Computing Massive 1→3 Splitting Kernels for Future Parton Shower Models

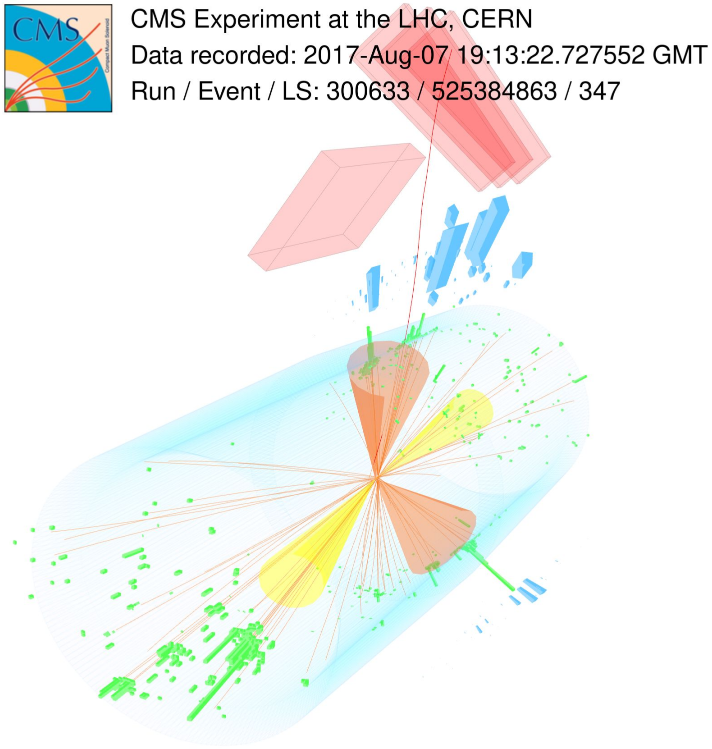
Stefan Höche[1], Matt LeBlanc[2], Jennifer Roloff[2], Grant Whitman[2]
2026 Phenomenology Symposium, Pittsburgh
[1]Fermi National Accelerator Laboratory, Batavia, IL, 60510, USA
[2]Physics Department, Brown University, Providence, RI, 02912, USA

BROWN
Department of Physics

Motivation

Re-examined massive 1→3 splitting kernels in the Scalar+Remainder scheme.

- Industry standard PS use computationally intensive subtraction algorithms.
    - NNLO complicates further
- Computation budget and the HL-LHC.
    - Efficiency, numerical stability, and precision
    - Mass effects
- Natural extension of theory.
    - Reorganization of recursive currents



CMS Event, HH→4b. Massive effects are significant, and factor into Higgs self-coupling measurements.

BROWN
Department of Physics

Monte-Carlo Event Generators

Splitting kernels analytically describe shower evolution at each step.

- MC Event Generators create simulated data, connecting theory and experiment.
- Factorized into component processes:
  - Hard Process
  - Parton Shower
  - Hadronization
- Parton shower is composed of DGLAP kernels.
  - Singularities in soft and collinear limits

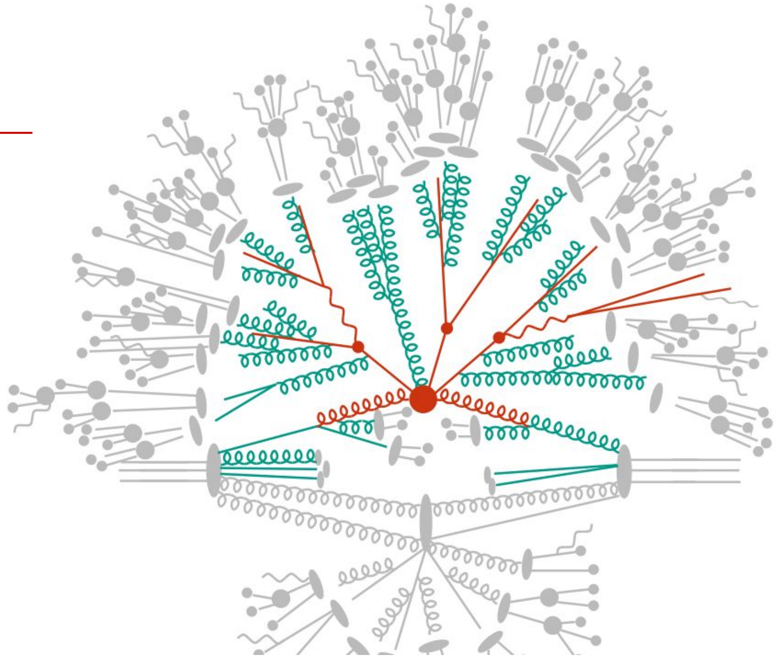


Image credit: Frank Krauss.
Illustration of a tt event.

BROWN
Department of Physics

Scalar+Remainder Calculations

[Hoppe, Knobbe, Preuss, Reichelt, Höche ‘25]

QCD Current can be split into scalar and remainder components, sectoring soft divergences.

- Gordon Decomposition [Gordon, ZeitPhys140(1928)630]
- Separate into terms matching Scalar QCD, and “magnetic” interactions + seagull-like term.

- Scalar current dominates behavior at amplitude level.
  - Leading-order soft divergences
- Allows simple dipole subtraction scheme.

$$\frac{\not{p} + \not{q}}{(p+q)^2} T^a_{ij} \gamma^\mu = T^a_{ij} \left[S^\mu(p,q) + \frac{i\sigma^{\nu\mu} q_\nu}{(p+q)^2} - \frac{\gamma^\mu \not{p}}{(p+q)^2} \right]$$

$$S^\mu(p,q) = \frac{(2p+q)^\mu}{(p+q)^2} \qquad \sigma^{\nu\mu} = \frac{i}{2}[\gamma^\nu, \gamma^\mu]$$

BROWN
Department of Physics

Scalar+Remainder Calculations

[Hoppe, Knobbe, Preuss, Reichelt, Höche ‘25]

QCD Current can be split into scalar and remainder components, sectoring soft divergences.

(a)

$$\frac{\not{p}+\not{q}}{(p+q)^2}T^a_{ij}\gamma^\mu = T^a_{ij}\left[S^\mu(p,q) + \frac{i\sigma^{\nu\mu}q_\nu}{(p+q)^2} - \frac{\gamma^\mu\not{p}}{(p+q)^2}\right]$$

$$S^\mu(p,q) = \frac{(2p+q)^\mu}{(p+q)^2} \qquad \sigma^{\nu\mu} = \frac{i}{2}[\gamma^\nu,\gamma^\mu]$$

BROWN
Department of Physics

Techniques

- Axial gauge.

$$d^{\mu\nu}(p,\bar{n}) = \sum_{\lambda=\pm} \varepsilon^{\mu}_{\lambda}(p,\bar{n})\varepsilon^{*\,\nu}_{\lambda}(p,\bar{n}) = -g^{\mu\nu} + \frac{p^{\mu}\bar{n}^{\nu} + p^{\nu}\bar{n}^{\mu}}{p\bar{n}}$$

- Sudakov parametrization.
    - On-shell condition

$$p_i^{\mu} = z_i \bar{p}^{\mu}_{1..m} + \tilde{k}^{\mu}_i - \frac{\tilde{k}_i^2}{z_i}\frac{\bar{n}^{\mu}}{2\bar{p}_{1..m}\bar{n}}\ , \quad \bar{p}^{\mu} = p^{\mu} - \frac{p^2 - m^2}{2p\bar{n}}\bar{n}^{\mu}$$

- Berends & Giele recursive currents.

$$\Psi_i(p_\alpha) = \sum_{\substack{\{\beta,\gamma\}\in\\ P(\alpha,2)}} g_s T^a_{ij} \frac{i\sigma^{\mu\nu}}{p_\alpha^2 - m_\alpha^2} p_{\gamma,\nu}\, J^a_\mu(p_\gamma, n)\Psi_j(p_\beta)\ ,$$

$$+ \sum_{\substack{\{\beta,\gamma\}\in\\ P(\alpha,2)}} \left[g_s T^a_{ij} S^{\mu}(p_\beta, p_\gamma) J^a_\mu(p_\gamma, n) - \sum_{\substack{\{\delta,\epsilon\}\in\\ OP(\gamma,2)}} \frac{g_s^2}{p_\alpha^2} \left\{T^a, T^b\right\}_{ij} J^{\mu,a}(p_\delta, n) J^b_\mu(p_\epsilon, n) \right] \Psi_j(p_\beta)$$

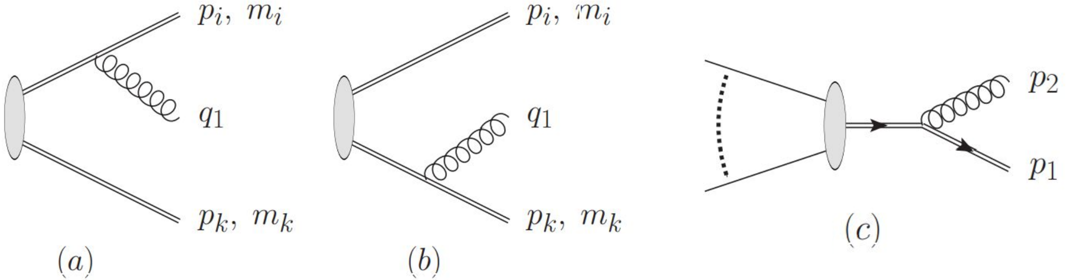
BROWN
Department of Physics

Q→Qg Example

Scalar components calculated separately; contain leading 1/z_2 soft singularities. Full QCD result is the sum.

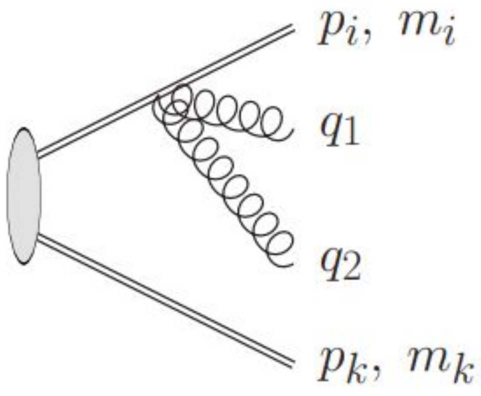
Scalar:
$$P_{\tilde{q}\to\tilde{q}}(p_1,p_2) = C_F\left[\frac{2z_1}{z_2}\left(1-\frac{p_1^2-m_1^2}{p_{12}^2-m_1^2}\frac{z_{12}}{z_1}-\frac{p_2^2}{p_{12}^2-m_1^2}\frac{z_{12}}{z_2}\right)-\frac{2m_1^2}{p_{12}^2-m_1^2}\right],$$

Remainder:
$$\langle P^{(\mathrm{f})}_{q\to q}(p_1,p_2)\rangle = C_F\,(1-\varepsilon)\left(\frac{z_2}{z_{12}}-\frac{z_2}{z_1}\frac{p_1^2-m_1^2}{p_{12}^2-m_1^2}-\frac{p_2^2}{p_{12}^2-m_1^2}\right)$$



Results

- Kernels match literature (Dhani 2025), and extend the massless forms (Höche 2025).
- Isolates soft structure, simplifying subtraction of phase-space overlap.
- Avoid quasi-collinear limit, commutativity issues.
- Compactness, yielding numerically stability.
    - Important for ML applications
- Novel calculation of scalar 2-gluon emission structure.



The seagull vertex is present in scalar QCD, but not in fermionic.

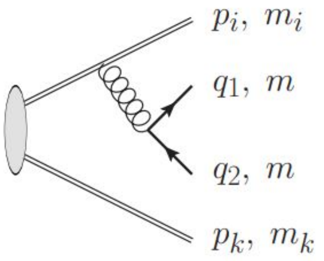
BROWN
Department of Physics

Massive Q→QQQ Example

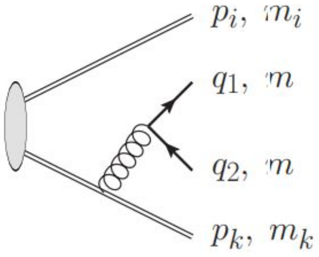
QQQ splitting kernel contains di-quark dipole emission, and g→QQ splitting components.

$$\langle P_{q\to\bar{q}'q'q}(p_1,p_2,p_3)\rangle = \frac{C_F T_R}{2}\frac{s_{123}-m_3^2}{s_{12}}\left[-\frac{t_{12,3}^2}{s_{12}(s_{123}-m_3^2)} + \frac{4z_3+(z_1-z_2)^2}{z_1+z_2} - \frac{4m_3^2}{s_{123}-m_3^2} + \left(1-2\varepsilon+\frac{4m_1^2}{s_{12}}\right)\left(z_1+z_2-\frac{s_{12}}{s_{123}-m_3^2}\right)\right].$$

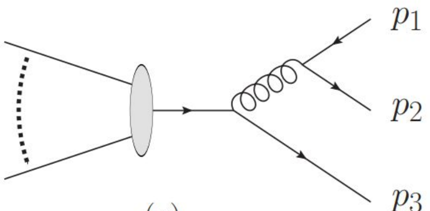
$$\mathcal{S}_{i;k}^{(q\bar{q})}(q_1,q_2;\bar{n}) = \frac{2}{\tilde{s}_{i12}\,\tilde{s}_{k12}}\left[\frac{2}{s_{12}}\left(\frac{z_i\tilde{s}_{k12}+z_k\tilde{s}_{i12}}{z_1+z_2}-\tilde{s}_{ik}\right)+1-\frac{t_{12,i}t_{12,k}}{s_{12}^2}\right].$$



(a)



(b)



(c)

BROWN
Department of Physics

Massive Q→QQQ Example

Form is significantly more compact, improving stability.

Us:

$$\langle P_{q\to\bar{q}'q'q}(p_1,p_2,p_3)\rangle = \frac{C_F T_R}{2}\frac{s_{123}-m_3^2}{s_{12}}\Bigg[-\frac{t_{12,3}^2}{s_{12}(s_{123}-m_3^2)}+\frac{4z_3+(z_1-z_2)^2}{z_1+z_2}-\frac{4m_3^2}{s_{123}-m_3^2} + \left(1-2\varepsilon+\frac{4m_1^2}{s_{12}}\right)\left(z_1+z_2-\frac{s_{12}}{s_{123}-m_3^2}\right)\Bigg].$$

Dhani:

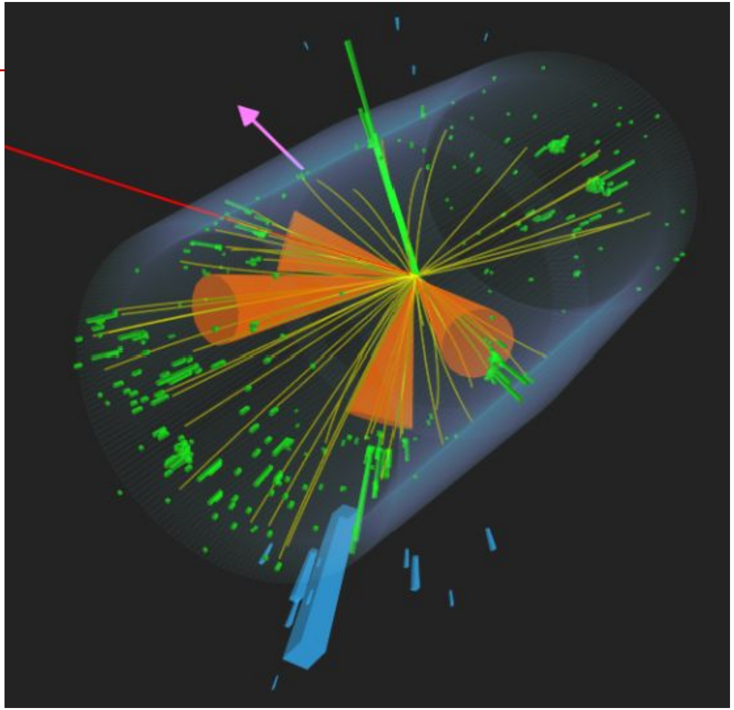
$$\langle \hat{P}^{(0)}_{\bar{Q}_1'Q_2'Q_3}\rangle = C_F T_R\Bigg\{\frac{\tilde{s}_{12}\tilde{s}_{123}}{2s_{12}^2}\left[-\frac{t_{12,3}^2}{\tilde{s}_{12}\tilde{s}_{123}}+\frac{4z_3+(z_1-z_2)^2}{1-z_3}+(1-2\epsilon)\left(z_1+z_2-\frac{\tilde{s}_{12}}{\tilde{s}_{123}}\right)\right]$$
$$+\frac{2m_{Q'}^2}{s_{12}^2}\Bigg[\frac{z_3\tilde{s}_{123}}{(1-z_3)^2}(1+2z_3-3z_3^2+4z_1z_2)-\frac{\tilde{s}_{23}}{1-z_3}(2-3z_1-5z_2+z_1^2+z_2^2)$$
$$-\frac{\tilde{s}_{13}}{1-z_3}(2-5z_1-3z_2+z_1^2+z_2^2)-\epsilon\Big(\tilde{s}_{123}(1-z_3)-\tilde{s}_{12}(1+z_3)\Big)\Bigg]-2\frac{m_Q^2\tilde{s}_{12}}{s_{12}^2}$$
$$+\frac{4m_{Q'}^4}{s_{12}^2}z_3\left[\epsilon+\frac{2z_1z_2}{(1-z_3)^2}+\frac{2z_3}{1-z_3}\right]-4\frac{m_Q^2m_{Q'}^2}{s_{12}^2}\Bigg\}. \qquad (5.2)$$

BROWN
Department of Physics

Conclusion

Calculated the massive 1→3 splitting kernels in the Scalar+Remainder scheme.

- Techniques are physically motivated, and consistent including masses.
    - Compact forms → speed and stability
- Soft divergences separate at leading order.
- Potential for further work:
    - Incorporating into MC (Massless successful in Alaric PS)
    - Loop diagrams & higher orders
    - Wilson line formalism
    - Stability from gauge terms



CMS Candidate Event, di-lepton ttH. Used in measuring top Yukawa, relies on top-initiated massive b-quark showers.

BROWN
Department of Physics

Backup Slides

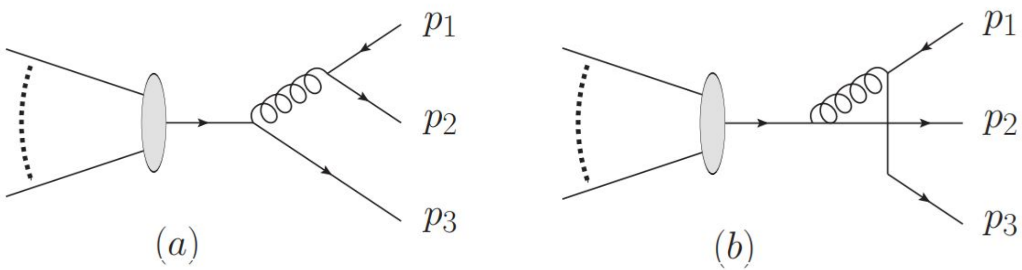
Function $\times s_{123}^{-2}$	Definition	Scaling behavior for $\lambda \to 0$			
		$p_1 \to \lambda p_1,\ p_2 \to \lambda p_2$	$p_1 \to \lambda p_1$	$\tilde{p}_{1,2} \to \lambda \tilde{p}_{1,2}$	$\tilde{p}_{2,3} \to \lambda \tilde{p}_{2,3}$
$P_{\bar{q} \to q' \bar{q}' \bar{q}}$	Eq. (32)	$\propto \lambda^{-4}$	–	$\propto \lambda^{-2}$	–
$\langle P^{(p)}_{q \to q \bar{q} q} \rangle$	Eq. (67)	$\propto \lambda^{-3}$	–	$\propto \lambda^{0}$	$\propto \lambda^{0}$
$P^{(\mathrm{ab})}_{\tilde{q} \to g g \tilde{q}}$	Eq. (37)	$\propto \lambda^{-4}$	$\propto \lambda^{-2}$	–	$\propto \lambda^{-2}$
$P^{(\mathrm{ab},p)}_{\tilde{q} \to g g \tilde{q}}$	Eq. (71)	$\propto \lambda^{-4}$	$\propto \lambda^{-1}$	–	$\propto \lambda^{0}$
$\langle P^{(\mathrm{ab})}_{q \to g g q} \rangle$	Eq. (36)	$\propto \lambda^{-4}$	$\propto \lambda^{-2}$	–	$\propto \lambda^{-2}$
$\langle P^{(\mathrm{ab},p,f)}_{q \to g g q} \rangle$	Eq. (72)	$\propto \lambda^{-2}$	$\propto \lambda^{-1}$	–	$\propto \lambda^{0}$
$P^{(\mathrm{nab})}_{\tilde{q} \to g g \tilde{q}}$	Eq. (39)	$\propto \lambda^{-4}$	$\propto \lambda^{-2}$	$\propto \lambda^{-2}$	–
$P^{(\mathrm{pnab},p)}_{\tilde{q} \to g g \tilde{q}}$	Eq. (75)	$\propto \lambda^{-4}$	$\propto \lambda^{-1}$	$\propto \lambda^{0}$	–
$\langle P^{(\mathrm{nab})}_{q \to g g q} \rangle$	Eq. (38)	$\propto \lambda^{-4}$	$\propto \lambda^{-2}$	$\propto \lambda^{-2}$	–
$\langle P^{(\mathrm{pnab},p,f)}_{q \to g g q} \rangle$	Eq. (76)	$\propto \lambda^{-2}$	$\propto \lambda^{-1}$	$\propto \lambda^{0}$	–
$P^{\mu\nu\,(\mathrm{ab})}_{g \to g q \bar{q}}$	Eq. (41)	–	$\propto \lambda^{-2}$	–	$\propto \lambda^{-2}$
$P^{\mu\nu\,(\mathrm{ab},p)}_{g \to g q \bar{q}}$	Eq. (78)	–	$\propto \lambda^{-1}$	–	$\propto \lambda^{0}$
$P^{\mu\nu\,(\mathrm{nab})}_{g \to g q \bar{q}}$	Eq. (42)	$\propto \lambda^{-4}$	$\propto \lambda^{-2}$	–	$\propto \lambda^{-2}$
$P^{\mu\nu\,(\mathrm{pnab},p)}_{g \to g q \bar{q}}$	Eq. (81)	$\propto \lambda^{-3}$	$\propto \lambda^{-1}$	–	$\propto \lambda^{0}$
$P^{\mu\nu}_{g \to g g g}$	Eq. (43)	$\propto \lambda^{-4}$	$\propto \lambda^{-2}$	$\propto \lambda^{-2}$	
$P^{\mu\nu\,(p)}_{g \to g g g}$	Eq. (84)	$\propto \lambda^{-3}$	$\propto \lambda^{-1}$	$\propto \lambda^{0}$	

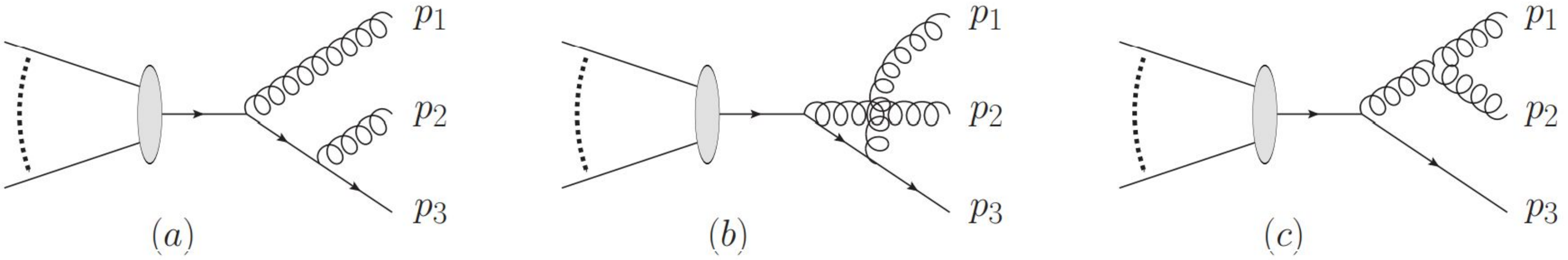
TABLE I. Scaling behavior of the tree-level splitting functions and their pure components. See the main text for details.

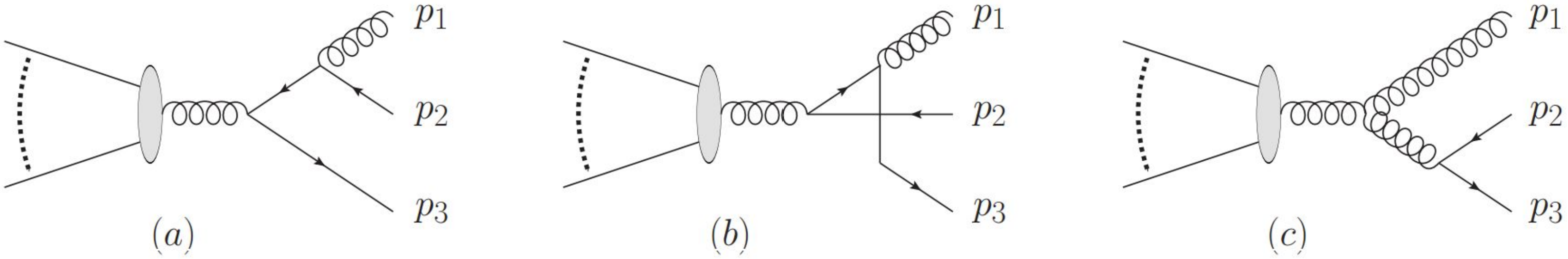
Höche et al. (2025)

BROWN
Department of Physics

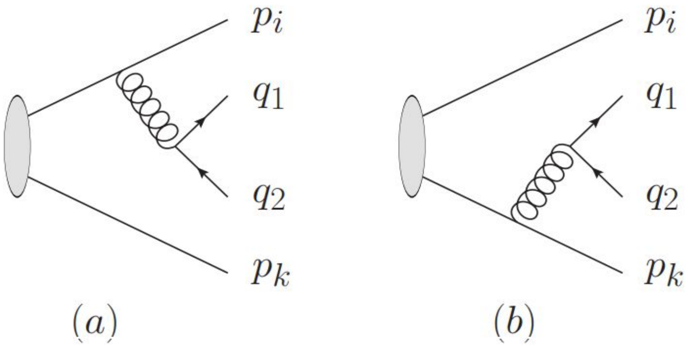
1->3 Diagrams


p_1
p_2
p_3
(a)
p_1
p_2
p_3
(b)


p_1
p_2
p_3
(a)
p_1
p_2
p_3
(b)
p_1
p_2
p_3
(c)


p_1
p_2
p_3
(a)
p_1
p_2
p_3
(b)
p_1
p_2
p_3
(c)

Dipole Diagrams



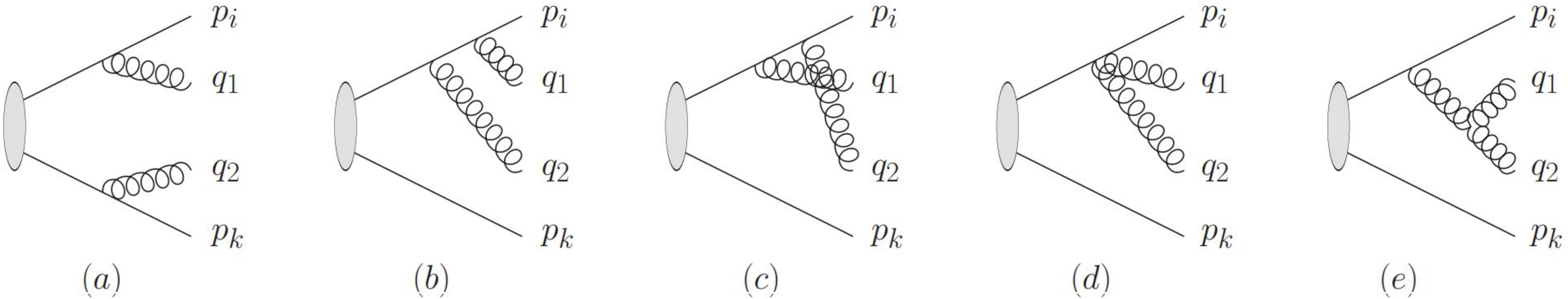


Image credit: Höche et al. (2025)

$$
\begin{aligned}
\langle P^{(\mathrm{id})}_{q\to\bar{q}qq}(p_1,p_2,p_3)\rangle = C_F\left(C_F-\frac{C_A}{2}\right)\Bigg\{ & -\frac{(s_{123}-m^2)^2}{s_{12}s_{13}}\frac{z_1}{2}\left[\frac{1+z_1^2}{(1-z_2)(1-z_3)}-\varepsilon\left(1+2\frac{1-z_2}{1-z_3}\right)-\varepsilon^2\right] \\
& +\frac{s_{123}-m^2}{s_{12}}\left[\frac{1+z_1^2}{1-z_2}-\frac{2z_2}{1-z_3}-\varepsilon\left(\frac{(1-z_3)^2}{1-z_2}+1+z_1-\frac{2z_2}{1-z_3}\right)-\varepsilon^2(1-z_3)\right]+(1-\varepsilon)\left(\frac{2s_{23}}{s_{12}}-\varepsilon\right) \\
& +\frac{m^2}{s_{12}s_{13}}\left[s_{23}\frac{(1+z_1)(z_1-z_2z_3)}{(1-z_2)(1-z_3)}+\varepsilon\left(z_1(s_{123}-m^2)+s_{23}+6s_{12}\right)\right] \\
& -\frac{2m^2}{s_{13}}\frac{(1+z_1)(1+z_2z_3)-(z_2-z_3)^2}{(1-z_2)(1-z_3)}+\frac{4m^4}{s_{12}s_{13}}\left(\frac{2z_2}{1-z_3}-\frac{z_1(1+z_2)}{1-z_2}-\varepsilon\right)\Bigg\}\,.
\end{aligned}
$$

$$\langle P^{(\mathrm{ab})}_{q\to ggq}(p_1,p_2,p_3)\rangle = C_F^2\Bigg\{\frac{\tilde{s}^2_{123}}{2\tilde{s}_{13}\tilde{s}_{23}}\,z_3\left[\frac{1+z_3^2}{z_1z_2}-\varepsilon\frac{z_1^2+z_2^2}{z_1z_2}-\varepsilon(1+\varepsilon)-\frac{4m^2}{\tilde{s}_{123}}\left(\frac{1-z_3}{z_1z_2}+(1-\varepsilon)\frac{z_1}{z_3}\right)\right]$$
$$+\frac{\tilde{s}_{123}}{\tilde{s}_{13}}\left[\frac{z_3(1-z_1)+(1-z_2)^3}{z_1z_2}+\varepsilon^2(1+z_3)-\varepsilon(z_1^2+z_1z_2+z_2^2)\frac{1-z_2}{z_1z_2}-\frac{4m^2}{\tilde{s}_{13}}\left(\frac{1-z_2}{z_2}+(1-\varepsilon)\frac{z_2}{2}\right)\right]$$
$$+\varepsilon(1-\varepsilon)-\frac{\tilde{s}_{23}}{\tilde{s}_{13}}(1-\varepsilon)^2+\frac{4m^2}{\tilde{s}_{13}}+\frac{1}{2}\left(\frac{2m^2}{\tilde{s}_{13}}+\frac{2m^2}{\tilde{s}_{23}}\right)^2\Bigg\}+(1\leftrightarrow 2)\,.$$

$$\langle P^{(\mathrm{nab})}_{q\to ggq}(p_1,p_2,p_3)\rangle = -\frac{C_A}{2C_F}P^{(\mathrm{ab})}_{q\to ggq}(p_1,p_2,p_3)+C_FC_A\Bigg\{\frac{1-\varepsilon}{4}\left(\frac{t^2_{12,3}}{s^2_{12}}+1\right)-(1-\varepsilon)^2\frac{\tilde{s}_{23}}{2\tilde{s}_{13}}$$
$$+\frac{\tilde{s}^2_{123}}{2s_{12}\tilde{s}_{13}}\left[\frac{(1-z_3)^2(1-\varepsilon)+2z_3}{z_2}+\frac{z_2^2(1-\varepsilon)+2(1-z_2)}{1-z_3}-\frac{4m^2}{\tilde{s}_{123}}\left(1+\frac{z_1^2}{z_2(1-z_3)}\right)\right]$$
$$+\frac{\tilde{s}_{123}}{2s_{12}}\left[(1-\varepsilon)\frac{z_1(2-2z_1+z_1^2)-z_2(6-6z_2+z_2^2)}{z_2(1-z_3)}+2\varepsilon\frac{z_3(z_1-2z_2)-z_2}{z_2(1-z_3)}+\frac{4m^2}{\tilde{s}_{123}}\right]$$
$$+\frac{\tilde{s}_{123}}{2\tilde{s}_{13}}\left[(1-\varepsilon)\frac{(1-z_2)^3+z_3^2-z_2}{z_2(1-z_3)}-\varepsilon\frac{2(1-z_2)(z_2-z_3)}{z_2(1-z_3)}-\varepsilon(1-\varepsilon)(1-z_1)\right.$$
$$\left.-\frac{4m^2}{\tilde{s}_{13}}\left(\frac{1-z_2}{z_2}+(1-\varepsilon)\frac{z_2}{2}\right)\right]+\frac{2m^2}{\tilde{s}_{13}}+\frac{2m^4}{\tilde{s}^2_{13}}+(1\leftrightarrow 2)\Bigg\}\,.$$

$$P^{\mu\nu\,(\mathrm{ab})}_{g\to gq\bar{q}}(p_1,p_2,p_3) = C_F T_R \Big\{ d^{\mu\nu}(p_{123},\bar{n}) \Big[2s_{123}\Big(\frac{\tilde{s}_{23}}{\tilde{s}_{12}\tilde{s}_{13}} - \frac{m^2}{\tilde{s}_{12}^2} - \frac{m^2}{\tilde{s}_{13}^2}\Big) + (1-\varepsilon)\Big(\frac{\tilde{s}_{12}}{\tilde{s}_{13}} + \frac{\tilde{s}_{13}}{\tilde{s}_{12}}\Big) - 2\varepsilon \Big]$$

$$+ \frac{4\tilde{s}_{123}}{\tilde{s}_{12}\tilde{s}_{13}}\Big(\tilde{p}^{\mu}_{2,13}\tilde{p}^{\nu}_{3,12} + \tilde{p}^{\mu}_{3,12}\tilde{p}^{\nu}_{2,13}\Big) + 8m^2\Big(\frac{\tilde{p}^{\mu}_{2,13}\tilde{p}^{\nu}_{2,13}}{\tilde{s}_{13}^2} + \frac{\tilde{p}^{\mu}_{3,12}\tilde{p}^{\nu}_{3,12}}{\tilde{s}_{12}^2}\Big) - (1-\varepsilon)\,\frac{4s_{123}}{\tilde{s}_{12}\tilde{s}_{13}}\,\tilde{p}^{\mu}_{1,23}\,\tilde{p}^{\nu}_{1,23} \Big\} \,.$$

$$P^{\mu\nu\,(\mathrm{nab})}_{g\to gq\bar{q}}(p_1,p_2,p_3) = -\frac{C_A}{2C_F}\,P^{\mu\nu\,(\mathrm{ab})}_{g\to gq\bar{q}}(p_1,p_2,p_3) + \frac{C_A T_R}{4}\Big\{ \frac{\tilde{s}_{123}}{s_{23}^2}\Big[-d^{\mu\nu}(p_{123},\bar{n})\frac{t^2_{23,1}}{\tilde{s}_{123}} - 16\,\frac{1-z_1}{z_1}\,\tilde{p}^{\mu}_{2,3}\tilde{p}^{\nu}_{2,3} \Big]$$

$$+ 16m^2\Big[\frac{\tilde{p}^{\mu}_{2,3}\tilde{p}^{\nu}_{2,3}}{z_1}\Big(\frac{z_1}{\tilde{s}_{12}^2} - \frac{2(1-z_1)}{s_{23}^2}\Big) - \frac{\tilde{p}^{\mu}_{1,23}\tilde{p}^{\nu}_{2,3} + \tilde{p}^{\mu}_{2,3}\tilde{p}^{\nu}_{1,23}}{1-z_1}\Big(\frac{z_2}{\tilde{s}_{13}^2} - \frac{1-z_1}{\tilde{s}_{13}s_{23}}\Big) + \frac{\tilde{p}^{\mu}_{1,23}\tilde{p}^{\nu}_{1,23}}{(1-z_1)^2}\Big(\frac{z_2}{\tilde{s}_{13}} - \frac{1-z_1}{s_{23}}\Big)^2 \Big]$$

$$+ d^{\mu\nu}(p_{123},\bar{n})\Big[\frac{2\tilde{s}_{13}}{\tilde{s}_{12}}(1-\varepsilon) + \frac{2\tilde{s}_{123}}{\tilde{s}_{12}}\Big(\frac{1-z_3}{z_1(1-z_1)} - 2\Big) + \frac{2s_{123}}{s_{23}}\frac{1-z_1+2z_1^2}{z_1(1-z_1)} - 1 - \frac{4m^2 s_{123}}{\tilde{s}_{13}^2}$$

$$- \frac{4m^2}{\tilde{s}_{13}}\Big(2 - \frac{1-z_2}{z_1(1-z_1)}\Big) \Big] + \frac{s_{123}}{\tilde{s}_{12}s_{23}}\Big[2s_{123}\,d^{\mu\nu}(p_{123},\bar{n})\frac{z_2(1-2z_1)}{z_1(1-z_1)} - 16\tilde{p}^{\mu}_{3,12}\tilde{p}^{\nu}_{3,12}\frac{z_2^2}{z_1(1-z_1)}$$

$$+ 8(1-\varepsilon)\tilde{p}^{\mu}_{2,13}\tilde{p}^{\nu}_{2,13} + 4(\tilde{p}^{\mu}_{2,13}\tilde{p}^{\nu}_{3,12} + \tilde{p}^{\mu}_{3,12}\tilde{p}^{\nu}_{2,13})\Big(\frac{2z_2(z_3-z_1)}{z_1(1-z_1)} + 1 - \varepsilon\Big) \Big] + (2 \leftrightarrow 3) \Big\} \,.$$

$$\mathcal{S}^{(\mathrm{ab})}_{i,k;l,m}(q_1,q_2;\bar{n}) = \frac{1}{4}\,\mathcal{S}_{i;l}(q_1;\bar{n})\mathcal{S}_{k;m}(q_2;\bar{n}) \;,$$

$$\mathcal{S}^{(\mathrm{ab})}_{i,k;l}(q_1,q_2;\bar{n}) = \frac{1}{\tilde{s}_{l12}}\frac{s_{il}s_{kl}}{\tilde{s}_{i1}\tilde{s}_{k2}}\left(\frac{\tilde{s}_{k1}}{s_{kl}\tilde{s}_{l1}} + \frac{\tilde{s}_{i2}}{s_{il}\tilde{s}_{l2}} - \frac{s_{12}}{\tilde{s}_{l1}\tilde{s}_{l2}}\right)$$
$$+ \frac{1}{\tilde{s}_{l12}}\left[\frac{z_l}{z_2}\left(\frac{\tilde{s}_{il}s_{12}-\tilde{s}_{i2}\tilde{s}_{l1}}{\tilde{s}_{i1}\tilde{s}_{l1}\tilde{s}_{l2}} - \frac{z_l s_{12}}{2z_1\,\tilde{s}_{l1}\tilde{s}_{l2}}\right) + \frac{z_i\tilde{s}_{l1}+z_l\tilde{s}_{i1}-z_1\tilde{s}_{il}}{z_2\,\tilde{s}_{i1}\tilde{s}_{l1}}\right.$$
$$\left. - \frac{\tilde{s}_{ij}}{2\tilde{s}_{i1}\tilde{s}_{j2}}\left(1+\frac{m_k^2-m_j^2}{\tilde{s}_{ij}} + \frac{m_k^2+m_j^2}{\tilde{s}_{ij}}\frac{2\tilde{s}_{i2}}{\tilde{s}_{k2}}\right) + \frac{s_{12}(s_{ik}+\tilde{s}_{ik})(m_j^2+m_k^2)}{2\tilde{s}_{i1}\tilde{s}_{j2}\tilde{s}_{k1}\tilde{s}_{k2}} + \begin{pmatrix} i\leftrightarrow k \\ 1\leftrightarrow 2\end{pmatrix}\right] ,$$

$$\mathcal{S}^{(\mathrm{ab})}_{i;k}(q_1,q_2) = \frac{(\tilde{s}_{i1}\tilde{s}_{k2}-\tilde{s}_{i2}\tilde{s}_{k1})^2 - 2s_{12}\tilde{s}_{ik}(\tilde{s}_{i1}\tilde{s}_{k2}+\tilde{s}_{i2}\tilde{s}_{k1}) + s_{12}^2\tilde{s}_{ik}^2}{\tilde{s}_{i12}\tilde{s}_{k12}\,\tilde{s}_{i1}\tilde{s}_{k1}\tilde{s}_{i2}\tilde{s}_{k2}}$$
$$+ \frac{2s_{12}}{\tilde{s}_{i12}\tilde{s}_{k12}}\left(\frac{m_i^2}{\tilde{s}_{i1}\tilde{s}_{i2}} + \frac{m_k^2}{\tilde{s}_{k1}\tilde{s}_{k2}}\right) + \frac{2(1-\varepsilon)}{\tilde{s}_{i12}\tilde{s}_{k12}} \;,$$

$$
\begin{aligned}
\mathcal{S}^{(\mathrm{nab})}_{i;k}(q_1,q_2;\bar n) &= \mathcal{S}^{(\mathrm{nab},p)}_{i;k}(q_1,q_2;\bar n) - 2(1-\varepsilon)\,\mathcal{S}^{\mu\nu}_{i;k}(p_{12};\bar n) D_\mu(p_1,p_2,\bar n) D_\nu(p_1,p_2,\bar n) \\
&+ \frac{1}{4}\Big(\bar{\mathcal{S}}_{i;i}(q_{12};\bar n) + \bar{\mathcal{S}}_{k;k}(q_{12};\bar n) - 2\bar{\mathcal{S}}_{i;k}(q_{12};\bar n)\Big)\Big[\mathcal{S}_{i;i}(q_2;\bar n) + \mathcal{S}_{1;1}(q_2;\bar n) - 2\mathcal{S}_{i;1}(q_2;\bar n) \\
&\qquad - \frac{1}{2}\Big(\mathcal{S}_{i;i}(q_2;\bar n) + \mathcal{S}_{k;k}(q_2;\bar n) - 2\mathcal{S}_{i;k}(q_2;\bar n)\Big) + (i\leftrightarrow k) + (1\leftrightarrow 2) + \begin{pmatrix} i\leftrightarrow k \\ 1\leftrightarrow 2\end{pmatrix}\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{S}^{(\mathrm{nab},p)}_{i;k}(q_1,q_2;\bar n) &= \left[\frac{3\tilde s_{ik}}{4\tilde s_{i1}\tilde s_{k1}} + \frac{2s_{12}}{z_{12}^2}\left(\frac{z_i}{\tilde s_{i12}} - \frac{z_k}{\tilde s_{k12}}\right)\left(\frac{z_{i12}}{\tilde s_{i12}} - \frac{z_{k12}}{\tilde s_{k12}}\right)\right]\left(\frac{2\tilde s_{i1}}{\tilde s_{12}\tilde s_{i2}} - \frac{\tilde s_{ik}}{\tilde s_{i2}\tilde s_{k2}}\right) \\
&+ \frac{1}{\tilde s_{i12}\tilde s_{k12}}\Bigg\{\frac{3}{2} + \frac{3}{2}\frac{\tilde s_{ik}^2}{\tilde s_{i1}\tilde s_{k1}} + \frac{\tilde s_{ik}}{2\tilde s_{12}} - \frac{z_2}{z_1} + \frac{\tilde s_{k1}\tilde s_{i2} - \tilde s_{ik}\tilde s_{12}}{2\tilde s_{i1}\tilde s_{k2}}\left(1 - \frac{\tilde s_{ik}}{\tilde s_{12}}\right) + \frac{\tilde s_{ik}}{\tilde s_{i1}}\left(1 - \frac{3\tilde s_{i12}}{\tilde s_{12}}\right) \\
&\quad - \frac{\tilde s_{i2}}{\tilde s_{i1}}\frac{z_i}{z_2}\left(2 + \frac{z_1}{z_i} + \frac{\tilde s_{i12}}{\tilde s_{i2}} + \frac{\tilde s_{k12}}{\tilde s_{i2}}\right) + \frac{2\tilde s_{k1}}{\tilde s_{12}}\left(\frac{z_1 - z_2}{z_1}\frac{z_2 - z_i}{z_2} - \frac{z_i\tilde s_{i12}}{z_2\tilde s_{i1}} + \frac{\tilde s_{i2}}{\tilde s_{i1}} - \frac{\tilde s_{i1}}{\tilde s_{i2}}\frac{2z_i + z_2}{z_1}\right) \\
&\quad - \frac{z_{12}^2}{2z_1 z_2}\frac{t_{12,i}t_{12,k}}{\tilde s_{12}^2} - \frac{t_{12,k}}{2\tilde s_{12}}\left[\frac{\tilde s_{i1}}{\tilde s_{i2}}\frac{z_{i2}}{z_1}\left(1 - \frac{z_i\tilde s_{12}}{z_2\tilde s_{i1}}\right) + \frac{2z_{12}}{z_1}\left(1 + \frac{\tilde s_{i1}}{\tilde s_{12}}\right) + \frac{3z_i}{z_2} - (1\leftrightarrow 2)\right] \\
&\quad - \frac{m_i^2}{\tilde s_{i1}}\left[\frac{2t_{12,k}}{s_{12}} - \frac{2\tilde s_{ik}s_{12}}{\tilde s_{i2}^2}\left(1 + \frac{s_{12}}{\tilde s_{k1}} + \frac{\tilde s_{i2}\tilde s_{k1}}{\tilde s_{i1}\tilde s_{k2}}\right)\right]\Bigg\} + \frac{z_i^2}{2\tilde s_{i1}\tilde s_{i2}z_1 z_2}\left(1 + \frac{\tilde s_{i1} - \tilde s_{i2}}{\tilde s_{i12}}\frac{\tilde s_{k1} - \tilde s_{k2}}{\tilde s_{k12}}\right) \\
&+ \frac{4m_i^2}{\tilde s_{i12}^2}\frac{\tilde s_{i1}\tilde s_{j2} + \tilde s_{i2}\tilde s_{j1} - \tilde s_{ij}\tilde s_{12}}{\tilde s_{12}\tilde s_{i1}\tilde s_{j1}} + \frac{m_i^2}{\tilde s_{i1}\tilde s_{i12}}\left[\frac{4}{\tilde s_{k2}}\left(\frac{s_{ik}}{\tilde s_{i2}} - \frac{\tilde s_{k1}}{s_{12}} + \frac{\tilde s_{k2}}{s_{12}}\frac{z_1}{z_2} - \frac{\tilde s_{k1}}{2\tilde s_{i1}}\right) - \frac{1}{\tilde s_{i2}}\frac{2z_i}{z_1} + \frac{1}{\tilde s_{i1}}\frac{2z_{i1}}{z_2} \right.\\
&\quad \left. - \frac{2\tilde s_{i1} + s_{12}}{\tilde s_{i1}}\left(\frac{m_i^2}{\tilde s_{i2}^2} + \frac{2(m_i^2 + m_k^2)}{\tilde s_{i2}\tilde s_{k2}}\right)\right] + (i\leftrightarrow k) + (1\leftrightarrow 2) + \begin{pmatrix} i\leftrightarrow k \\ 1\leftrightarrow 2\end{pmatrix}.
\end{aligned}
$$