MaRTIn and four-loop operator mixing

(Towards an NNNLO prediction of ϵ_K)

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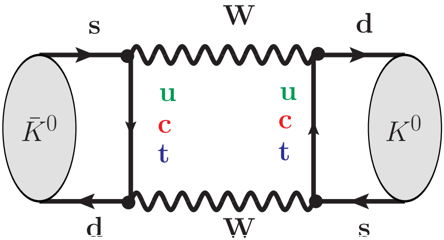
PHENO 2026, Pittsburgh

May 12th, 2025

Test short-distance dynamics with kaons

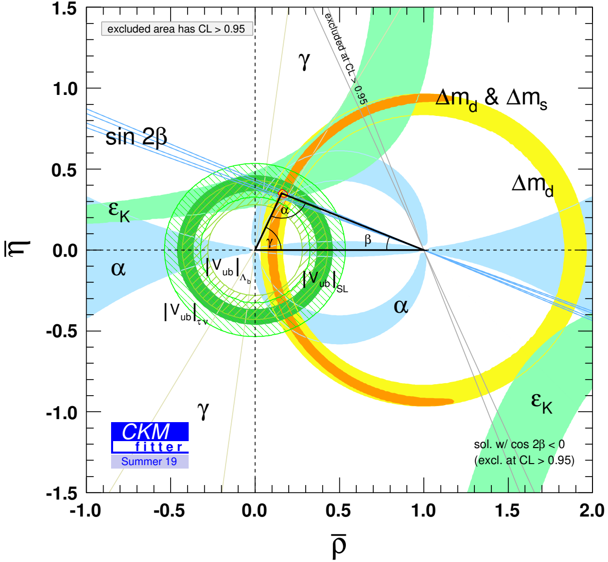
- Most kaon observables are long-distance dominated
- Few exceptions (mostly CP violation and rare decays)
- CKM structure and QCD effects make kaons sensitive probes of high-energy scales
    - In the SM
    - Beyond the SM
- Need both precise measurements and solid theory predictions

Reminder: neutral kaons



$\bar{K}^0 \sim [s\bar{d}]$ $\qquad$ $K^0 \sim [\bar{s}d]$

$$|\epsilon_K| = (2.228 \pm 0.011) \times 10^{-3}$$



excluded area has CL > 0.95
excluded at CL > 0.95
γ
Δm_d & Δm_s
sin 2β
Δm_d
ε_K
α
β
γ
$|V_{ub}|_{\Lambda_b}$
$|V_{ub}|_{SL}$
$|V_{ub}|_{\tau\nu}$
CKM fitter
Summer 19
sol. w/ cos 2β < 0
(excl. at CL > 0.95)
$\bar\eta$
$\bar\rho$
1.5
1.0
0.5
0.0
-0.5
-1.0
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0.5
1.0
1.5
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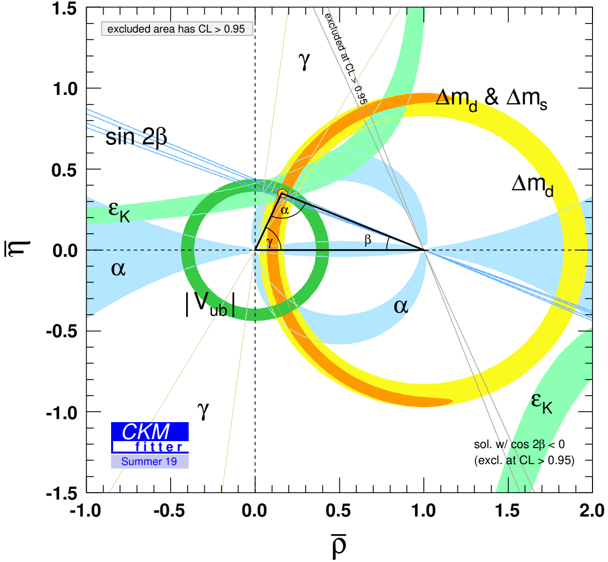
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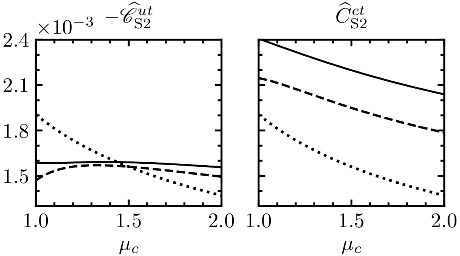
$\bar{\rho}$

SD contribution to ϵ_K – charm

- The use of "u-t unitarity" was proposed in 2012
  [Christ et al., 1212.5931; see also Barbieri 2007]

- Application to ϵ_K short-distance contribution [Brod, Gorbahn, Stamou 1911.06822]
  - Can "recycle" NNLO anomalous dimensions and matching conditions

- "Simple" rearrangement of effective Hamiltonian has reduced perturbative uncertainty (@ NNLO!) from $\sim 30\%$ to $\sim 1\%$

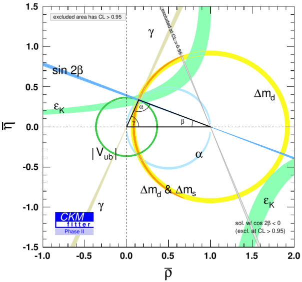


Z. Ligeti, "Flavor Physics in the New Era"

Towards theory at the per mille level

- Theory uncertainty of γ is negligible (2nd order EW)
  World average $\gamma = (66.4 \pm 2.8)^\circ$

  Theory uncert. of everything else nontrivial at $\sim 1\%$ level
- Most precise so far: $\sin 2\beta = 0.710 \pm 0.011$ $(22.62 \pm 0.45)^\circ$
  Precision of γ may overtake that of β
- Can the precision of β keep up with statistics?
  There are claims of possibly large V_{ub} ("penguin") contamination
- Which other measurements can reach such precision?
  (Both theory and experiment)

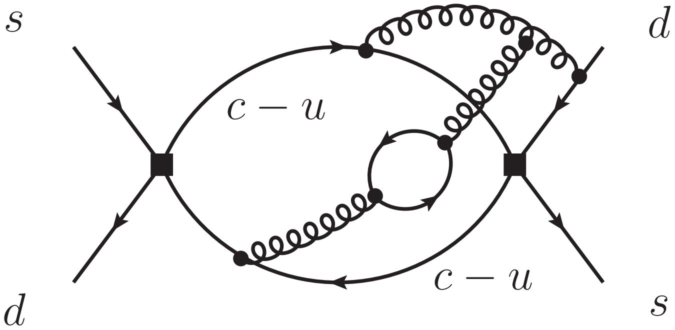


[2006.04824 — expectation early 2040s]

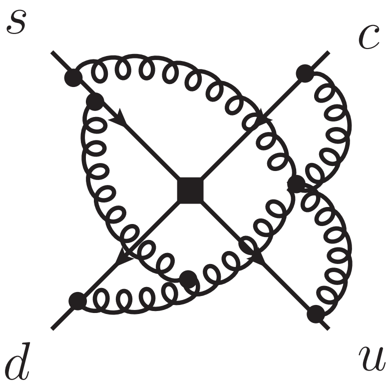
"To boldly go where no one has gone before"

(. . . i.e., to sub-percent precision for kaon mixing)

Double insertions at NNNLO



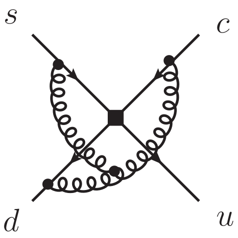
$|\Delta S| = 1$ mixing at NNNLO



[Brod, Stamou, Steudtner, 2604.16691]

Evanescent scheme dependence

- We calculate in $d = 4 - 2\epsilon$
  - $E = (\bar{s}_L \gamma_{\mu_1\mu_2\mu_3} c_L)(\bar{u}_L \gamma^{\mu_1\mu_2\mu_3} d_L) - 16(\bar{s}_L \gamma_\mu c_L)(\bar{u}_L \gamma^\mu d_L) = \mathcal{O}(\epsilon)$
  - $E = (\bar{s}_L \gamma_{\mu_1\mu_2\mu_3} c_L)(\bar{u}_L \gamma^{\mu_1\mu_2\mu_3} d_L) - 16(\bar{s}_L \gamma_\mu c_L)(\bar{u}_L \gamma^\mu d_L) + \epsilon Q_{\text{phys}} = \mathcal{O}(\epsilon)$
- $\epsilon \times (1/\epsilon^2) = 1/\epsilon$
- Can use this to “engineer” anomalous dimensions starting at two-loop



$|\Delta S| = 1$ mixing at NNNLO

$$\gamma^{(3)}_{Q_+^{qq'} \to Q_+^{qq'}} = \frac{250492828261063}{101768400} - \frac{1023760733731}{19081575} N_f - \frac{37339436464}{6360525} N_f^2 - \frac{68}{27} N_f^3 - \left(\frac{1428841}{243} + \frac{3671912}{729} N_f - \frac{57952}{243} N_f^2 - \frac{64}{27} N_f^3 \right) \zeta_3 + \left(\frac{616}{5} + \frac{104}{45} N_f - \frac{16}{27} N_f^2 \right) \pi^4 - \left(\frac{1563260}{81} - \frac{35920}{9} N_f \right) \zeta_5 \,,$$

$$\gamma^{(3)}_{Q_-^{qq'} \to Q_-^{qq'}} = \frac{2677463242787}{50884200} - \frac{75148351018}{19081575} N_f + \frac{491735576}{6360525} N_f^2 + \frac{136}{27} N_f^3 + \left(\frac{2807306}{243} + \frac{4756048}{729} N_f - \frac{50624}{243} N_f^2 - \frac{128}{27} N_f^3 \right) \zeta_3 - \left(\frac{616}{5} + \frac{544}{45} N_f - \frac{32}{27} N_f^2 \right) \pi^4 + \left(\frac{1771960}{81} - \frac{53920}{9} N_f \right) \zeta_5 \,.$$

Multiloop calculations with MaRTIn

- Roughly 250,000 four-loop Feynman diagrams
- Calculated with updated version of MaRTIn:
  [Brod, Hüdepohl, Stamou, Steudtner, 2401.04033]
  - Improved Dirac algebra
  - Utilizes FMFT [Pikelner 1707.01710] and master integrals from [Czakon 0411261]
  - Reduce first, then calculate all master integrals in one go
  - Store results in integral tables
- Setup will be made public in the near future

SD contribution to ϵ_K – Outlook

- NNLO QCD corrections to the top contribution
  [Gorbahn, Sondermann, Stamou, W.I.P.]
  - Residual error expected below percent level

- N^3LO QCD corrections to the charm contribution
  [Brod, Stamou, Steudtner, W.I.P.]
  - ✓ 4-loop operator mixing (current-current operators)
  - ⇒ 4-loop operator mixing (double insertions)
  - 3-loop weak-scale initial conditions
  - Full RG analysis
  - Residual error expected below percent level