Extensions Beyond 4-Point Amplitudes in Constructive QED

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MHV Hints at Efficient Alternatives

- n-gluon scattering expression from Parke and Taylor using MHV (1986).
- Single compact expression compared with potentially millions of Feynman diagram contributions.
- Suggests amplitudes have hidden structure, which is obfuscated by Feynman diagrams.

$$A_n(1^+2^+\ldots i^-\ldots j^-\ldots n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}.$$

Spinors

Boldface - Massive Spin Spinor.

Non-boldface - Massless Helicity Spinor.

Building blocks of constructive calculations.

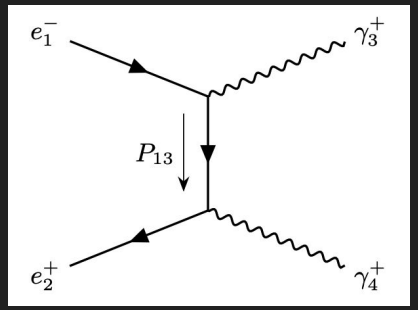
$$|\mathbf{i}\rangle^I = \begin{pmatrix} c\sqrt{E+p} & -s^*\sqrt{E-p} \\ s\sqrt{E+p} & c\sqrt{E-p} \end{pmatrix}$$

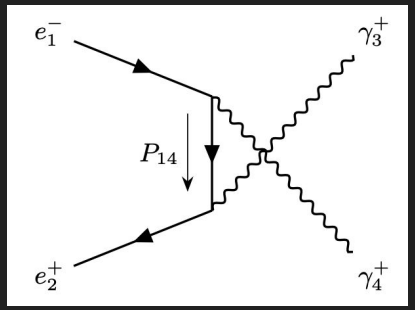
$$|i\rangle = \sqrt{2E}\begin{pmatrix} c \\ s \end{pmatrix}$$

$$c \equiv \cos(\theta/2) \text{ and } s \equiv \sin(\theta/2)e^{i\phi}.$$

Constructive Approach to 4-Point Case

- Agrees with Feynman diagrams.
- Already dramatically simpler.
- Question: How does this extend to higher multiplicities?

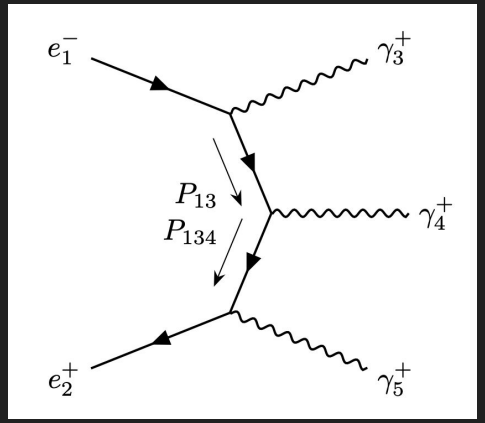




$$\mathcal{M}^{++} = \frac{-e^2 m_e \langle \mathbf{12} \rangle [34]^2}{(s_{13} - m_e^2)(s_{14} - m_e^2)}$$

Preliminary Results: 5-Point Case Final Simplification

- One constructive diagram combines multiple Feynman diagrams.
- Repeated for all possible photon orderings.



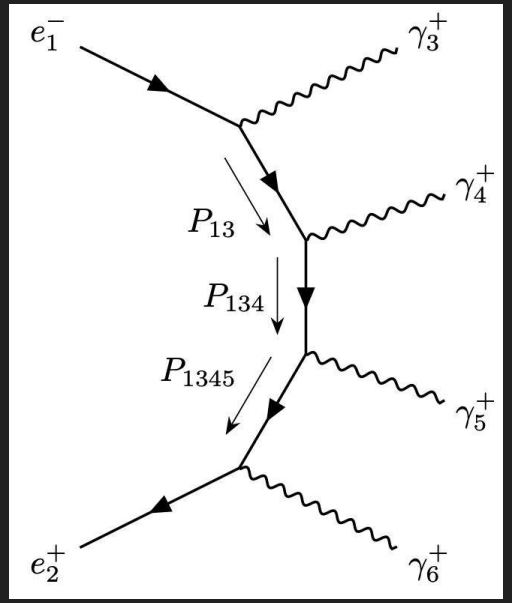
$$\mathcal{M}_{345}^{+++} = \frac{-e^3 m_e \langle \mathbf{12} \rangle [34]^2 [5|p_1 p_2|5]}{(s_{13} - m_e^2)(s_{14} - m_e^2)(s_{15} - m_e^2)(s_{25} - m_e^2)}.$$

Preliminary Results: 5-Point Final Form

- Only six total terms.
- Only three distinct spin structures.

$$\begin{aligned}\mathcal{M}^{+++} = e^3 m_e \langle \mathbf{12} \rangle \Bigg(& \frac{[45]^2 [3|p_1 p_2|3]}{(s_{13} - m_e^2)(s_{23} - m_e^2)} \left[\frac{1}{(s_{24} - m_e^2)(s_{25} - m_e^2)} - \frac{1}{(s_{14} - m_e^2)(s_{15} - m_e^2)} \right] \\ & + \frac{[35]^2 [4|p_1 p_2|4]}{(s_{14} - m_e^2)(s_{24} - m_e^2)} \left[\frac{1}{(s_{23} - m_e^2)(s_{25} - m_e^2)} - \frac{1}{(s_{13} - m_e^2)(s_{15} - m_e^2)} \right] \\ & + \frac{[34]^2 [5|p_1 p_2|5]}{(s_{15} - m_e^2)(s_{25} - m_e^2)} \left[\frac{1}{(s_{23} - m_e^2)(s_{24} - m_e^2)} - \frac{1}{(s_{13} - m_e^2)(s_{14} - m_e^2)} \right] \Bigg)\end{aligned}$$

Preliminary Results: 6-Point Case Calculation



$$\mathcal{M}_{3456}^{++++} \propto \frac{\langle \mathbf{12} \rangle [35]^2 [46]^2}{(s_{13} - m_e^2)(s_{15} - m_e^2)(s_{135} - m_e^2)(s_{24} - m_e^2)(s_{26} - m_e^2)}$$

Preliminary Results: 6-Point Final Form

- Only three unique spin structures in the numerator.
- No remaining momentum insertions.
- Propagator denominators follow a clear pattern.

$$\mathcal{M}^{++++} = -e^4 m_e^3 \langle \mathbf{12} \rangle \Big([34]^2 [56]^2 (D_{34,56} + D_{56,34}) + [35]^2 [46]^2 (D_{35,46} + D_{46,35}) + [36]^2 [45]^2 (D_{36,45} + D_{45,36}) \Big)$$

$$D_{ij,kl} = \frac{1}{(s_{1i} - m_e^2)(s_{1j} - m_e^2)(s_{1ij} - m_e^2)(s_{2k} - m_e^2)(s_{2l} - m_e^2)}$$

Conclusions

- Constructive methods reveal hidden simplicity in scattering amplitudes.
- Even at 5- and 6-point levels, results are significantly more compact than Feynman diagrams.
- Future work: validate against Feynman diagrams, extend to higher points and other helicity combinations.

Auxiliary Slides for Possible Questions

Constructive Approach to 4-Point Case

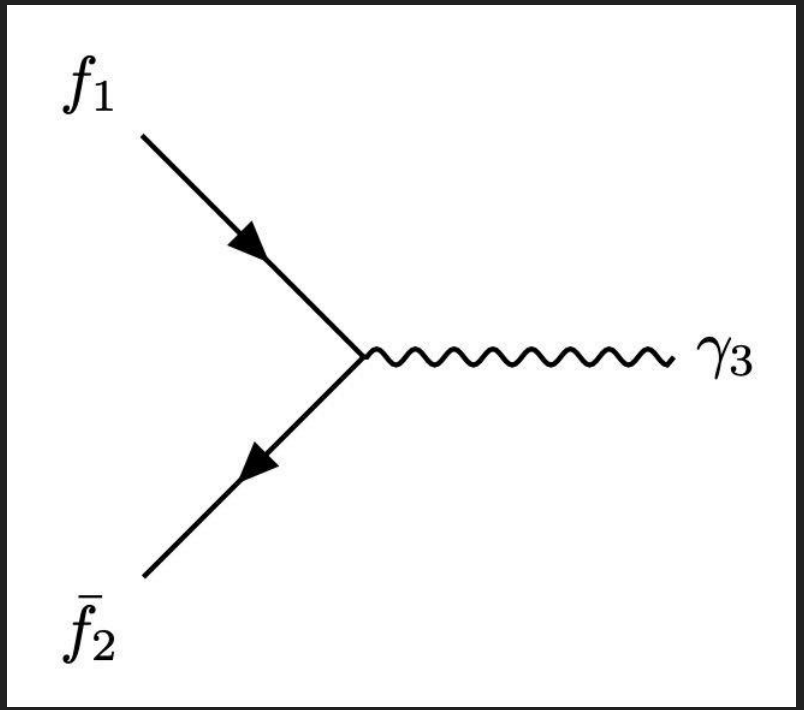
- Agrees with Feynman diagrams.
- Already dramatically simpler.
- Question: How does this extend to higher multiplicities?

$$\mathcal{M}^{++} = \frac{-e^2 m_e \langle \mathbf{12} \rangle [34]^2}{(s_{13} - m_e^2)(s_{14} - m_e^2)}$$

3-Point Vertex of Constructive QED

- Fermion, Antifermion, Photon
- Photon helicity determines structure.

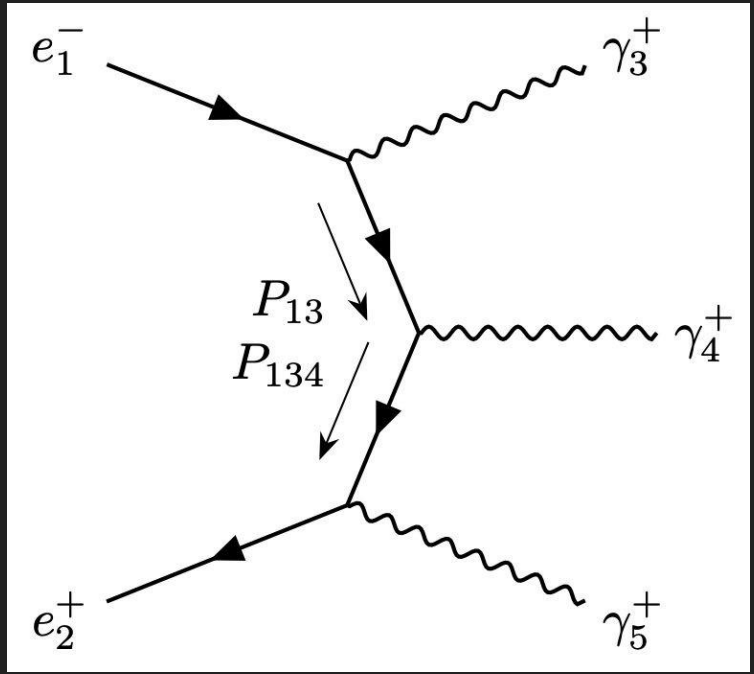
Helicity configuration	Vertex factor
$f_1 \bar{f}_2 \gamma_3^+$	$-ie\, x_{1,2}\, \langle \mathbf{1}\, \mathbf{2} \rangle$
$f_1 \bar{f}_2 \gamma_3^-$	$-ie\, \tilde{x}_{1,2}\, [\mathbf{1}\, \mathbf{2}]$



Preliminary Results: 5-Point Case Setup

- 6 diagram orderings.
- 2 internal propagators per diagram.
- All momenta are incoming.
- Vertices listed below:

$$
\begin{aligned}
&-iex_{1,245}\langle \mathbf{1P}_{245I}\rangle \\
&-iex_{13,25}\langle \mathbf{P}^{I}_{13}\mathbf{P}_{25J}\rangle \\
&-iex_{134,2}\langle \mathbf{P}^{J}_{134}\mathbf{2}\rangle
\end{aligned}
$$



Preliminary Results: 5-Point Case Calculation

- Obtain relevant vertices from the diagram and table.
- Multiply the vertices together.
- Contract the spin indices.

$$
\begin{aligned}
&- iex_{1,245}\langle \mathbf{1P}_{245I}\rangle \\
&- iex_{13,25}\langle \mathbf{P}^{I}_{13}\mathbf{P}_{25J}\rangle \\
&- iex_{134,2}\langle \mathbf{P}^{J}_{134}\mathbf{2}\rangle
\end{aligned}
$$

$$
i\mathcal{M}^{+++}_{345} = \frac{(-ie)^3 i^2 x_{1,245}x_{13,25}x_{134,2}\langle \mathbf{1P}_{245I}\rangle\langle \mathbf{P}^{I}_{13}\mathbf{P}_{25J}\rangle\langle \mathbf{P}^{J}_{134}\mathbf{2}\rangle}{(s_{13}-m_e^2)(s_{25}-m_e^2)}
$$

Preliminary Results: 5-Point Case Calculation

- Apply momentum conservation.
- Apply mass identity.

$$\langle \mathbf{1P}_{245I} \rangle = -\langle \mathbf{1P}_{13I} \rangle$$
$$\langle \mathbf{P}^J_{134} \mathbf{2} \rangle = -\langle \mathbf{P}^J_{25} \mathbf{2} \rangle,$$

$$|\mathbf{i}\rangle_I \langle \mathbf{i}|^I = -m_i$$

$$\mathcal{M}^{+++}_{345} = \frac{-e^3 m_e^2 x_{1,245} x_{13,25} x_{134,2} \langle \mathbf{12} \rangle}{(s_{13} - m_e^2)(s_{25} - m_e^2)}$$

Preliminary Results: 5-Point x-factors

- Required non-trivial mathematical tool for constructive QED.
- Prepare the expression for final simplification.

$$x_{i,j} = \frac{\langle \xi | p_j | k]}{m \langle \xi k \rangle},$$

$$\tilde{x}_{i,j} = \frac{[\xi | p_j | k \rangle}{m [\xi k]}.$$

$$x_{1,245} = \frac{\langle \xi | P_{245} | 3]}{m_e \langle \xi 3 \rangle},$$

$$x_{13,25} = \frac{\langle \zeta | P_{25} | 4]}{m_e \langle \zeta 4 \rangle},$$

$$x_{134,2} = \frac{\langle \chi | p_2 | 5]}{m_e \langle \chi 5 \rangle}.$$

Replacing Reference Spinors

- Choose an appropriate reference spinor.
- Simplify the momentum insertions in the numerator.
- Evaluate the inner product in the denominator.

$$\langle \xi | = [3|p_2$$

$$x_{1,245} = \frac{[3|p_2 P_{245}|3]}{m_e [3|p_2|3\rangle}.$$

$$x_{1,245} = \frac{[3|p_1 p_2|3]}{m_e (s_{23} - m_e^2)}.$$

Preliminary Results: 5-Point Case Calculation

- Choose reference spinors that lead to a smooth simplification.

$$x_{1,245} = \frac{[3|p_1p_2|3]}{m_e(s_{23} - m_e^2)}.$$

$$x_{13,25} = \frac{-[4|p_1p_3|4]}{m_e(s_{14} - m_e^2)}.$$

$$x_{134,2} = \frac{[5|p_1p_2|5]}{m_e(s_{15} - m_e^2)}.$$

$$\mathcal{M}_{345}^{+++} = \frac{e^3\langle \mathbf{12}\rangle[3|p_1p_2|3][4|p_1p_3|4][5|p_1p_2|5]}{m_e(s_{13} - m_e^2)(s_{14} - m_e^2)(s_{15} - m_e^2)(s_{23} - m_e^2)(s_{25} - m_e^2)}$$

Final Simplification

- Apply a Schouten identity to the spinor products involving particles 3 and 4.
- Do momentum reversals on both terms.
- Second term vanishes.

$$[3|p_1p_2|3][4|p_1p_3|4] = -[4|p_3p_1p_1p_2|3][43] + [4|p_3p_1|3][4|p_1p_2|3]$$

Term 1

$$-2p_2 \cdot p_3 m_e^2 [43]^2$$

Term 2

$$2p_1 \cdot p_3 [43][4|p_1p_2|3] - [4|p_1p_3|3][4|p_1p_2|3]$$

$$-m_e^2(s_{23} - m_e^2)[34]^2.$$

Preliminary Results: 5-Point Case Final Simplification

- Only two momentum insertions.
- Clean propagator structure.

$$\mathcal{M}_{345}^{+++} = \frac{-e^3 m_e \langle \mathbf{12} \rangle [34]^2 [5|p_1 p_2|5]}{(s_{13} - m_e^2)(s_{14} - m_e^2)(s_{15} - m_e^2)(s_{25} - m_e^2)}.$$

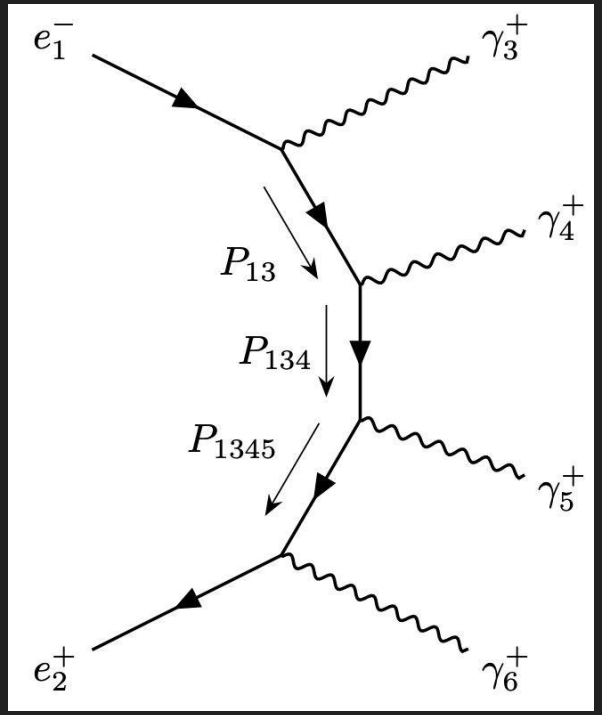
Preliminary Results: 5-Point Case Pattern

- Only three independent numerators.
- Six propagator structures.
- Highly symmetric pairing.

Permutation	Matching Expression	Spin Structure
$\mathcal{M}_{345}$	$\mathcal{M}_{435}$	$[34]^2[5\|p_1p_2\|5]$
$\mathcal{M}_{354}$	$\mathcal{M}_{534}$	$[35]^2[4\|p_1p_2\|4]$
$\mathcal{M}_{453}$	$\mathcal{M}_{543}$	$[45]^2[3\|p_1p_2\|3]$

Preliminary Results: 6-Point Case Setup

- 24 possible diagram orderings.
- 3 internal propagator denominators per diagram.
- All momenta are incoming.



Preliminary Results: 6-Point Case Pattern

- Spinor products grow with particle number.
- Propagator-denominator grouping persists.

Spin Structure	Electron-Line Channel
$[34]^2[56]^2$	$s_{134},\ s_{156}$
$[35]^2[46]^2$	$s_{135},\ s_{146}$
$[36]^2[45]^2$	$s_{136},\ s_{145}$

Overall Emergent Patterns

- On-shell simplifications before summing diagrams.
- Only a small number of independent spin structures.
- Clear photon-number symmetry.
- Propagator clustering by photon-number.