Real-Time Nucleation in High-Temperature Quantum Field Theories

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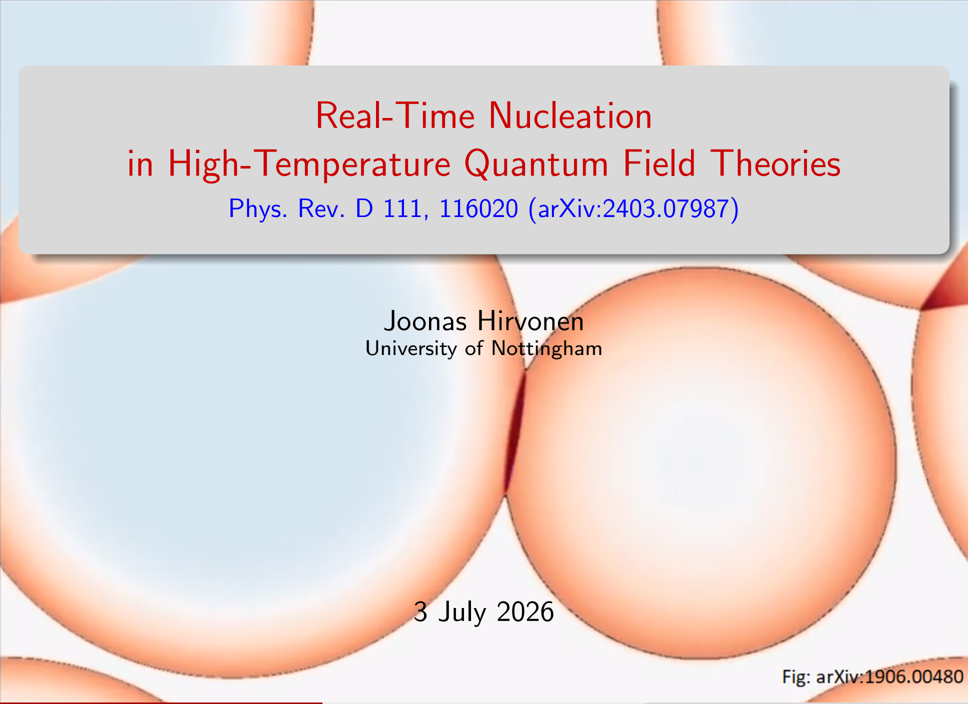


Fig: arXiv:1906.00480

Overview of the talk

0 Nucleation and cosmological phase transitions

1 Field and Boltzmann eqs. for high-T QFTs

2 Nucleation rate from a Hamiltonian description

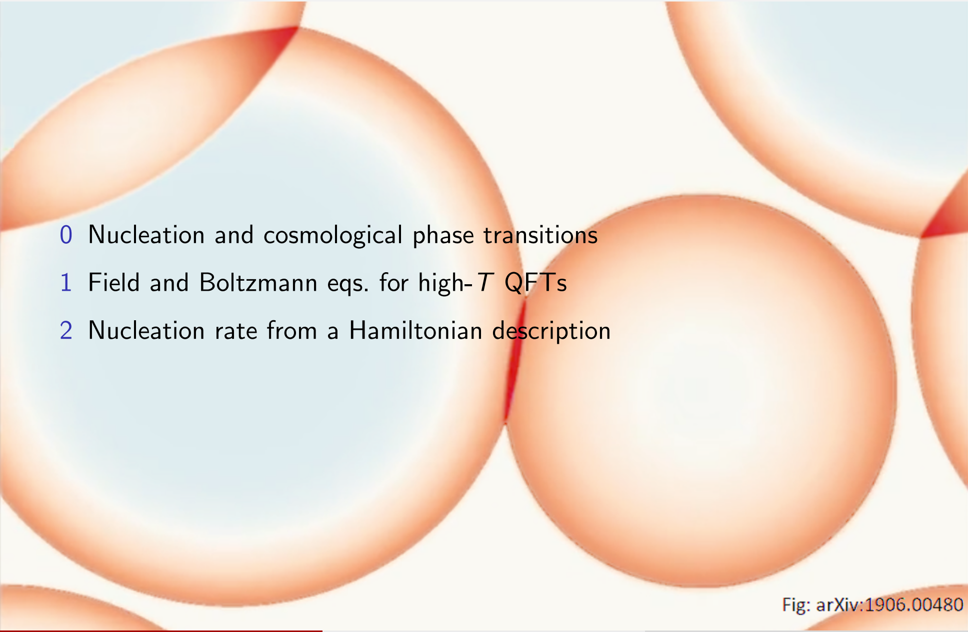


Fig: arXiv:1906.00480

Nucleation

- System trapped in ϕ_1
- Escaping locally with bubble nucleation to ϕ_2

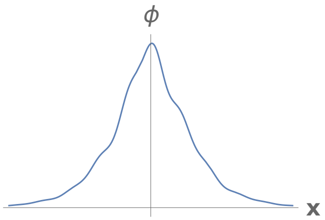


Figure: Cross section of a bubble

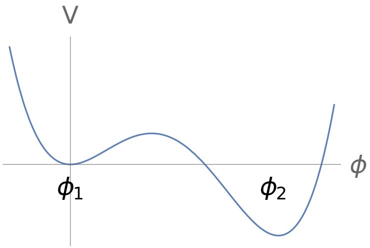


Figure: Potential with a transition

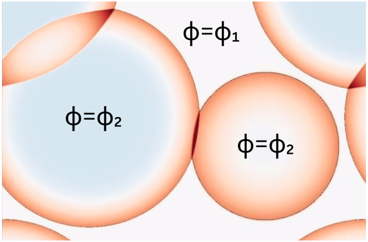


Figure: Growing bubbles [Cutting et al. '20]

Gravitational waves

- Plasma shocks from bubbles
  $\rightarrow$ Gravitational waves
- Possibly observable with e.g. LISA
  - BSM signal

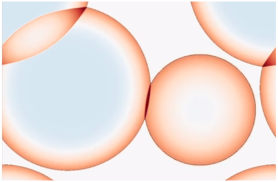


Figure: Growing bubbles (arXiv:1906.00480)

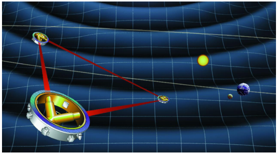


Figure: Gravitational waves and LISA (ESA)

Nucleation rates and cosmological phase transitions

- Nucleation rate, Γ, changes in time
- Duration

$$\Delta t \sim \left(\frac{\mathrm{d}}{\mathrm{d}t} \ln \Gamma \right)^{-1}$$

- Transition temperature

$$\Delta t^4 \, \Gamma \sim 1 \Rightarrow T$$

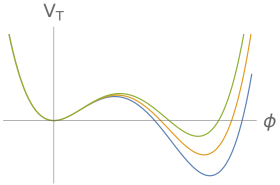


Figure: T-dependent potential

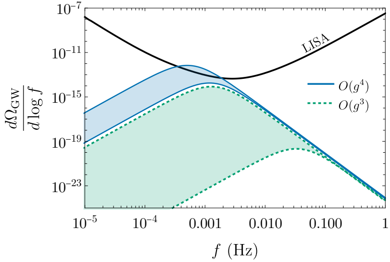


Figure: GW uncertainties [Gould, Tenkanen '21]

Nucleation out of equilibrium

- Previously equilibrium plasma particles assumed
- Relaxing the assumption
  - Corroborates equilibrium methods up to a certain validity
  - Future studies with SM extensions

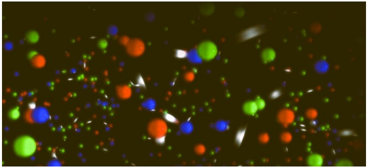


Figure: Plasma particles (Dr Rene Bellwied)

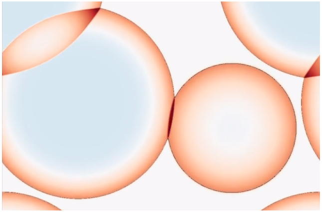
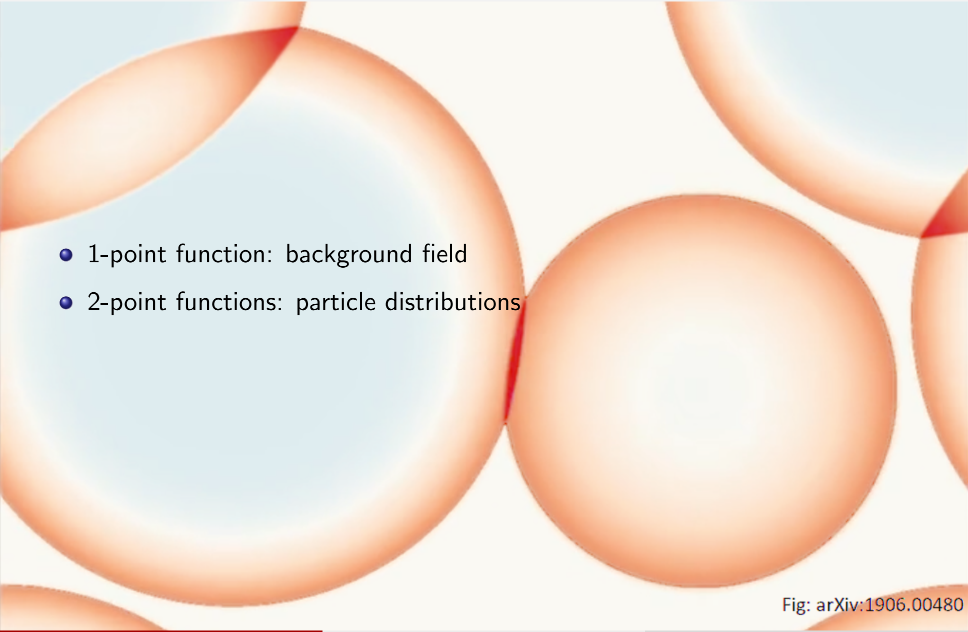


Figure: Growing bubbles (arXiv:1906.00480)

Boltzmann and field eqs following [Blaizot & Iancu '01]



- 1-point function: background field
- 2-point functions: particle distributions

Fig: arXiv:1906.00480

Thermal plasma in high-T QFTs

- Scale hierarchy: $T \gg m$
- Thermal particles
  - $E \sim T$
- Long-range bosonic fields
  - $E \ll T$
  - Bose enhanced, $f_b \gg 1$
    - ⋆ Classical
  - Nucleation

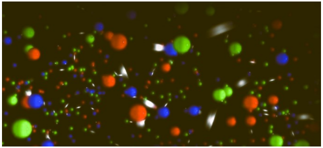


Figure: Plasma particles

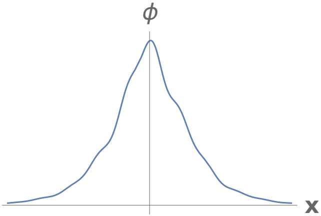


Figure: A bubble in a long-range scalar field

Field equation – brief sketch from contours

QCD review [Ghiglieri et al. '20]

- Real scalar, ϕ
- Scale hierarchy
  $\rightarrow$ approx. classical
- EoM from integrating over ϕ_{a}

$$\phi_{\mathrm{r}} \sim \mathcal{O}\left(n_{\mathrm{B}}^{1/2}\right) \sim \mathcal{O}\left((T/m)^{1/2}\right)$$
$$\gg \phi_{\mathrm{a}} \sim \mathcal{O}\left(n_{\mathrm{B}}^{0}\right) \sim \mathcal{O}\left((T/m)^{0}\right)$$

$$\int \mathcal{D}\phi_{\mathrm{r}}\mathcal{D}\phi_{\mathrm{a}} \, \exp\left\{ i \int \mathrm{d}^D x \phi_{\mathrm{a}} \frac{\delta S[\phi_{\mathrm{r}}]}{\delta \phi_{\mathrm{r}}} + \mathcal{O}\left(\phi_{\mathrm{a}}^3\right) \right\}$$
$$\approx \int \mathcal{D}\phi_{\mathrm{r}} \prod_{\mathbf{x},t} 2\pi\delta\left(\frac{\delta S[\phi_{\mathrm{r}}]}{\delta \phi_{\mathrm{r}}}\right)$$

Background field evolution from operators

- Real scalar, Φ
- Averaging the operator EOM
  - $\phi \equiv \langle \Phi \rangle$
  - $\delta\Phi \equiv \Phi - \phi$
- Two-point function, $\langle \delta\Phi^2 \rangle$
  - On-shell particles

$$\Box\Phi = -V'(\Phi)$$

$$\Rightarrow \Box\phi \approx -V'(\phi) - \frac{1}{2} V'''(\phi) \langle \delta\Phi^2 \rangle$$

$$= -V'(\phi) - \frac{1}{2} \frac{\mathrm{d}m^2}{\mathrm{d}\phi} \langle \delta\Phi^2 \rangle$$

Correlators of interest

- 2 indep. correlators
- Choosing
  - Lesser Green's function, $G^<$
  - Spectral function, ρ

$$G^<(x, y) = \left\langle \delta\phi_2(x)\delta\phi_1(y) \right\rangle$$

$$\rho(x, y) = \left\langle \delta\phi_1(x)\delta\phi_2(y) - \delta\phi_2(x)\delta\phi_1(y) \right\rangle$$

Simple kinetic equation with Wigner transformation

- EoMs for the correlators
- Higer orders
  - Collisions
  - Decay
- Wigner transform:
  - Difference equation
  - $\rightarrow$ Kinetic equation

$$(\Box_x + m^2(x))G^<(x,y) = \text{higher orders}$$

$$(\Box_y + m^2(y))G^<(x,y) = \text{higher orders}$$

$$(2\partial_s \cdot \partial_x + \underbrace{s \cdot \partial_x m^2(x)}_{m^2(x+\frac{s}{2})-m^2(x-\frac{s}{2})\approx})G^<(x+\frac{s}{2}, x-\frac{s}{2}) = \text{h.o.}$$

$$(\underbrace{k \cdot \partial_x}_{\text{free streaming}} + \underbrace{\frac{1}{2}\partial_x m^2(x) \cdot \partial_k}_{\text{force term}})G^<(x,k) = \text{h.o.}$$

Quasiparticle approximation

- Spectral function delta spikes
  - Width higher order
- $G^<$: on-shell particle distributions, $f(x, \mathbf{k})$
- Collisionless Boltzmann equation

$$\rho(x,k) = 2\pi \text{sign}(k_0)\delta(k^2 - m^2(x))$$

$$G^<(x,k) = 2\pi\delta(k^2 - m^2(x)) \times \Big[\theta(k_0) f(x,\mathbf{k}) + \theta(-k_0)\big(1 \pm f(x,-\mathbf{k})\big)\Big]$$

$$(\underbrace{E\partial_t + \mathbf{k}\cdot\boldsymbol{\nabla}}_{\text{free streaming}} + \underbrace{\frac{1}{2}\boldsymbol{\nabla} m^2(x)\cdot\partial_{\mathbf{k}}}_{\text{force}}) f(x,\mathbf{k}) = 0$$

Particles onto background

- EoM for scalar
- Two-point function
  - Thermal particle part
  - Vacuum part
- Closed system of equations!
  - $\phi(x)$
  - $f(x, \mathbf{k})$

$$\Box\phi = -V'(\phi) - \frac{1}{2}\frac{\mathrm{d}m^2}{\mathrm{d}\phi}\langle\delta\Phi^2\rangle$$

$$\langle\delta\Phi^2\rangle = \int \frac{\mathrm{d}^D k}{(2\pi)^D} G^<(x, k)$$

$$= \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d E}\left(f(x, \mathbf{k}) + \frac{1}{2}\right)$$

Separating off-equilibrium particles, δf

$$\delta f = f - \overbrace{\frac{1}{e^{\beta E} \mp 1}}^{\equiv f_{\text{eq}}},$$

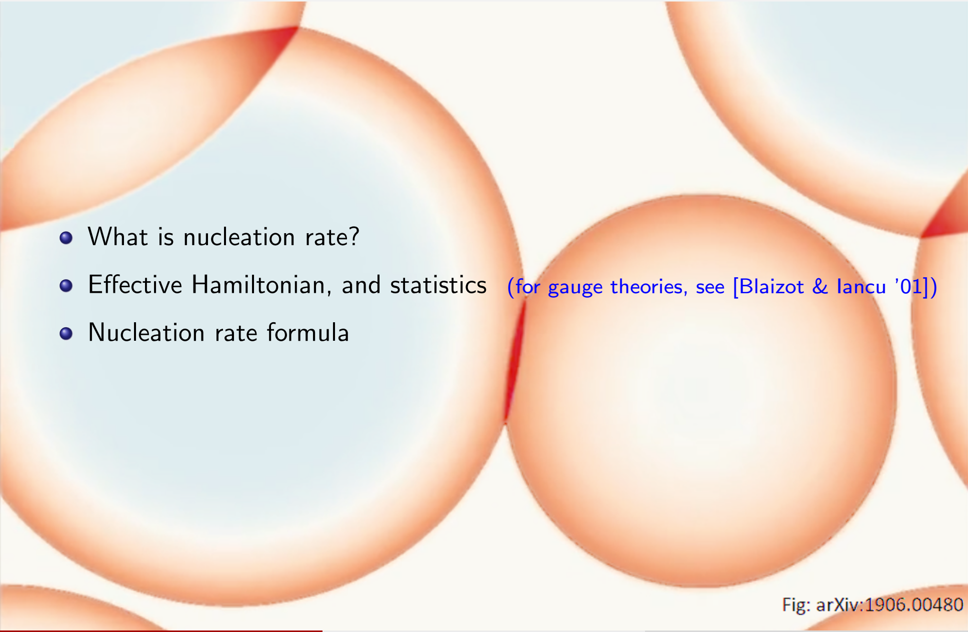
$$E = \sqrt{\mathbf{p}^2 + m^2(\phi(x))}$$

- Equilibrium in thermal potential, V_T
  - Thermal phase structure
- $f_{\text{eq},a}$ time-dependent
  - $m^2(\phi(x))$
  - Sources δf
- Force-term h.o. for light particles, $m^2 \ll T^2$

$$\Box\phi + V_T'(\phi) = -\frac{\mathrm{d}m^2}{\mathrm{d}\phi}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2E}\delta f$$

$$(E\partial_t + \mathbf{k}\cdot\boldsymbol{\nabla})\delta f = -\frac{f_{\text{eq}}'(E)}{2}\frac{\mathrm{d}m^2}{\mathrm{d}\phi}\partial_t\phi$$

Real-time nucleation



- What is nucleation rate?
- Effective Hamiltonian, and statistics (for gauge theories, see [Blaizot & Iancu '01])
- Nucleation rate formula

Fig: arXiv:1906.00480

Nucleation rate

- Barrier between the phases
- Rate, Γ: probability leak
  - Find nucleating distribution function $\rho_{\mathsf{nucl}}[\phi, \pi, \delta f_a]$

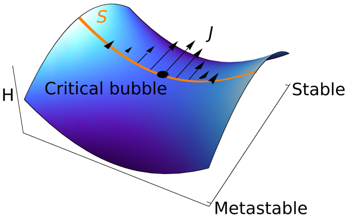


Figure: Barrier between the phases

$$\Gamma = \frac{1}{P_{\mathsf{meta}}} \frac{\mathrm{d}P_{\mathsf{meta}}}{\mathrm{d}t} = \frac{1}{\int \mathcal{D}\Gamma_{\mathsf{meta}} \rho_{\mathsf{nucl}}} \int \mathcal{D}S \cdot \pi \rho_{\mathsf{nucl}}$$

Effective Hamiltonian, H_{eff}

- Structure
  - Basic field terms
  - Particle-term related to quantum statistics
- Conserved under time evolution

$$H_{\text{eff}} = \int \mathrm{d}^d\mathbf{x} \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V_T(\phi) + \sum_a \frac{1}{2} \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^3} \frac{\delta f_a^2}{\left| f'_{\text{eq},a} \right|} \right)$$

- Time evolution from Poisson brackets
  - Not relevant for our analysis

$$\{\phi(\mathbf{x}), \pi(\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y}),$$

$$\{\delta f_a(\mathbf{x},\mathbf{p}), \delta f_a(\mathbf{y},\mathbf{q})\} = (2\pi)^3 f'_{\text{eq},a}\delta(\mathbf{p}-\mathbf{q}) \times \mathbf{v}\cdot\nabla\delta(\mathbf{x}-\mathbf{y}),$$

$$\{\pi(\mathbf{x}), \delta f_a(\mathbf{y},\mathbf{q})\} = \frac{f'_{\text{eq},a}}{2E}\frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\delta(\mathbf{x}-\mathbf{y})$$

Equilibrium distribution function

- Given by the Hamiltonian
- Equilibirum distribution for ϕ
  - *High-temperature dimensional reduction*
  - Connection to equilibrium methods [Gould & Hirvonen '21]
- Bose-Einstein and Fermi-Dirac statistic
  - White noise

$$
\begin{aligned}
\rho_{\text{eq}} &\propto e^{-\beta H_{\text{eff}}} \\
&= \exp\left(-\beta \int \mathrm{d}^d\mathbf{x} \frac{1}{2}\pi^2\right) \\
&\quad \times \exp\left(-\beta \int \mathrm{d}^d\mathbf{x} \left(\frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + V_T(\phi)\right)\right) \\
&\quad \times \exp\left(-\beta \int \mathrm{d}^d\mathbf{x} \sum_a \frac{1}{2} \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^3} \frac{\delta f_a^2}{-f'_{\text{eq},a}}\right)
\end{aligned}
$$

Nucleating distribution

- Thermal flow from metastable phase

$$\rho_{\text{nucl}} = \underbrace{\theta(u)}_{\substack{\text{flow from}\\ \text{metastable}\\ \text{phase}}} \underbrace{Z_{\text{meta}}^{-1}\, e^{-\beta H_{\text{eff}}}}_{\substack{\text{in thermal}\\ \text{equilibrium}}}$$

- (Solves the Liouville equation)

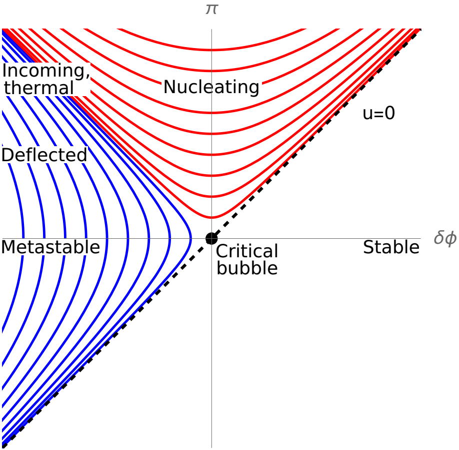


Figure: 2D slice of the phase space

Form of the results

- Matches Langer
- Equilibrium part unchanged!
  [Gould & Hirvonen '21]
- Particles affect κ
  - Exponential growth rate

$$\frac{\Gamma}{V} = \frac{\kappa}{2\pi}\left(\frac{H_{\rm CB}}{2\pi T}\right)^{\frac{3}{2}} \left|\frac{\det\left(-\nabla^2 + V''_{\rm meta}\right)}{\det'\left(-\nabla^2 + V''_{\rm CB}\right)}\right|^{\frac{1}{2}} e^{-\beta H_{\rm CB}},$$

$$H_{\rm CB} = H_{\rm eff}[\phi_{\rm CB}] - H_{\rm eff}[\phi_{\rm meta}]$$

Exponential growth

- In EoMs, $\partial_t \to \kappa$:

$$\partial_t\left(\overline{\delta\phi}\, e^{\kappa t}\right) = \kappa\left(\overline{\delta\phi}\, e^{\kappa t}\right)$$

- Friction from off-equilibrium particles

$$\kappa^2\overline{\delta\phi} = (\boldsymbol{\nabla}^2 - V''_{\rm CB})\overline{\delta\phi} - \sum_a \frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\int\frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^3 2E}\overline{\delta f_a}$$

$$\kappa\overline{\delta f_a} = -\mathbf{v}\cdot\boldsymbol{\nabla}\overline{\delta f_a} - \frac{f'_{\mathrm{eq},a}}{2E}\frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\kappa\overline{\delta\phi}$$

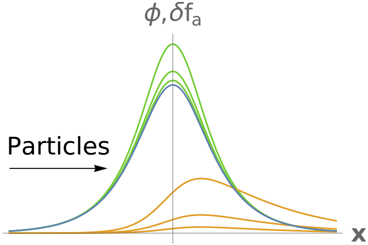


Figure: Exponentially growing configuration, δf_a comes from the left

Summary

- Boltzmann equations arise from real-time QFTs
- Hamiltonian allows solving for nucleation rate
- Results
  - Particles affect the exponential growth rate, κ
  - Corroborates equilibrium computations

$$\frac{\Gamma}{V} = \frac{\kappa}{2\pi}\left(\frac{H_{\mathrm{CB}}}{2\pi T}\right)^{\frac{3}{2}}\left|\frac{\det\left(-\nabla^2 + V''_{\mathrm{meta}}\right)}{\det'\left(-\nabla^2 + V''_{\mathrm{CB}}\right)}\right|^{\frac{1}{2}} e^{-\beta H_{\mathrm{CB}}}$$

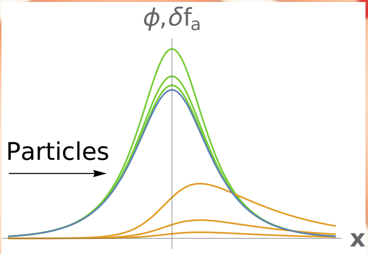


Fig: arXiv:1906.00480

Outlook

$$\Delta t = \left(\frac{\mathrm{d}}{\mathrm{d}t} \ln \Gamma \right)^{-1}$$

$$\Delta t^4 \Gamma \approx 1 \Rightarrow T_\mathrm{n}$$

- Realistic models, and effects on GWs through
  - Duration of the transition
  - Nucleation temperature
  - Benoit Laurent, Perimeter Institute

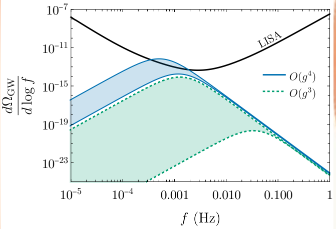


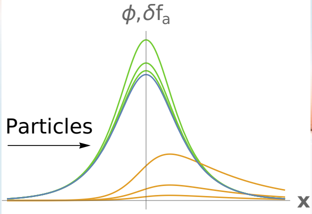
Figure: GW uncertainties [Gould, Tenkanen '21]

Fig: arXiv:1906.00480

Thanks for listening!

$$\frac{\Gamma}{V} = \frac{\kappa}{2\pi}\left(\frac{H_{\mathrm{CB}}}{2\pi T}\right)^{\frac{3}{2}} \left|\frac{\det\left(-\nabla^2 + V''_{\mathrm{meta}}\right)}{\det'\left(-\nabla^2 + V''_{\mathrm{CB}}\right)}\right|^{\frac{1}{2}} e^{-\beta H_{\mathrm{CB}}}$$

- Boltzmann equations from QFTs
- Hamiltonian for nucleation rate
- Results
  - Off equilibrium in κ
  - Otherwise equilibrium
- Future: effects on GWs



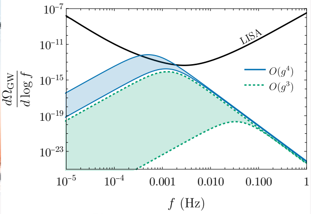


Figure: [Gould, Tenkanen '21]

$u = 0$ surface

- Not originating from either of the phases
- E.g. exponential growth:

  $$\delta\phi = \overline{\delta\phi}\, e^{\kappa t}, \quad \delta f_a = \overline{\delta f_a}\, e^{\kappa t}$$

  - Asymptotically from the critical bubble, $t \to -\infty$

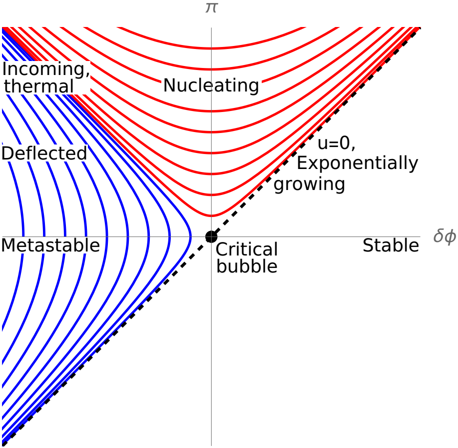


Figure: 2D slice of the phase space

Why no drastic changes

- Nucleation rate is exponentially rare
  - Fluctuations have to conspire to get there
- Extra fluctuations of plasma particles exponentially suppressed still

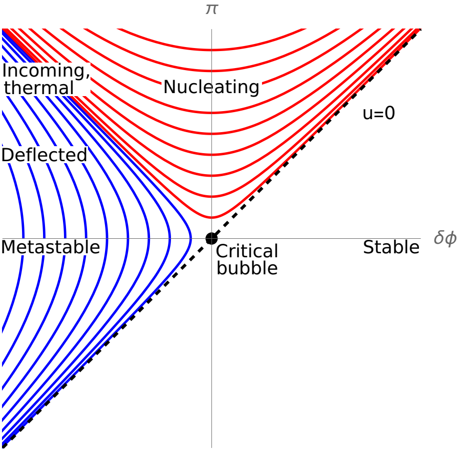


Figure: 2D slice of the phase space

Bosonic IR divergence

- Source of off equilibrium has $f'_{\mathrm{eq},a}$
  - Dominant fermionic: $E \sim T$
  - Dominant bosonic: $E \ll T$
- Bosonic fields damp more than the particles

$$\kappa\overline{\delta f_a} = -\mathbf{v}\cdot\boldsymbol{\nabla}\overline{\delta f_a} - \frac{f'_{\mathrm{eq},a}}{2E}\frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\kappa\overline{\delta\phi}$$

$$\Rightarrow \overline{\delta f_a} \propto \frac{f'_{\mathrm{eq},a}}{E}$$

$$\int_{\Lambda_{\mathrm{IR}}}^{\infty}\mathrm{d}E\, f'_{\mathrm{eq,ferm}} = -\frac{1}{2} + \mathcal{O}\left(\frac{\Lambda_{\mathrm{IR}}}{T}\right)$$

$$\int_{\Lambda_{\mathrm{IR}}}^{\infty}\mathrm{d}E\, f'_{\mathrm{eq,bos}} = -\frac{T}{\Lambda_{\mathrm{IR}}} + \frac{1}{2} + \mathcal{O}\left(\frac{\Lambda_{\mathrm{IR}}}{T}\right)$$

Collisions are OK!

- Problem: Fluctuations vanishing
- No fluctuations on exponential configuration
- Conjecture: rate formula holds
  - Langer's universality
  - Fluctuation-dissipation relation
- Could change the rate drastically

$$\kappa^2\overline{\delta\phi} = (\nabla^2 - V''_{\mathrm{CB}})\overline{\delta\phi} - \sum_a \frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^3 2E}\overline{\delta f_a}$$

$$\kappa\overline{\delta f_a} = -\mathbf{v}\cdot\nabla\overline{\delta f_a} - \frac{f'_{\mathrm{eq},a}}{2E}\frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\kappa\overline{\delta\phi} + C_{\mathrm{lin}}[\overline{\delta f_a}]$$

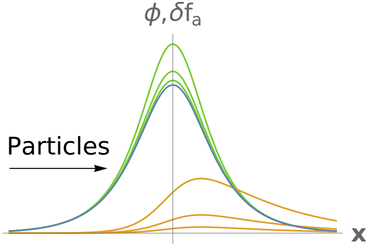


Figure: Exponentially growing configuration, δf_a comes from the left