Self-consistent computation of pair production from non-relativistic effective field theories in the Schwinger-Keldysh formalism

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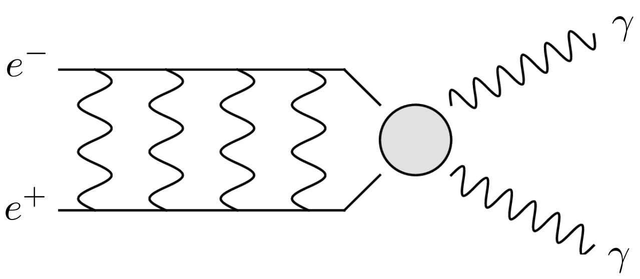
@ Contours 2026 - Effective Field Theories Meet the Schwinger–Keldysh Formalism

Positronium example

Bound-state decay:

$$\Gamma_n = (\sigma v)_0 \times |\psi_n(r=0)|^2$$

Pirenne & Wheeler 1946



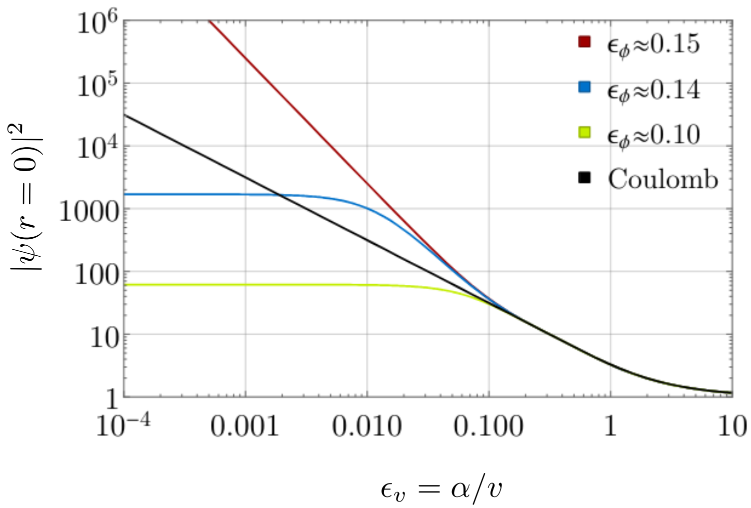
Sommerfeld enhancement:

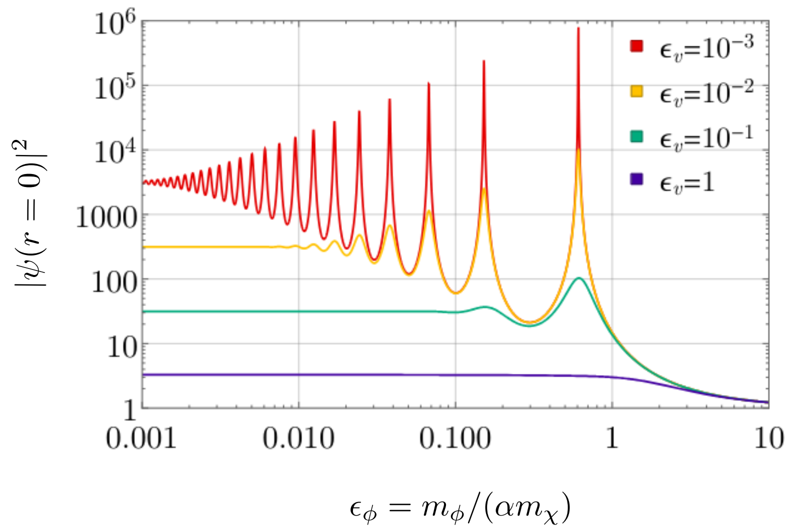
$$(\sigma v) = (\sigma v)_0 \times |\psi_v(r=0)|^2$$
$$\propto (\sigma v)_0 \, (\alpha/v)\,, \text{ for } v \ll \alpha.$$

Sakharov 1948
(Sommerfeld 1931)

Resonances in Sommerfeld enhanced annihilation

Instead of a Coulomb, consider a Yukawa potential: $V(r) = -\frac{\alpha}{r} e^{-m_\phi r}$





Close to such resonances (nearly zero-energy bound states),
the Sommerfeld-enhanced annihilation cross section $(\sigma v) = (\sigma v)_0 |\psi(r=0)|^2$
can exceed the partial-wave unitarity bound for 2→ 2 inelastic processes:

$$(\sigma v) \leq (\sigma v)_{\rm uni} = \frac{4\pi}{m_\chi^2 v} \text{ (for s-wave)}$$

Self-consistent Sommerfeld enhancement

[Blum, Ryosuke and Slatyer, 2016, “Self-consistent Calculation of the Sommerfeld Enhancement“]

$$\left[E + \frac{\Delta_{\mathbf{r}}}{m} - V(r) + ic\delta(\mathbf{r})\right]\psi(\mathbf{r}) = 0$$

Perturbative treatment (Born approx.)

$\langle\psi_0|\, 2\Im V\, |\psi_0\rangle = (\sigma v)_0 |\psi_0(0)|^2$

violates partial-wave unitarity at resonances.

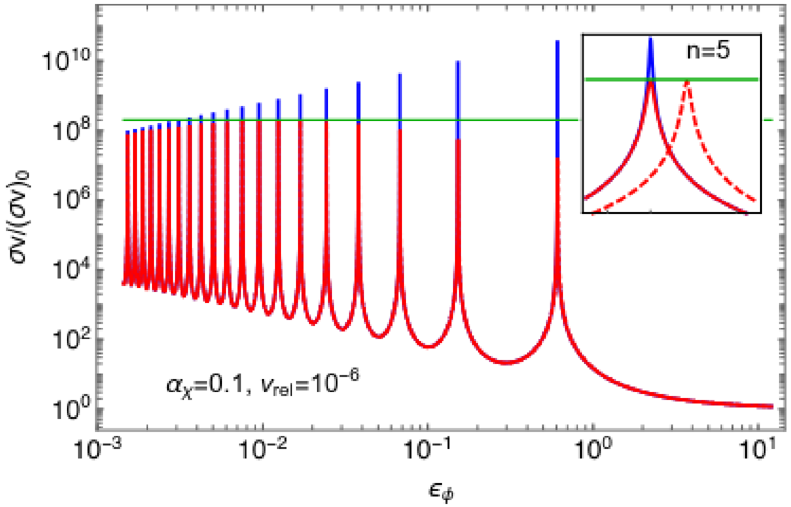
Use Lippmann-Schwinger equation,

$$\begin{aligned}\psi(\mathbf{r}) &= \psi_0(\mathbf{r}) - \int \mathrm{d}^3 r' G(E;\mathbf{r},\mathbf{r}')c\delta^3(\mathbf{r}')\psi(\mathbf{r}') \\ &= \psi_0(\mathbf{r}) - cG(E;\mathbf{r},0)\psi(0)\ ,\end{aligned}$$

to obtain **„self-consistent“** solution:

$$|\psi(0)|^2 = \frac{|\psi_0(0)|^2}{|1 + cG(E;0,0)|^2} \approx \frac{|\psi_0(0)|^2}{\left(1 + \frac{(\sigma v)_0 |\psi_0(0)|^2}{(\sigma v)_{\mathrm{uni}}}\right)^2}$$

Consistent with unitarity bound:



Motivation

- Unitarization of Sommerfeld effect in the in-in formalism
- Self-consistent computation of pair annihilation *and* creation
  - Reverse process is temperature dependent, allows for equilibration
- Bound states
  - on-shell or Breit-Wigner energy profile?

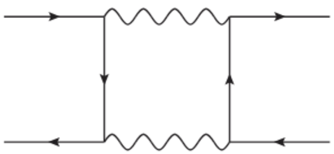
→ **NR EFTs in Schwinger-Keldysh.**

NR and pNRQED in vacuum

[Caswell & Lepage 1986, Pineda & Soto 1997, Brambilla et al. 1999]

Hierarchy of scales: $m \gg \alpha m \gg \alpha^2 m$

Integrate out hard and soft scale: $\mathcal{L} \overset{m}{\rightarrow} \mathcal{L}^{\rm NR} \overset{\alpha m}{\rightarrow} \mathcal{L}^{\rm pNR}$



$$\mathcal{L}^{\rm NR} = \eta^\dagger \left[iD_0 + \frac{\Delta}{2m} + ... \right] \eta + \xi^\dagger \left[iD_0 + \frac{\Delta}{2m} + ... \right] \xi + ic\eta^\dagger \xi \xi^\dagger \eta + ... \,.$$

Projecting NREFT into two-particle subspace + multipole expansion of gauge field lieds to pNREFT:

$$\mathcal{L}^{\rm pNR} = \int {\rm d}^3 r \; S^\dagger(\mathbf{x}, \mathbf{r}, t) \left[i\partial_t + \frac{\nabla_{\mathbf{x}}^2}{4m} + \frac{\nabla_{\mathbf{r}}^2}{m} - V(r) + \; \mathbf{r} \cdot g\mathbf{E}(\mathbf{x}, t) \right] S(\mathbf{x}, \mathbf{r}, t)$$

Remark: To get the right EFT at finite T, start from the **full** Lagrangian on the CTP contour and rederive matching coefficients in the respective EFT.

Outline

- Basic idea
  - From NREFT on CTP derive **number density equation**
  - Coupled system of 2- and 4-point functions due to annihilation
  - Solve **4-point functions** „self-consistently“
- Simplifications
  - Neglect ultrasoft
  - No Hubble expansion
  - No significant finite T potential corrections ($T \ll \alpha m$)

NR effective action on CTP contour

[TB, L. Covi, K. Mukaida, PRD, 2018] [see also TB, Edward Wang, 2026]

Consider NR scalar species, which can annihilate and scatter:

$$S_{\rm NR}[\eta,\xi] = \int_{x\in\mathcal{C}} \eta^\dagger\left[i\partial_t + \frac{\Delta}{2m}\right]\eta + \xi^\dagger\left[i\partial_t + \frac{\Delta}{2m}\right]\xi + \int_{x,y\in\mathcal{C}} \underbrace{\eta^\dagger(x)\xi(x) i\Gamma(x,y)\xi^\dagger(y)\eta(y)}_{\text{“annihilation”}} + \text{static potential},$$

- For pair annihilation only (+ reverse later):

$$\Gamma(x,y) = \frac{(\sigma v)_0}{2}\delta^4(x-y)\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$$

- Static potential particle and anti-particle number conserving.

Particle number density equation

From NREFT on CTP we derive kinetic equation,
from which we get:

$$\dot{n} = -(\sigma v)_0 G^{++--}_{\eta\xi}(x,x,x,x)$$

$$n(x) \equiv \langle \eta^\dagger(x)\eta(x)\rangle, \; G_{\eta\xi}(x,y,z,w) \equiv \langle T_{\mathcal{C}}\eta(x)\xi^\dagger(y)\xi(w)\eta^\dagger(z)\rangle$$

- n is the particle number density.
- **Goal: solve 4-point function „self-consistently“**
- Free case: $G^{++--}_{\eta\xi}(x,x,x,x) \simeq G^{+-}_{\eta}(x,x)G^{+-}_{\xi}(x,x) = n^2$

$\longrightarrow$ $\dot{n} = -(\sigma v)_0 n^2 \,, \quad \textit{matches expectation from Boltzmann eq.}$ ✓

„Self-consistent" computation of $G_{\eta\xi}^{++--}$

[TB, Edward Wang, 2026]

- KMS relation w/ finite chemical potential fails
  - Annihilation operator breaks particle and anti-particle number conservation
  - Self-consistent spectral function shows different unitarization and leads to Breit-Wigner energy profile of bound states
- Direct computation of 4-point correlator:

  1.) NREFT approach: truncate the correlator hierarchy at the four-point function level, solve the EoM

  2.) pNREFT approach: closed system of 4 point correlators

  Both approaches give consistent results. **pNREFT** approach in the following.

pNREFT on CTP contour

Projecting NREFT into two-particle subspace gives pNREFT:

$$S_{\mathrm{pNR}} = \int\limits_{x^0 \in \mathcal{C}} \mathrm{d}^4x \mathrm{d}^3r \; S^\dagger(x,\mathbf{r})[i\partial_{x^0} - h]S(x,\mathbf{r}) + i \int\limits_{x^0, y^0 \in \mathcal{C}} \mathrm{d}^4x \mathrm{d}^4y \mathrm{d}^3r \; S^\dagger(x,\mathbf{r})\delta(\mathbf{r})\Gamma(x,y)S(y,\mathbf{r})$$

where $h = -\dfrac{\Delta_{\mathbf{r}}}{m} - \dfrac{\Delta_{\mathbf{x}}}{4m} + V(r)$.

The EoM of the two-body field correlator, $G(x,y;\mathbf{r},\mathbf{r}') \equiv \langle T_{\mathcal{C}} S(x,\mathbf{r}) S^\dagger(y,\mathbf{r}') \rangle$,

are closed and given by:

$$(i\partial_{x^0} - h_x)G(x,y;\mathbf{r},\mathbf{r}') = i\delta^4_{\mathcal{C}}(x-y)\delta(\mathbf{r}-\mathbf{r}') - i \int\limits_{z^0 \in \mathcal{C}} \mathrm{d}^4z \delta(\mathbf{r})\Gamma(x,z)G(z,y;\mathbf{r},\mathbf{r}') \; ,$$

$$(-i\partial_{y^0} - h'_y)G(x,y;\mathbf{r},\mathbf{r}') = i\delta^4_{\mathcal{C}}(x-y)\delta(\mathbf{r}-\mathbf{r}') - i \int\limits_{z^0 \in \mathcal{C}} \mathrm{d}^4z G(x,z;\mathbf{r},\mathbf{r}')\Gamma(z,y)\delta(\mathbf{r}')$$

Solution via integral form

Let G_0 be the solution of $(i\partial_{x^0} - h_x)G_0(x,y;\mathbf{r},\mathbf{r}') = i\delta^4_{\mathcal{C}}(x-y)\delta(\mathbf{r}-\mathbf{r}')$

w/o annihilation. Then, the formal solution is:

$$G(x,y;\mathbf{r},\mathbf{r}') = G_0(x,y;\mathbf{r},\mathbf{r}') - \int\limits_{z^0,w^0\in\mathcal{C}} \mathrm{d}^4z\mathrm{d}^4w\mathrm{d}^3\bar{\mathbf{r}}\; G_0(x,z;\mathbf{r},\bar{\mathbf{r}})\delta(\bar{\mathbf{r}})\Gamma(z,w)G(w,y;\bar{\mathbf{r}},\mathbf{r}')\;.$$

In Wigner coordinates, Fourier space, and LO gradient expansion:

$$G^{+-}(P;\mathbf{r},\mathbf{r}') = G_0^{+-}(P;\mathbf{r},\mathbf{r}') - c\big[G_0^R(P;\mathbf{r},0)G^{+-}(P;0,\mathbf{r}') - G_0^{+-}(P;\mathbf{r},0)G^A(P;0,\mathbf{r}')\big]\;,$$
$$G^A(P;\mathbf{r},\mathbf{r}') = G_0^A(P;\mathbf{r},\mathbf{r}') + cG_0^A(P;\mathbf{r},0)G^A(P;0,\mathbf{r}')\;.$$

Solution is stationary and given by:

$$G^{+-}(P;0,0) = \frac{1}{\big|1+cG_0^R(P;0,0)\big|^2} \times G_0^{+-}(P;0,0)\;.$$

Unitarization of s-wave Sommerfeld effect consistent with

[Blum, Ryosuke and Slatyer, 2016, “Self-consistent Calculation of the Sommerfeld Enhancement“]

Solutions

For bound states, need to retain time/energy gradients:

$$G^{+-}(T,\omega,\mathbf{P};0,0) = G_0^{+-}(T,\omega,\mathbf{P};0,0) -$$
$$c\big[G_0^R(T,\omega,\mathbf{P};0,0)e^{-i\diamond}G^{+-}(T,\omega,\mathbf{P};0,0) - G_0^{+-}(T,\omega,\mathbf{P};0,0)e^{-i\diamond}G^A(T,\omega,\mathbf{P};0,0)\big] \; .$$

Neglecting coherence, the solution to all order in time/energy gradients is:

$$G^{+-}(T,P;0,0) = \sum_n e^{-\Gamma_n^{\mathrm{dec}}T} f_n(T_0, 2m+\omega)(2\pi)\delta(\omega - \mathbf{P}^2/(4m) - E_n)|\psi_n(0)|^2 \; , \; \text{for } E<0.$$

- Boltzmann equation for bound states emerges dynamically from 4-point function under semi-classical assumptions.
- Bound states are **on-shell** in their out-of-equilibrium decay.

Including pair creation

Environment is assumed to be time translational invariant.

$$\Gamma = \Gamma(x-y) = \begin{pmatrix} \Gamma^{++}(x-y) & \Gamma^{+-}(x-y) \\ \Gamma^{-+}(x-y) & \Gamma^{--}(x-y) \end{pmatrix}$$

We analytically solve the coupled set of spectral function and $G^F = \frac{1}{2}\langle\{S(x,\mathbf{r}), S^\dagger(y,\mathbf{r}')\}\rangle$

$$(i\partial_{x_0} - h_x)\, G^F(x,y;\mathbf{r},\mathbf{r}') = -i\int_{t_0}^{x^0} \mathrm{d}z^0 \int \mathrm{d}^3 z \delta(\mathbf{r})\Gamma^\rho(x-z)G^F(z,y;\mathbf{r},\mathbf{r}') + i\int_{t_0}^{y^0} \mathrm{d}z^0 \int \mathrm{d}^3 z \delta(\mathbf{r})\Gamma^F(x-z)G^\rho(z,y;\mathbf{r},\mathbf{r}')\ ,$$

$$(i\partial_{x_0} - h_x)\, G^\rho(x,y;\mathbf{r},\mathbf{r}') = -i\int_{y^0}^{x^0} \mathrm{d}z^0 \int \mathrm{d}^3 z \delta(\mathbf{r})\Gamma^\rho(x-z)G^\rho(z,y;\mathbf{r},\mathbf{r}')\ ,$$

All other correlators follow from the relation

$$G(x,y;\mathbf{r},\mathbf{r}') = G^{-+}(x,y;\mathbf{r},\mathbf{r}')\theta_{\mathcal{C}}(x^0-y^0) + G^{+-}(x,y;\mathbf{r},\mathbf{r}')\theta_{\mathcal{C}}(y^0-x^0) = G^F(x,y;\mathbf{r},\mathbf{r}') + \frac{1}{2}\mathrm{sign}_{\mathcal{C}}(x^0-y^0)G^\rho(x,y;\mathbf{r},\mathbf{r}')\ ,$$

For 2-point functions, see:

[Anisimov, Buchmuller, Drewes, Mendizabal, 2019, “Nonequilibrium Dynamics of Scalar Fields in a Thermal Bath“]

Analytic solution

$$G^F(x^0, y^0, \mathbf{P}; \mathbf{r}, \mathbf{r}') = G_h^F(x^0, y^0, \mathbf{P}; \mathbf{r}, \mathbf{r}') + G_m^F(x^0, y^0, \mathbf{P}; \mathbf{r}, \mathbf{r}')$$

Homogeneous part contains initial condition:

$$G_h^F(x^0, y^0, \mathbf{P}; \mathbf{r}, \mathbf{r}') = \int \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 G^\rho(x^0, \mathbf{P}; \mathbf{r}, \mathbf{r}_1) G^F(0, 0, \mathbf{P}; \mathbf{r}_1, \mathbf{r}_2) [G^\rho(y^0, \mathbf{P}; \mathbf{r}', \mathbf{r}_2)]^\dagger$$

„Memory term“ independent of initial condition:

$$G_m^F(x^0, y^0, \mathbf{P}; \mathbf{r}, \mathbf{r}') = \int_0^{x^0} \mathrm{d}t_1 \int_0^{y^0} \mathrm{d}t_2 G^\rho(x^0 - t_1, \mathbf{P}; \mathbf{r}, 0) \Gamma^F(t_1 - t_2, \mathbf{P}) G^\rho(t_2 - y^0, \mathbf{P}; 0, \mathbf{r}')$$

Homogeneous part

Negleting coherence in initial condition we obtain:

$$G_h^F(T,\omega,\mathbf{P};0,0) = \sum_n e^{-\Gamma_n T}(2\pi)\delta(E-\Re E_n)|\psi_n(0)|^2\left[\frac{1}{2}+f_n(\mathbf{P})\right]$$
$$+\theta(E)\frac{m^2}{2\pi}\sqrt{\frac{E}{m}}\frac{S(v)}{|1+\Gamma^R(P)G_0^R(E;0,0)|^2}\left[\frac{1}{2}+f_{\mathbf{p}}(\mathbf{P})\right].$$

- Out-of-equilibrium bound states decay on-shell. Consistent with our gradient method.
- Unitarization of Sommerfeld effect now temperature dependent.

Large time T limit

$$
\begin{aligned}
G^{+-}(T,P;0,0) &= G^F(T,P;0,0) - \frac{1}{2}G^\rho(P;0,0) \\
&= \frac{G_0^\rho(E;0,0)}{|1+\Gamma^R(P)G_0^R(E;0,0)|^2}\left[f_{\mathbf{p}}(\mathbf{P}) - f_B^{\mathrm{eq}}(2m+\omega)\right] + f_B^{\mathrm{eq}}(2m+\omega)G^\rho(P;0,0)\ , \\
G^{-+}(T,P;0,0) &= G^F(T,P;0,0) + \frac{1}{2}G^\rho(P;0,0) \\
&= \frac{G_0^\rho(E;0,0)}{|1+\Gamma^R(P)G_0^R(E;0,0)|^2}\left[(1+f_{\mathbf{p}}(\mathbf{P})) - (1+f_B^{\mathrm{eq}}(2m+\omega))\right] + (1+f_B^{\mathrm{eq}}(2m+\omega))G^\rho(P;0,0)\ ,
\end{aligned}
$$

- Bound states satisfy KMS relation as they equilibrate in large time limit.
- Stationary scattering states satisfy KMS relation for equilibrium phase-space distribution.

Number density equation

At LO in gradient expension:

$$\dot{n}_\eta = \int \frac{d^4P}{(2\pi)^4}\Gamma^{+-}(P)G^{-+}(T,P;0,0) - \Gamma^{-+}(P)G^{+-}(T,P;0,0)$$

$$= \int \frac{d^4P}{(2\pi)^4}\frac{G_0^\rho(E;0,0)}{|1+\Gamma^R(P)G_0^R(E;0,0)|^2}\left[\Gamma^{+-}(P)(1+f_{\mathbf{p}}(\mathbf{P})) - \Gamma^{-+}(P)f_{\mathbf{p}}(\mathbf{P})\right].$$

- All equilibrium parts canceled. In this way, full spectral function does not enter collision term. However, the **correlator** satisfies the KMS for equilibrium phase-space density.

- Example: $\Phi^\dagger\Phi\phi^\dagger\phi$

$$\Gamma^{+-}(P) = (\sigma v)_0 f_B^{\rm eq}(m+P^0/2) f_B^{\rm eq}(m+P^0/2)\ , \quad \Gamma^{-+}(P) = (\sigma v)_0 \left(1+f_B^{\rm eq}(m+P^0/2)\right)^2\ ,$$

$$\Gamma^R(P) = \frac{1}{2}(\sigma v)_0\left(1+2f_B^{\rm eq}(m+P^0/2)\right)\ .$$

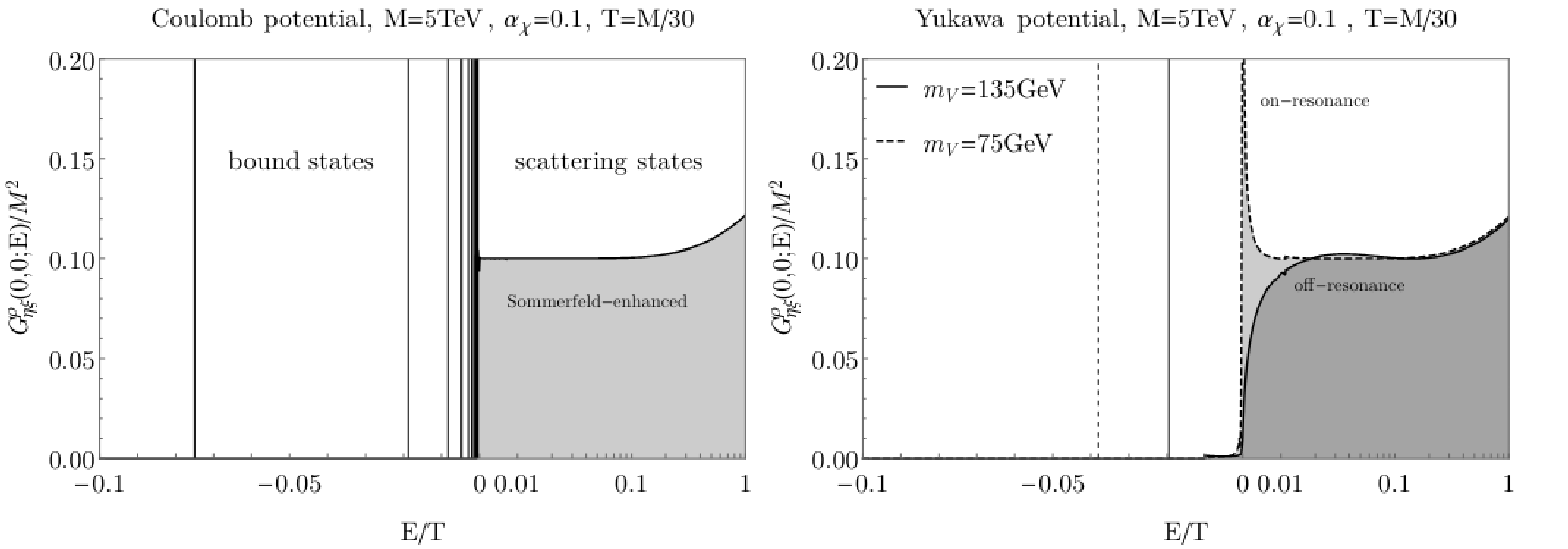
Unitarization temperature dependent, exponentially small correction

Summary

- Confirmed „self-consistent“ computation of Sommerfeld enhancement from thermal field theory (NREFT & pNREFT at finite T).
- Unitarization temperature dependent, but correction exponentially small.
- Bound states are on-shell in their out-of-equilibrium decay and satisfy KMS relation in large time limit with Breit-Wigner energy shape.
- Scattering states are stationary. Chemical evolution is not determined by the four-point function alone and requires, in addition, the two-point equation for the particle number density.

Spectral function

[TB, L. Covi, K. Mukaida, PRD, 2018]



Solution for grand-canonical ensemble

Switching off annihilation, leads to particle and anti-particle number conservation.

$$G_{\eta\xi,0}(x,y,z,w) \equiv G_{\eta\xi}(x,y,z,w)\big|_{\Gamma=0}$$

Grand canonical state $\rho \propto e^{-\beta(\hat{H}_{\mathrm{NR},0}-\mu_\eta \hat{N}_\eta - \mu_\xi \hat{N}_\xi)}$ leads to KMS relation:

$$G^{++--}_{\eta\xi,0}(x,x,x,x) = e^{-2\beta(m-\mu)} \int \frac{\mathrm{d}^3\mathbf{P}}{(2\pi)^3} e^{-\beta\mathbf{P}^2/(4m)} \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{(2\pi)} e^{-\beta E} G^{\rho}_{\eta\xi,0}(E;\mathbf{0},\mathbf{0}).$$

$$\left[E + i\epsilon + \frac{\Delta_{\mathbf{r}}}{m} - V(r)\right] G^R_{\eta\xi,0}(E;\mathbf{r},\mathbf{r}') = i\delta(\mathbf{r}-\mathbf{r}')\ .$$

$$G^R_{\eta\xi,0}(E;\mathbf{r},\mathbf{r}') = i\sum_{\mathcal{B}} \frac{\psi_{\mathcal{B}}(\mathbf{r})\psi^\star_{\mathcal{B}}(\mathbf{r}')}{E - E_{\mathcal{B}} + i\epsilon} + i\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\psi_{\mathbf{q}}(\mathbf{r})\psi^\star_{\mathbf{q}}(\mathbf{r}')}{E - q^2/m + i\epsilon}$$

$$G^{\rho}_{\eta\xi,0}(E;0,0) = 2\Im i G^R_{\eta\xi,0}(E;0,0) = \sum_n |\psi_n(0)|^2 (2\pi)\delta(E-E_n) + \theta(E)\frac{m^2}{2\pi}\sqrt{\frac{E}{m}}S(v)$$

Matches standard Boltzmann equations with Sommerfeld enhancement and bound states (in ionization eq.).

First attempt for „self-consistent“ computation

Grand canonical state $\rho \propto e^{-\beta(\hat{H}_{\mathrm{NR},0}-\mu_\eta \hat{N}_\eta - \mu_\xi \hat{N}_\xi)}$ leads to KMS relation:

$$G^{++--}_{\eta\xi,0}(x,x,x,x) = e^{-2\beta(m-\mu)} \int \frac{\mathrm{d}^3\mathbf{P}}{(2\pi)^3} e^{-\beta \mathbf{P}^2/(4m)} \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{(2\pi)} e^{-\beta E} G^{\rho}_{\eta\xi,0}(E;\mathbf{0},\mathbf{0}).$$

Suppose, we naively replace $G^{\rho}_{\eta\xi,0} \to G^{\rho}_{\eta\xi}$, computed from

$$\left[E + i\epsilon + \frac{\Delta_{\mathbf{r}}}{m} - V(r) + ic\delta^3(\mathbf{r})\right] G^R_{\eta\xi}(E;\mathbf{r},\mathbf{r}') = i\delta(\mathbf{r}-\mathbf{r}') \ .$$

Then, there occur two inconsistencies for scatter and bound states:

$$\frac{|\psi_0(0)|^2}{\left(1 + \frac{(\sigma v)_0 |\psi_0(0)|^2}{(\sigma v)_{\mathrm{uni}}}\right)^2} \neq \frac{|\psi_0(0)|^2}{\left(1 + \frac{(\sigma v)_0 |\psi_0(0)|^2}{(\sigma v)_{\mathrm{uni}}}\right)}$$

$$G^{\rho}_{\eta\xi}(E \to E_n; 0,0) = |\psi_n(0)|^2 \frac{\Gamma_n}{(E-E_n)^2 + (\Gamma_n/2)^2}$$

Collision term diverges as exponential wins.

KMS relation with finite chemical potential not allowed to use, as annihilation breaks the conservation of particle and anti-particle number density.