

Effective action for dissipative fluids from Schwinger–Keldysh holography

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Outline

- **Motivation:** hydrodynamic EFT, partially on-shell holography, all orders in gradients
- Hydrodynamic effective field theory (QFT & holographic pictures)
- **Holographic derivation of the effective action** (general scheme)
- **Schwinger–Keldysh formalism:** matching conditions & SK effective action
- **Perturbations of AdS–SK black branes:** channels, Heun functions, full-spacetime solutions, effective action
- Conclusions & prospects

Motivation

Holography maps strongly-coupled finite- T dynamics to classical AdS gravity; in the long-wavelength regime it is captured by *hydrodynamics*:

$$\nabla_{\mu} T_{\text{hydro}}^{\mu\nu} = 0, \quad T_{\text{hydro}}^{\mu\nu} = (\epsilon + P)u^{\mu}u^{\nu} + Pg_r^{\mu\nu} - \eta_0\sigma^{\mu\nu} + \dots$$

We build the **effective action of a dissipative neutral fluid** from gravity, with three guiding principles:

- **Schwinger–Keldysh formalism**: closed time path \Rightarrow fluctuation *and* dissipation built in.
- **Partially on-shell (p.o.s.) prescription**: solve only the *dynamical* bulk equations, relax the constraints \Rightarrow keep hydrodynamic d.o.f. X^{μ} dynamical, integrate out gapped modes.
- **All orders in the gradient expansion**: solutions are obtained exactly (Heun functions), not order-by-order.

Hydrodynamic EFT: field-theory perspective

Generating functional minimally coupled to an external metric,

$$Z[g_{\mu\nu}] = \int_{\rho_0} [D\psi] e^{i \int d^4y \mathcal{L}[\psi, g_{\mu\nu}]}, \quad Z[\bar{g}(\bar{x})] = Z[g(x)] \Rightarrow \nabla_\mu T^{\mu\nu} = 0.$$

Gapless (hydro) modes \Rightarrow promote the diffeomorphism parameter to a dynamical field

$$x^\mu \rightarrow X^\mu(y^\alpha),$$

giving the invariant building block

$$\mathcal{G}_{\alpha\beta}(X) = g_{\mu\nu}(X) \frac{\partial X^\mu}{\partial y^\alpha} \frac{\partial X^\nu}{\partial y^\beta}, \quad X^\mu \rightarrow X^\mu - \xi^\mu, \quad g_{\mu\nu} \rightarrow g_{\mu\nu} + \nabla_{(\mu} \xi_{\nu)}.$$

EOM for $X^\mu \equiv$ conservation of $T^{\mu\nu}$.

[Crossley–Glorioso–Liu, 1511.03646]

Hydrodynamic EFT: SK doubling & holographic RG

Schwinger–Keldysh doubling: $g \rightarrow g_{\downarrow}, g_{\uparrow}$, $X \rightarrow X_{\downarrow}, X_{\uparrow}$, $\mathcal{G} \rightarrow \mathcal{G}_{\downarrow}, \mathcal{G}_{\uparrow}$; (r, a) basis,

$$Z[g_{\downarrow}, g_{\uparrow}] = \int [DX_{\downarrow}][DX_{\uparrow}] e^{iS_{\text{eff}}[\mathcal{G}_{\downarrow}, \mathcal{G}_{\uparrow}]}, \quad S_{\text{eff}} = \int d^4y \sqrt{-\mathcal{G}_r} \tilde{\mathcal{L}}_{\text{eff}}[\mathcal{G}_r, \mathcal{G}_a].$$

Symmetries constraining S_{eff} : normalization, Z_2 reflection, $\text{Im } S_{\text{eff}} \geq 0$, fluid reparametrization, dynamical KMS.

Holographic Wilsonian RG

$Z_{\text{CFT}} = Z_{\text{AdS}}$; integrate out bulk modes dual to gapped CFT modes:

$$Z_{\text{AdS}} \simeq \int [D\tau][DX^{\mu}][DG_{\alpha\beta}] e^{iS[G_{\alpha\beta}]} = \int [DX^{\mu}] e^{iS[G_{\alpha\beta}[X^{\mu}]]|_{\text{p.o.s.}}}$$

Partially on-shell: only dynamical equations fix $G_{\alpha\beta}$; constraints (energy–momentum conservation) left free.

Holographic action

$$S = S_0 + S_{\text{ct}} + S_{\text{GH}}, \quad S_0 = \frac{1}{2} \int_M \sqrt{|G|} (R - 2\Lambda),$$
$$S_{\text{GH}} = \int_{\partial M} \sqrt{|G|} \mathcal{K}_I - \int_{\Sigma} \sqrt{|G|} \mathcal{K}_H, \quad \mathcal{K} \equiv \Pi_b^a \nabla_a s^b, \quad s_a \equiv \partial_a s(x)$$

Π_b^a is a projector on the hypersurface: $\Pi_b^a s_a = 0$

Boundary term for *any* surface type

The GHY form above holds for timelike, spacelike **and null** boundaries Parattu–Chakraborty–Padmanabhan
1602.07546

Coordinate-independent description. UV and IR boundaries given by the general equations

$$\partial M : s_I(x) = \infty, \quad \Sigma : s_H(x) = 0.$$

No gauge/coordinates fixed at this stage.

Variation, conjugate momentum, EOM

$$\delta S_0 = \frac{1}{2} \int_M d^{d+1}x \sqrt{|G|} (E_{ab} \delta G^{ab} + G^{ab} \delta R_{ab}), \quad \int_M d^{d+1}x \sqrt{|G|} G^{ab} \delta R_{ab} = \int_M d^{d+1}x \sqrt{|G|} \nabla_c \omega^c.$$

$$\int_M d^{d+1}x \sqrt{|G|} \nabla_c \omega^c = \int_{\partial M} d^d y \sqrt{|G|} s_{Ic} \omega^c - \int_{\Sigma} d^d y \sqrt{|G|} s_{Hc} \omega^c,$$

Using $\sqrt{|G|} s_c \omega^c = \partial_a (\sqrt{|G|} \Pi_b^a \delta s^b) - 2\delta(\sqrt{|G|} \mathcal{K}) + \sqrt{|G|} Q_{ab} \delta G^{ab}$,

$$\delta S_0 + \delta S_{\text{GH}} = -\frac{1}{2} \int_M \sqrt{|G|} E^{ab} \delta G_{ab} - \frac{1}{2} \int_{\partial M} \sqrt{|G|} Q^{ab} \delta G_{ab} + \frac{1}{2} \int_{\Sigma} \sqrt{|G|} Q^{ab} \delta G_{ab}.$$

The GHY term cancels $\delta(\mathcal{K}\sqrt{|G|})$; the *conjugate momentum* Q^{ab} is the relevant boundary datum (related to Israel junction conditions). With $\delta G_{ab}|_{\partial M, \Sigma} = 0$:

$$\boxed{E^{ab}[G] = 0}$$

Gauge & constraints: dynamical vs. constraint equations

A gauge fixes $d+1$ metric components: $G_{1a} = F_a(G_{\mu\nu})$, i.e. $\frac{\delta G_{1a}}{\delta G_{\mu\nu}} = f_a^{\mu\nu}$, hence $h_{1a} = f_a^{\mu\nu} h_{\mu\nu}$.

$$E^{ab} \delta G_{ab} = (E^{11} f_1^{\mu\nu} + 2E^{1\alpha} f_\alpha^{\mu\nu} + E^{\mu\nu}) \delta G_{\mu\nu} = 0$$

Dynamical equations of motion

$$E^{11} f_1^{\mu\nu} + 2E^{1\alpha} f_\alpha^{\mu\nu} + E^{\mu\nu} = 0$$

In the p.o.s. formalism we solve only these. The remaining E^{1a} are *constraints*, left un-imposed. The combinations $E^{ab} \delta G_{ab}$ and $h_{ab} \delta E^{ab}$ are **gauge invariant** and vanish, even though individual components E^{ab} , δE^{ab} may be nonzero for the p.o.s case.

Effective action: general scheme

$$S_{\text{eff}} = (S_0 + S_{\text{GH}})|_{\text{p.o.s.}} + S_{\text{ct}}, \quad G_{ab} = G_{ab}^{(0)} + h_{ab}, \quad \gamma_{\mu\nu} = \gamma_{\mu\nu}^{(0)} + s_I^2 B_{\mu\nu}, \quad h_{\mu\nu}|_{\partial M} = s_I^2 B_{\mu\nu}.$$

Taylor's formula for variational derivatives (also expand the perturbed horizon $s_H = s_H^{(0)} + s_H^{(1)}$):

$$S_{\text{eff}}[G^{(0)} + h, s_H] = S_{\text{eff}}[G^{(0)}, s_H] + \frac{\delta S_{\text{eff}}}{\delta G_{ab}}[G^{(0)}, s_H] h_{ab} + \frac{1}{2} h_{ab} \frac{\delta^2 S_{\text{eff}}}{\delta G_{ab} \delta G_{cd}}[G^{(0)}, s_H] h_{cd} + \dots$$

Adapted to p.o.s. — two terms vanish

$$E^{ab}[G^{(0)}] h_{ab} = 0, \quad h_{ab} \frac{\delta E^{ab}}{\delta G_{cd}}[G^{(0)}] h_{cd} = 0$$

for both the fully on-shell and the partially on-shell cases.

Hence the general formulas for $S_{\text{eff}}^{(n)}$ are identical; only the h_{ab} and $s_H^{(1)}$ used differ.

Effective action: contributions order by order

0th order (background, $R - 2\Lambda = \frac{4\Lambda}{d-1}$): bulk + counterterm + GHY on $\partial M, \Sigma_0$.

1st order: split into boundary and horizon pieces

$$S_{\text{eff}}^{(1)}|_{\partial M} = \frac{1}{4} \int_{\partial M} \{E^{\mu\nu}[\gamma^{(0)}] - 2K_I^{\mu\nu} + 2\gamma^{(0)\mu\nu} K_I\} h_{\mu\nu} \sqrt{|\gamma^{(0)}|} d^d y.$$

2nd order (boundary part):

$$\begin{aligned} S_{\text{eff}}^{(2)}|_{\partial M} = & -\frac{1}{4} \int_{\partial M} \left\{ (-2\gamma^{(0)ae} \gamma^{(0)df} \gamma^{(0)bc} + \gamma^{(0)ae} \gamma^{(0)bf} \gamma^{(0)cd} + \gamma^{(0)ce} \gamma^{(0)df} \gamma^{(0)ab}) \nabla_c s_{Id} h_{ab} h_{ef} - \right. \\ & \left. -\frac{1}{2} (\gamma^{(0)bc} \gamma^{(0)ad} - \gamma^{(0)ab} \gamma^{(0)cd}) h_{ab} (s_I^k (\nabla_k h_{cd} - 2\nabla_c h_{kd}) + \nabla_c s_{Id} h) \right\} \sqrt{-G^{(0)}} \\ & + \frac{1}{16} \int_{\partial M} \sqrt{|\gamma^{(0)}|} \{ \gamma^{(0)\alpha\beta} R^{\mu\nu} - 2\gamma^{(0)\alpha\mu} (E^{\beta\nu} + R^{\beta\nu}) + E^{\alpha\beta} \gamma^{(0)\mu\nu} \} h_{\alpha\beta} h_{\mu\nu}, \end{aligned}$$

$$\gamma^{(0)ab} = G^{(0)ab} - \frac{s_I^a s_I^b}{s_I^2}, \quad K_I^{ab} \text{ is extrinsic curvature}$$

Horizon contributions $S_{\text{eff}}^{(n)}|_{\Sigma}$ involve Q_H^{ab} , \mathcal{K}_H and $s_H^{(1)}$.

Schwinger–Keldysh formalism: complexified geometry

SK contour \Rightarrow glue two Lorentzian copies M^\uparrow, M^\downarrow with a Euclidean cap.

Matching rule Skenderis–van Rees, 0805.0150, 0812.2909: bulk fields (G_{ab}) and their conjugate momenta (Q_{ab}) are continuous across the mixed-signature manifold.

Liu prescription Glorioso–Crossley–Liu, 1812.08785: the Euclidean insertion reduces to an **analytic continuation of the radial coordinate along a circle** around the horizon $r = r_h$ — one only stitches the two Lorentzian sheets over the future horizon.



Complex r -plane: two conformal boundaries $\infty_{1,2}$ and the horizon $r = r_h$.

Junction conditions from the principle of least action

Varying S over $M^\uparrow \cup M^\downarrow$ glued along the null Σ gives a jump term. The matching condition follows **directly from least action**:

$$[Q^{ab} \delta G_{ab} \sqrt{-G}]_\Sigma = 0.$$

Choosing a continuous metric across Σ (we do *not* fix G_{ab} on the horizon) gives $[Q^{ab}] = 0$, with $[Q^{ab}] s_a = 0$ automatically.

Adapted to the p.o.s. formalism

$$\left[(Q^{11} f_1^{\mu\nu} + 2Q^{1\alpha} f_\alpha^{\mu\nu} + Q^{\mu\nu}) \delta G_{\mu\nu} \sqrt{-G} \right]_\Sigma = 0.$$

Matching for the metric correction

Same Taylor expansion of variational derivatives. Since $Q^{ab}[G^{(0)}]$ is continuous, the linearized matching is

$$\left[\frac{\delta K^{ab}}{\delta G_{cd}} [G^{(0)}, s_H^{(0)}] h_{cd} \right] = 0,$$

and its **p.o.s. (dynamical) form**

$$\left[\delta Q^{11} f_1^{\mu\nu} + 2\delta Q^{1\alpha} f_\alpha^{\mu\nu} + \delta Q^{\mu\nu} \right] = 0, \quad \delta Q^{ab} = \frac{\delta K^{ab}}{\delta G_{cd}} [G^{(0)}, s_H^{(0)}] h_{cd}.$$

Ultimately: continuity of h_{ab} and their conjugate momenta along the entire complex r -contour.

$$S_{\text{eff}} = S_{\text{eff}}^{\uparrow}|_{\partial M^{\uparrow}} - S_{\text{eff}}^{\downarrow}|_{\partial M^{\downarrow}} + \underbrace{S_{\text{eff}}^{\uparrow}|_{\Sigma} - S_{\text{eff}}^{\downarrow}|_{\Sigma}}_{= 0 \text{ (up to 2nd order)}} .$$

- **0th order:** $S_{\text{eff}}^{(0)} = 0$ (the \uparrow/\downarrow geometries coincide).
- **1st order :**

$$S_{\text{eff}}^{(1)\uparrow}|_{\partial M^{\uparrow}} - S_{\text{eff}}^{(1)\downarrow}|_{\partial M^{\downarrow}} = \frac{1}{4} \int_{\partial M} \{ E^{\mu\nu}[\gamma^{(0)}] - 2K_I^{\mu\nu} + 2\gamma^{(0)\mu\nu} K_I \} (h_{\mu\nu}^{\uparrow} - h_{\mu\nu}^{\downarrow}) \sqrt{|\gamma^{(0)}|} .$$

- **2nd order :**

$$S_{\text{eff}}^{(2)} = S_{\text{eff}}^{(2)\uparrow}|_{\partial M^{\uparrow}} - S_{\text{eff}}^{(2)\downarrow}|_{\partial M^{\downarrow}} .$$

Horizon pieces cancel because \mathcal{K}_H, Q_H^{ab} agree on the two sheets at the background level.

AdS–SK black brane: coordinates & gauge

Background ($d+1 = 5$, $\Lambda = -6$), $f = 1 - (r_h/r)^4$:

$$\text{SS: } ds_0^2 = \frac{dr^2}{r^2 f} - r^2 f dt^2 + r^2 \delta_{ij} dx^i dx^j, \quad \text{EF: } ds_0^2 = 2 dv dr - r^2 f dv^2 + r^2 \delta_{ij} dx^i dx^j,$$

related by $dt = dv - \frac{dr}{r^2 f}$.

Strategy

- **Radial gauge** $s^a \delta G_{ab} = 0$ throughout.
- **Solve the dynamical EOM in SS coordinates** — time-reversal \mathcal{T} acts most simply there.
- **Write final solutions & matching conditions in EF coordinates** (regular at the horizon).

Residual gauge ξ^a (from $\nabla_1 \xi_a + \nabla_a \xi_1 = 2f_a^{\mu\nu} \nabla_\mu \xi_\nu$) yields pure-gauge perturbations; one subclass realizes the *fluid reparametrization symmetries* at the horizon.

Junction conditions in EF coordinates

From the gauge condition $\delta G^{ra} = 0$ we get $[Q_{\mu\nu} \delta G^{\mu\nu}] = 0 \Leftrightarrow [Q_{\mu\nu}] = 0$, when the dynamical, linearized matching conditions are

$$(s_H^{(0)})^2 (q_{\alpha\mu}^{(0)} q_{\beta\nu}^{(0)} - q_{\mu\nu}^{(0)} q_{\alpha\beta}^{(0)}) [\partial_r h^{\alpha\beta}] = 0,$$

$$q^{ab} \equiv G^{ab} + l^a s^b + s^a l^b, \quad l^a \text{ is auxiliary null vector}$$

In EF coordinates:

$$\boxed{f [\partial_r h_{vi}] = 0, \quad f [\partial_r h_{ij}] = 0, \quad [\partial_r h_{vv}] = 0}$$

These tell us *which* quantities must be continuous along the full complex contour; non-regular (e.g. outgoing) modes make them nontrivial.

Perturbations: three channels

$\delta(ds^2) = r^2 \tilde{H}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu$, Fourier $\propto e^{-i\omega t + iqx}$, momentum along x . The remaining $O(2)_{yz}$ symmetry splits the linearized EOM into three decoupled channels:

Tensor: $\{\tilde{H}_{yz}, \tilde{H}_- = \tilde{H}_{yy} - \tilde{H}_{zz}\}$

Shear: $\{\tilde{H}_{ty}, \tilde{H}_{xy}\}$ ($\tilde{H}_{ry} = 0$)

Scalar (sound): $\{\tilde{H}_{tt}, \tilde{H}_{tx}, \tilde{H}_{xx}, \tilde{H}_+ = \tilde{H}_{yy} + \tilde{H}_{zz}\}$

Time-reversal \mathcal{T} (compatible with the radial Schwarzschild gauge) relates ingoing/outgoing solutions and acts as $q^\mu = (\omega, q, 0, 0) \rightarrow \bar{q}^\mu = (-\omega, q, 0, 0)$.

Gauge-invariant master variables

In each channel the dynamical EOM reduce to a *single* equation for a gauge-invariant quantity
Kovtun–Starinets, hep-th/0506184:

$$Z_1 = q\tilde{H}_{ty} + \omega\tilde{H}_{xy} \text{ (shear),} \quad Z_2 = 4\omega q\tilde{H}_{tx} + 2\omega^2\tilde{H}_{xx} + (q^2(2-f) - \omega^2)\tilde{H}_+ + 2q^2\tilde{H}_{tt} \text{ (sound),}$$

and $Z_3 \equiv \tilde{H}_T$ (tensor).

p.o.s. \Rightarrow a source on the right-hand side

We do *not* solve the constraints, so the master equations acquire a nonzero RHS built from the constraint components δE^{ar} , e.g. in the shear channel

$$\left\{ \frac{d^2}{dr^2} + \dots \right\} Z_1 = \frac{4q\omega(1-f)}{rf(q^2f - \omega^2)} 2i \delta E^{ry}.$$

Although unfixed, the radial dependence of δE^{ar} is **not arbitrary**: the contracted Bianchi identity $\nabla_a \delta E^{ab} = 0$ gives $\{d/dr + 5/r\} \delta E^{ry} = 0 \Rightarrow \delta E^{ry} = w_{||}/r^5$.

Heun functions

The radial equations for the tensor and shear channels reduce to the general Heun equation

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a} \right) y' + \frac{\alpha\beta z - k}{z(z-1)(z-a)} y = 0, \quad \varepsilon = \alpha + \beta + 1 - \gamma - \delta,$$

with four regular singular points $\{0, 1, a, \infty\}$, P denotes a set of parameters.

- Fuchs–Frobenius local solutions $HI(P; z) = \sum_j c_j z^j$, $c_0 = 1$, three-term recurrence.
- Bases at $z = 0, 1, a$ related by Wronskian connection coefficients $C_{ij} = W[\cdot, \cdot]/W[\cdot, \cdot]$.
- When indicial exponents differ by an integer ($\gamma \in \mathbb{Z}$) **logarithmic solutions** appear:
 $y_1 = \sum_s b_s z^s + w y_2 \ln z$.

These provide *exact*, all-order-in-gradient solutions in the the tensor and shear channels channels.

Tensor & shear channels via Heun

Tensor: after $\tilde{H}_T = (u^2 - 1)^{i\tilde{\omega}/4} (u^2 + 1)^{\tilde{\omega}/4} y_T$, $u = r_h/r$, $\tilde{\omega} = \omega/r_h$, the equation becomes Heun. Ingoing solution, valid on $u \in [0, 1]$:

$$\begin{aligned} \tilde{H}_T^{(\text{in})}(u) = & (u^2 - 1)^{-\frac{i\tilde{\omega}}{4}} (u^2 + 1)^{\frac{\tilde{\omega}}{4}} \text{Hl} \left(P_{H2}^T; \frac{1 - u^2}{2} \right) \theta(u - u_1) + \\ & + (u^2 - 1)^{\frac{i\tilde{\omega}}{4}} (u^2 + 1)^{\frac{\tilde{\omega}}{4}} (C_{21}^T y_{I1}^T(u) + C_{22}^T y_{I2}^T(u)) \theta(u_1 - u), \end{aligned}$$

Shear: reduces to Heun for $\frac{d}{du} \tilde{H}_{ty}$; ingoing/outgoing are on-shell ($w_{\parallel} = 0$), the **polynomial** solution solves the *sourced* equation ($w_{\parallel} \neq 0$). \tilde{H}_{xy} follows algebraically.

Both channels are solved *analytically*; the sound channel uses a Frobenius/connection-coefficient construction for Z_2 (numerics cross-checked).

General structure of the solutions per channel

$$\text{Tensor: } H_T = c_T H_T^{(in)} + d_T H_T^{(out)}$$

$$\text{Shear: } H_{\parallel} = c_{\parallel} H^{(in)} + d_{\parallel} H^{(out)} + p_{\parallel} H^{(pg)} + n_{\parallel} H^{(pn)}$$

$$\text{Sound: } H_{\bullet} = c_s H_{\bullet}^{(in)} + d_s H_{\bullet}^{(out)} + \sum_{k=1}^3 (p_{s_k} H_{\bullet}^{(pn_k)} + n_{s_k} H_{\bullet}^{(pg_k)})$$

Off-shell content

Ingoing/outgoing = on-shell ($\delta E^{ar} = 0$); pure-gauge = trivial. **Only the “polynomial” solutions are genuinely off-shell** — they are the particular solutions of the *sourced* master equations and are the ones that do *not* satisfy the constraints. Setting their coefficients to zero \Leftrightarrow going on-shell.

From SS to EF: analytic continuation & matching

The Schwarzschild time carries a radial factor,

$$t = v - \zeta(r), \quad \zeta(r) = \int_{\infty}^r \frac{dy}{y^2 f(y)}, \quad e^{i\omega\zeta} = \left(\frac{1-u}{1+u}\right)^{\frac{i\bar{\omega}}{4}} \left(\frac{u-i}{u+i}\right)^{-\frac{\bar{\omega}}{4}} e^{i\omega C}.$$

Complexifying r makes $\zeta_{\uparrow} \neq \zeta_{\downarrow}$, with the jump at the horizon:

$$t^{\uparrow} - t^{\downarrow} = \zeta_{\downarrow} - \zeta_{\uparrow} = \oint_{|y-r_h|=\varepsilon} \frac{dy}{y^2 f(y)} = -\frac{i\beta}{2}.$$

Full SK geometry

EF solutions $H_{\alpha\beta} = \tilde{H}_{\alpha\beta} e^{i\omega\zeta}$; the in/out relation $H^{(out)} = \sigma \bar{H}^{(in)} e^{2i\omega\zeta}$, ($\sigma = \pm 1$). **Matching occurs at a singular point of the EOM**, so continuation of $\zeta(r)$ around the horizon + the matching conditions yield a solution on the *entire* SK contour. We follow the procedure from Bu–Demircik–Lublinsky, 2012.08362.

Solutions in the full spacetime

Continuity of H and of $f[\partial_u H]$ relates the \uparrow / \downarrow coefficients, e.g. tensor channel:

$$d_T^\uparrow = d_T^\downarrow e^{\beta\omega} \equiv d_T, \quad c_T^\uparrow = c_T^\downarrow \equiv c_T,$$

$$H_T(u) = c_T H_T^{(in)}(u) - d_T \bar{H}_T^{(in)}(u) e^{2i\omega\zeta_1(u)}, \quad \zeta_1(u) \equiv -\int_{0^\uparrow}^u \frac{d\tilde{u}}{r_h f(\tilde{u})}, \quad u \in (0^\uparrow, 0^\downarrow),$$

$$c_T = \frac{1}{2} B_T^{(a)} \coth \frac{\beta\omega}{2} + B_T^{(r)}, \quad d_T = \frac{B_T^{(a)}}{1 - e^{-\beta\omega}}.$$

Shear: same plus the polynomial coefficients $n_{\parallel}^{\uparrow/\downarrow}$; *on-shell* \Leftrightarrow *polynomial coefficient* $\rightarrow 0$. Sound channel: 16 matching conditions for 16 coefficients (in/out, $3 \times \text{pn}$, $3 \times \text{pg}$ on each sheet), fixed by AdS boundary data $B_{\bullet}^\uparrow, B_{\bullet}^\downarrow$.

The finite contribution to the action is the u^3 coefficient $t_{\alpha\beta}$ of $\frac{d}{du} H_{\alpha\beta}$, with $\sigma t_{\alpha\beta}^{(out)} = \bar{t}_{\alpha\beta}^{(in)}$.

Effective action in EF coordinates

General result,

$$\begin{aligned} S_{\text{eff}}^{(0)} &= 0, \\ S_{\text{eff}}^{(1)} &= \int \frac{r^4}{4} \{ (3B_{vv} + B_{ii})|_{\infty\downarrow}^{\infty\uparrow} \} d^4y, \\ S_{\text{eff}}^{(2)} &= \frac{r^4}{16} \int \{ 3B_{vv}^2 - 12B_{vi}^2 - 2B_{ij}^2 + 10B_{vv}B_{ii} + B_{ii}B_{jj} \}|_{\infty\downarrow}^{\infty\uparrow} + \\ &+ \frac{r^5}{16} \int \{ f\partial_r (H_{ii}H_{jj} - (H_{ij})^2) + 2\partial_r ((H_{vi})^2 - H_{vv}H_{ii}) \}|_{\infty\downarrow}^{\infty\uparrow} d^4y + O\left(\frac{1}{r^6}\right) \end{aligned}$$

Channel decomposition with t_{ab} = coefficient of u^3 in $\frac{d}{du}H_{ab}$:

$$\begin{aligned} (S_{\text{eff}}^{(2)})_T &= \frac{r^4}{16} \int \{ B_- t_- + 4B_{yz} t_{yz} \}|_{\infty\downarrow}^{\infty\uparrow} d^4y + \dots, \quad (S_{\text{eff}}^{(2)})_{\parallel} = \frac{r^4}{4} \int \{ B_{xy} t_{xy} + B_{xz} t_{xz} - B_{vy} t_{vy} - B_{vz} t_{vz} \}|_{\infty\downarrow}^{\infty\uparrow} d^4y + \dots, \\ (S_{\text{eff}}^{(2)})_s &= -\frac{r^4}{16} \int \{ B_+ (\frac{1}{2}t_+ + 2t_{xx}) + (\frac{1}{2}B_+ + 2B_{xx})t_+ \}|_{\infty\downarrow}^{\infty\uparrow} d^4y - \frac{r^4}{8} \int \{ 2B_{vx} t_{vx} - B_{vv}(t_{xx} + t_+) - (B_{xx} + B_+)t_{vv} \}|_{\infty\downarrow}^{\infty\uparrow} d^4y + \dots \end{aligned}$$

Should satisfy all the SK/KMS symmetry requirements

From $B_{\mu\nu}$ to $A_{\mu\nu}$ and π^μ

The building block is reorganized into external source $A_{\mu\nu}$ and dynamical field π^μ :

$$B_{sab} = A_{sab} + \partial_a \pi_{sb} + \partial_b \pi_{sa} + A_{sav} \partial_b \pi_s^\nu + \dots, \quad g_{s\mu\nu} = \eta_{\mu\nu} + A_{s\mu\nu}, \quad X_s^\mu = \delta_\alpha^\mu y^\alpha + \pi_s^\mu, \quad s = \uparrow, \downarrow.$$

Physical/dynamical variables:

$$X_r^\mu \equiv \frac{1}{2}(X_\downarrow^\mu + X_\uparrow^\mu), \quad X_a^\mu \equiv X_\downarrow^\mu - X_\uparrow^\mu \Rightarrow X_r^\mu = \delta_\alpha^\mu y^\alpha + \pi_r^\mu, \quad X_a^\mu = \pi_a^\mu.$$

Passing to physical spacetime $x^\mu \equiv X_r^\mu(y)$ and reorganizing by powers of sources/fields, the action separates:

$$S_{\text{eff}} = S_s + S_{sd} + S_d = \int d^4 y [\mathcal{L}_s + \mathcal{L}_{sd} + \mathcal{L}_d]$$

\mathcal{L}_s : source only; \mathcal{L}_{sd} : source \times field; \mathcal{L}_d : dynamical field only.

Conclusions

- A holographic derivation of the dissipative-fluid effective action on the SK contour, built on a **partially on-shell** prescription: only dynamical bulk equations are solved, constraints are relaxed and remain Bianchi-constrained.
- Coordinate-independent set-up with a boundary term valid for *any* (incl. null) surface; general first- and second-order effective-action formulas adapted for p.o.s. case.
- SK matching (fields *and* conjugate momenta continuous) implemented via analytic continuation of $\zeta(r)$ around the horizon; horizon contributions to S_{eff} cancel.
- Exact, all-order-in-gradient solutions: **Heun functions** in tensor/shear channels that allow for the computation of transport coefficients to all orders in derivatives; only “polynomial” solutions are genuinely off-shell.

- Complete the analysis of the sound channel
- Go **beyond linearization**: take the (inhomogeneous) fluid–gravity solution as the mean field and add noise/interactions self-consistently — fluid-gravity as the mean-field part of hydro EFT.
- Extend **beyond thermal equilibrium**; explore the structure of noise correlators beyond the first-order stochastic model.
- Extension to higher-derivative gravity (Gauss-Bonnet, etc.)
- Apply the p.o.s. + SK machinery to charged fluids; incorporate bulk Maxwell fields to capture charge diffusion and conductivity; .

Thank you!