Schwinger-Keldysh Hydrodynamics of the SYK Lattice

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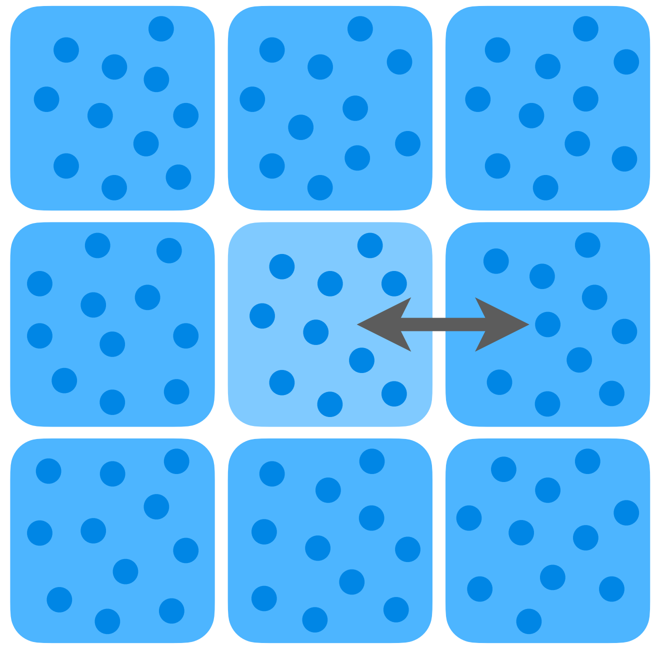
EFT and Hydrodynamics

- Effective field theory (EFT) is a powerful tool to describe collective macroscopic phenomena in complex many-body systems, largely *agnostic* of microscopic details.
- Conventional EFTs only apply at zero temperature or in thermal equilibrium.
- Hydrodynamics describes collective dynamics of many-body systems operating *"close"* to thermal equilibrium — *local* thermal equilibrium.



- Hydrodynamics is *"almost"* an EFT — *local fields* (velocity, temperature, density, etc.), whose dynamics is dictated by universal symmetry principles.
- However, conventional hydrodynamics is *not* based on an effective action principle.

Schwinger-Keldysh EFT

Hydrodynamics Across Scales

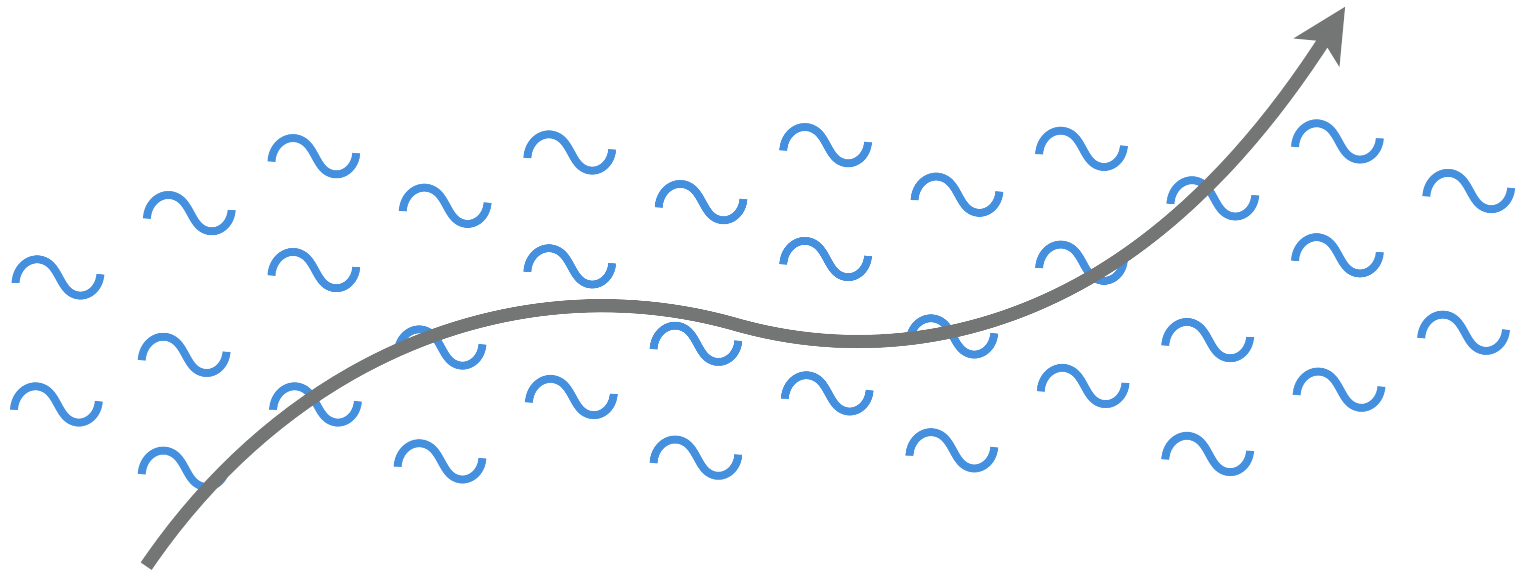
- ➤ Hydrodynamics is very universal — it applies to almost* all interacting physical systems in a highly excited near-equilibrium state.
- ➤ **Working assumption:** all “high-energy” excitations die out at macroscopic scales. Only collective “low-energy” excitations protected by symmetries remain:

Conserved Quantities

e.g. energy, momenta, charge, particle number

Goldstones of Spontaneously Broken Symmetries

e.g. superfluid phase, phonons



- ➤ Hydrodynamics prescribes the dynamics of low-energy excitations through universal conservation laws derived from symmetries.

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* The fine print!

Classical Hydrodynamics

- Classical hydrodynamics is *deterministic:* dynamics is derived from conservation laws.

$$\text{Energy Conservation}: \quad \dot{\epsilon} + \nabla \cdot \mathbf{J} = 0$$

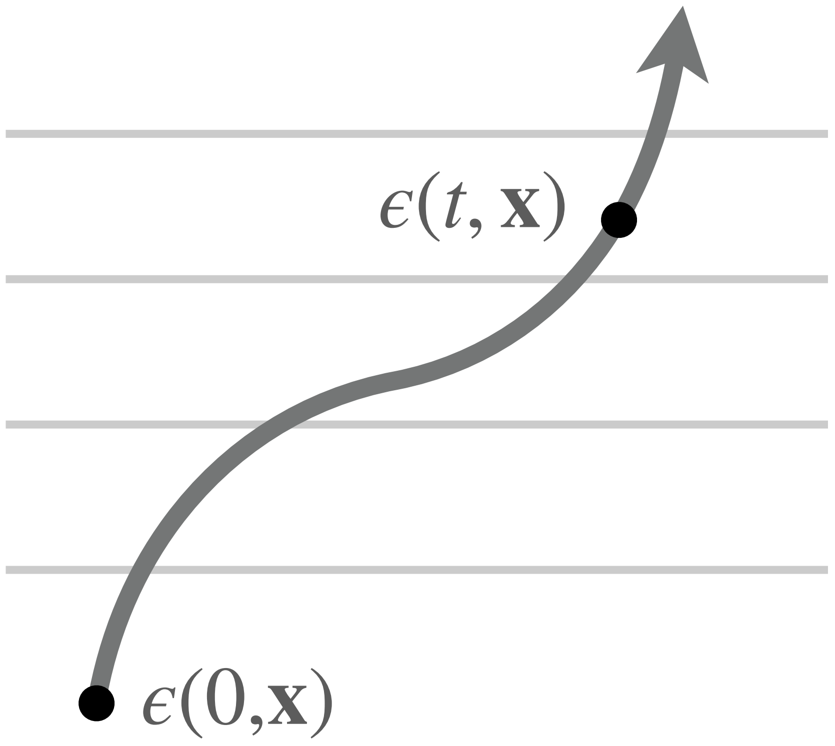
easily generalisable
to conserved particles
or momenta.

- Diffusive constitutive relations:

$$\mathbf{J} = -\,\kappa(T)\,\nabla T \;+\; \text{higher-derivatives}$$

thermal
conductivity

local temperature
$T(\epsilon)$ given by the equation of state



- Energy diffusion:

$$\dot{\epsilon} = D\,\nabla^2\epsilon \;+\; \text{non-linear} \;+\; \text{higher-derivatives}$$

$$D = \frac{\kappa}{c_V} \qquad c_V = \frac{\partial \epsilon}{\partial T}$$

Schwinger-Keldysh EFTs

- Effective action for energy diffusion

$$\mathscr{L} = \boxed{\Big[\,\dot{\epsilon}(T) + \nabla \cdot \mathbf{J}(T, \nabla T, \ldots)\,\Big] X_a} + \boxed{iO\big(X_a^2\big) + O\big(X_a^3\big) + iO\big(X_a^4\big) + \ldots}$$

classical part

fluctuation part

$$\frac{\delta S}{\delta X_a} = 0 \quad \Longrightarrow \quad \dot{\epsilon} + \nabla \cdot \mathbf{J} = \# X_a \qquad\qquad \frac{\delta S}{\delta T} = 0 \quad \Longrightarrow \quad \# X_a = 0$$

- Fluctuation part contributes to correlation functions, such as

$$\Big\langle \epsilon(t, \mathbf{x})\, \epsilon(t', \mathbf{x}') \ldots \Big\rangle \qquad \Big\langle \mathbf{J}(t, \mathbf{x})\, \mathbf{J}(t', \mathbf{x}') \ldots \Big\rangle \qquad \Big\langle \epsilon(t, \mathbf{x})\, \mathbf{J}(t', \mathbf{x}') \ldots \Big\rangle$$

Schwinger-Keldysh EFTs

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classical part

fluctuation part

severely constrained by **symmetries**, **microscopic unitarity**, and the **initial state**

thermal state:
fluctuation-dissipation relations

[Crossley, Glorioso, Liu 2015] [Haehl, Loganayagam, Rangamani 2015] [Jensen, Pinzani-Fokeeva, Yarom 2017]

Schwinger-Keldysh EFTs

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Schwinger-Keldysh

Effective Field Theory

[Crossley, Glorioso, Liu 2015] [Haehl, Loganayagam, Rangamani 2015] [Jensen, Pinzani-Fokeeva, Yarom 2017]

Schwinger-Keldysh EFT for Energy Diffusion

$$\mathscr{L} = \boxed{-\epsilon(T)\,\dot{X}_a + \kappa(T)\,\nabla T \cdot \nabla X_a + \ldots} + \boxed{iT^2\kappa(T)\,\nabla X_a \cdot \nabla X_a + \ldots}$$

- SK EFTs are *not* manifestly unitary ($\operatorname{Im} S \neq 0$) — strongly constrained by microscopic unitarity

$$S\left[T, X_a = 0\right] = 0 \qquad S\left[T, -X_a\right] = -S^*\left[T, X_a\right] \qquad \operatorname{Im} S\left[T, X_a\right] \geq 0$$

- Initial thermal states — KMS symmetry ($\beta_0 \equiv 1/T_0$)

$$T(t) \to T(-t) + O\left(\beta_0^2\right) \qquad X_a(t) \to -X_a(-t) - \frac{i}{T(-t)} + i\beta_0 + O\left(\beta_0^2\right)$$

- Energy diffusion — time-translation symmetry (doubled in SK EFT)
$\Longrightarrow$ responsible for the conservation of energy (and its SK noise partner).

Goal for Today

To explicitly derive the SK-EFT for a simple solvable microscopic model.

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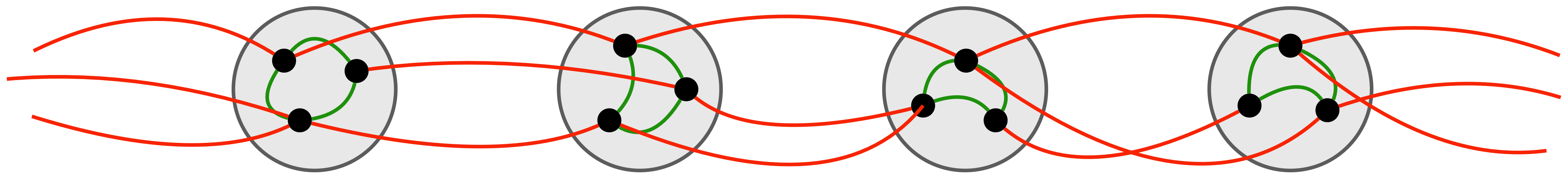
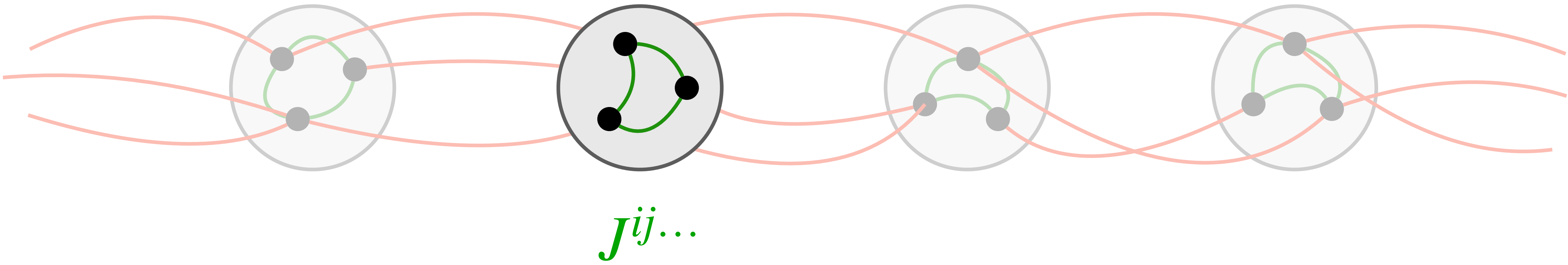


SYK Chain

The SYK Dot



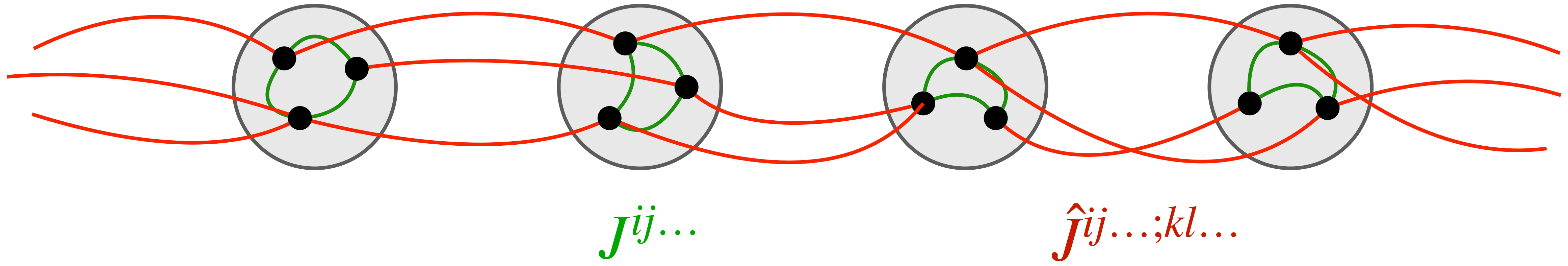
- ➤ SYK dot — 0+1d model of N Majorana fermions with local p-body random interactions.

[Sachdev, Ye 1992] [Kitaev 2015]
[Maldacena, Stanford 2016]
[Polchinski, Rosenhaus 2016]

Gaussian-random couplings

$$S_{\text{SYK}} = i\sum_{i=1}^{N}\Psi_i\partial_t\Psi_i - i^{p/2}\sum_{i,j\ldots=1}^{N} J^{ij\ldots}\left(\underbrace{\Psi_i\;\Psi_j\;\ldots}_{p\text{ fields}}\right)$$

The SYK Chain



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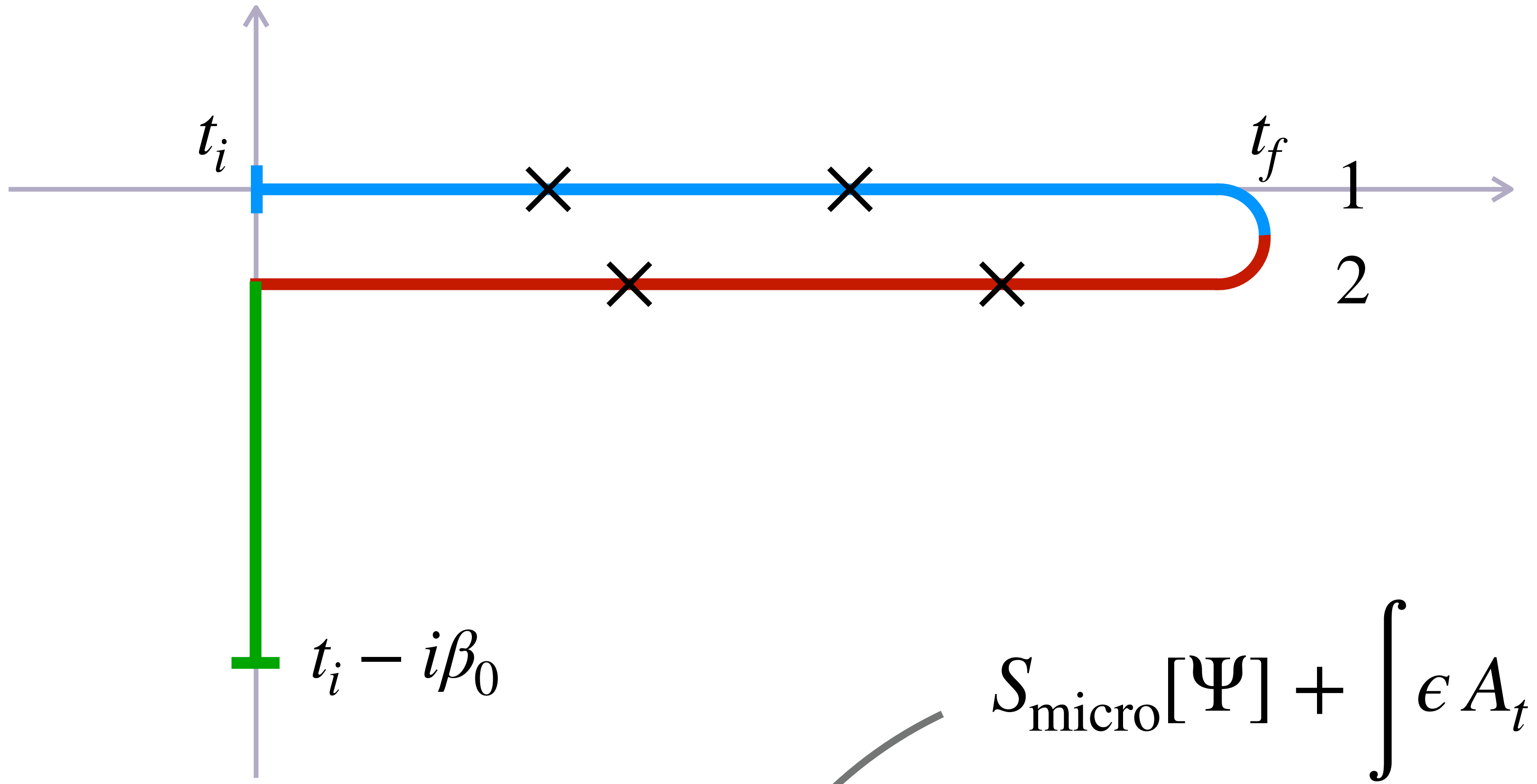
- SYK chain — 1+1d chain of SYK dots with random nearest-neighbour interactions.

[Gu, Qi, Stanford 2017]

$$S_{\text{SYK-chain}} = \sum_x \Bigg[S_{\text{SYK}|\text{x}} - i^{p/2} \sum_{i,j\dots;k,l,\dots=1}^{N} \hat{J}^{ij\dots;kl\dots}_{x,x+1} \Big(\underbrace{\Psi_{i|x} \; \Psi_{j|x} \; \dots}_{p/2 \text{ fields}} \Big)\Big(\underbrace{\Psi_{k|x+1} \; \Psi_{l|x+1} \; \dots}_{p/2 \text{ fields}} \Big) \Bigg]$$

Schwinger-Keldysh Path Integral

$$\mathrm{tr}\Big(\mathrm{e}^{-\beta H}\,\tilde{\mathsf{T}}\big[\epsilon(t')\ldots\big]\,\mathsf{T}\big[\epsilon(t)\ldots\big]\Big)$$



$$S_{\mathrm{micro}}[\Psi] + \int \epsilon\, A_t + \mathbf{J}\cdot\mathbf{A} + \ldots$$

$$\mathcal{Z}\big[A_1, A_2\big] = \int \mathcal{D}\Psi_1\,\mathcal{D}\Psi_2\,\mathcal{D}\Psi_{\mathrm{E}}\,\exp\Big(i\,S_{\mathrm{micro}}\big[\Psi_1; A_1\big] - i\,S_{\mathrm{micro}}\big[\Psi_2; A_2\big] - S^{\mathrm{E}}_{\mathrm{micro}}\big[\Psi_{\mathrm{E}}\big]\Big)$$

$\Psi_1(t_f) = \Psi_2(t_f)$

$\Psi_2(t_i) = \Psi_{\mathrm{E}}(t_i)$

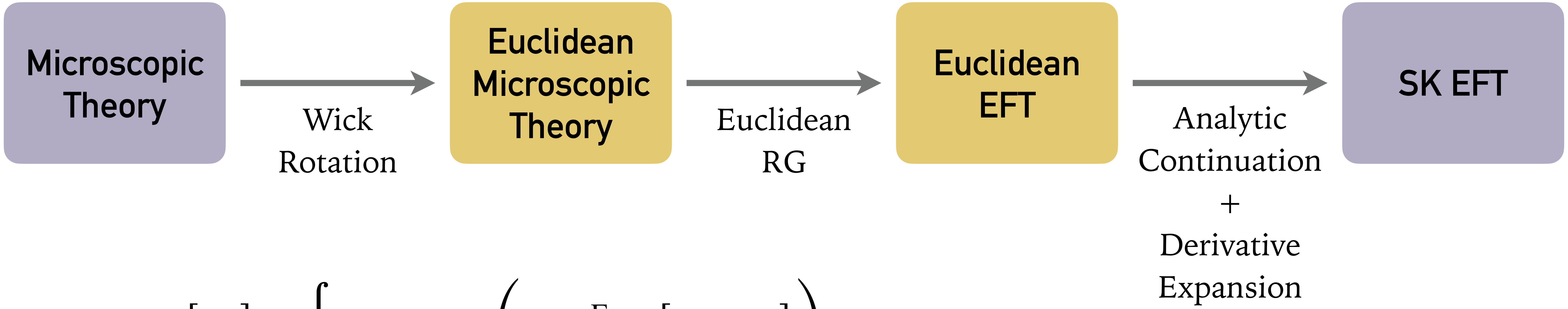
$\Psi_1(t_i) = \Psi_{\mathrm{E}}(t_i - i\beta_0)$

$$\Big\downarrow \text{RG}$$

$$\mathcal{Z}\big[A_1, A_2\big] = \int \mathcal{D}F_1\,\mathcal{D}F_2\,\exp\Big(i\,S[F_1, F_2; A_1, A_2]\Big)$$

$F_1(t_f) = F_2(t_f)$

SK EFT via the Euclidean Formalism



Microscopic Theory → Wick Rotation → **Euclidean Microscopic Theory** → Euclidean RG → **Euclidean EFT** → Analytic Continuation + Derivative Expansion → **SK EFT**

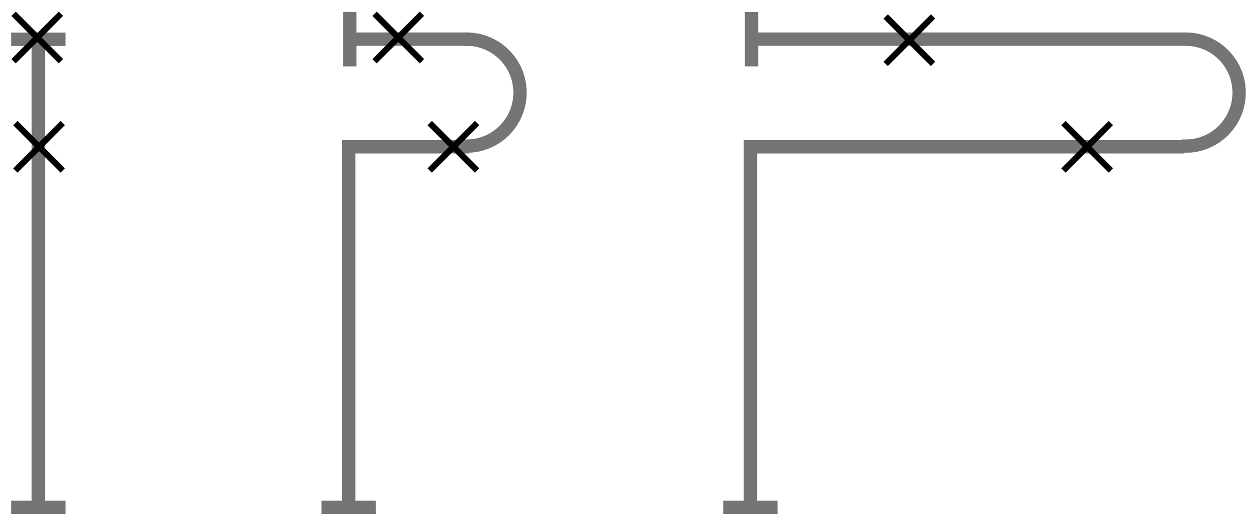
$$\mathcal{Z}_{\mathrm{E}}[A_{\mathrm{E}}] = \int \mathcal{D}\Psi_{\mathrm{E}} \exp\Big(- S^{\mathrm{E}}_{\mathrm{micro}}[\Psi_{\mathrm{E}}, A_{\mathrm{E}}] \Big)$$

$$\downarrow$$

$$\mathcal{Z}_{\mathrm{E}}[A_{\mathrm{E}}] = \int \mathcal{D}F_{\mathrm{E}} \exp\Big(i\, S^{\mathrm{E}}[F_{\mathrm{E}}; A_{\mathrm{E}}] \Big)$$

$$\downarrow$$

$$\mathcal{Z}[A_1, A_2] = \int \mathcal{D}F_1\, \mathcal{D}F_2 \exp\Big(i\, S[F_1, F_2; A_1, A_2] \Big)$$



Schwarzian Theory

- Upon disorder-averaging over couplings, the SYK fermions can be traded for the collective field

[Maldacena, Stanford 2016]

$$G(\tau,\tau') \equiv \frac{1}{N}\sum_{i=1}^{N} \mathsf{T}_{\mathrm{E}}\Psi_{i,\mathrm{E}}(\tau)\Psi_{i,\mathrm{E}}(\tau') = \frac{1}{2}\mathrm{sgn}(\tau-\tau')\left(1+\frac{g(\tau,\tau')-g_0}{p}+O\left(1/p^2\right)\right)$$

- For $N \gg p^2 \gg 1$, the Euclidean path-integral is dominated by the saddle

$$\exp\left(g_*(\tau,\tau')\right) = \frac{1}{\cos^2\left(\pi v\left(1/2-|\tau-\tau'|/\beta_0\right)\right)} \qquad \beta_0 J \sim \frac{\pi v}{\cos\left(\pi v/2\right)}$$

- For $\beta_0 J \gg 1$, low-energy perturbations are governed by the Schwarzian mode

$$\exp\left(g(\tau,\tau')\right) \sim \frac{\dot{F}_{\mathrm{E}}(\tau)\,\dot{F}_{\mathrm{E}}(\tau')}{\sin^2\left(\pi F_{\mathrm{E}}(\tau)-\pi F_{\mathrm{E}}(\tau')\right)}$$

Soft Mode Action

- For sufficiently weak interactions between nearest neighbours, the low-energy description of the SYK chain is given by an spatially-interacting Schwarzian theory

[Maldacena, Qi 2018]
[Altland, Bagrets, Kamenev 2019]
[Almheiri, Milekhin, Swingle 2019]
[Bucca, Mezei 2024]

$$S^{\mathrm{E}}_{\text{Sch-Chain}} = \int \mathrm{d}\tau\, L^{\mathrm{E}}_{\mathrm{Sch}}[F(\tau)] + \int \mathrm{d}\tau\, \mathrm{d}\tau'\, L^{\mathrm{E}}_{\mathrm{int}}\left[F(\tau), F(\tau')\right]$$

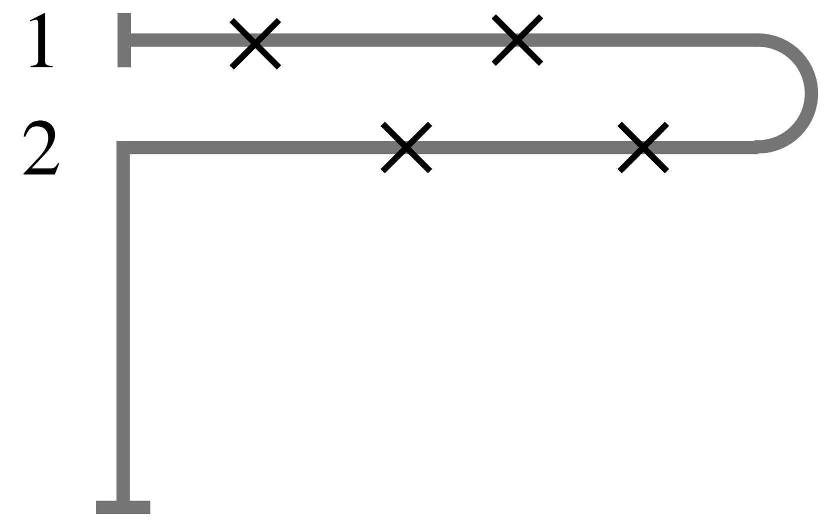
$$L^{\mathrm{E}}_{\mathrm{Sch}}[F(\tau)] = -\mathcal{N} \sum_{x} \left(2\pi^2 \dot{F}(\tau,x)^2 + \frac{\dddot{F}(\tau,x)}{\dot{F}(\tau,x)} - \frac{3}{2}\frac{\ddot{F}(\tau,x)^2}{\dot{F}(\tau,x)^2} \right)$$

$$L^{\mathrm{E}}_{\mathrm{int}}\left[F(\tau), F(\tau')\right] = -\mathcal{N}' \sum_{x} \left(\sqrt{\frac{\dot{F}(\tau,x)\,\dot{F}(\tau',x)}{\sin^2\left(\pi F(\tau,x) - \pi F(\tau',x)\right)}} \times (x \to x + a) \right)$$

- The soft mode action is bilocal in time.

SK-EFT for SYK Chains

- Analytic continuation to closed-time contour:



$$S = \int \mathrm{d}t \left(L_{\mathrm{Sch}}\left[F_1(t)\right] - L_{\mathrm{Sch}}\left[F_2(t)\right] \right)$$

$$+ \int \mathrm{d}t\, \mathrm{d}t' \left(L_{\mathrm{int}}\left[F_1(t), F_1(t')\right] - L_{\mathrm{int}}\left[F_1(t), F_2(t')\right] - L_{\mathrm{int}}\left[F_2(t), F_1(t')\right] + L_{\mathrm{int}}\left[F_2(t), F_2(t')\right] \right)$$

- For $|t - t'| \gg \beta_0/\pi,$ we can perform the t' integral to yield the SK EFT

$$\boxed{S = \int \mathrm{d}t \left(L_{\mathrm{Sch}}[F_1(t)] - L_{\mathrm{Sch}}[F_2(t)] + L_{\mathrm{int}}[F_1(t), F_2(t)] \right)}$$

Results at Leading Order in Derivatives

- At leading order* in derivatives, we recover

$$\mathscr{L} = -\,\epsilon(T)\,\dot{X}_a + \kappa(T)\,\nabla T \cdot \nabla X_a + iT^2\kappa(T)\,\nabla X_a \cdot \nabla X_a + \ldots$$

$$T = \frac{\dot{F}_1 + \dot{F}_2}{2} \qquad -T\,X_a = F_1 - F_2$$

$$\epsilon = 2\pi^2 \mathcal{N}\,T^2 \qquad \kappa = 8\pi^3\,\mathcal{N}'\,T$$

Can be extended to arbitrary order in X_a and derivatives.

- Unitarity constraints and KMS symmetry:

$$S[F,F] = 0 \qquad F_1(t) \to -\,F_1(-t)$$

$$S[F_2,F_1] = -\,S^*[F_1,F_2] \qquad F_2(t) \to -\,F_2(-t-i\beta) - i$$

$$\operatorname{Im} S[F_1,F_2] \geq 0$$

(order-by-order in derivatives)

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* We use the scaling $\partial_t \sim \nabla^2$ for derivatives and $\dot{F}, Q \sim O(\nabla^0)$ for fields.

Classical and Noise Energy Current

- The theory admits two conserved energy currents: one for each Lorentzian fold.

$$\epsilon = \frac{\epsilon_1 + \epsilon_2}{2} \qquad \mathbf{J} = \frac{\mathbf{J}_1 + \mathbf{J}_2}{2} \qquad\qquad \epsilon_a = \frac{\epsilon_1 - \epsilon_2}{\hbar} \qquad \mathbf{J}_a = \frac{\mathbf{J}_1 - \mathbf{J}_2}{\hbar}$$

classical energy current

quantum/thermal noise

- The difference conservation is identically solved by $F_1 = F_2 \equiv F$.
  The remaining average conservation yields the classical energy diffusion equation.
- The theory also admits an entropy current with non-negative divergence

$$\dot{s} + \nabla \cdot \mathbf{J}_s \geq 0 \quad \text{(onshell)}$$

Unitarity constraints + KMS $\Longrightarrow$ local entropy production.

Symmetries of the SK EFT

- Average time- and space-translation symmetry — Noether currents are purely noise:

$$F_{1,2}(t,\mathbf{x}) \to F_{1,2}(t+\xi,\mathbf{x}) \qquad F_{1,2}(t,\mathbf{x}) \to F_{1,2}(t,\mathbf{x}+\mathbf{c})$$

- Difference time-translation symmetry — associated with average energy conservation:

$$F_1(t,\mathbf{x}) \to F_1(t+\zeta,\mathbf{x}) \qquad F_2(t,\mathbf{x}) \to F_2(t-\zeta,\mathbf{x})$$

  Spontaneously broken (Goldstone X_a) — **strong-to-weak SSB** by the thermal state $\mathrm{e}^{-\beta H}$.

- Spatially-local SL(2,$\mathbb{R}$) symmetry: [Blake, Lee, Liu 2018] [Haehl, Rozali 2018]

$$F_{1,2}(t,\mathbf{x}) \to F_{1,2}(t,\mathbf{x}) + a(\mathbf{x}) + b_+(\mathbf{x})\,\mathrm{e}^{2\pi F_{1,2}(t,\mathbf{x})} + b\_(\mathbf{x})\,\mathrm{e}^{-2\pi F_{1,2}(t,\mathbf{x})} + O(b^2)$$

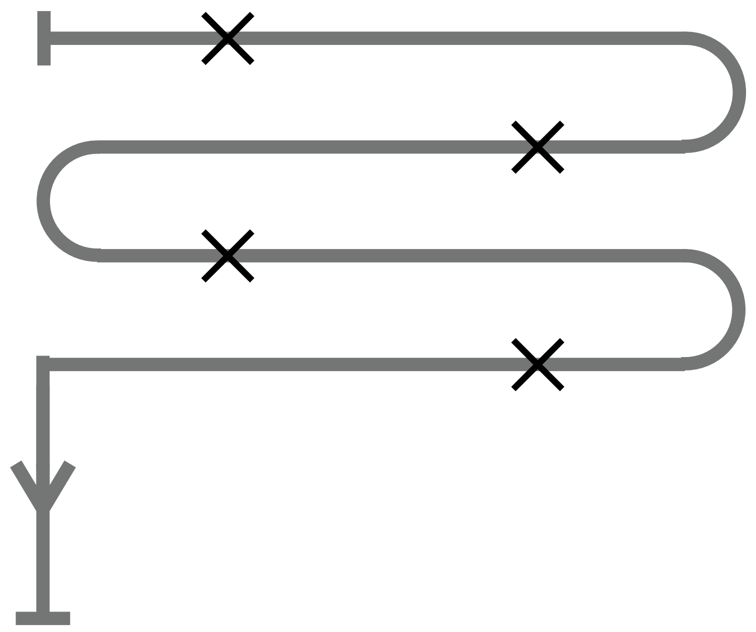
  The $b_\pm$ parts of the symmetry are broken by the derivative expansion.

- KMS symmetry: $F_1(t,\mathbf{x}) \to -F_1(-t,\mathbf{x}) \qquad F_2(t,\mathbf{x}) \to -F_2(-t-i\beta_0,\mathbf{x}) - i$

SK EFT for OTOCs

- SK EFT for OTOCs require multi-fold closed-time contours. [Mishra, Wang, Pappalardi, Delacretaz 2026]

$$\Big\langle W(t)\, V(0)\, W(t)\, V(0) \Big\rangle \sim e^{\lambda\, t} \quad \text{(early time)}$$



$$S^{(2n)}[F_1, \ldots, F_{2n}] = \int dt \sum_{i=1}^{2n} (-1)^{i+1} L_{Sch}\big[F_i(t)\big] + \int dt\, dt' \sum_{i,j=1}^{2n} (-1)^{i+j} L_{int}\big[F_i(t), F_j(t')\big]$$

$$= \sum_{j>i=1}^{2n} (-1)^{i+j+1} S^{(2)}\big[F_i, F_j\big]$$

- **WIP:** Do OTOCs of energy admit Lyapunov growth?
- **WIP:** Large-p SYK is maximally chaotic ($\lambda = 2\pi/\beta_0$). EFT for sub-maximal chaos at finite p?

Summary

- Derivation of dissipative SK EFT for energy diffusion from a local unitary microscopic model — SYK chain.
- Explicit identification of the low-energy variables and symmetries.
- Includes (non-Gaussian) fluctuations compatible with fluctuation-dissipation relations — realised by the (quantum) KMS symmetry.
- Access to the expectation values and correlation functions of energy and entropy currents in the SYK chain, at arbitrarily high derivative orders.
- Includes several stochastic transport coefficients not fixed by classical energy diffusion.
- Future: conserved momenta/particles, OTOCs, sub-maximal chaos, holography.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THANK YOU

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