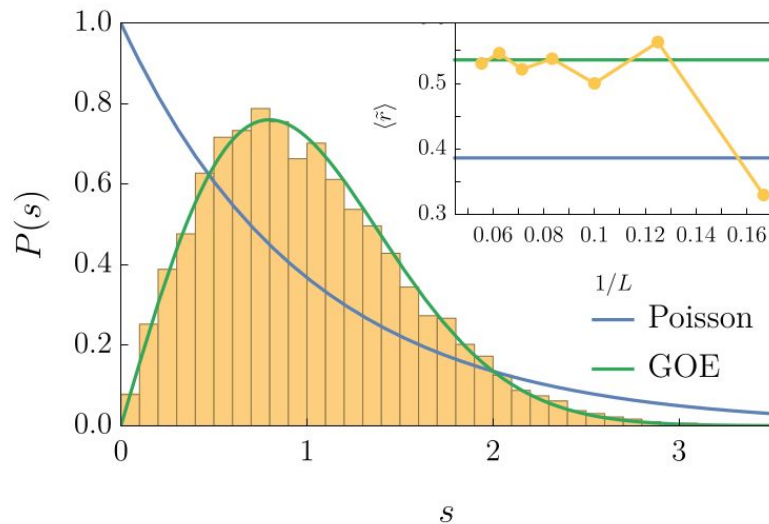


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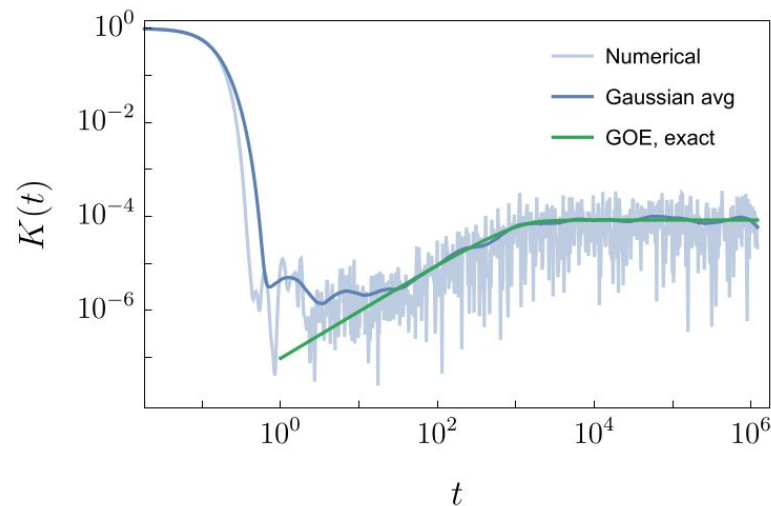


What explains the slow transverse spin dynamics? Integrability? No.

### Level spacing statistics



### Spectral form factor



$\tilde{H}$  has “quantum group” symmetry

# Quantum group symmetry of XXZ model

$$H_0 = \sum_i \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{1}{2}(q + q^{-1}) \sigma_i^z \sigma_{i+1}^z \right)$$

where  $\Delta = \frac{1}{2}(q + q^{-1})$  has the quantum group symmetry  $U_q(SU(2))$  [Pasquier Saleur '90]

when  $q \rightarrow 1$ , recover  $SU(2)$  [which also protects transverse spin!] with densities:

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$$S^\pm = \sum_i \dots \otimes q^{\sigma_z} \otimes q^{\sigma_z} \otimes \sigma_i^\pm \otimes \mathbf{1} \otimes \mathbf{1} \otimes \dots \xrightarrow{q \rightarrow 1} S^\pm = \sum_i \sigma_i^\pm$$

$\dots$  

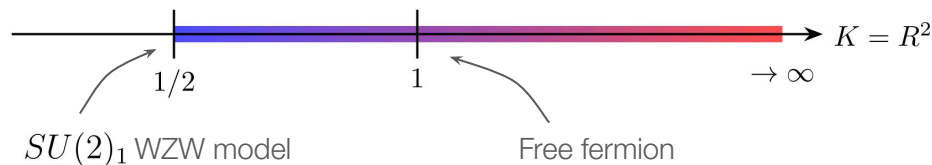
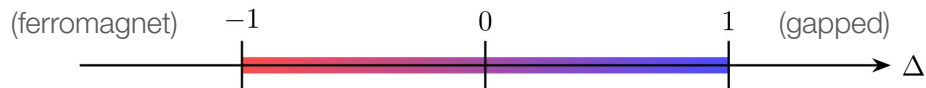
QG-symmetric Hamiltonians commute with charges whose densities are nonlocal

# Quantum group symmetry of XXZ model

Continuum intuition for nonlocal density:

XXZ model  $H = \sum_i \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z$  flows to compact boson CFT with:

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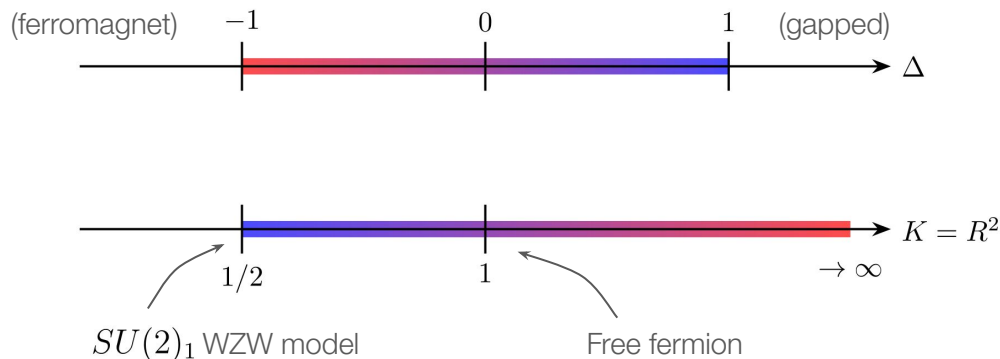
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At  $SU(2)$  point, vertex operator becomes an additional current:

$$e^{\pm i\phi} = j^{\pm}$$

Away from that point, that operator still exists, but is non-local, because ill-quantized ( $\phi \sim \phi + 2\pi R$ )



# Constructing chaotic QG symmetric models

The Hamiltonian density of the XXZ model is a QG singlet: (“Hecke algebra element”)

$$h_i = \frac{1}{2} \left[ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q+q^{-1}}{2} \underbrace{\left( \sigma_i^z \sigma_{i+1}^z + 1 \right)}_{(\text{const})} - \frac{q-q^{-1}}{2} \left( \underbrace{\sigma_i^z - \sigma_{i+1}^z}_{(\text{total derivative})} - \underbrace{2}_{(\text{const})} \right) \right]$$

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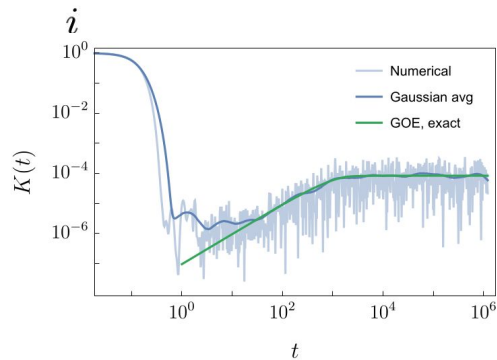
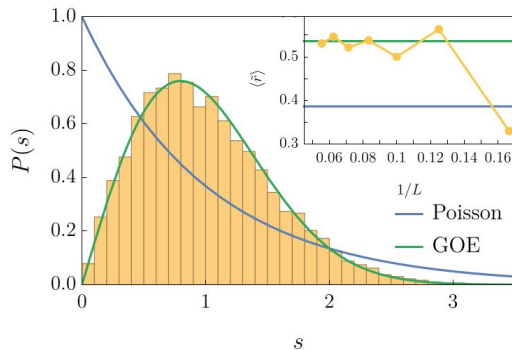
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Can easily generate further QG invariants using  $h_i$ , e.g.:  $\tilde{H} = \sum h_i + \lambda h_i h_{i+1}$

$\tilde{H}$  still has  $U_q(SU(2))$  symmetry, but is chaotic!

(QG symmetry sectors must be taken into account before comparing statistics to RMT)



# Is QG a good generalization of continuous symmetry?

[Gabai Gorbenko Qiao Zan Zhabin '24]

[Gorbenko Zhabin '26]

What is a symmetry of a QFT?

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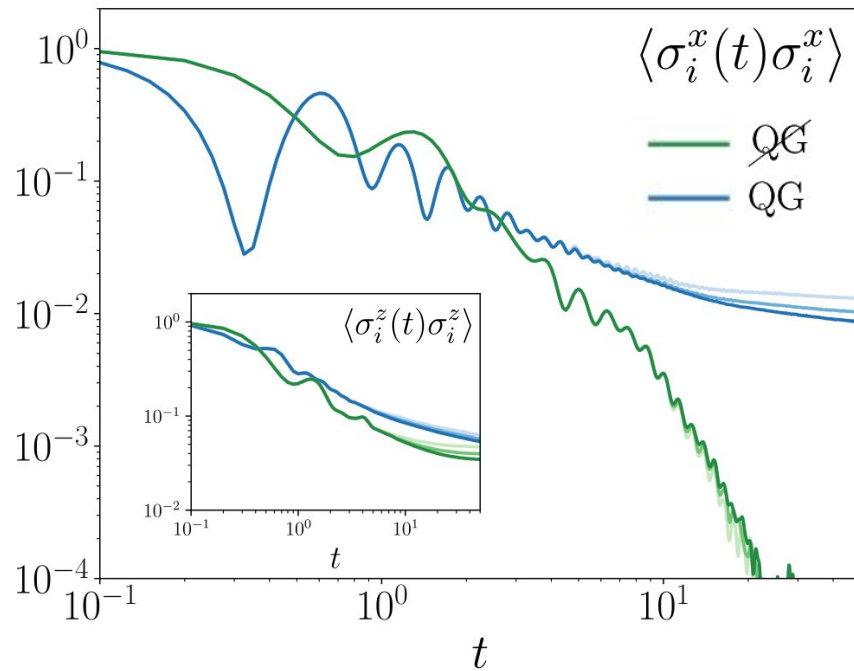
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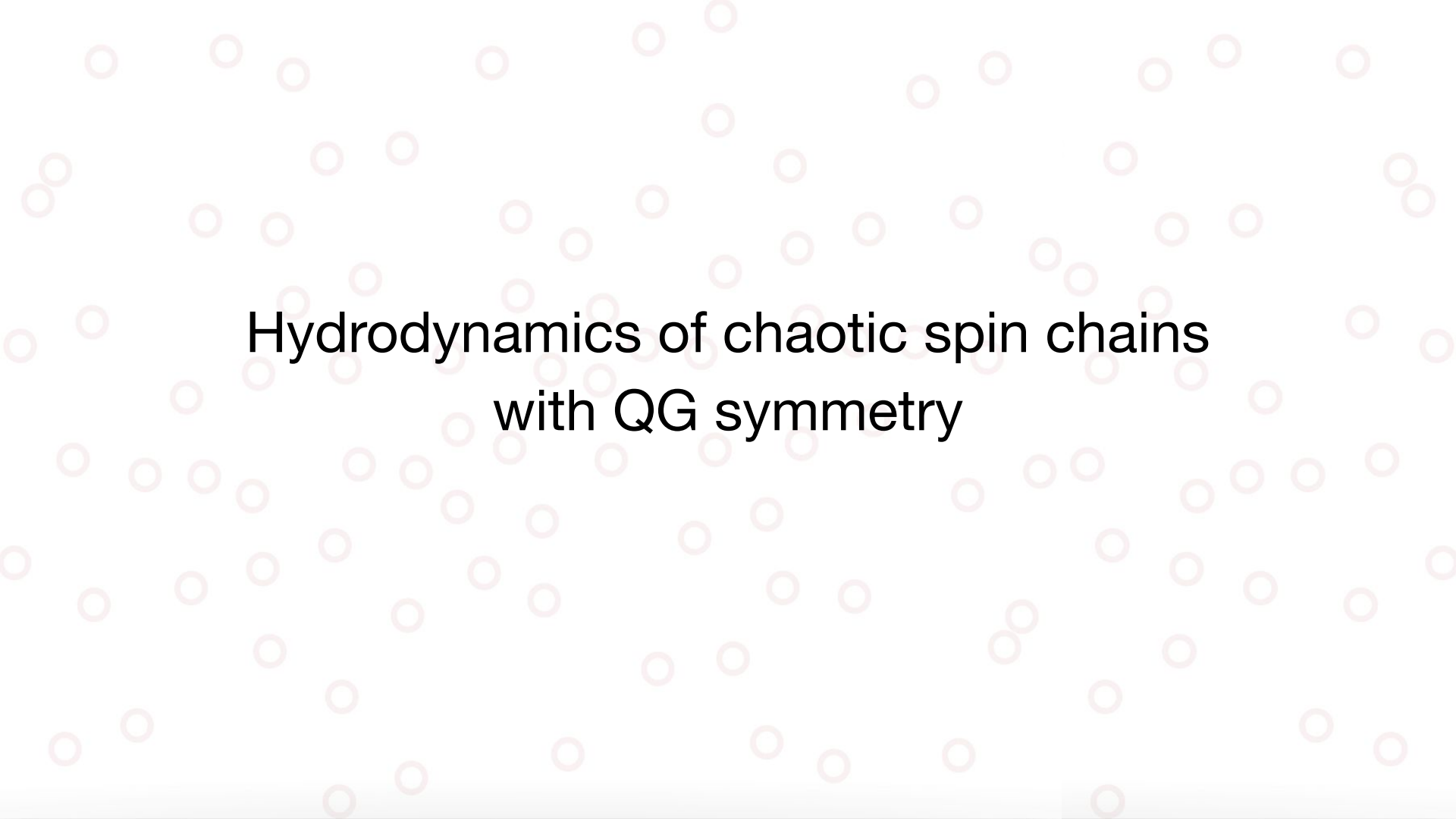
What is a symmetry of a QFT?

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Hydrodynamic protection!



The background of the slide is white with a pattern of light pink circles scattered across it. The circles vary in size and are distributed somewhat randomly, creating a subtle, decorative texture.

# Hydrodynamics of chaotic spin chains with QG symmetry

# Hydrodynamic tails in Hamiltonian models

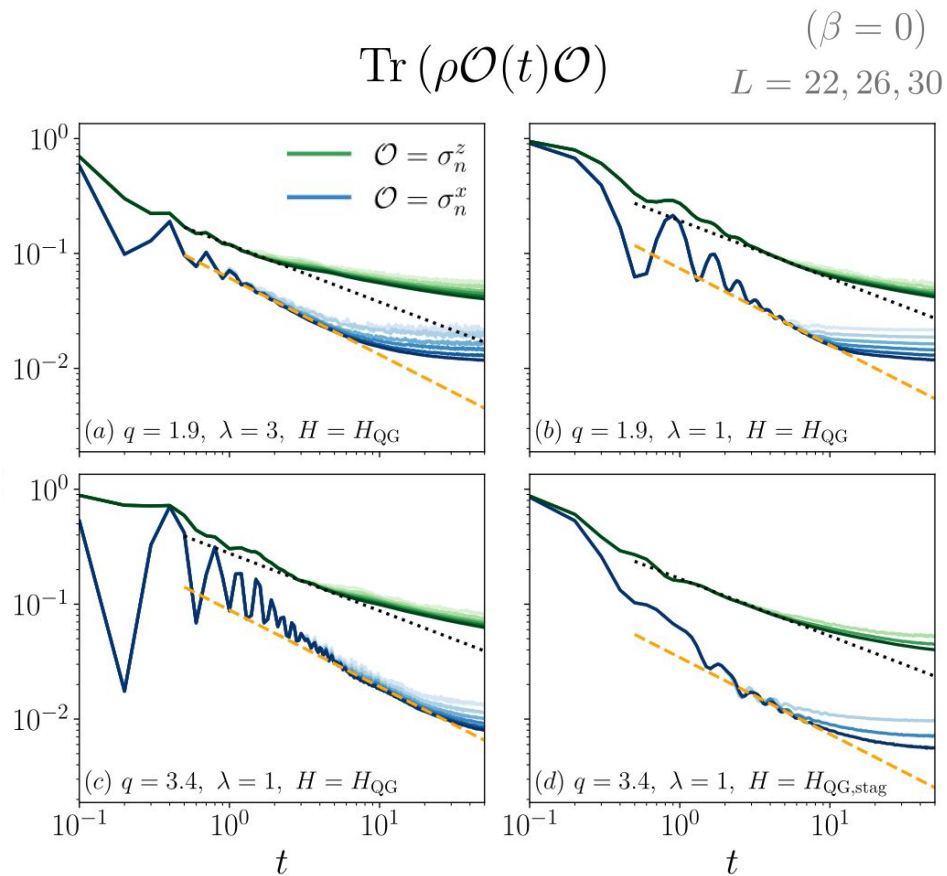
Study  $\tilde{H} = \sum_i h_i + \lambda h_i h_{i+1}$

and  $\tilde{H}_{\text{stag}} = \sum_i h_i + \lambda(-1)^i h_i h_{i+1}$

for several values of  $\lambda, q$

Hydrodynamic protection of transverse spin, similar power-laws throughout

Notice that  $SU(2)$  is strongly broken

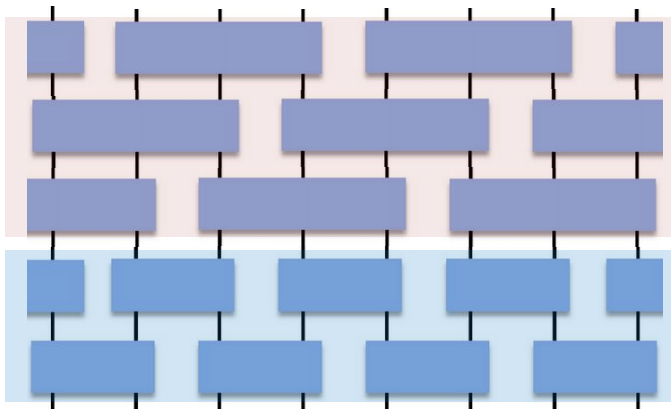


# Hydrodynamic tails in Floquet models

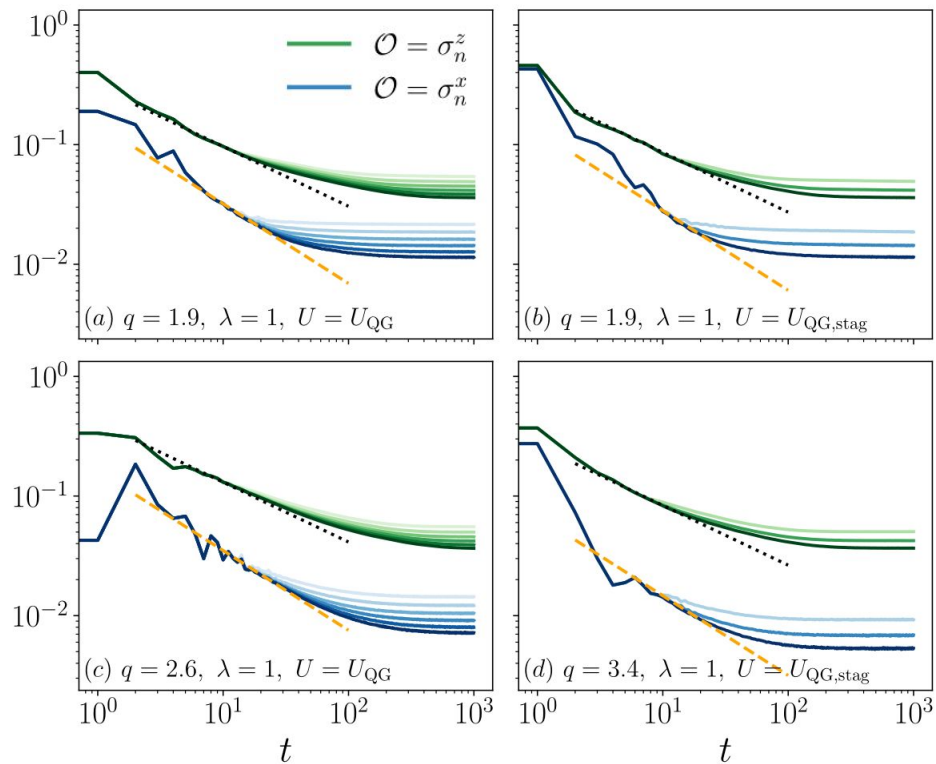
Can also make QG invariant Floquet unitaries:

$$U_0 = \prod_i^{L/2} e^{-ih_{2i}}$$

$$U_\lambda = \prod_i^{L/3} e^{-i\lambda h_{3i} h_{3i+1}}$$



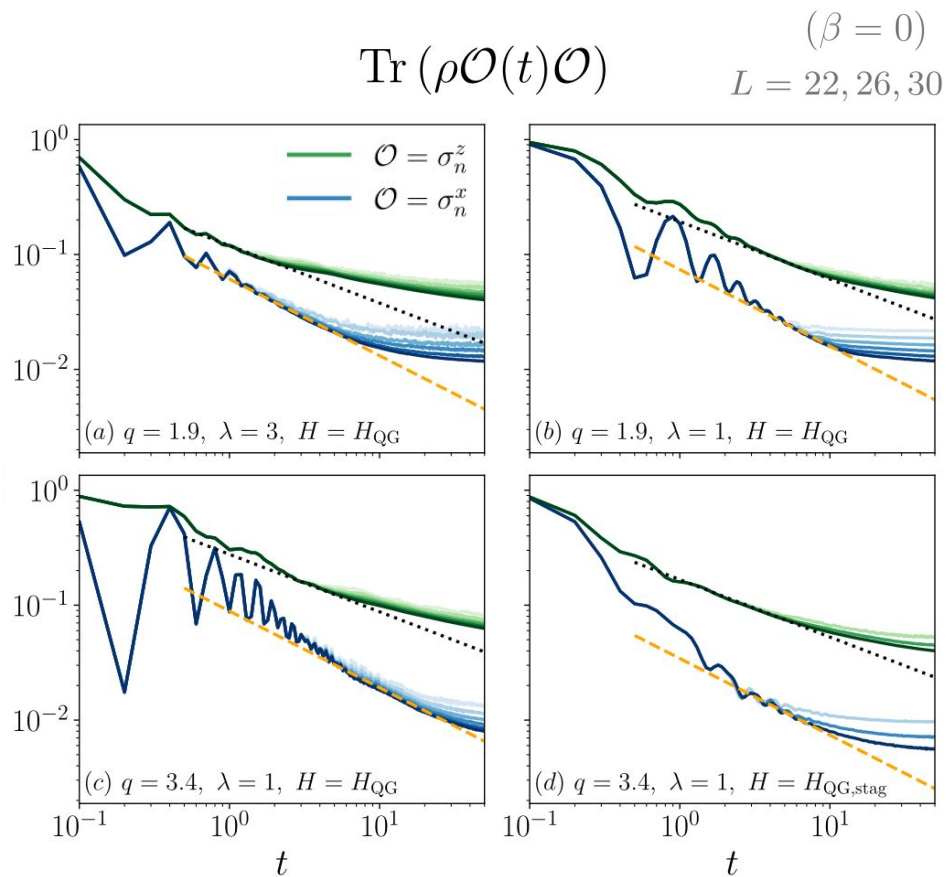
$\text{Tr}(\rho \mathcal{O}(t) \mathcal{O})$  ( $\beta = 0$ )  
 $L = 22, 26, 30$



# Superdiffusion?

Find hydrodynamic protection across QG-symmetric models!

Even more excitingly, hydrodynamics appears to be *superdiffusive*



# Superdiffusion?

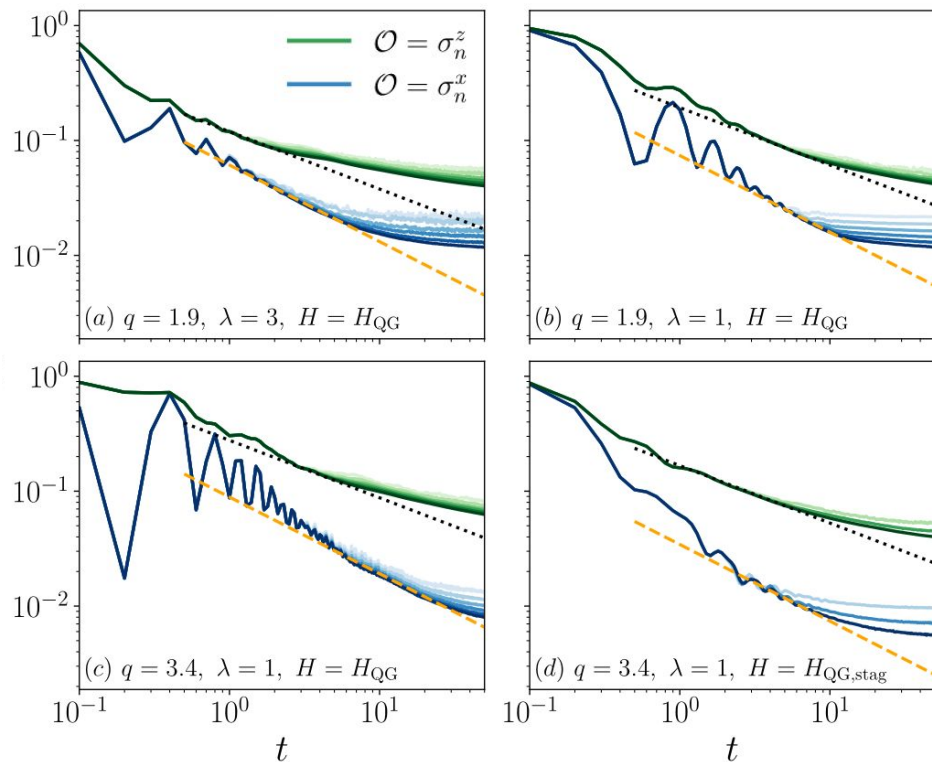
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$$1/t^{1/z} \quad z < 2$$

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# Superdiffusion?

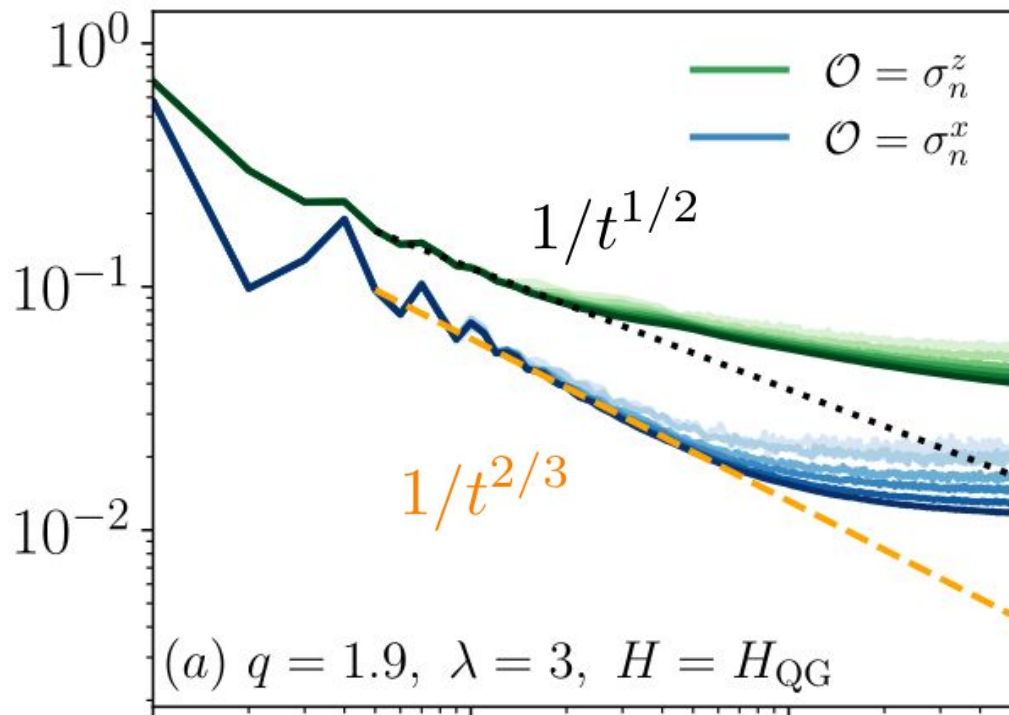
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# Superdiffusion?

Probe dynamic critical exponent and universality class with position-resolved correlators:

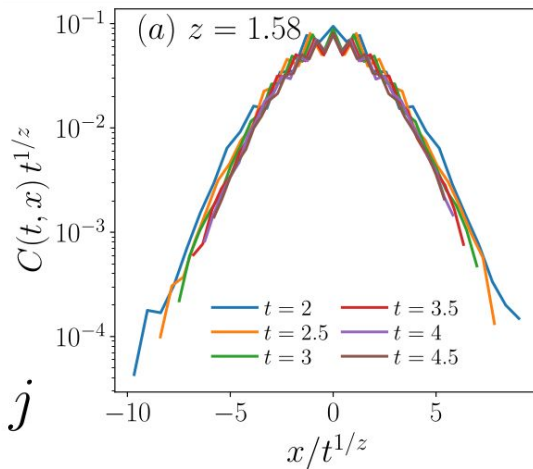
$$C(t, x) = \langle \sigma_i^+(t) \sigma_j^- \rangle, \quad x = i - j$$

Collapses onto universal scaling function

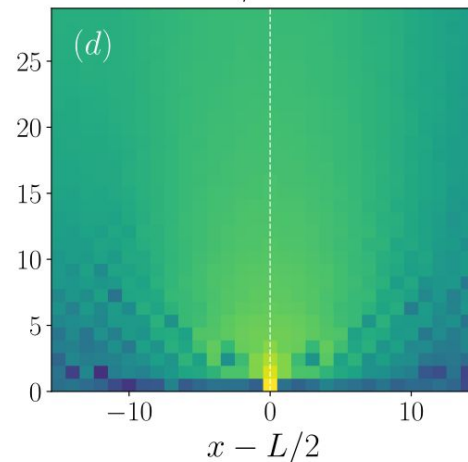
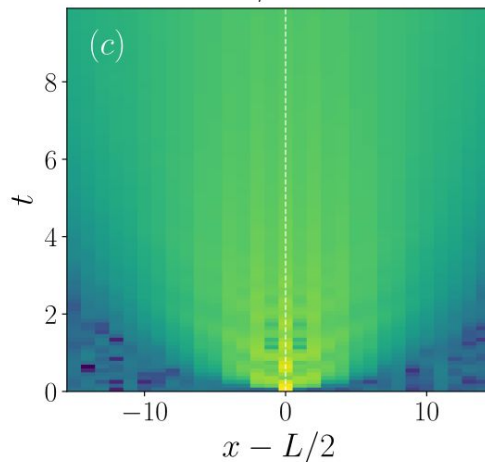
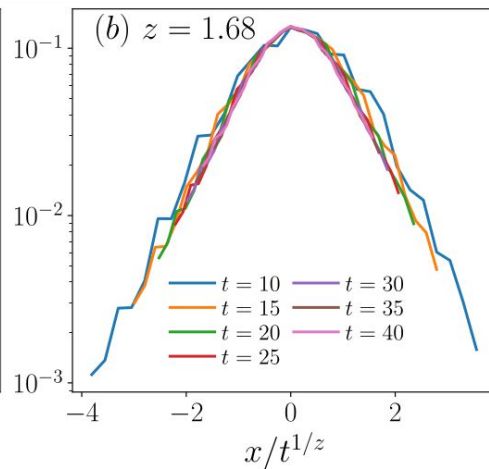
$$\langle \sigma_i^+(t) \sigma_j^-(0) \rangle \simeq \frac{1}{t^{1/z}} F\left(x/t^{1/z}\right)$$

with  $z \approx 1.6$

(Hamiltonian)



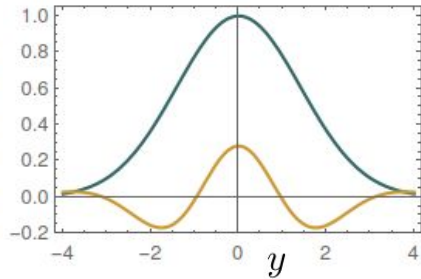
(Floquet)



# EFT systematics

$$y \equiv \frac{x}{\sqrt{Dt}}$$

$$\langle n(t, x)n \rangle = \frac{\chi T}{(4\pi Dt)^{d/2}} \left[ F_{0,0}(y) + \frac{1}{t} F_{0,1}(y) + \frac{1}{t^2} F_{0,2}(y) + \dots \right. \\ \left. + \frac{1}{t^{d/2}} \left( F_{1,0}(y) + \frac{1}{t} F_{1,1}(y) + \frac{1}{t^2} F_{1,2}(y) + \dots \right) \right. \\ \left. + \frac{1}{t^d} \left( F_{2,0}(y) + \frac{1}{t} F_{2,1}(y) + \frac{1}{t^2} F_{2,2}(y) + \dots \right) + \dots \right]$$

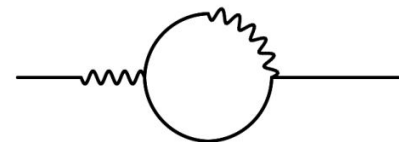


Up to a handful of Wilson coefficients, EFT predicts entire *scaling functions*  $F_{\ell,n}$

$$F_{0,0}(y) = e^{-y^2/4}$$

$$F_{1,0}(y) = \frac{\chi D'^2}{D^{5/2}} \left[ \frac{4 + y^2}{8\sqrt{\pi}} e^{-y^2/2} + \frac{y(y^2 - 10)}{16} e^{-y^2/4} \text{Erf}(y/2) \right]$$

(d=1)



# Let's be careful

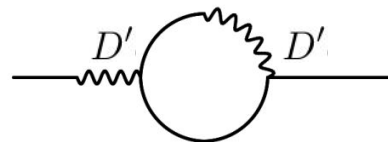
Power-law corrections to diffusion make generic diffusive systems appear to be *superdiffusive* at intermediate times

$$\langle n(0, t)n \rangle = \frac{\chi}{\sqrt{4\pi Dt}} \left( 1 + \frac{\chi D'^2}{4\sqrt{\pi} D^{5/2}} \frac{1}{\sqrt{t}} + O\left(\frac{\log t}{t}\right) \right)$$

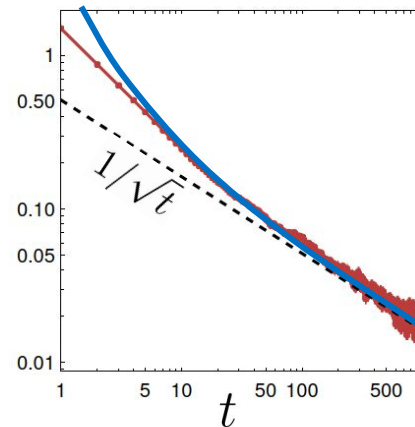
[Michailidis Abanin LVD '23]

$$\begin{array}{c} \uparrow \\ \geq 0 \end{array}$$

Due to a cubic vertex in the EFT of fluctuating diffusion

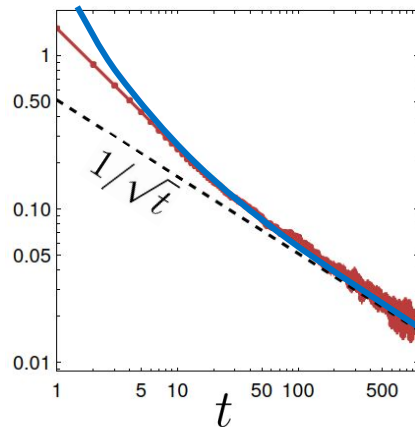


(These are quite large in the related classical Heisenberg model! [Glorioso LVD Chen Nandkishore Lucas '20])



# Let's be careful

Power-law corrections to diffusion make generic diffusive systems appear to be *superdiffusive* at intermediate times

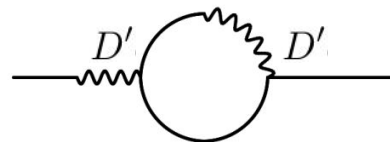


$$\langle n(0, t)n \rangle = \frac{\chi}{\sqrt{4\pi Dt}} \left( 1 + \frac{\chi D'^2}{4\sqrt{\pi} D^{5/2}} \frac{1}{\sqrt{t}} + O\left(\frac{\log t}{t}\right) \right)$$

[Michailidis Abanin LVD '23]

$$\geq 0$$

Due to a cubic vertex in the EFT of fluctuating diffusion



Thankfully, this same cubic vertex enters in the 3pt function, at leading order

$$\langle n(2t)n(t)n \rangle_c = \text{diagram} + \dots \Rightarrow \text{can measure it to rule out diffusion!}$$

# Ruling out diffusion

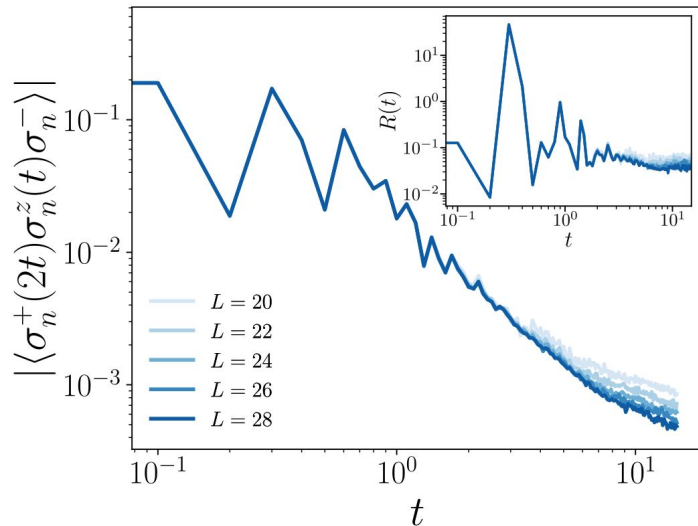
The size of power-law corrections coming from non-Gaussianities

$$\langle n(t)n \rangle \sim \frac{1}{\sqrt{t}} \left( 1 + \sqrt{\frac{\tau_{\text{loop}}}{t}} + \dots \right)$$

can be estimated by the dimensionless ratio

$$\sqrt{\frac{\tau_{\text{loop}}}{t}} \sim R(t) \equiv \frac{\langle n(2t)n(t)n \rangle_c^2}{\langle n(t)n \rangle^3} \approx 4\text{-}5\%$$

This cannot explain 20% deviation from  $z = 2$   
( $z \approx 1.6$ )



What is the theory of QG hydrodynamics?

# Mechanism for superdiffusion

Superdiffusion usually arises from “convective” nonlinearities in 1+1d fluctuating hydrodynamics: [Forster Nelson Stephen '77] (Noisy Burger's equation)

$$\partial_t n + c(n)\partial_x n + D\partial_x^2 n + \dots = 0$$

Nonlinearities  $dc(n)/dn$  are relevant in 1d, and drive to KPZ universality ( $z = \frac{3}{2}$ )

An equation of this form leads to superdiffusion in momentum conserving systems [Spohn '14], quantum Hall edges [LVD Glorioso '20], and the integrable XXX chain [De Nardis Gopalakrishnan Vasseur '22].

# Mechanism for superdiffusion


In QG symmetric spin chains,  $U(1)$  symmetry forbids this nonlinearity for  $\sigma^\pm$  alone.

However, it does allow for nonlinearities involving the  $U(1)$  density  $n = \sigma^z$

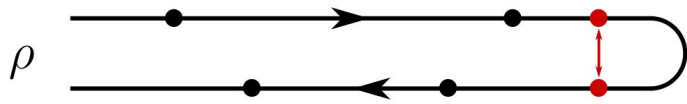
$$\partial_t \sigma^+ + \sigma^+ \partial_x n + D \partial_x^2 \sigma^+ + \dots = 0$$

Such an equation seems plausible given the expression for the nonlocal density:

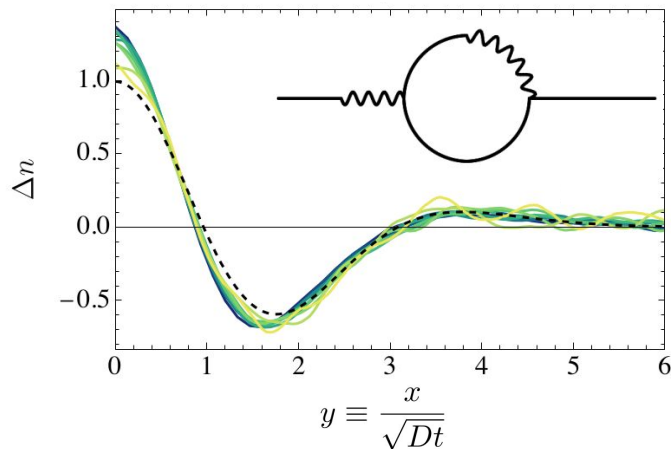
$$S^\pm = \sum_i \dots \otimes q^{\sigma^z} \otimes q^{\sigma^z} \otimes \sigma_i^\pm \otimes \mathbf{1} \otimes \mathbf{1} \otimes \dots$$

... 

## Tools: Precision physics in thermalizing systems



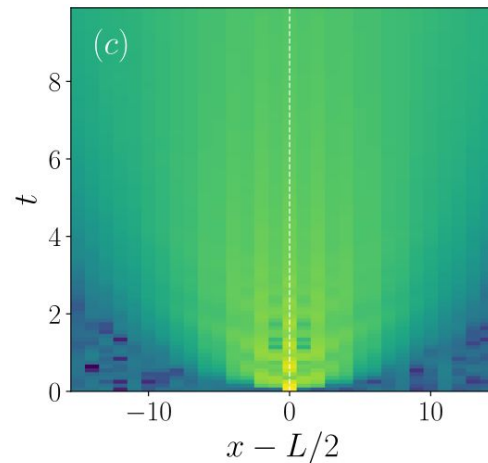
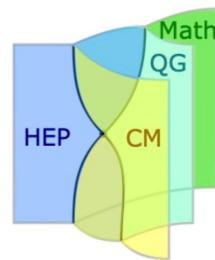
$$Z = \int D\phi_a D\mu_r e^{iS_{\text{eff}}[\phi_a, \mu_r]}$$



## Discovery: Hydrodynamics beyond conventional symmetry

$$S^\pm = \sum_i \dots \otimes q^{\sigma_z} \otimes q^{\sigma_z} \otimes \sigma_i^\pm \otimes \mathbf{1} \otimes \mathbf{1} \otimes \dots$$

$z < 2$   
universality



Extra slides

# Funny finite size effects

At late times in finite systems  $t \gg L^\#$ , autocorrelation functions saturate

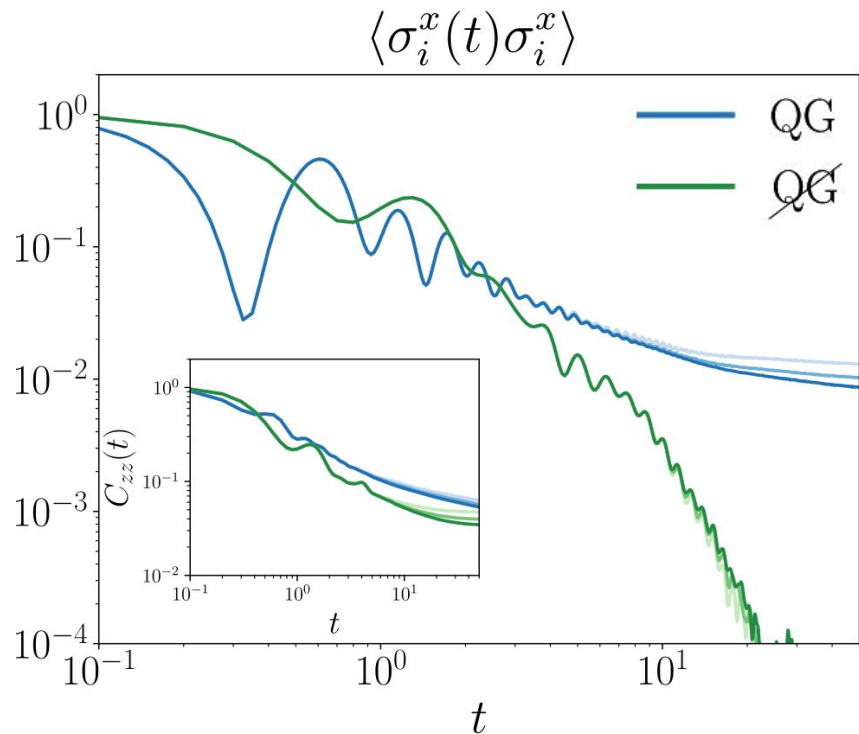
For non-conserved operators, expect

$$\lim_{t \rightarrow \infty} \langle \mathcal{O}(t) \mathcal{O} \rangle \sim e^{-L^d}$$

Instead, for operators overlapping with hydro modes:

$$\lim_{t \rightarrow \infty} \langle \mathcal{O}(t) \mathcal{O} \rangle \sim 1/L^d$$

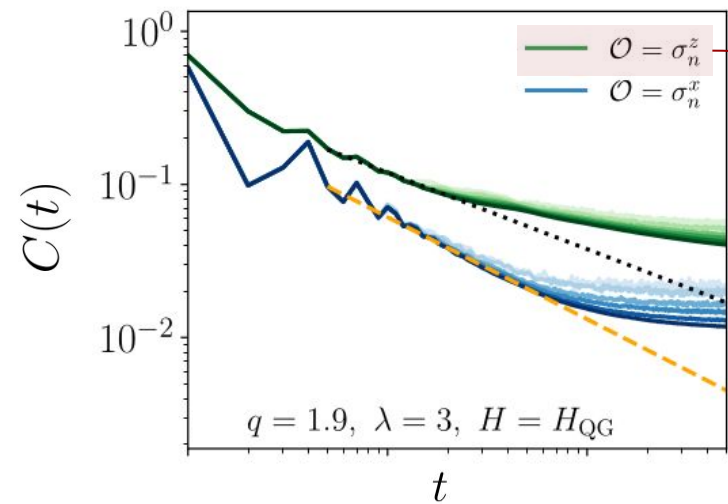
Transverse spin in QG-symmetric models appears to belong to the 2nd category



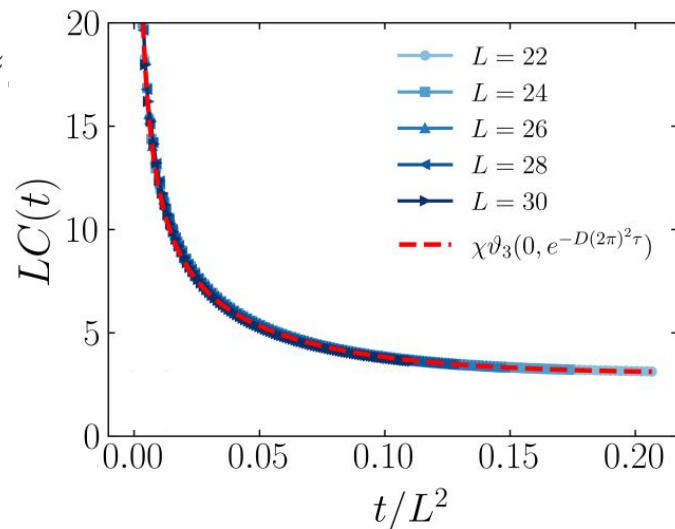
# Funny finite size effects

The entire finite size data is expected to collapse to a scaling function, that is a universal property of the hydro universality class + boundary conditions:

$$\langle n(t, x = 0)n \rangle = \frac{1}{L^d} F(t/L^z)$$



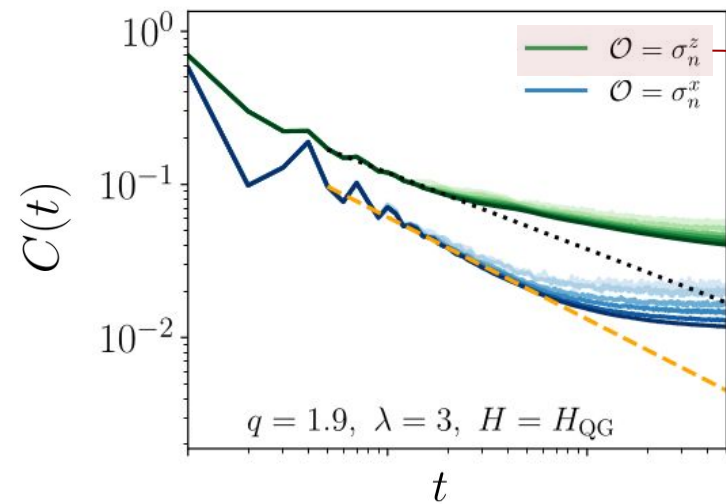
Collapse works well  
for longitudinal spin  $\sigma_n^z$ ,  
which diffuses



# Funny finite size effects

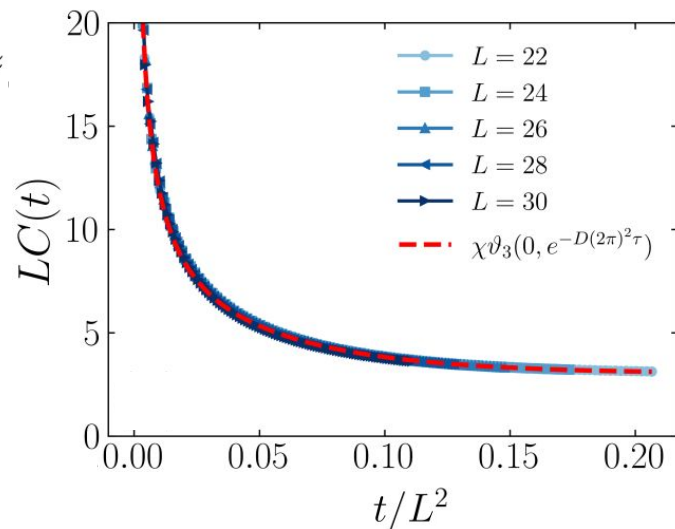
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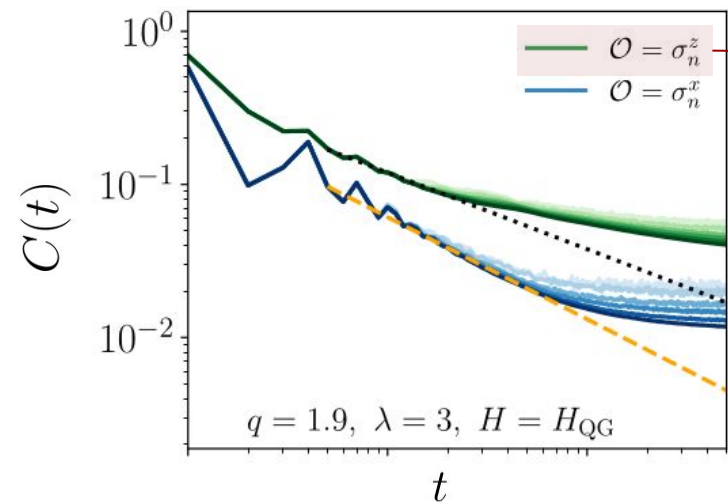
Try to similarly collapse  $\sigma_n^x$  data, and confirm  $z$ ?



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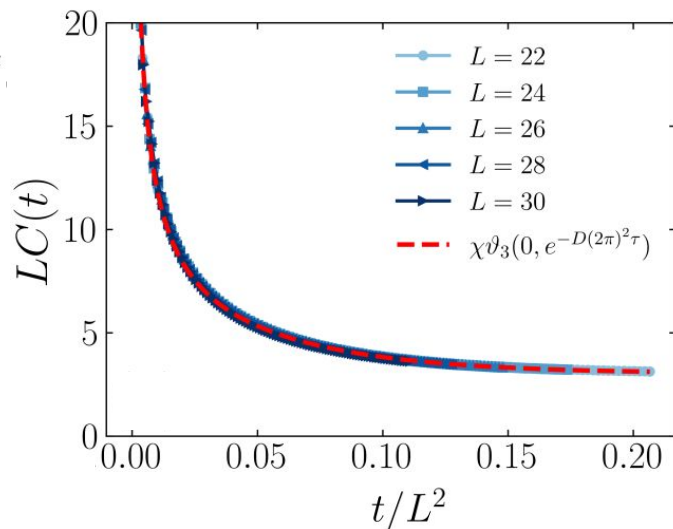
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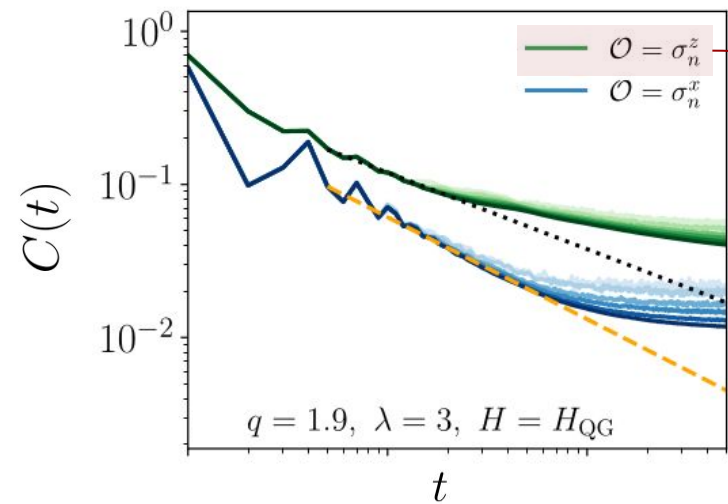
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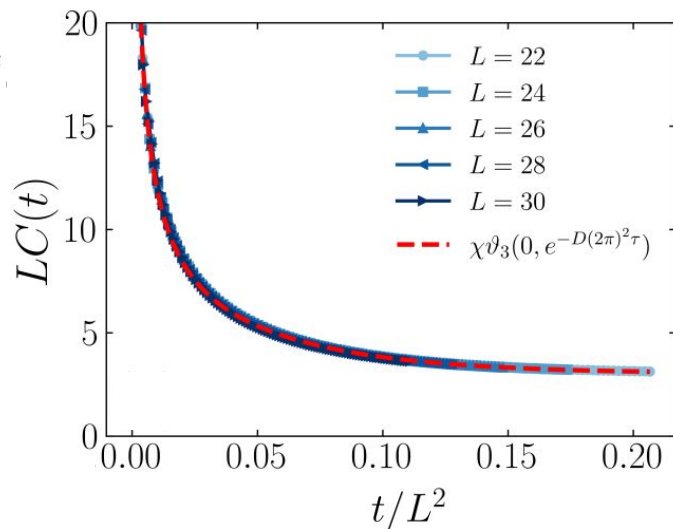
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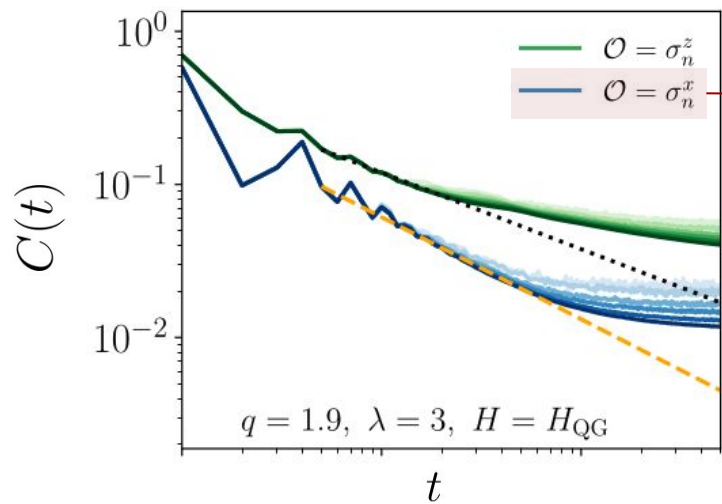
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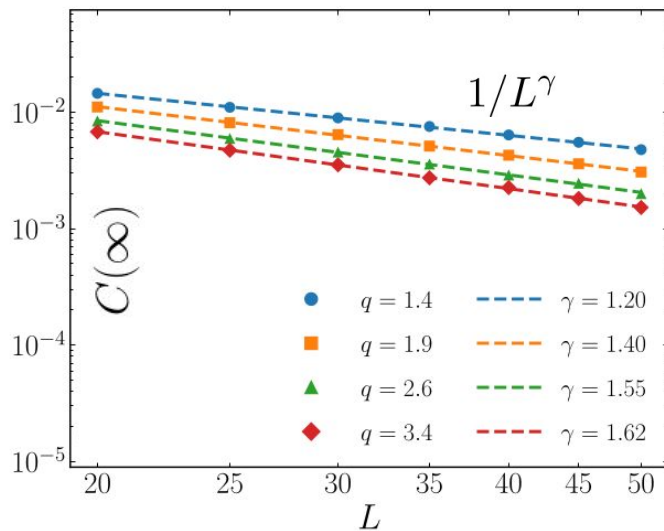
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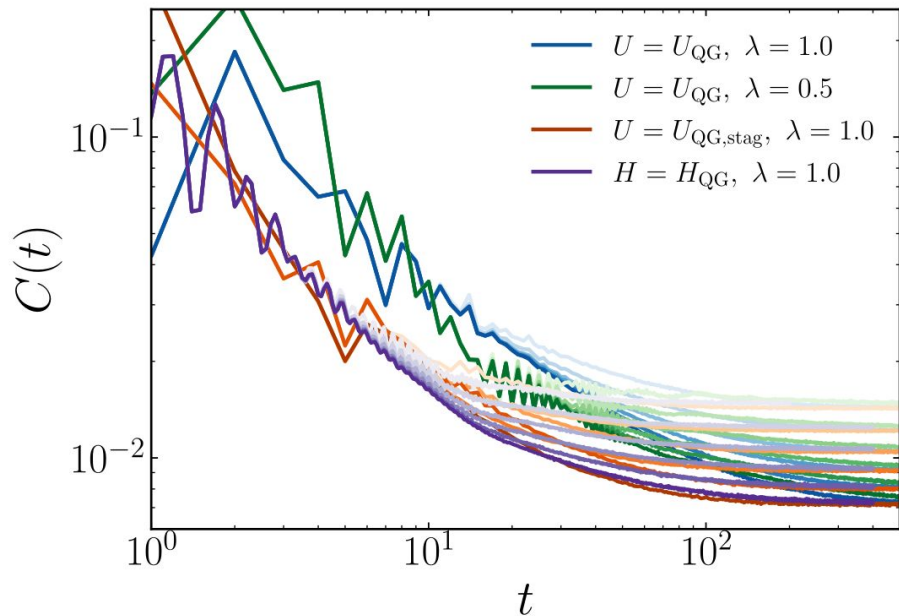


# Funny finite size effects

Nevertheless, finite size saturated value appears to be universal:

$$\lim_{t \rightarrow \infty} C(t) = f(q, L)$$

What is the value of  $f(q, L)$ ?



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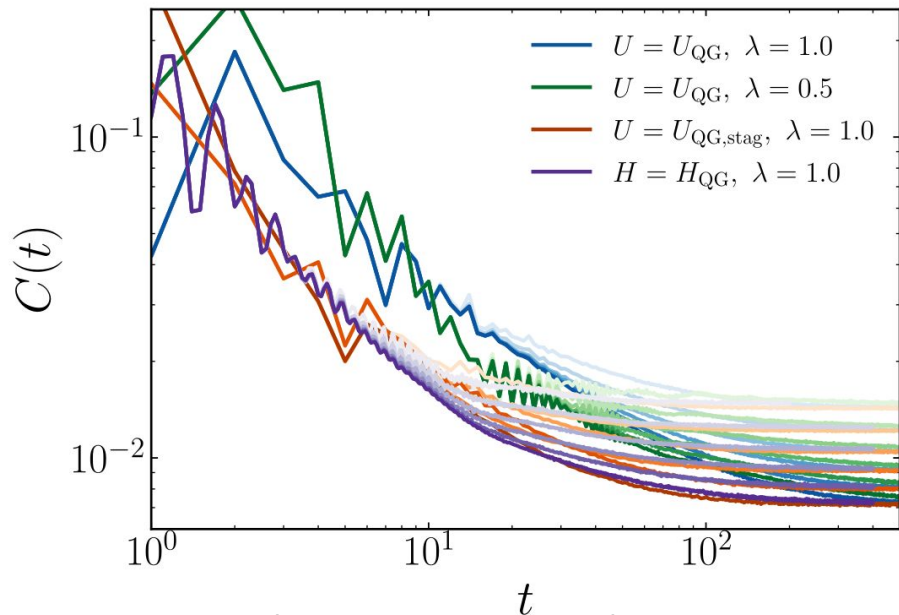
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Overlap with conserved charges provides a lower bound (Mazur inequality)

$$C(\infty) = \lim_{t \rightarrow \infty} \langle \sigma_i^+(t) \sigma_i^- \rangle \geq \frac{\langle \sigma^+ Q \rangle \langle Q^\dagger \sigma^- \rangle}{\langle Q^\dagger Q \rangle} \quad \text{for } [H, Q] = 0$$

Use QG charge  $Q = S_{\text{QG}}^\pm = \sum_i \cdots \otimes q^{\sigma_z} \otimes q^{\sigma_z} \otimes \sigma_i^\pm \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots$  

Find:  $C(\infty) \geq \frac{1}{L} \left( \frac{q + q^{-1} + 2}{2(q + q^{-1})} \right)^L$  (interpolates between  $1/L$  and  $2^{-L}$ )



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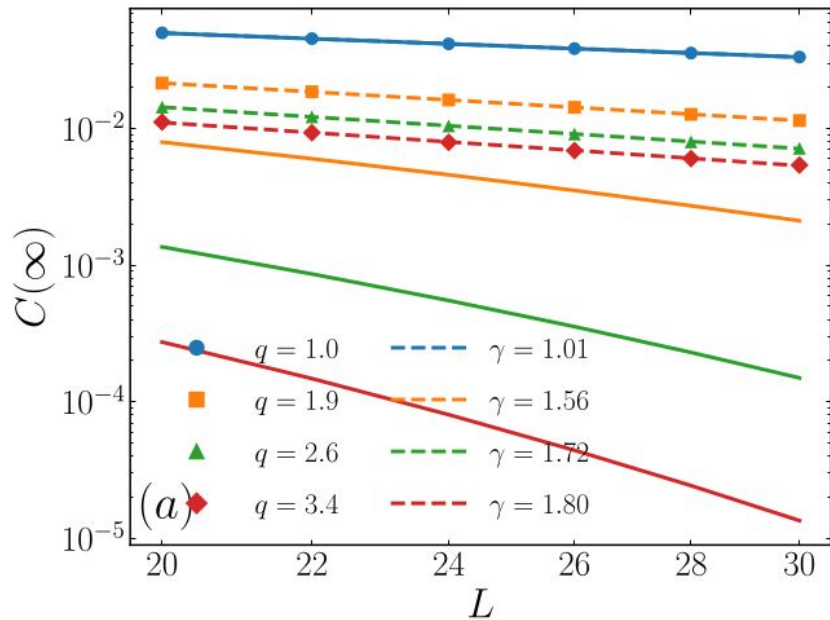
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# Classical model with QG symmetry

Classical Heisenberg model:  $H = \frac{1}{2} \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$ ,  $|\vec{S}_i| = 1$

thermalizes and has  $SU(2)$  symmetry.

One can deform it to obtain model with  $U_q(SU(2))$  symmetry, also superdiffusive

$$H = - \sum_{i=1}^{L-1} \left[ K_i^{-1/2} (E_{i+1} F_i + E_i F_{i+1}) K_{i+1}^{1/2} - \frac{1}{\eta \sinh \eta} K_i^{-1} K_{i+1} + \frac{1}{\eta \tanh \eta} (K_i^{-1} + K_{i+1}) \right]$$

$$K_j = e^{\eta S_j^z}, \quad E_j = f(S_j^z) S_j^+, \quad F_j = \frac{-\eta}{\sinh \eta} f(S_j^z) S_j^-$$

