Atsuhisa Ota (Chongqing University)

Stability analysis of first-order hydrodynamics as an IR effective theory

Based on arXiv:2410.23837, to appear in PRD and PRDL.

Evolve dissipative relativistic fluids as a closed PDE system

Long-standing goal

Numerically evolve a relativistic fluid with dissipation as a system of partial differential equations that is closed by the conservation laws alone.

$$\nabla_\mu T^{\mu\nu} = 0, \qquad \nabla_\mu N^\mu = 0$$

energy–momentum & charge, locally conserved

The problem

The naïve first-order theory cannot be evolved consistently in time.

- instabilities
- acausal propagation
- ill-posed initial value problem

Refs: Landau–Lifshitz, Fluid Mechanics

Plane-wave analysis of the first-order theory

1

Perturb equilibrium

Linear plane-wave perturbations about a homogeneous equilibrium.

$$\delta\Phi \propto e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$$

2

Insert constitutive relations

Substitute the first-order constitutive relations into the conservation laws.

3

Read off the dispersion

Solve for the dispersion relation; inspect its complex roots.

$$\omega = \omega(\mathbf{k})$$

Classic conclusion

- **Unstable modes appear**
- **Causality is violated**
- **Deemed generically pathological**

Reference: Hiscock & Lindblom 1985

Refs: Hiscock – Lindblom, Phys. Rev. D 31, 725 (1985); Phys. Rev. D 35, 3723 (1987)

Two established routes to a well-behaved PDE system

Müller–Israel–Stewart

Promote the dissipative fluxes to independent dynamical variables and add relaxation equations for them.

$$\pi^{\mu\nu}, \quad q^{\mu}, \quad \Pi$$

extra relaxing d.o.f.

Bemfica-Disconzi-Noronha-Kovtun first-order theory

Choose the hydrodynamic frame judiciously so that a first-order PDE system is stable, causal and well-posed — with no extra variables.

- stability
- causality
- well-posedness

Where they sit: both MIS and BDNK are ***UV prescriptions*** for treating first-order hydrodynamics as a good PDE system.

Refs: Israel–Stewart (1979) · Kovtun arXiv:1907.08191 · BDNK arXiv:2009.11388, 2004.04102

Hydrodynamics is not fundamental — it is an IR effective theory

Analogy: chiral perturbation theory

Exactly inverting a derivative expansion truncated at finite order can produce spurious poles on the UV side.

$$p \ll \Lambda_{\mathrm{EFT}}$$

trust the expansion only here

The UV pole is simply not regarded as a physical degree of freedom.

The same should hold for first-order hydrodynamics

Don't over-extend

The truncated equations should not be treated as exact PDEs valid at all ω and all k.

Don't reject on UV poles

Pathological poles in the UV are no reason to discard the theory as an IR description.

IR is all you need

Nonlinear response and loop corrections only require a controlled IR pole under the derivative expansion.

Can first-order hydrodynamics be made consistent as an IR EFT?

Refs: Dubovsky–Hui–Nicolis–Son arXiv:1107.0731 · Endlich–Nicolis–Porto–Wang arXiv:1211.6461

Two different “frames” in relativistic fluid mode analysis

1 Hydrodynamic frame

A field-redefinition freedom in how the fluid variables are defined.

$$\beta^\mu \to \beta^\mu + \delta\beta^\mu, \qquad \mu \to \mu + \delta\mu$$

The same conserved currents, expressed through different fluid variables — a field redefinition.

- **Landau frame** — flow defined by energy flux
- **Eckart frame** — flow defined by particle flux

2 Inertial frame

A choice of which inertial observer describes the same fluid background.

$$k_\mu \longrightarrow k'_\mu = \Lambda_\mu{}^\nu\, k_\nu$$

- **Comoving frame** — observer moving with the background
- **Non-comoving frame** — observer boosted relative to the fluid

These two boosts realize Lorentz symmetry very differently.

Hydrodynamic frame ≠ Inertial frame

Refs: Kovtun arXiv:1907.08191 · Mitra arXiv:2106.08510

Stability only for one special combination

Stable

Landau frame + comoving inertial frame

In this one combination, every mode is stable.

"A peculiar singular limit"

— how the Landau frame came to be regarded

Move away — instabilities return at once

- Shift the hydrodynamic frame off Landau
- Shift the inertial frame off comoving

But a physical IR pole should...

- not depend on redefinition of the fluid variables
- not depend on the choice of inertial frame

Refs: Hiscock–Lindblom (1985) · Denicol et al. arXiv:0807.3120 · Pu–Koide–Rischke arXiv:0907.3906

Fluid modes as Stückelberg fields

Global equilibrium thermal vector

A local fluid configuration is generated from the constant thermal vector by a diffeomorphism Stückelberg field:

$$\beta^\mu(x) = e^{\mathscr{L}_{\pi_R}}\, \beta^\mu_c$$

β^J_o : constant global-equilibrium thermal vector

π_R^μ the physical hydrodynamic field

Schwinger–Keldysh: a second copy

The SK (closed-time-path) formalism introduces a second Stückelberg field associated with noise diffeomorphisms.

$\pi_{A\mu}$ Stückelberg field for noise diffeomorphisms

The SK effective action is then built systematically from the nonlinear realization of diffeomorphisms on the background fields:

$$I_{SK} = I_{SK}[\pi^\mu_R,\ \pi^\mu_A]$$

Refs: Crossley–Glorioso–Liu arXiv:1511.03646 · Nicolis–Penco–Rosen arXiv:1307.0517

Dissipation and fluctuation from one effective action

$$I_{IR} = \int d^4x \left[B^M \pi_{A,M} + \pi_{A,M} E^M{}_N \pi_R^N + \frac{i}{2} \pi_{A,M} N^{MN} \pi_{A,N} \right]$$

B^M
background equation-of-motion term

$E^M{}_N$
response / EOM kernel

N^{MN}
noise kernel

Inherited from the UV generating functional

- conjugation condition
- unitarity
- KMS symmetry

B, E, N are not independent — the SK structure constrains them

$$\text{Unitarity} + \text{KMS} \implies \text{local 2nd law}$$
$$\text{KMS} \implies \text{thermodynamic relations}$$
$$\text{KMS} \implies \text{fluctuation–dissipation}$$

Refs: Crossley–Glorioso–Liu arXiv:1511.03646, 1701.07817 · Glorioso–Liu arXiv:1612.07705

Background conservation from the noise-field variation

Action at first order in the noise fields (R/A notation)

$$I_d^{(1,0,0)} = -\int d^4x \left[(\rho - \mu n)\dot{\pi}_{A0} + P\,\partial_i \pi_{Ai} - n\,\dot{\theta}_A\right]$$

vary w.r.t. the noise fields

Background conservation

$$\dot{\rho} = 0, \qquad \dot{n} = 0$$

Keep the derivative expansion when eliminating constraints

From the $O(\pi_A, \pi_R)$ terms

The quadratic action gives the linearized equations of motion — but not all of them are independent evolution equations.

$$\frac{\delta I}{\delta \pi_A^0} = 0, \qquad \frac{\delta I}{\delta \theta_A} = 0$$

energy & charge conservation constraints

Landau frame stays local

Solving the constraints generally produces non-local expressions in a generic hydrodynamic frame, even starting from local first-order relations.

In the Landau frame the constraints stay local in the derivative expansion.

Solve perturbatively, not exactly

After eliminating constraints the dispersion relation may formally contain higher powers of ω — but solving it exactly in a truncated EFT is not justified.

$$\omega = \omega_0 + \varepsilon\, \omega_1 + O(\varepsilon^2)$$

Non-hydrodynamic roots never appear at any order of the derivative expansion.

Validity of the fluid Stückelberg expansion (nonlinear realization):

$$\frac{\partial}{M_{SSB}} \ll 1, \qquad M_{SSB} \sim (\rho + P)^{1/4}$$

Refs: Ota arXiv:2410.23837 · Abbasi–Rischke arXiv:2410.07929

Start from the most general charged first-order fluid

Most general U(1)-charged first-order constitutive tensors

$$T^{\mu\nu} = \left(\rho + \rho^{(1)}\right)u^\mu u^\nu + \left(P + P^{(1)}\right)\gamma^{\mu\nu} + q^\sigma u^{(\mu}\gamma^{\nu)}{}_\sigma - \Sigma^{\mu\nu}$$

$$J^\mu = \left(n + n^{(1)}\right)u^\mu + \gamma^{\mu\nu} j_\nu$$

Scalar sector — all first-order coefficients retained

$$\rho^{(1)} = \beta^{-1}\epsilon\, u^\rho \nabla_\rho \beta - \lambda_1 \nabla_\rho u^\rho - \nu_4 u^\rho \nabla_\rho (\beta\mu)$$

$$P^{(1)} = \beta^{-1}\lambda_2\, u^\rho \nabla_\rho \beta - \zeta \nabla_\rho u^\rho - \nu_5 u^\rho \nabla_\rho (\beta\mu)$$

$$n^{(1)} = \nu_1 u^\rho \nabla_\rho \beta - \beta\nu_2 \nabla_\rho u^\rho - \beta\nu_3 u^\rho \nabla_\rho (\beta\mu)$$

Vector & tensor sector

$$q^\mu = \beta^{-1}\kappa_1 \nabla^\perp_\mu \beta - \kappa_2 u^\rho \nabla_\rho u_\mu - \tau_3 \nabla^\perp_\mu (\beta\mu) - \beta\tau_4 u^\rho F_{\rho\mu}$$

$$j^\mu = \tau_1 \nabla^\perp_\mu \beta - \beta\tau_2 u^\rho \nabla_\rho u_\mu - \beta\chi_1 \nabla^\perp_\mu (\beta\mu) - \beta^2 \chi_2 u^\rho F_{\rho\mu}$$

$$\Sigma^{\mu\nu} = 2\eta\, \sigma^{\mu\nu}, \qquad \nabla^\perp_\mu \equiv \gamma_\mu{}^\nu \nabla_\nu$$

No specialization to Landau frame · KMS not yet imposed

Longitudinal sector

Eliminating the constraints gives, formally:

$$\sum_{m=0}^{6} d_m\, \omega^m = 0$$

Solve along the derivative expansion:

Leading order:

$$\omega_0 = \pm c_{s/n}\, k, \qquad c^2_{s/n} = \left(\frac{\partial P}{\partial \rho}\right)_{s/n}$$

Frame-dependent inputs → a pole built from invariants only

At first sight — many frame-dependent coefficients:

$$\omega_1 = -ik^2\, F\big(\epsilon, \lambda_{1,2}, \zeta, \nu_{1\ldots5}, \kappa_{1,2}, \tau_{1\ldots4}, \chi_{1,2}, \eta\big)$$

But it collapses to:

$$\boxed{\omega_1 = -i\,\Gamma_L\, k^2}$$

Longitudinal sound attenuation

$$\Gamma_L = \frac{1}{2(\rho+P)}\left[\frac{4}{3}\eta + \tilde{\zeta} + \left(\frac{\partial P}{\partial \rho}\right)_n \tilde{\lambda} + \left(\frac{\partial P}{\partial n}\right)_\rho \tilde{\nu} + f^2 \left(\frac{\partial P}{\partial \rho}\right)_{s/n}\left(\tilde{\chi}_1 - \frac{n\,\tilde{\tau}_2}{\beta(\rho+P)}\right) + \beta\left(\frac{\partial P}{\partial n}\right)_\rho (\tilde{\tau}_1 - \tilde{\tau}_2)\right]$$

Every quantity appearing is a frame invariant

$$\tilde{\lambda} = \lambda_2 - \left(\frac{\partial P}{\partial \rho}\right)_n \epsilon - \left(\frac{\partial P}{\partial n}\right)_\rho \nu_1$$

$$\tilde{\zeta} = \zeta - \left(\frac{\partial P}{\partial \rho}\right)_n \lambda_1 - \left(\frac{\partial P}{\partial n}\right)_\rho \nu_2$$

$$\tilde{\nu} = \nu_5 - \left(\frac{\partial P}{\partial \rho}\right)_n \nu_4 - \left(\frac{\partial P}{\partial n}\right)_\rho \nu_3$$

$$\tilde{\tau}_1 = \tau_1 - \frac{n}{\beta(\rho+P)}\kappa_1, \qquad \tilde{\tau}_2 = \tau_2 - \frac{n}{\beta(\rho+P)}\kappa_2$$

$$\tilde{\chi}_1 = \chi_1 - \frac{n}{\beta(\rho+P)}\tau_3$$

Even starting from a generic frame-dependent constitutive theory, the pole is expressible through frame invariants alone.

KMS symmetry and unitarity force stability

Conditions from KMS + unitarity

$$\tilde{\tau}_1 = \tilde{\tau}_2, \qquad \eta \geq 0$$

$$\tilde{\zeta} + \left(\frac{\partial P}{\partial \rho}\right)_n \tilde{\lambda} + \left(\frac{\partial P}{\partial n}\right)_\rho \tilde{\nu} \geq 0$$

$$\tilde{\chi}_1 - \frac{n}{\beta(\rho+P)} \tilde{\tau}_2 \geq 0$$

plus $c_s^2 \geq 0$ *and* $f^2 > 0$

The consequence

$\rho + P > 0$

physical positivity of enthalpy

$\Gamma_L \geq 0$

attenuation non-negative

$\mathrm{Im}\,\omega \leq 0$

every mode decays — stable

$$\rho + P > 0 \implies \Gamma_L \geq 0 \implies \mathrm{Im}\,\omega \leq 0$$

Fluid modes realize boosts non-linearly

Unlike a linear representation

For a mode that sits in a linear Lorentz representation (e.g. the photon), different inertial frames are related simply by boosting the wave-vector:

$$k_\mu \longrightarrow k'_\mu = \Lambda_\mu{}^\nu\, k_\nu$$

Boosting k alone is not enough

Because fluid modes realize boosts non-linearly, a linear boost of the wave-vector does not identify the same dynamical degree of freedom.

Never identify a non-linear fluid mode by a linear boost of k alone.

The non-linear transformation

Under an infinitesimal t–x boost:

$$\xi^\mu = v\,(x\,\delta^\mu_0 + t\,\delta^\mu_1)$$

$$\tilde{\pi}^0 = \pi^0 + vx, \qquad \tilde{\pi}^1 = \pi^1 + vt$$

$$\dot{\tilde{\pi}}^0 = \dot{\pi}^0, \qquad \dot{\tilde{\pi}}^1 = \dot{\pi}^1 + v$$

Sound mode = density wave in the local rest frame

Using the original π^m in a comoving frame mixes the density wave with mass transport by the mean flow — a different d.o.f.

Apparent comoving-frame instability is not an instability of the IR theory.

First-order hydrodynamics is sound as an IR effective theory

Frame-independent pole

The physical IR pole does not depend on the hydrodynamic frame.

KMS + unitarity ⇒ stable

Under KMS symmetry and unitarity, $\rho + P > 0$ implies $\mathrm{Im}\,\omega \leq 0$.

Inertial-frame independent

Realizing Lorentz boosts non-linearly makes mode identification frame-independent too.

$$\rho + P > 0 \implies \mathrm{Im}\,\omega \leq 0$$

For simulation: this is separate from the PDE question — MIS / BDNK completions remain useful in practice.

As an IR EFT: sufficient for linear response, fluctuations, loops and renormalization — no need to extend to second order.

Refs: Ota arXiv:2410.23837

Where this goes next

1 · Curved spacetime & other symmetry backgrounds

Minkowski → curved / time-dependent backgrounds

- Couple fluid modes to gravitational perturbations
- Extend to FLRW, Einstein static universe, black-hole backgrounds
- NG modes and stability under different spacetime symmetries
- Effect of background curvature and time-dependence on the IR pole

2 · Non-linear fluctuations & loop corrections

quadratic action → non-linear SK effective action

- Corrections from fluid modes running in loops
- Infrared renormalization of transport coefficients
- Long-time tails and non-analytic response
- Hydrodynamic-frame-invariant correlators and effective action

Next goal: *a frame-independent, non-linear, quantum effective theory of fluids.*