PURE GAUGE THEORIES IN 1+1D

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Overview

- Analytically solve 1+1D compact U(1) theory exactly
  - Partition function & Wilson loops with full θ dependence
    - Poisson summation to map into instanton sectors
    - Non-zero θ → cancellations among various instanton configs
- Non-relativistic QM to calculate electric dipole moment
  - Deconfinement at θ = π, EDM diverges → Strong C/P problem
- Partition function for non-abelian SU(N) and SU(N)/$\mathbb{Z}_N$
  - Instantons as non-contractible loops on maximal torus mod Weyl grp
  - Path integral formulation of *Z* using instantons to reproduce Poisson summation result from canonical form

Only N = 2, 3 for this talk

A.A. Migdal, Recursion equations in gauge field theories, Sov. Phys. JETP 42 (1975) 413

Compact U(1) in 1+1D

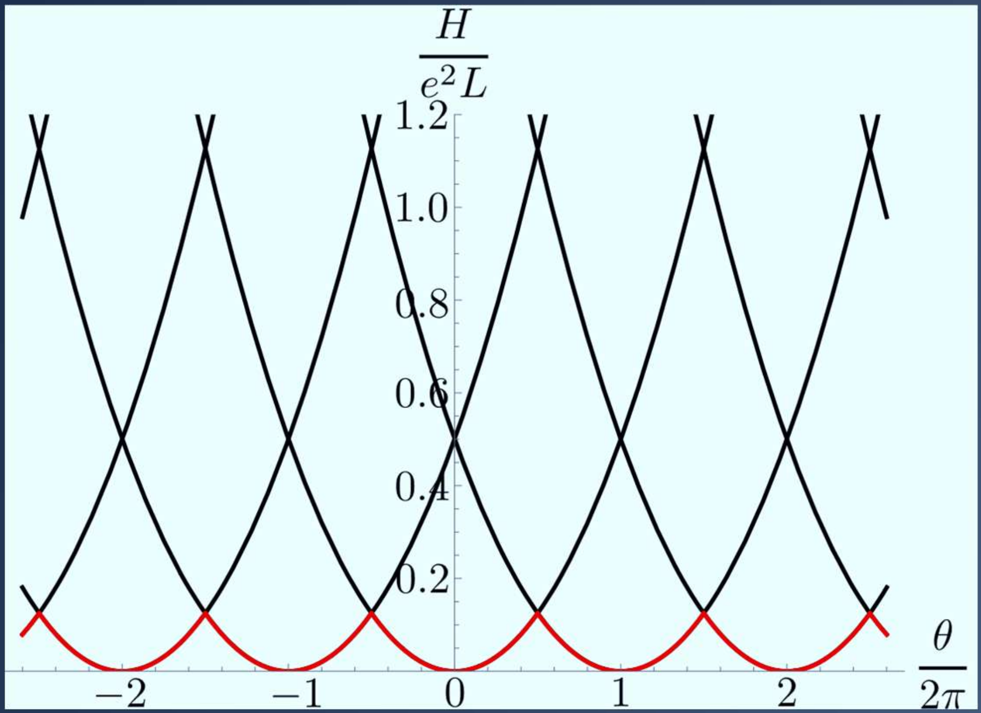
$[e] = 1$

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{4\pi} \epsilon^{\mu\nu} F_{\mu\nu}$$

- Theta term and confinement like in 4D pure Yang-Mills
- Put it on a circle: $A_x(x, t) = a(t)$ by gauge transformation
  - QFT → 1D QM: $S = L \int dt(-\frac{1}{2e^2}\dot{a}^2 + \frac{\theta}{2\pi}\dot{a})$
- Only gauge invariant is Wilson loop $W = e^{i \oint Adx} = e^{iaL}$
  - Periodicity $a \approx a + \frac{2\pi}{L}$ as *W* doesn't change
  - QM in a finite box → quantized momenta $p_a = -\frac{L}{e^2}\dot{a} + L\frac{\theta}{2\pi}$

Hamiltonian Eigenvalues

$$H = \frac{e^2}{2L}\left(-i\frac{d}{da} - L\frac{\theta}{2\pi}\right)^2 = \frac{e^2 L}{2}\left(n - \frac{\theta}{2\pi}\right)^2$$



- Periodic in θ by multiples of 2π, discrete energy spectrum ($n \in \mathbb{Z}$)
- Phase transitions whenever θ is an odd multiple of π
- At finite temp β = 1/kT

$$Z = \sum_n e^{-\beta \frac{e^2 L}{2}\left(n - \frac{\theta}{2\pi}\right)^2}$$

Partition Function

- Modular transformation of Jacobi elliptic theta functions

  i.e. *Poisson summation* with conjugate $m \in \mathbb{Z}$

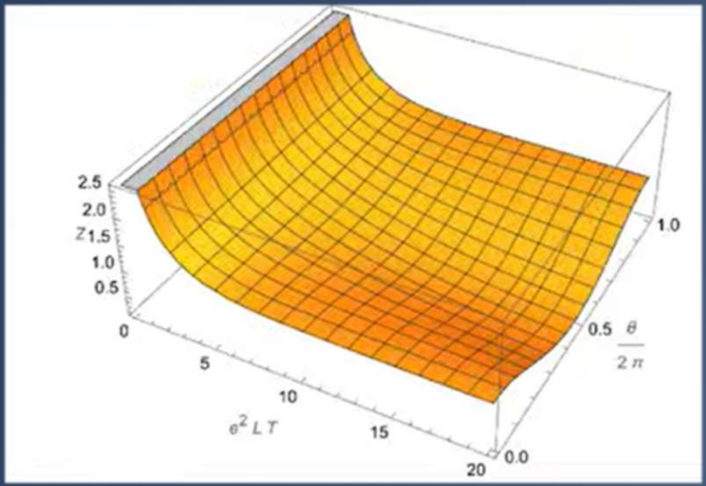
Instanton action weighted by θ

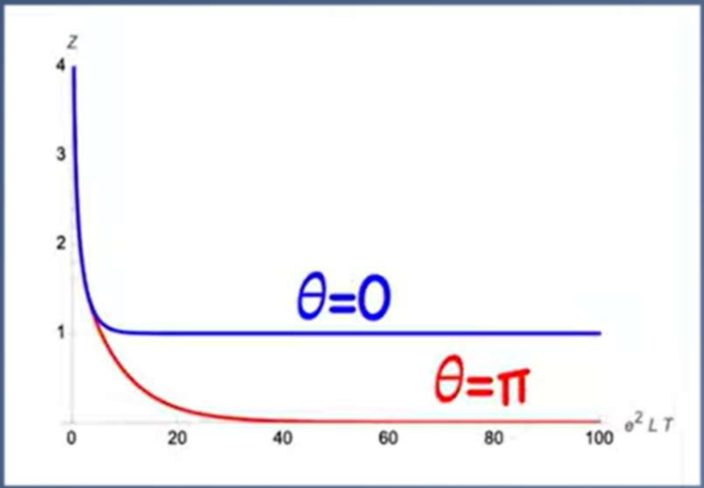
$$Z = \sum_n e^{-\beta \frac{e^2 L}{2}\left(n - \frac{\theta}{2\pi}\right)^2} \to \left(\frac{2\pi}{e^2 \beta L}\right)^{1/2} \sum_m e^{\frac{1}{2e^2 L \beta}(2\pi m)^2 + im\theta}$$

  - Analytic check for num calc via lattice sim of QED_2 with sign problem

- Cancellations due to sum with alternating phases when θ > 0

E.g. $Z = 0$ when θ = π & coupling is large



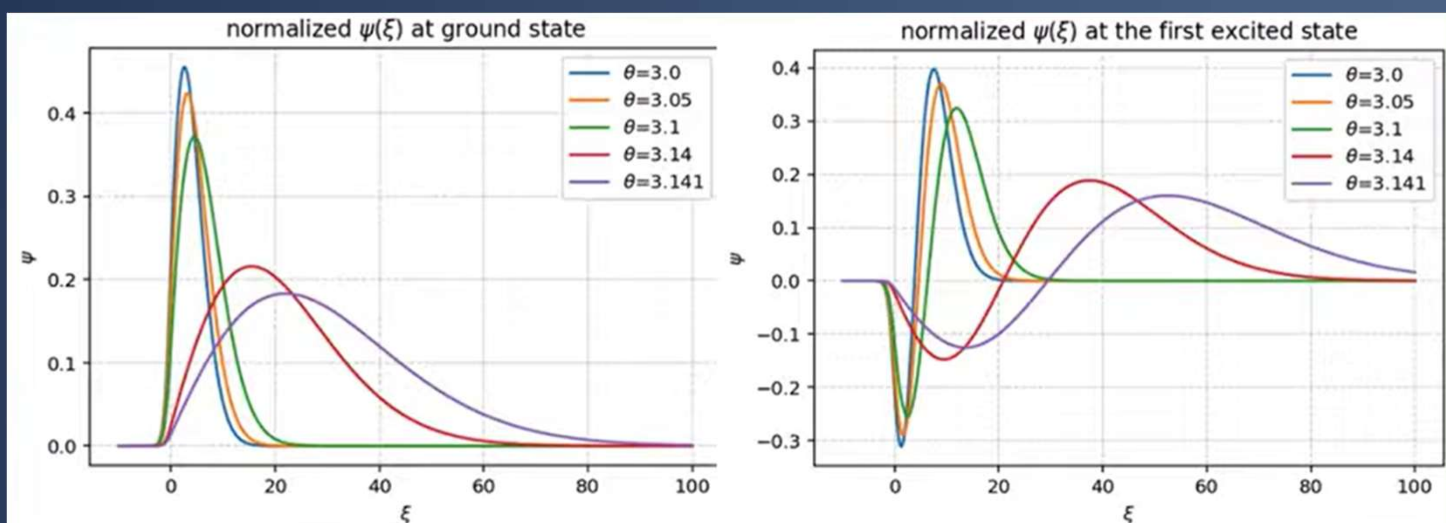


Confinement

- Couple NR heavy (mass >> e) charge pairs e^+e^- or $q\bar{q}$ to the U(1)
  - Rising linear potential when charges are separated by x
  - Theta term depends on sign of *x*
    → acts like a constant E field

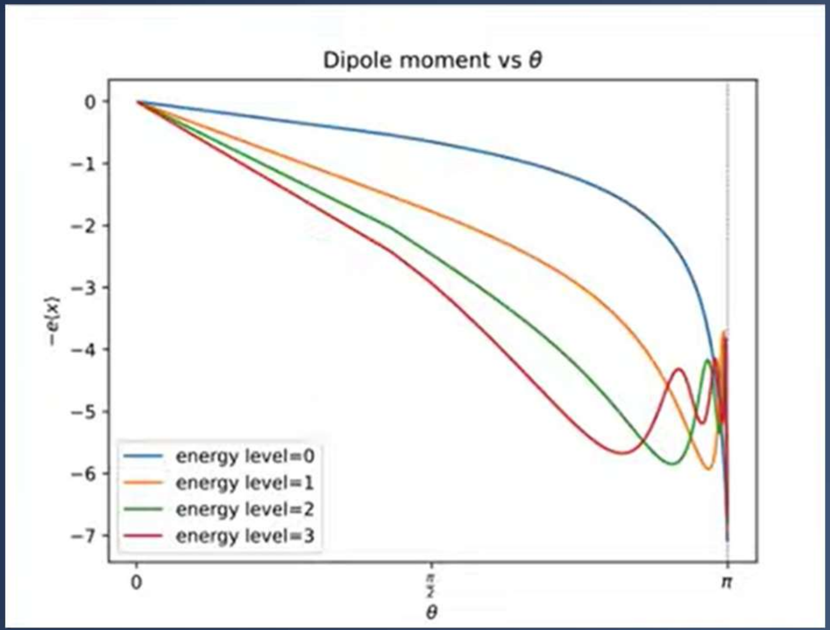
$$H = \frac{p^2}{2\mu} + \frac{e^2}{2}\left(|x| + \frac{\theta}{\pi}x\right)$$

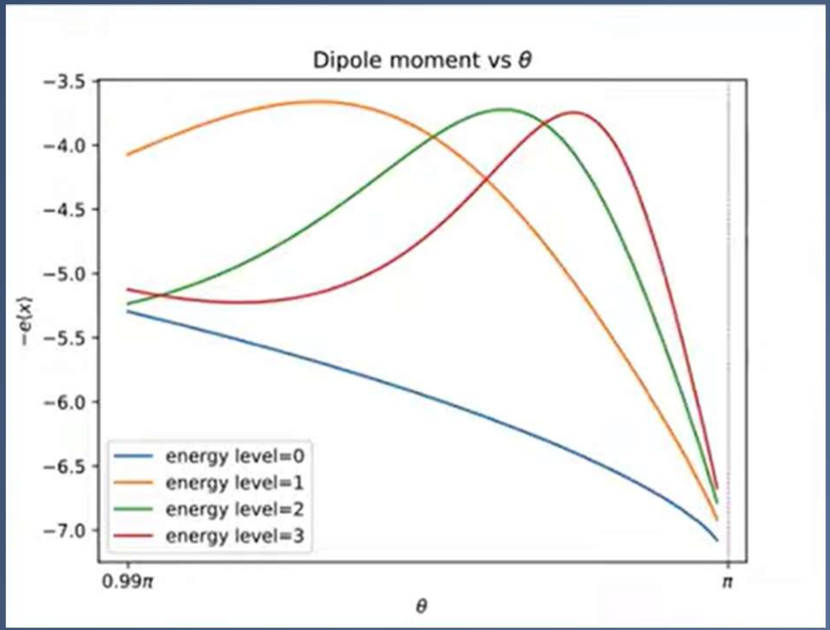
- Solve Schrodinger eq for ψ(pair)
  - e^+e^- start to separate when θ turned on → EDM like in QCD
  - θ = π: Potential 0 for x < 0 → e^+e^- displace to ∞ → Deconfinement



Electric Dipole Moment

- EDM of neutral bound states is induced by θ
  - Strong C/P problem: θ violates C & P but not CP
  - Diverges to - ∞ as $\theta \to \pi$
- EDM of nth energy level has n dips as a function of θ before it diverges





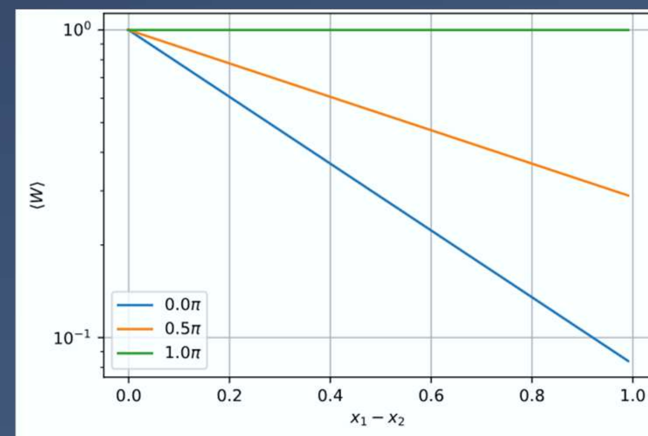
$$\vartheta(z,q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}$$

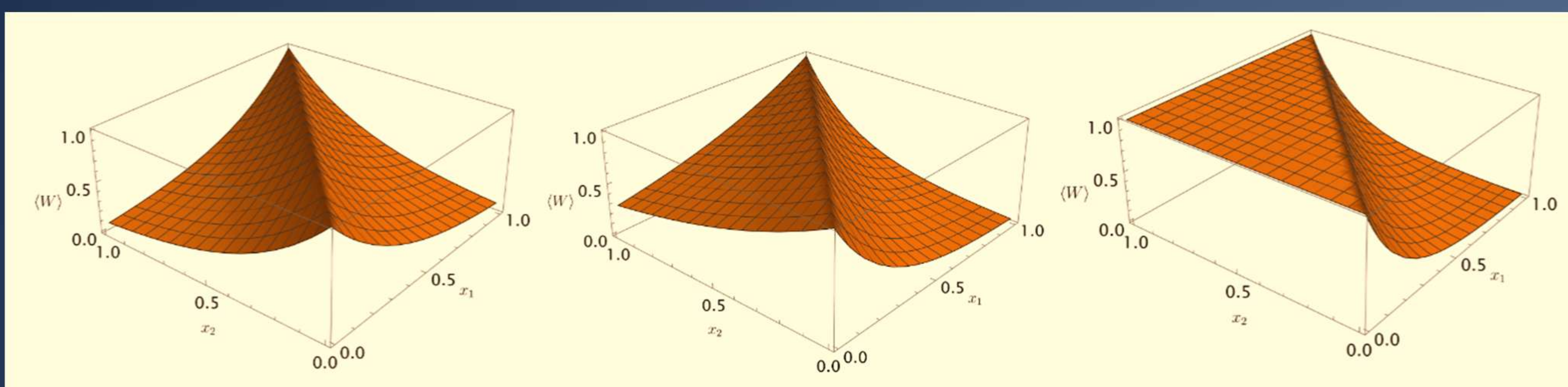
Wilson Loops

$$\langle W \rangle \equiv \frac{1}{Z} \int \mathcal{D}A_\mu e^{-S_E} e^{i \oint A_\mu dx^\mu} = \frac{e^{\frac{-e^2|\Delta x|\Delta\tau(LT-|\Delta x|\Delta\tau)}{2LT}} \vartheta\left(-\frac{\theta}{2} - \frac{\pi \Delta x \Delta\tau}{LT}, e^{-\frac{2\pi^2}{e^2 LT}}\right)}{\vartheta\left(-\frac{\theta}{2}, e^{-\frac{2\pi^2}{e^2 LT}}\right)}$$

- In the limit that either space or time dim. is large ($e^2 LT >> 1$)
    - Area Law
    - $\langle W \rangle = 1$ at θ = π

$$\langle W \rangle \approx e^{e^2 \frac{\theta}{2\pi}(x_2-x_1)(\tau_2-\tau_1) - \frac{1}{2}e^2(x_2-x_1)(\tau_2-\tau_1)}$$





θ = 0 θ = π/2 θ = π

Non-abelian Gauge Theories in 1+1D

- Hilbert space: characters of reps $$HW_R = \frac{g^2 L}{2} C_2(R) W_R$$
  - L^2 functions on the space of Wilson loops -- exact eigenstates of *H*
- Axial gauge fix: all dynamics in one diagonalized, Weyl invariant transition function $U = e^{2\pi i n_k H_k x/L}$ on S^1

Physical states reside on the maximal torus T/W

$$L = \int dx \frac{1}{g^2} \dot{a}_i \dot{a}_j \mathrm{tr} H_i H_j$$
$$A = a_i H_i$$

- SU(N)/$\mathbb{Z}_N$ theories in 3 + 1D have θ ∈ [0, 2πN]

  → multiple definitions connected by shifting θ by N
  - 1 + 1D: no θ-term, similar effect by Stiefel-Whitney class w_2
- 1/N instantons ($w_2 \neq 0$) that can't lift to universal enveloping grp SU(N)

$$Z = \sum_{\mathrm{R\ with}\ N\text{-ality}\ k} e^{-\frac{g^2}{2} C_2(R) LT} = \sum_{n=0}^{N-1} e^{2\pi i k n/N} (-1)^{n(N-1)} \int \mathcal{D}A_n e^{-S[A]}$$

$$+ \text{ boundary terms}$$

Sum over the gauge bundles with w_2 = n (mod N)

Highest weight is $\sum_{m=1}^{N-1} n_m \mu_m$ and $\sum_{m=1}^{N-1} m\, n_m$ (mod N) = *k*
Wavefunctions in R transform by ω_N^k under large gauge transformation with U(x=L) = ω_N

E. Sharpe, Undoing decomposition, 1911.05080
G. Aminov, Spontaneous symmetry breaking in pure 2D Yang-Mills theory, 1911.03494

Example: N = 2

- SU(2) $$Z = \sum_{j\in\frac{1}{2}\mathbf{Z}_{\geq 0}}^{\infty} e^{-g^2 LTj(j+1)/2} = e^{g^2LT/8}\frac{1}{2}\left[-1+\left(\frac{8\pi}{g^2LT}\right)^{1/2}\sum_{m=-\infty}^{\infty} e^{-8\pi^2m^2/g^2LT}\right]$$

Non-topological solution
$a = \frac{2\pi}{LT}t$ to Euclidean EOM

$$S = \int dxdt \frac{1}{g^2}\mathrm{Tr}\dot{a}^2 = LT\frac{1}{g^2}2\left(\frac{2\pi}{LT}\right)^2 = \frac{8\pi^2}{g^2LT}$$

- $(SU(2)/\mathbb{Z}_2)_{k=0} = SO(3)_+$
  - Wavefunc transform by $(-1)^{2j = \text{even}}$ → periodic BC
  - Weight the path integral by $(-1)^{kw_2} = 1$

$$Z = \sum_{j\in\mathbf{Z}_{\geq 0}}^{\infty} e^{-g^2LTj(j+1)/2} = e^{g^2LT/8}\left[\frac{1}{2}\left(\frac{2\pi}{g^2LT}\right)^{\frac{1}{2}}\sum_{m=-\infty}^{\infty}(-1)^m e^{-2\pi^2m^2/g^2LT}\right]$$

½ instanton ($w_2 \neq 0$) sol. $a = \frac{\pi}{LT}t$ when m = odd

- $(SU(2)/\mathbb{Z}_2)_{k=1} = SO(3)_-$
  - Wavefunc transform by $(-1)^{2j = \text{odd}}$ → antiperiodic BC
  - Weight the PI by $(-1)^{kw_2}$ which induces relative signs

$$Z = \sum_{j=\text{half-odd}>0}^{\infty} e^{-g^2LTj(j+1)/2}$$
$$= e^{g^2LT/8}\frac{1}{2}\left[-1+\left(\frac{2\pi}{g^2LT}\right)^{1/2}\sum_{m=-\infty}^{\infty} e^{-2\pi^2m^2/g^2LT}\right]$$

Interpreting the Path Integral

$$(\vec{n}_2,\sigma_2)(\vec{n}_1,\sigma_1) = (\sigma_2(\vec{n}_1)+\vec{n}_2, \sigma_2\sigma_1)$$

- Instantons: straight-line-non-contractible loops on $T/W = \mathbb{R}^r/(\tilde{\Lambda}_w \rtimes W)$
  - 1-instantons: $\forall\, \vec{n} \in \widetilde{\Lambda}_w$, $(\vec{n}, e)^k(\vec{x}) = \vec{x} + k\vec{n}$ is a straight line on $\mathbb{R}^r$
  - For $(\vec{n},\sigma)$ with $\sigma \neq e \in W$ and $\sigma^k = e$, we get a $1/k$ instanton starting at $\overrightarrow{x_0}$ and ending at $\overrightarrow{x_1} = (\vec{n},\sigma)(\overrightarrow{x_0}) = \sigma(\overrightarrow{x_0}) + \vec{n}$ iff

Straight line

$\sigma^j(\vec{n}) \in \widetilde{\Lambda}_w$ → $k(\overrightarrow{x_1} - \overrightarrow{x_0}) \in \widetilde{\Lambda}_w$ and $\sigma(\overrightarrow{x_1} - \overrightarrow{x_0}) = \overrightarrow{x_1} - \overrightarrow{x_0}$

- SU(N) Example: $\sigma = (1,2,\ldots,k)$, $W = S_N$ and $\widetilde{\Lambda}_w$ = root lattice (simply laced)

$$\vec{x}_0 = \Big(\underbrace{0, \tfrac{1}{k}, \cdots, \tfrac{k-1}{k}}_{k}, \underbrace{0, \cdots, 0}_{N-k-1}, -\tfrac{1}{2}(k-1)\Big) \qquad \vec{n} = \Big(\underbrace{0, \cdots, 0, 1}_{k}, \underbrace{0, \cdots, 0}_{N-k-1}, -1\Big)$$

$$(123\cdots k)(\vec{x}_0) = \Big(\underbrace{\tfrac{1}{k}, \cdots, \tfrac{k-1}{k}, 0}_{k}, \underbrace{0, \cdots, 0}_{N-k-1}, -\tfrac{1}{2}(k-1)\Big) \qquad \vec{x}_1 - \vec{x}_0 = \Big(\underbrace{\tfrac{1}{k}, \cdots, \tfrac{1}{k}}_{k}, \underbrace{0, \cdots, 0}_{N-k-1}, -1\Big)$$

Like this, find all instantons for each σ

$$Z = \frac{1}{|W|}\sum_{\sigma}(-1)^{\mathrm{sgn}(\sigma)} L_\sigma \sum_{\vec{n}\in\Lambda_\sigma} e^{-S(\vec{n},\sigma)}$$

- Path integral term over the instanton is weighted by +(-)1 when σ is even(odd)

Example: SU(3)

$$\rho = \mu_1 + \mu_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right) + \left(0, \sqrt{\frac{2}{3}}\right)$$

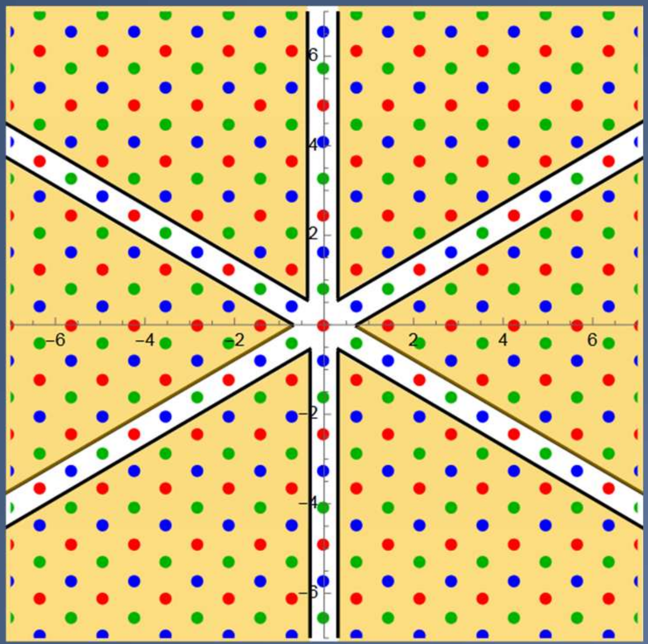
- To perform Poisson summation, need to extend sum over dominant weights to the whole weight lattice

$$Z = \sum_{\mu=\text{dominant}} e^{-g^2 LTC_2/2}$$

$$2C_2 = \langle \mu, \mu + 2\rho \rangle = \langle \mu + \rho, \mu + \rho \rangle - \langle \rho, \rho \rangle$$

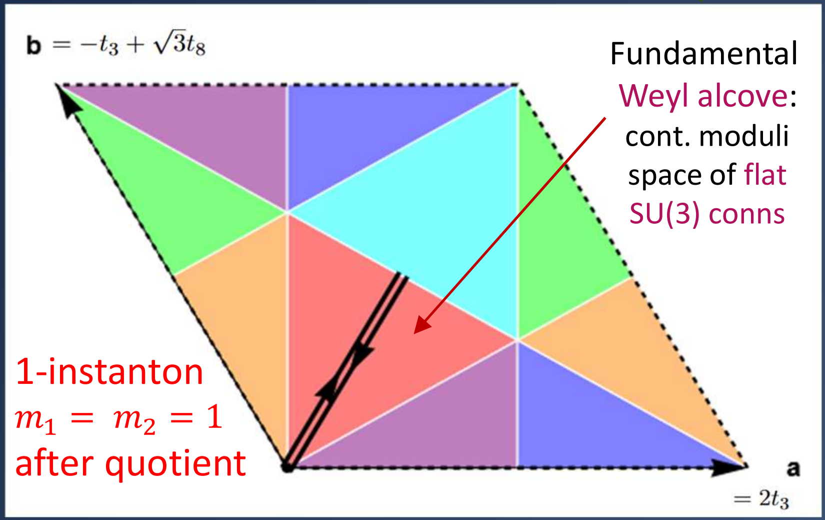
$$= \frac{1}{3!}\left[\sum_\mu e^{-g^2 LT\mu\cdot(\mu+2\rho)/4} - 3\sum_{n_2=-\infty}^{\infty} e^{-g^2 LT(-\mu_1+n_2\mu_2)\cdot(-\mu_1+n_2\mu_2+2\rho)/4} + 2\right]$$

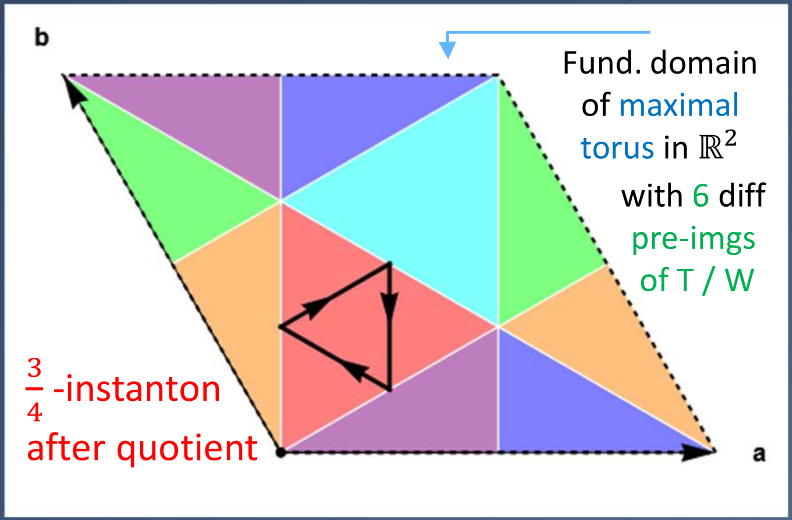
- Yellow upper right sextant is the dominant Weyl chamber offset by ρ
- Divide sum by 6 as other Weyl chambers give same contribution
- Subtract 3 line-gaps b/w Weyl chambers from the sum and add back 2 origins
- Red, blue, and green weights have triality zero, one, and two, respectively.



SU(3) Dual Picture

$$Z = \frac{1}{3!} e^{g^2 LT/2} \left[\frac{4\sqrt{3}\pi}{g^2 LT} \sum_{m_1, m_2 = -\infty}^{\infty} e^{-\frac{8\pi^2}{g^2 LT}(m_1^2 - m_1 m_2 + m_2^2)} - 3\sqrt{\frac{6\pi}{g^2 LT}} \sum_{m=-\infty}^{\infty} e^{-\frac{3}{4}\frac{8\pi^2}{g^2 LT} m^2} + 2 \right]$$



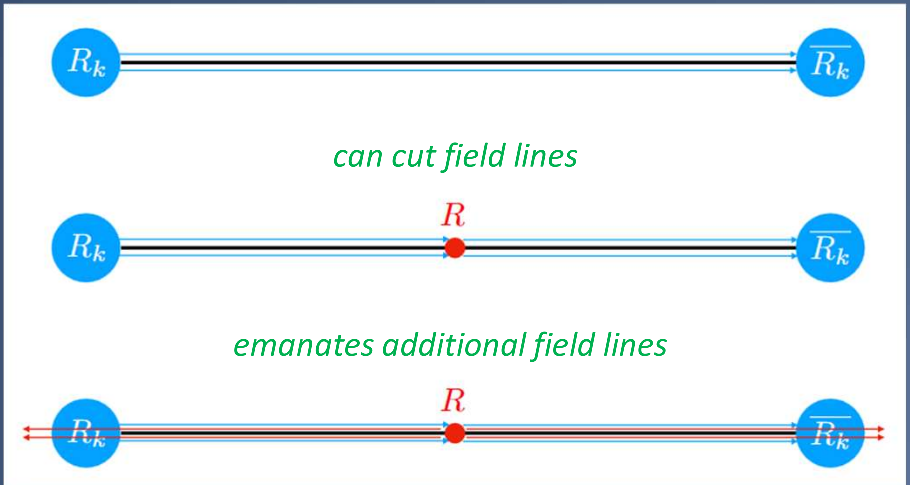


- $S_3 = \{id, R_{120}, R_{240}, R_1, R_2, R_3\}$ and $\widetilde{\Lambda}_w = \{(m_1, m_2) \in \mathbb{Z}^2\}$ *i.e. all instantons*
- The $\overrightarrow{x_1} - \overrightarrow{x_0}$ that satisfy $k(\overrightarrow{x_1} - \overrightarrow{x_0}) \in \widetilde{\Lambda}_w$ and $\sigma(\overrightarrow{x_1} - \overrightarrow{x_0}) = \overrightarrow{x_1} - \overrightarrow{x_0}$ are
  - $\sigma = id$: $\{(m_1, m_2)\}$
  - $\sigma = R_1$: $\{(m, 2m)\}$
  - $\sigma = R_2$: $\{(2m, m)\}$
  - $\sigma = R_3$: $\{(m, -m)\}$
  - $\sigma = R_{120}$: $\{(0, 0)\}$
  - $\sigma = R_{240}$: $\{(0, 0)\}$

Line Operators & Confinement

$\equiv e^{-E_R T}$ (E_R is energy of free color charge in rep R)

- $Z = f(g^2 LT)$ invariant under $L \leftrightarrow T \Rightarrow$ $W_R = \sum_{R'} \langle R' | \mathrm{Tr}_R P e^{i \oint A dx} | R' \rangle e^{-g^2 LTC_2(R')/2}$
  - Like Polyakov loop: If $W_R = 0$ in inf vol limit (G.S. dominant), then theory confines
- SU(N): G.S. ($E \propto C_2(R) = 0$) is χ of singlet (= const) $\Rightarrow W_R = 0 \; \forall \; R$
  - At finite imaginary t (temp > 0), excited states $\rightarrow W_R \neq 0$ for 0 N-ality R *(e.g. adj)*
- $(\mathrm{SU(N)}/\mathbb{Z}_\mathrm{N})_k$: G.S. is χ_k of R_k with highest wt μ_k $\langle k | \mathrm{Tr}_R P e^{i \oint A dx} | k \rangle = \int da \chi_k^* \chi_R \chi_k$
  - Counts no. of R in $R_k \otimes \bar{R}_k \Rightarrow$ *color in adj. confined only when k = 0*
- Color charges R_k, $\bar{R}_k$ at spatial infs in vacuum with color-electric flux energy $LC_2(R_k)/g^2$ b/w them
- Color charge of $R \notin R_k \otimes \bar{R}_k$ has inf energy cost in large vol limit, Polyakov loop is zero → confined



Summary

- Exactly solved 1+1D compact pure U(1) theory for analytic Partition function & Wilson loops with full θ dependence → good test for sims with sign problem
- Deconfinement at θ = π, EDM diverges → Strong C/P problem
- Poisson sum canonical Z of non-abelian SU(N) & SU(N)/$\mathbb{Z}_N$ with N-ality and Stiefel-Whitney class → Path integral formulation of Z using instantons
    - Non-contractible loop instanton subspaces on T/W or T/ $\mathbb{Z}_N$/W by scanning the affine Weyl grp elements $(\vec{n}, \sigma)$ $L_\sigma = \frac{(2\pi)^{\dim(\Lambda_\sigma)}}{\sqrt{\det(g^2 LTC^{-1}|_{\Lambda_\sigma}/2)}}$
    - SU(N) dual picture: $Z = \frac{1}{|W|} e^{g^2 LTN(N^2-1)/48} \sum_{\sigma \in W} (-1)^{\mathrm{sgn}(\sigma)} L_\sigma \sum_{\vec{x_{10}} \in \overline{\Lambda}_\sigma} e^{-S(\vec{x_{10}})}$
- Polyakov loop with physical states as characters of reps
    - All R for SU(N) and only $R \notin R_k \otimes \bar{R}_k$ for $(\mathrm{SU(N)}/\mathbb{Z}_\mathrm{N})_k$ confine

Thank You