Free mutual information and higher-point OTOCs

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2509.13406 with Jinzhao Wang

Background and Motivations

$\rightarrow$ Schwinger-Keldysh effective field theory and quantum information theory are complementary tools to probe strongly interacting many-body dynamics.

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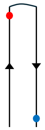

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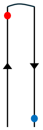

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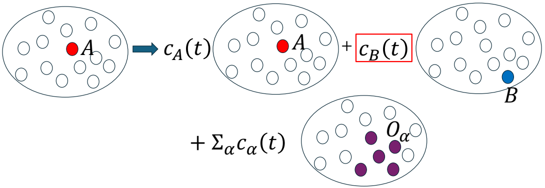

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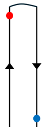

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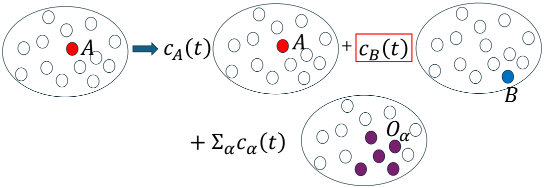

→ This talk: new concept from QI that explains physical interpretation of observables associated with arbitrary number of contours.

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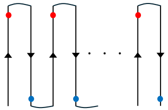

?

OTOCs and operator size

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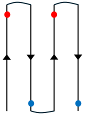

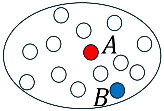

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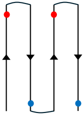

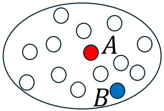

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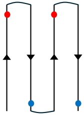

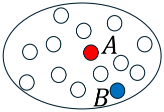

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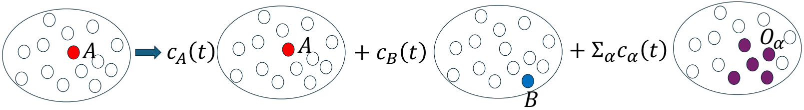

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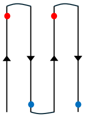

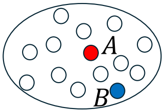

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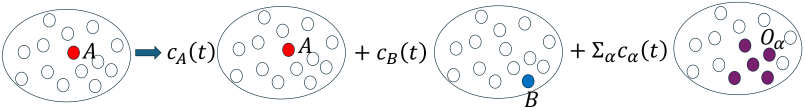

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Roberts-Stanford-Streicher '18

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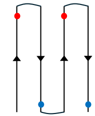

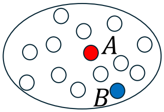

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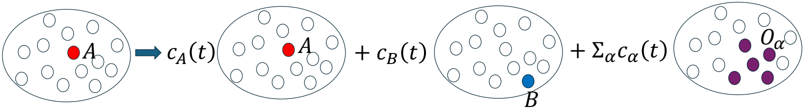

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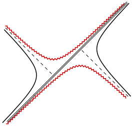
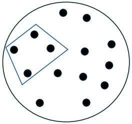
- Chaos leads to universal growth of operator size, and decay of OTOCs.

Various approaches have helped understand universal behavior of OTOCs:

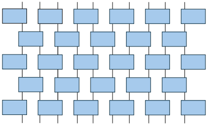
- Explicit computations in solvable models:



AdS/CFT



SYK model



Random unitary circuits

Shenker-Stanford '13, Kitaev '14
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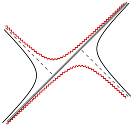

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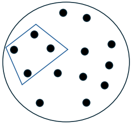

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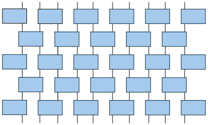

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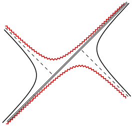
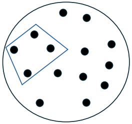
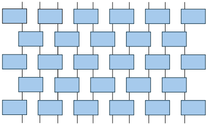

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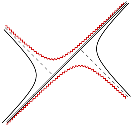

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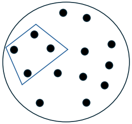

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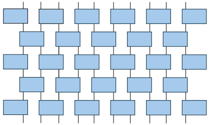

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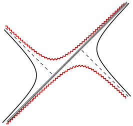

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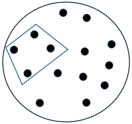

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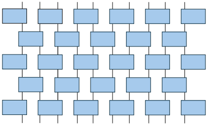

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  → generic chaotic systems with conservation laws. Mishra-Wang-Pappalardi-Delacretaz '25

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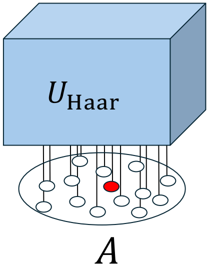

$$\langle (U^\dagger A U) B (U^\dagger A U) A \rangle_{\mathrm{Haar}}$$

$$= \langle AA \rangle \langle B \rangle \langle B \rangle + \langle A \rangle \langle A \rangle \langle BB \rangle$$
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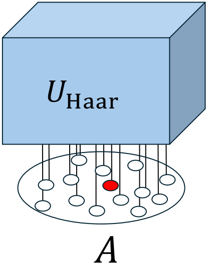

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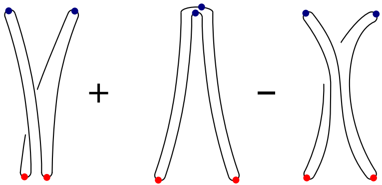
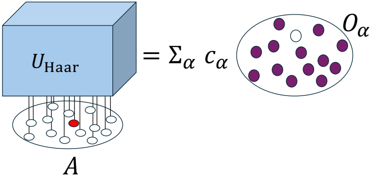
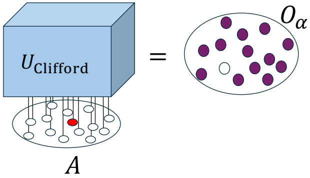
Haar-random value of OTOC can be successfully reproduced by "scramblon EFT."

Stanford-Yang-Yao '21



- OTOCs do not capture the full story about chaos and operator spreading.
- Random Clifford unitaries lead to much simpler operator evolution, but have identical behavior of OTOCs to Haar-random unitaries.





$$\langle (U^\dagger A U) B (U^\dagger A U) A \rangle_{\text{Haar}} = \langle (U^\dagger A U) B (U^\dagger A U) A \rangle_{\text{Clifford}}$$

Higher-point OTOCs

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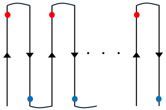

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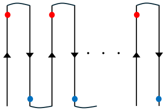

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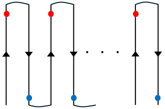
$$\langle A(t)B\,A(t)B\ \ldots A(t)B\rangle =$$ 

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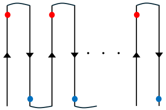

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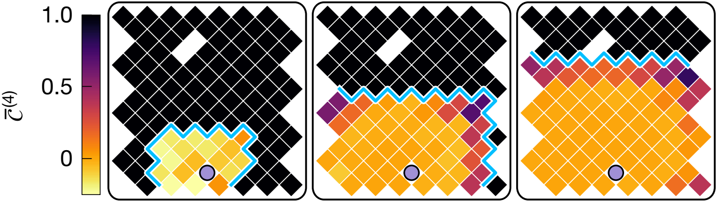
- Freely independent variables are maximally non-commuting, indicating complete lack of correlation between $UAU^\dagger$ and B.

Terence Tao, *Topics in Random Matrix Theory*

Higher-point OTOCs are not merely abstract mathematical quantities.

Recently measured in an experiment on Google's quantum processor:



Google Quantum AI '25

Drawbacks of higher-point OTOCs

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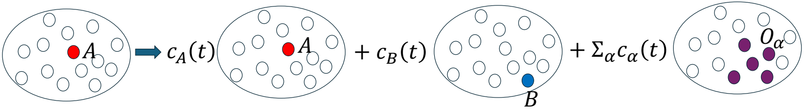

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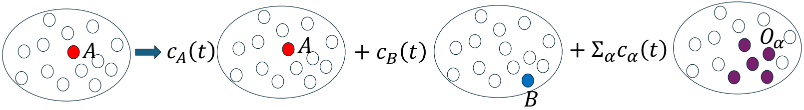

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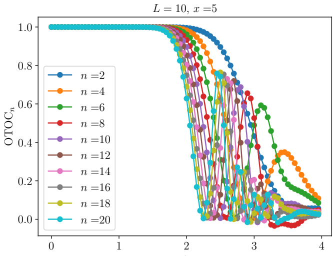
- For increasing number of contours, they seem to show increasingly erratic and non-universal time-evolution. For example, in a chaotic spin chain,



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We will introduce a quantity called the free mutual information (FMI), which satisfies each of the above conditions.

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We want a measure of quantum many-body chaos that:

- Similar to higher-point OTOCs, is sensitive to more fine-grained features of Heisenberg evolution than growth of operator size.
- Similar to higher-point OTOCs, captures the approach to "freeness" of $A(t)$ and B.
- Unlike higher-point OTOCs, has a good physical interpretation in terms of chaos and ergodicity.
- Unlike higher-point OTOCs, shows a simple, universal behavior in its time-evolution.

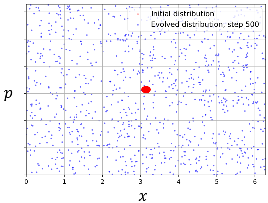
We will introduce a quantity called the free mutual information (FMI), which satisfies each of the above conditions.

We will further show how this quantity has a precise relation to a sum over all higher-point OTOCs, which captures their universal physical content.

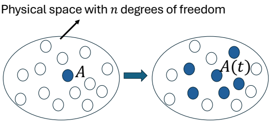
FMI as a measure of spreading in operator space

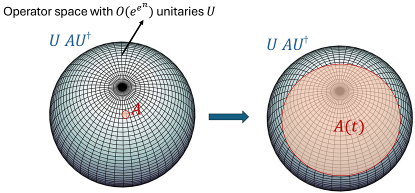
In classical chaos, ergodicity is defined as spreading of classical phase space distributions.

No direct analog in generic quantum many-body systems.



- Spreading of operators in the space of degrees of freedom can be seen as one generalization to quantum chaos, captured by OTOCs.
- The FMI captures a different notion of spreading in the abstract space of all possible time-evolved operators.





Plan

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How much does $A(t) = e^{iHt} A e^{-iHt}$ spread out with time in the space of all possible time-evolved operators?

Need to clarify:

1. What we mean by "the space of all possible time-evolved operators."

2. What it means for $A(t)$ to spread out in this space.

Space of all possible time-evolved operators

- For a given initial operator A in a d-dimensional Hilbert space, consider the set of all operators that can be obtained by any unitary evolution:

$$\mathcal{S}_A = \{UAU^\dagger \text{ for all } U \in \mathbf{U}(d)\}$$

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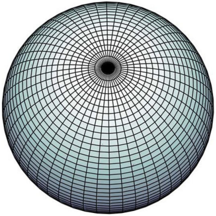
- $\mathbf{U}(d)$ is a compact manifold of dimension d^2.
- The manifold $\mathcal{S}_A$ has slightly reduced dimension, which is still $O(d^2)$.
- We can associate a natural volume measure with $\mathcal{S}_A$ and its subsets.

$$\text{Vol}(\mathcal{S}_A) = \int_{\mathbf{U}(d)/\text{Stab}_{\mathbf{U}(d)}(A)} dU$$

- To better understand the size of the set $\mathcal{S}_A$, we can discretize it with an ϵ-net $\mathcal{S}_{A,\epsilon}$:

$$n_{\mathcal{S}_{A,\epsilon}} \propto \left(\frac{1}{\epsilon}\right)^{d^2}$$



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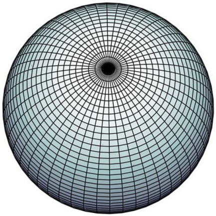

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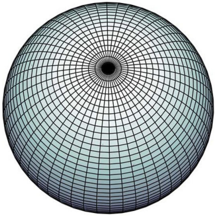

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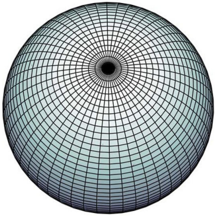

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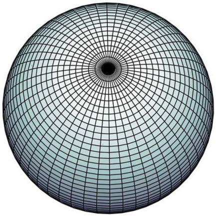

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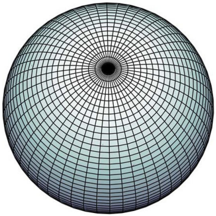

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- Need to coarse-grain $A(t)$ in some way to obtain a larger set.

Coarse-graining $A(t)$ through expectation values

- We will define our notion of coarse-graining using some reference operator B (some simple known operator) and some reference state ρ.

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- Now define $\mathcal{S}_{A(t)|B}$ as the set of all operators $\tilde{A} \in \mathcal{S}_A$ such that joint moments of $\tilde{A}$ and B are the same as joint moments of $A(t)$ and B:

$$\mathrm{Tr}[\rho A(t)^{m_1} B^{n_1} A(t)^{m_2} B^{n_2} \ldots] \approx \mathrm{Tr}[\rho \tilde{A}^{m_1} B^{n_1} \tilde{A}^{m_2} \tilde{B^{n_2}} \ldots]$$

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- Similar idea to coarse-graining procedure in "Jaynes entropy."

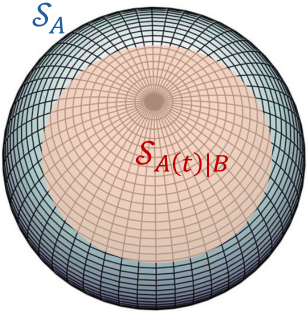
- $\mathcal{S}_{A(t)|B}$ is a set of coarse-grained operators for $A(t)$ with respect to B (we will set $\rho = \mathbf{1}/d$).

- We can now precisely state our question about spreading of $A(t)$ in $\mathcal{S}_A$:

  Does the ratio of volumes

$$f_{A(t)} = \frac{\mathrm{Vol}(\mathcal{S}_{A(t)|B})}{\mathrm{Vol}(\mathcal{S}_A)}$$

  grow with time, and approach 1 at late times in chaotic systems?



- The free mutual information can be defined in terms of this volume ratio:

$$I_{\text{free}}(A(t):B) = -\frac{4}{d^2}\log\left(\frac{\text{Vol}(\mathcal{S}_{A(t)|B})}{\text{Vol}(\mathcal{S}_A)}\right)$$

$$= \lim_{\epsilon\to 0} \log n_{\mathcal{S}_{A,\epsilon}} - \log n_{\mathcal{S}_{A(t)|B,\epsilon}}$$

$\log n_{\mathcal{S}_{A,\epsilon}}$ → free entropy of $A(t)$

$\log n_{\mathcal{S}_{A(t)|B,\epsilon}}$ → conditional free entropy of $A(t)$, given the set of joint moments with B

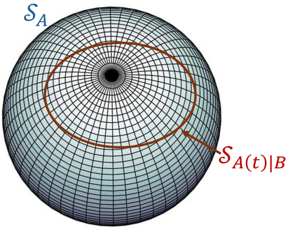
- Increase of volume fraction from 0 to 1 $\leftrightarrow$ Decrease of FMI from ∞ to 0.

- Closely related to free mutual information in mathematics literature Voiculesciu, Hiai, Petz, but modified to physics setup.

Two extreme limits

- If $A(t)$ and B commute,

$$\frac{\mathrm{Vol}(\mathcal{S}_{A(t)|B})}{\mathrm{Vol}(\mathcal{S}_A)} = 0 \text{ and } I_{\mathrm{free}} = \infty.$$



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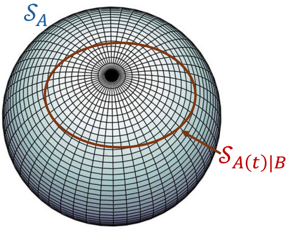
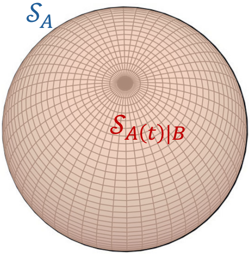


- Take the time-evolution operator to be a typical instance of a Haar-random unitary from $\mathbf{U}(d)$.

  In the limit $d \to \infty$,

$$I_{\mathrm{free}}(A : U_H A U_H^\dagger) = 0\,.$$

  A and $U_H A U_H^\dagger$ are "freely independent" in this limit.



Expectations for FMI in chaotic systems

- Haar-random unitaries in the large d limit are often used as the simplest models for chaotic dynamics at late times.

- Consider a generic chaotic many-body system, where A and B commute at $t = 0$, so the FMI is ∞ at $t = 0$.

- Does the FMI show universal decay with time to a small late-time value?

  We need an explicit formula to address this question.

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Free mutual information for Pauli strings

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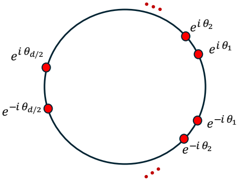
- Say the eigenvalues $A(t)B$ are $\{e^{\pm i\theta_j}\}_{j=1}^{d/2}$.



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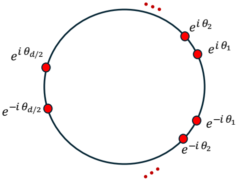

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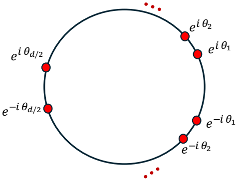

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→ $I_{\text{free}} = \infty$ if there are degeneracies in spectrum of $AB(t)$, and finite otherwise.

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Starting from the formula for I_{free} and using resolvent techniques, we find that for Pauli operators A and B:

$$\boxed{I_{\mathrm{free}}(A(t) : B) = \sum_{n=1}^{\infty} \frac{2}{n} \mathrm{OTOC}_n(t)^2} \qquad \mathrm{OTOC}_n(t) \equiv \frac{1}{d} \mathrm{Tr}[(A(t)B)^n]$$

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Comments:

- I_{free} has a clearer physical interpretation than any given OTOC_n for $n > 2$, and helps interpret the latter.

- OTOC_n are often easier to calculate than the spectrum of $AB(t)$, including in holographic CFTs Haehl-Rozali '17, and can be experimentally measured Google Quantum AI '25.

  General relation allows us to infer FMI from the OTOCs.

Questions about time-evolution

$$\boxed{I_{\mathrm{free}}(A(t):B) = \sum_{p=1}^{\infty} \frac{2}{n} \mathrm{OTOC}_n(t)^2}$$

- At $t = 0$, if A and B commute, RHS diverges as each $\mathrm{OTOC}_n = 1$.
- On what time scale does I_{free} start to converge, and how is this related to time scales for decay of OTOC_n? What is the n-dependence of the decay time scale?

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Models for chaotic systems:

- Random GUE Hamiltonians. ($\overline{\mathrm{OTOC}_n}$ are analytically tractable. No locality.)
- Random unitary circuits. (Partial analytic control over OTOC_n.)
- Chaotic spin chains. (Both locality and energy conservation, FMI and OTOC_n can be accessed numerically.)

Overall conclusions from chaotic systems:

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Overall conclusions from chaotic systems:

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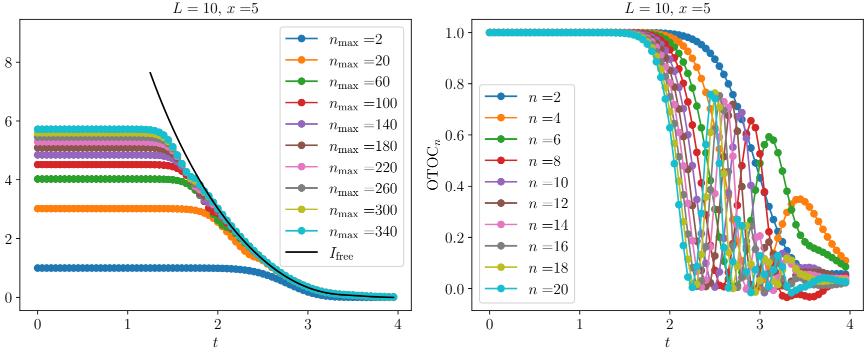
2. At intermediate times, $\mathrm{OTOC}_n(t)$ for $n > 2$ show oscillations as functions of both t and n.
3. The FMI, and the partial sums $\sum_{n=1}^{n_{\max}} \mathrm{OTOC}_n(t)^2$, show monotonic decay with t.

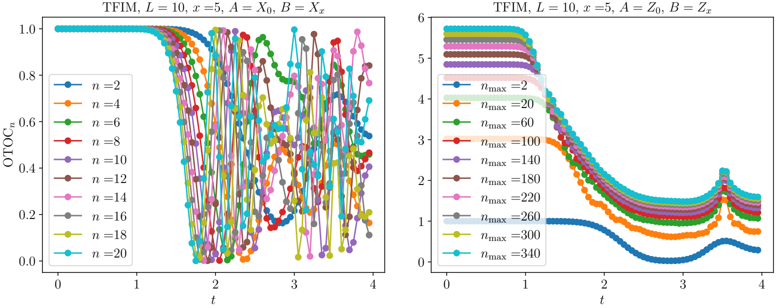
Chaotic spin chain

$$H = \sum_i (Z_i Z_{i+1} + gX_i + hZ_i), \quad g = 0.5, \quad h = -1.05$$



Transverse field Ising model

- Using the mapping to free fermions, we can show that for any t and any Pauli operators A, B, $A(t)B$ has only two distinct eigenvalues.
- Due to large degeneracy, $I_{\text{free}}(t) = \infty$ for all times.



Summary and future directions

- We interpreted the free mutual information as a natural measure of spreading in operator space.
- Gave an explicit formula for two Pauli operators, and related it to a sum over higher-point OTOCs.
- Saw convergence and monotonic decay of the OTOC partial sums and FMI in various chaotic systems. No decay in Cliffords, PFC, free fermions.

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Questions

- Developing finite temperature version of the free mutual information
- Effects of energy conservation and other conservation laws
- Understanding behavior of OTOC sum in holography and other exactly solvable models, and developing EFT approaches.

Thank you!

Comments on physical cutoffs

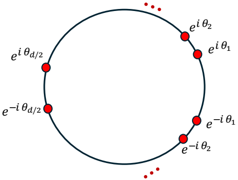
$$I_{\text{free}}(A(t):B) = -\frac{4}{d^2}\log\left(\frac{\text{Vol}(\mathcal{S}_{A(t)|B})}{\text{Vol}(\mathcal{S}_A)}\right)$$

- Closely related to free mutual information in mathematics literature.
  Voiculesciu, Hiai, Petz
- In the physical setup with large but finite t and fixed time-evolution operator, we need to introduce certain cutoffs:

$$|\mathrm{Tr}[\rho A(t)^{m_1} B^{n_1} A(t)^{m_2} B^{n_2} ...] - \mathrm{Tr}[\rho \tilde{A}^{m_1} B^{n_1} \tilde{A}^{m_2} \tilde{B}^{n_2} ...]| < \delta,$$
$$\sum_i m_i + \sum_i n_i < N$$

For large d, we can choose N large enough, δ small enough, such that at leading order, FMI is independent of N, δ.

- Say the eigenvalues $A(t)B$ are $\{e^{\pm i\theta_j}\}_{j=1}^{d/2}$.



Since we use the Haar measure for volumes in $\mathcal{S}_A$:

$$\frac{\mathrm{Vol}(\mathcal{S}_{A(t)|B})}{\mathrm{Vol}(\mathcal{S}_A)} = \underset{U}{\text{Prob}_{\text{Haar-random}}}\left(\text{eigenvalues of } UAU^\dagger B \text{ are } \{e^{\pm i\theta_j}\}\right)$$

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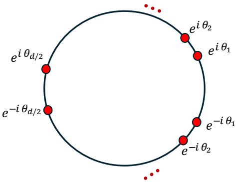

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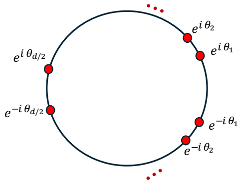

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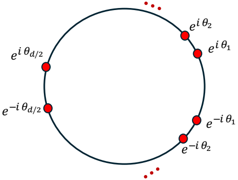
- Probability distribution for eigenvalues $\propto$ Vandermonde determinant.
- This leads to

$$I_{\mathrm{free}}(A(t):B) = -\frac{4}{d^2} \sum_{1\leq i<j\leq d/2} \log|\cos\theta_i - \cos\theta_j| - \log 2 + \mathcal{O}\left(\frac{1}{d^z}\right)$$

- Say the eigenvalues $A(t)B$ are $\{e^{\pm i\theta_j}\}_{j=1}^{d/2}$.



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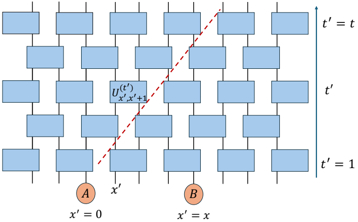
$\rightarrow$ $I_{\mathrm{free}} = \infty$ if there are degeneracies in spectrum of $AB(t)$, and finite otherwise.

Random unitary circuits



- Information travels at characteristic velocity v_B determined by local Hilbert space dimension q.
- For $v \lesssim v_B$, it is well-known that Nahum-Vijay-Haah, von Keyserlingk-Rakovszky-Pollmann-Sondhi

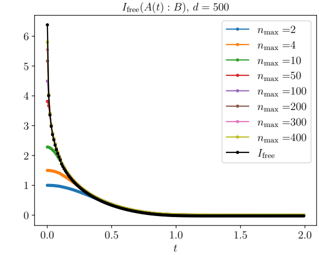
$$\mathrm{OTOC}_2(t, v) \approx e^{-\frac{(v-v_B)^2}{2D}t}$$

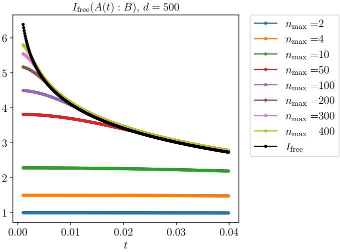
- With many assumptions about membrane picture for higher n and some combinatorial facts, we conjecture at late times and $v \lesssim v_B$,

$$\mathrm{OTOC}_n(t, v) \approx e^{-n\frac{(v-v_B)^2}{2D}t}, \quad n \ \ \mathrm{even}$$

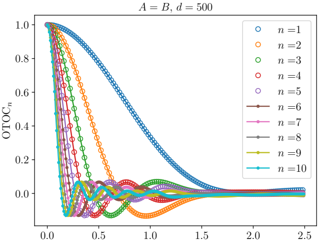
Random GUE Hamiltonians

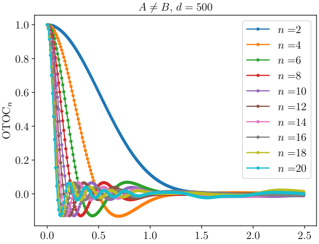
Coulomb gas formula and partial sums





Oscillations in individual OTOC_n





Even though the OTOC_n are analytically tractable, we get increasingly complicated expressions for higher n: (here $\eta(t) \equiv J_1(2t)/t$)

n	**Even NC Partitions** π	$\overline{\mathrm{OTOC}_n(t)}$
1	$(1,2)$, $c=1$, $\|\pi^c\| = 1^2$	$\eta(t)^2$
2	$(12)(34)$, $c=2$, $\|\pi^c\| = 1^2 2^1$ (4321), $c=1$, $\|\pi^c\| = 1^4$	$2\eta(t)^2\eta(2t) - \eta(t)^4$
3	$(12)(34)(56)$, $c=2$, $\|\pi^c\| = 1^3 3^1$ $(12)(36)(45)$, $c=3$, $\|\pi^c\| = 1^2 2^2$ $(12)(3456)$, $c=6$, $\|\pi^c\| = 1^4 2^1$ (123456), $c=1$, $\|\pi^c\| = 1^6$	$2\eta(t)^3\eta(3t) + 3\eta(t)^2\eta(2t)^2 -$ $6\eta(t)^4\eta(2t) + 2\eta(t)^6$
4	$(12)(34)(56)(78)$, $c=2$, $\|\pi^c\| = 1^4 4^1$ $(12)(34)(58)(67)$, $c=8$, $\|\pi^c\| = 1^3 2^1 3^1$ $(14)(23)(58)(67)$, $c=4$, $\|\pi^c\| = 1^2 2^3$ $(1234)(56)(78)$, $c=4$, $\|\pi^c\| = 1^5 3^1$ $(1234)(58)(67)$, $c=4$, $\|\pi^c\| = 1^4 2^2$ $(1256)(34)(78)$, $c=8$, $\|\pi^c\| = 1^4 2^2$ $(1234)(5678)$, $c=4$, $\|\pi^c\| = 1^6 2^1$ $(12)(345678)$, $c=8$, $\|\pi^c\| = 1^6 2^1$ (12345678), $c=1$, $\|\pi^c\| = 1^8$	$2\eta(t)^4\eta(4t) + 8\eta(t)^3\eta(2t)\eta(3t) +$ $4\eta(t)^2\eta(2t)^3 - 8\eta(t)^5\eta(3t) -$ $20\eta(t)^4\eta(2t)^2 + 20\eta(t)^6\eta(2t) - 5\eta(t)^8$
5	$(12)(34)(56)(78)(910)$, $c=2$, $\|\pi^c\| = 1^5 5^1$ $(12)(34)(56)(710)(89)$, $c=10$, $\|\pi^c\| = 1^4 2^1 4^1$ $(12)(34)(510)(67)(89)$, $c=5$, $\|\pi^c\| = 1^4 3^2$ $(12)(34)(510)(69)(78)$, $c=20$, $\|\pi^c\| = 1^3 2^2 3^1$ $(12)(310)(49)(58)(67)$, $c=5$, $\|\pi^c\| = 1^2 2^4$ $(1234)(510)(69)(78)$, $c=50$, $\|\pi^c\| = 1^4 2^3$ $(1234)(56)(710)(89)$, $c=60$, $\|\pi^c\| = 1^5 2^1 3^1$ $(1234)(56)(78)(910)$, $c=10$, $\|\pi^c\| = 1^6 4^1$ $(1234)(56710)(89)$, $c=35$, $\|\pi^c\| = 1^6 2^2$ $(1234)(5678)(910)$, $c=10$, $\|\pi^c\| = 1^7 3^1$ $(123456)(710)(89)$, $c=35$, $\|\pi^c\| = 1^6 2^2$ $(123456)(78)(910)$, $c=10$, $\|\pi^c\| = 1^7 3^1$ $(1234)(5678910)$, $c=10$, $\|\pi^c\| = 1^8 2^1$ $(12)(345678910)$, $c=10$, $\|\pi^c\| = 1^8 2^1$ (12345678910), $c=1$, $\|\pi^c\| = 1^{10}$	$2\eta(t)^5\eta(5t) + 5\eta(t)^2\eta(2t)^4 +$ $5\eta(t)^4\eta(3t)^2 + 10\eta(t)^4\eta(2t)\eta(4t) +$ $20\eta(t)^3\eta(2t)^2\eta(3t) + 14\eta(t)^{10} -$ $70\eta(t)^8\eta(2t) + 105\eta(t)^6\eta(2t)^2 -$ $50\eta(t)^4\eta(2t)^3 + 30\eta(t)^7\eta(3t) -$ $60\eta(t)^5\eta(2t)\eta(3t) - 10\eta(t)^6\eta(4t)$

Non-chaotic systems: Clifford unitaries and PFC ensemble

- Random unitaries from the Clifford group U_C send initial Pauli strings to Pauli strings under time-evolution, but increase their size.

$$X \otimes I \otimes Z \otimes Y \otimes X \otimes Z \otimes Z \otimes Y \otimes X$$

$$\uparrow$$

$$I \otimes I \otimes I \otimes I \otimes X \otimes I \otimes I \otimes I \otimes I$$

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- The behaviour of I_{free} and higher-point OTOCs under random cliffords is strikingly different from Haar-random unitaries:

$$I_{\mathrm{free}}(U_C A U_C^\dagger : B) = \infty$$

$$\frac{1}{d}\overline{\mathrm{Tr}[(A U_C B U_C^\dagger)^{4n}]} = 1 \text{ for all integer } n$$

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- By adding a random permutation and random phase, we get the more powerful PFC ensemble, which forms *approximate* t-designs for higher t Metger et. al. '24. We can still find A, B for which $I_{\mathrm{free}}(t) = \infty$.