Pheno 2026: Fermionic Electroweak Two-Loop Corrections to Drell-Yan and Related Processes

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Need for precision physics

At colliders such as FCC-ee or CEPC, **statistical/systematic** uncertainty likely be exceeded by **theoretical**

For instance, statistical uncertainty for $e^+e^- \to \mu^+\mu^-$ above Z-resonance: $\mathcal{O}(0.01\%)$

To extract cross-sections, need theoretical precision near experimental precision

To achieve this goal, reduce uncertainties from electroweak sector

NNLO electroweak corrections

At HL-LHC, process of interest would be **Drell-Yan scattering** ($pp \to Z/\gamma \to l\bar{l}$) [ATLAS-CONF-2018-037]

Similarly, at FCC-ee, measuring A_4 parameter or electroweak mixing angle in **fermion pair production** ($e^+e^- \to f\bar{f}$) avenue to non-resonant new physics

Known corrections:

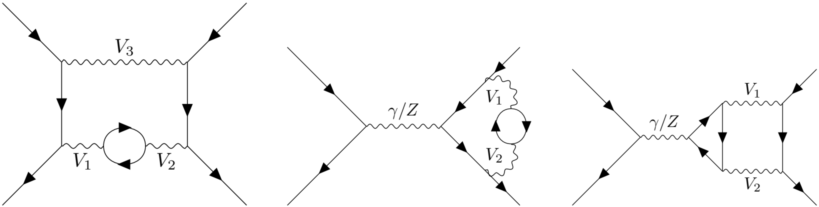
- NLO [hep-ph/0108274]
- NNLO QCD [1007.2351]
- NNLO EW-QCD [2412.16095, 2203.11237, 2106.11953]
- N^{3}LO QCD [2111.10379]

Now also need second-order electroweak

NNLO Electroweak Corrections from Closed Fermion Loops

In particular, examine contributions from two-loop diagrams with closed fermion loops

Expect to be large due to top quark mass, number of light fermions in loops



Method of Calculation

Want to avoid challenges presented by analytical calculation [hep-ph/0102033] due to multiple mass scales

However, also wanted to avoid multi-dimensional numerical integration

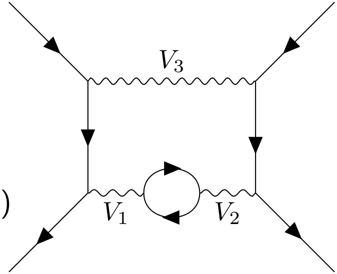
Use a semi-numerical method to simplify calculation to up to a **two-dimensional** numerical integral

Results in a calculation that can be performed in minutes per diagram

Use dispersion relation techniques!

In general a two-loop diagram with fermionic self-energy sub-loop reads,

$$\int \frac{d^D k}{i\pi^{d/2}} \frac{N(k)}{\prod_i [(k+p_i)^2 - m_i^2 + i\epsilon]} \hat{\Sigma}_{\mu\nu}^{V_1,V_2}(k^2)$$



Can replace transverse component of self-energy with dispersion relation

$$\Sigma(k^2, m_1^2, m_2^2) = \frac{1}{\pi} \int_{(m_1+m_2)^2}^{\infty} d\sigma \frac{\Delta\Sigma(\sigma, m_1^2, m_2^2)}{\sigma - k^2 - i\epsilon}.$$

$$\Delta\Sigma(k^2, m_1^2, m_2^2) = \text{Im}\Sigma(k^2, m_1^2, m_2^2)$$

[hep-ph/9409388]

Triangle sub-loop

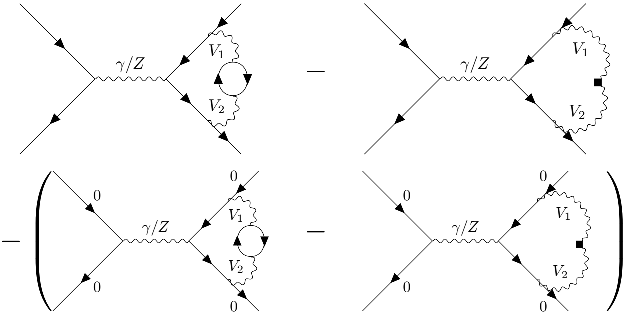
For triangular fermionic subloops, easier to express as self-energies using a Feynman parameter [hep-ph/0608099]:

$$
\begin{aligned}
I^{\mathrm{v}} &= \int_{-\infty}^{\infty} d^D q_2 \, \frac{q_2^\mu \cdots q_2^\nu}{[(q_1+q_2)^2 - m_f^2]\,[q_2^2 - m_f^2]\,[(q_2 - p_1 - p_2)^2 - m_f^2]} \\
&= \int_0^1 dx \int_{-\infty}^{\infty} d^D q_2 \, \frac{q_2^\mu \cdots q_2^\nu}{[(q_1+q_2)^2 - m_f^2]\left[(q_2 - p')^2 - m'^2\right]^2} \\
&= \int_0^1 dx \, \frac{\partial}{\partial (m')^2} \int_{-\infty}^{\infty} d^D q_2 \, \frac{q_2^\mu \cdots q_2^\nu}{[(q_1+q_2)^2 - m_f^2]\left[(q_2 - p')^2 - m'^2\right]} \\
&= \int_0^1 dx \sum_k N_k^{\mu\cdots\nu} \, B_k(q_1^2, m_f^2, m'^2),
\end{aligned}
$$

$$
p' = (p_1 + p_2)(1-x), \qquad m' = \sqrt{m_f^2 - (p_1+p_2)^2 x(1-x)}
$$

UV Divergences

Diagrams can have divergences in both the sub-loop and the global loop



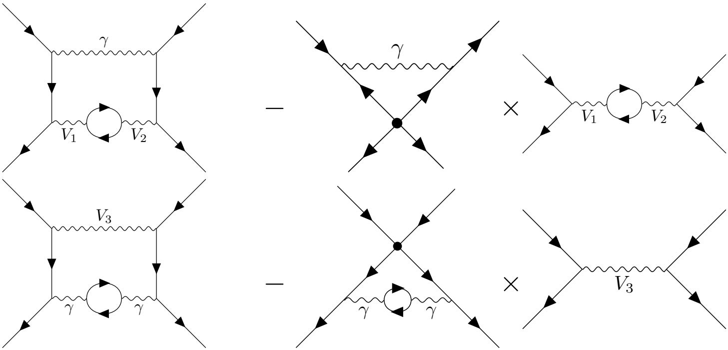
For sub-loops, handled by subtracting term with fixed sub-loop momenta

In the global loop, subtract diagram with external momentum set to 0 (vacuum diagram) [hep-ph/0608099]

IR subtractions

IR divergences occur when internal photons become soft

Subtract universal QED singularities [Annals Phys. 13 (1961) 379–452] to avoid double counting



For technical purposes, use a non-zero photon mass in the integration

Residual fermion mass dependence in photon self-energy diagram

Different Renormalization Schemes

$\alpha(0)$ *scheme:* The electromagnetic coupling defined through Thomson scattering

$$\Delta\alpha \equiv 1 - \alpha(0)/\alpha(m_Z),$$

$\alpha(m_Z)$ *scheme:* resums the running effects from $\Delta\alpha$

G_μ *scheme:*

$$G_\mu = \frac{\pi\alpha}{\sqrt{2}(1 - m_W^2/m_Z^2)m_W^2}(1 + \Delta r)$$

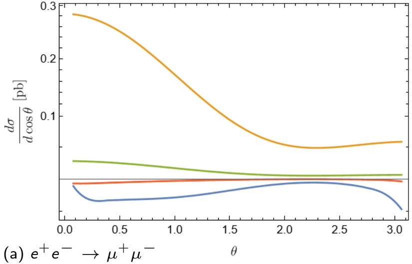
Implementation

Use `Mathematica` to generate analytical expressions

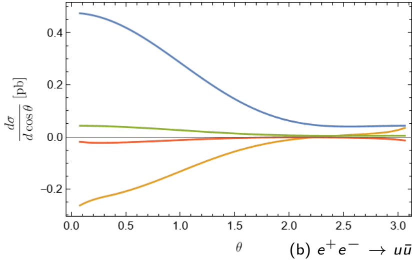
- FeynArts 3.11 [hep-ph/0012260] generates expression based on topology of process
- FeynCalc 9.3 [2001.04407] handles Dirac algebra and conversion into Passarino-Veltman functions
- Change sub-loops to dispersion relations

Numerical integral is performed in `C++` (`C`)

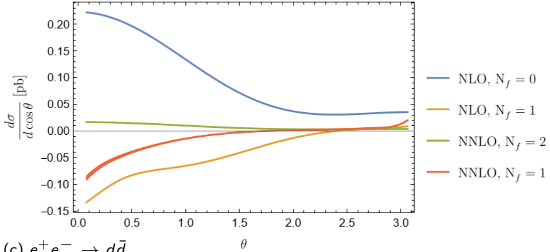
- LoopTools 2.16 [hep-ph/9807565] calculates numerical values of Passarino-Veltman functions
- GNU Scientific Library (http://www.gnu.org/software/gsl/) is used to perform numerical integration
- QuadMath (https://gcc.gnu.org/onlinedocs/libquadmath/) is used for higher precision calculations



(a) $e^+e^- \to \mu^+\mu^-$



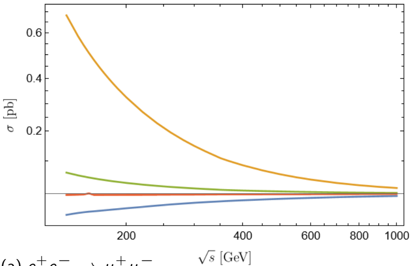
(b) $e^+e^- \to u\bar{u}$



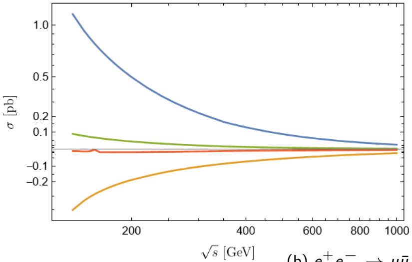
(c) $e^+e^- \to d\bar{d}$

Differential unpolarized cross-section,

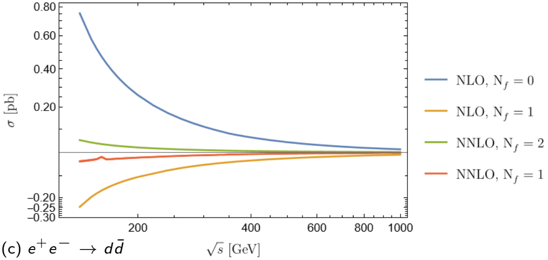
$\alpha(0)$ scheme, $\sqrt{s} = 240$ GeV



(a) $e^+e^- \to \mu^+\mu^-$



(b) $e^+e^- \to u\bar{u}$



(c) $e^+e^- \to d\bar{d}$

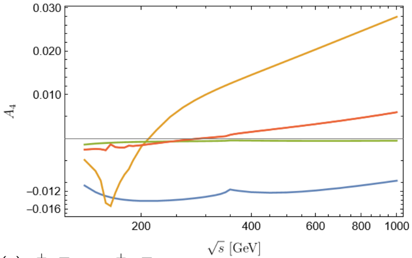
Total unpolarized cross-section,

$\alpha(0)$ scheme

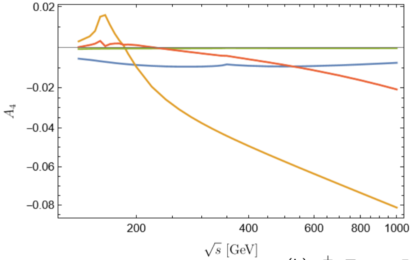
Total Cross Section (cont.)

	Scheme	$e^+e^- \to \mu^+\mu^-$		
		$\alpha(0)$	$\alpha(m_Z)$	G_μ
$\sqrt{s} =$ 150 GeV	LO	5.206	5.880	5.581
	+NLO	5.731	5.773	5.821
	+NNLO	5.783	5.789	5.817
240 GeV	LO	1.797	2.030	1.927
	+NLO	1.990	2.007	2.022
	+NNLO	2.010	2.012	2.021
1 TeV	LO	0.099	0.112	0.107
	+NLO	0.109	0.110	0.111
	+NNLO	0.110	0.110	0.110

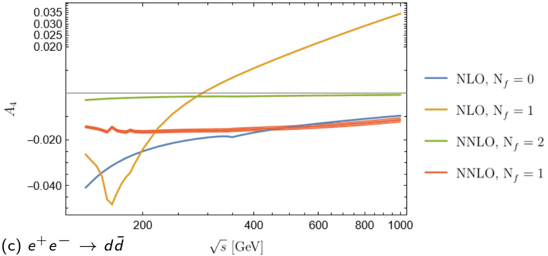
Table: Partonic cross-sections in pb in different renormalization schemes



(a) $e^+e^- \to \mu^+\mu^-$



(b) $e^+e^- \to u\bar{u}$



(c) $e^+e^- \to d\bar{d}$

A_4 parameter,
$\alpha(0)$ scheme,

details on calculation in backup

Conclusions

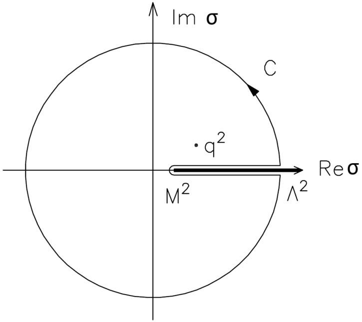
Dispersion relations offer a powerful tool for evaluating NNLO corrections to processes of interest to future colliders, allowing these types of diagrams to be evaluated within minutes

Found that the fermionic NNLO corrections to $e^+e^- \to \mu^+\mu^-/u\bar{u}/d\bar{d}$ would be around 1% of the NLO corrections (see [2512.15700])

Thank you!

Backup

Can be convenient to evaluate some function f at q^2 by integrating around q^2 over complex σ plane along the following contour:



Can evaluate f by integrating along contour

$$f(q^2) = \frac{1}{2\pi i} \oint d\sigma \frac{f(\sigma)}{\sigma - q^2}$$

Get dispersion relation:

$$f(q^2) = \frac{1}{\pi} \int_{M^2}^{\infty} \frac{\mathrm{Im} f(\sigma)}{\sigma - q^2 - i\epsilon}$$

To ensure integral is finite, subtract an additional term

$$f(q^2) = f(q_0)^2 + \frac{q^2 - q_0^2}{\pi} \int_{M^2}^{\infty} \frac{d\sigma}{\sigma - q_0^2} \frac{\mathrm{Im} f(\sigma)}{\sigma - q^2 - i\epsilon}$$

Something easy to calculate analytically

hep-ph/9607255

More specifically,

$$\hat{\Sigma}^{AA/AZ}_{\mu\nu}(k^2) = (g_{\mu\nu}k^2 - k_\mu k_\nu)\hat{\Pi}^{AA/AZ}_T(k^2),$$

$$\hat{\Sigma}^{VV}_{\mu\nu}(k^2) = \frac{1}{k^2}\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\hat{\Sigma}^{VV}_T(k^2),$$

$$\hat{\Pi}^{AA}_T(k^2) = \Pi^{AA}_T(k^2) + \delta Z^{AA} = \frac{k^2}{\pi}\int_0^\infty d\sigma\ \frac{\mathrm{Im}\{\Pi^{AA}_T(\sigma)\}}{\sigma(\sigma - k^2 - i\epsilon)}\ ,$$

$$\hat{\Pi}^{AZ}_T(k^2) = \Pi^{AZ}_T(k^2) + \tfrac{1}{2}\delta Z^{AZ} = \frac{k^2 - m_Z^2}{\pi}\fint_0^\infty d\sigma\ \frac{\mathrm{Im}\{\Pi^{AZ}_T(\sigma)\}}{(\sigma - m_Z^2)(\sigma - k^2 - i\epsilon)}\ ,$$

$$\hat{\Sigma}^{VV}_T(k^2) = \Sigma^{VV}_T(k^2) - \delta m_V^2 + \delta Z^{VV}\,(k^2 - m_V^2) = \frac{(k^2 - m_V^2)^2}{\pi}\fint_0^\infty d\sigma\ \frac{\mathrm{Im}\{\Sigma^{VV}_T(\sigma)\}}{(\sigma - m_V^2)^2(\sigma - k^2 - i\epsilon)}\ ,$$

$\fint$ denotes a principle value integral and δ are renormalization counterterms to make the integral finite.

Can find the imaginary parts through optical theorem/Cutkosky rules

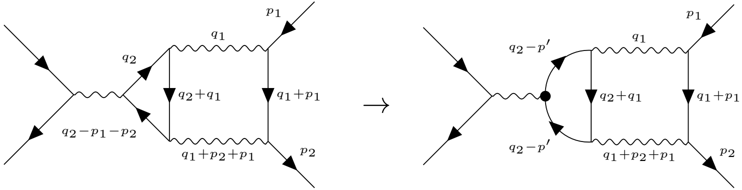
$$\mathrm{Im}\{\Pi_T^{AA}(\sigma)\} = \mathrm{Im}\{\Pi_T^{AZ}(\sigma)\} = \frac{N_c g_1 g_2}{12\pi}\left(1 + \frac{2m_f^2}{\sigma}\right)\sqrt{1 - \frac{4m_f^2}{\sigma}}\,\Theta(\sigma - 4m_f^2)$$

$$g_1 g_2 = e^2 Q_f^2 \quad (AA)$$

$$g_1 g_2 = \frac{e^2 Q_f (2s_W^2 Q_f - I_{3f})}{2 s_W c_W} \quad (AZ)$$

The light fermions can be treated as 0 in most cases, but for AA are needed to regulate IR singularities.

Triangle sub-loop (cont.)

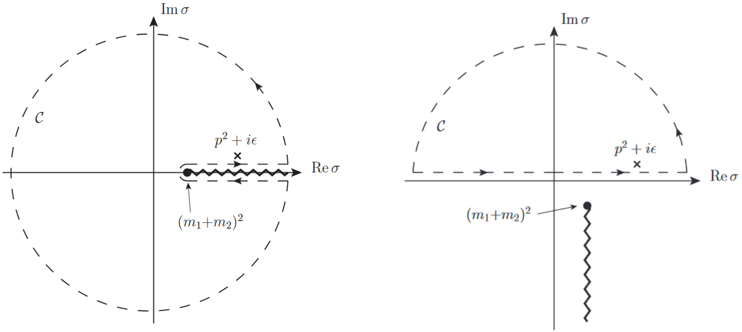


B_k functions are then expressed via dispersion relation:

$$B_k(q_1^2, m_f^2, m'^2) = \frac{1}{\pi}\int_0^\infty d\sigma\, \frac{\mathrm{Im}\left\{B_k(\sigma, m_f^2, m'^2)\right\}}{\sigma - q_1^2 - i\epsilon} \quad \text{(for } m'^2 > 0\text{)},$$

$$B_k(q_1^2, m_f^2, m'^2) = \frac{1}{2\pi i}\int_{-\infty}^\infty d\sigma\, \frac{B_k(\sigma, m_f^2, m'^2)}{\sigma - q_1^2 - i\epsilon} \quad \text{(for any } m'^2\text{)}.$$

Why Two Dispersion Relations?



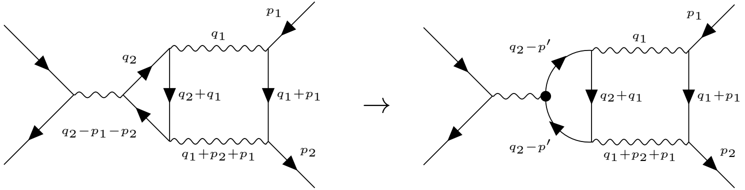
[2101.00308]

$(m_1 + m_2)^2$ can now be complex valued, since m' can be complex

Can no longer find dispersion relation as imaginary part of function

Instead, use full function and integrate over entire real number line

Triangle sub-loop (cont.)



With full diagram, get dispersion relation multiplied by scalar integral with two numerical integrations.

$$\int_0^1 dx \int d^D q_1 \frac{1}{(q_1-p_1)^2-m_{V_1}^2} \frac{1}{q_1^2-m_f^2} \frac{1}{(q_1+p_2)^2-m_{V_2}^2} \int \frac{d\sigma}{\pi} \frac{\partial_{m'^2}\Delta B_0(\sigma, m'^2, m_f^2)}{\sigma-(p'+q_1-p_1)^2}$$

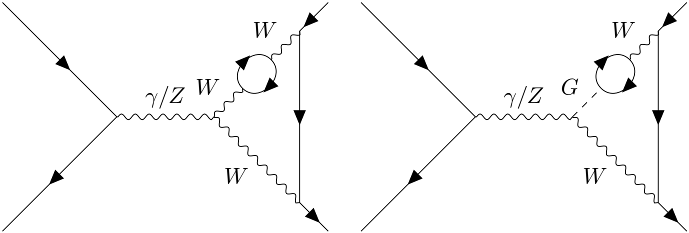
$$\to \int_0^1 dx \int d\sigma D_0(p_1^2, p_2^2, p'^2, (p_1+p_2-p')^2, m_{V_1}, m_f^2, m_{V_2}, \sigma)\partial_{m'^2}\Delta B_0(\sigma, m'^2, m_f^2)$$

In general, can contain higher-rank terms ($C_1/C_2/\ldots$, $B_1/B_{11}/\ldots$) depending on form of numerator.

Longitudinal Contributions

While most diagrams only provide contributions from transverse component, a few have longitudinal contributions

Typically suppressed for vector bosons, since coupling to external fermions is proportional to mass from Goldstone equivalence theorem



Sub-loop divergences cancel in diagram with equivalent Goldstone

UV Divergences

Diagrams can have divergences in both the sub-loop and the global loop

For the self-energy subloops, handled by the renormalization counterterms added to the numerical integral

In the triangle subloops, handled by subtracting term where momenta in q_2 integral are set to 0 [2209.07612]:

$$\int_0^1 dx \, \frac{\partial}{\partial(\mu^2)} \int_{-\infty}^{\infty} d^D q_2 \frac{q_2^\mu \cdots q_2^\nu}{\left[q_2^2 - m_f^2\right] \left[q_2^2 - \mu^2\right]}$$

In the global loop, subtract diagram with external momentum set to 0 (vacuum diagram) [hep-ph/0608099]

Handling traces with γ^5 (e.g. $\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^5) = 4i\epsilon^{\alpha\beta\gamma\delta}$) becomes non-trivial when working in D dimensions

Cannot hold both previous equation and $\{\gamma^\mu, \gamma^5\} = 0$
Conveniently, most of our calculations of interest can be done in 4 dimensions

For triangle subloop diagrams, terms proportional to ϵ are UV-finite at this order.

Can evaluate in two parts, first by calculating UV-divergent pieces in D dimensions with naively anti-commuting γ^5, then finite pieces resulting in ϵ tensors are calculated in four dimensions, then combine
[hep-ph/0608099]

Renormalization

Use on-shell renormalization scheme

Incorporate counterterms, which are well-known, and cancel with analytically re-added terms
[hep-ph/0202131]

Additionally, because on-shell masses are defined through propagator pole, on-shell W and Z masses receive an NNLO correction from the decay width

$$m_Z = m_Z^{\mathrm{exp}}[1 + (\gamma_Z^{\mathrm{exo}}/m_Z^{\mathrm{exp}})^2]^{-1/2} \approx m_Z^{\mathrm{exp}} - 34\,\mathrm{MeV},$$
$$m_W = m_W^{\mathrm{exp}}[1 + (\gamma_W^{\mathrm{exo}}/m_W^{\mathrm{exp}})^2]^{-1/2} \approx m_W^{\mathrm{exp}} - 27\,\mathrm{MeV}.$$

[2012.11642]

IR divergences occur when internal photons become soft

These universal QED singularities can be evaluated via Monte Carlo along with real radiation and so we subtract them from the calculation to avoid double counting

In particular, box diagrams with $V_3 = A$ have the following universal IR-singular contribution:

$$\mathcal{M}^{V_1=A}_{\mathrm{box}(2)}(s,t) \to \mathcal{M}^{V_1=A}_{\mathrm{box}}(s,t) - \frac{2\alpha}{\pi} Q_f Q_{f'} \left[B_{(1)}(t) - B_{(1)}(u)\right] \mathcal{M}_{\mathrm{se}(1)}(s)$$

and box diagrams with $V_{1,2} = A$ have the following universal IR-singular contribution:

$$\mathcal{M}^{V_{2,3}=A}_{\mathrm{box}(2)}(s,t) \to \mathcal{M}^{V_{2,3}=A}_{\mathrm{box}(2)}(s,t) - \frac{2\alpha}{\pi} Q_f Q_{f'} \left[B_{(2)}(t) - B_{(2)}(u)\right] \mathcal{M}_{\mathrm{tree}}(s),$$

where $\mathcal{M}_{se(1)}$ is the one-loop fermion self-energy from $V_{1,2}$, $\mathcal{M}_{tree}$ is the tree-level fermion process, $B_{(1)}$ is the universal eikonal one-loop vertex factor (Annals Phys. 13 (1961) 379–452), and $B_{(2)}$ is two-loop QED vertex factor

IR subtractions

$$B_{(1)}(Q^2) = \int \frac{d^D q}{i(2\pi)^{D-2}} \; \frac{(q+2p_1)\cdot(q-2p_2)}{[q^2+i\epsilon][(q+p_1)^2-m_f^2+i\epsilon][(q-p_2)^2-m_f^2+i\epsilon]}$$

$$B_{(2)}(Q^2) = \int \frac{d^D q}{i(2\pi)^{D-2}} \; \frac{(q+2p_1)\cdot(q-2p_2)}{[q^2+i\epsilon][(q+p_1)^2-m_f^2+i\epsilon][(q-p_2)^2-m_f^2+i\epsilon]} \, \Pi_T^{AA}(q^2).$$

Other implementation specifics

LoopTools can have numerical issues in these integrals, especially for negative momenta

Resolve by adding small imaginary displacement to dispersion variable

Avoid numerical instabilities by instituting cutoff at large values of dispersion parameter

In general, fermion masses are set to 0, except for when they are needed to regularize collinear divergences.

SM values

Parameter	Value
m_Z	91.1535 GeV
m_W	80.358 GeV
m_H	125.1 GeV
α^{-1}	137.03599976
$\Delta\alpha$	0.059
G_μ	1.1663787×10^{-5} GeV^{-2}
m_τ	1.777 GeV
m_μ	105 MeV
m_e	0.511 MeV
m_t	173.2 GeV
$\hat{m}_b$	4.18 GeV
$\hat{m}_c$	1.27 GeV
$\bar{m}_s$	342^{+123}_{-123} MeV
$\bar{m}_{u,d}$	246^{+113}_{-113} MeV

Parameters used for above results

Threshold masses of quarks calculated in [1712.09146].

$$A_4 = \frac{8}{3}\,\frac{\sigma_{\rm F} - \sigma_{\rm B}}{\sigma_{\rm F} + \sigma_{\rm B}}, \qquad \sigma_{\rm F} = \int_0^1 d\cos\theta\,\frac{d\sigma}{d\cos\theta}, \quad \sigma_{\rm B} = \int_{-1}^0 d\cos\theta\,\frac{d\sigma}{d\cos\theta}.$$

Expanding this in perturbative orders yields

$$A_{4,\rm LO} = \frac{8}{3}\,\frac{\sigma_{\rm F,LO} - \sigma_{\rm B,LO}}{\sigma_{\rm F,LO} + \sigma_{\rm B,LO}},$$

$$A_{4,\rm NLO} = \frac{16(\sigma_{\rm B,LO}\sigma_{\rm F,NLO} - \sigma_{\rm F,LO}\sigma_{\rm B,NLO})}{3(\sigma_{\rm F,LO} + \sigma_{\rm B,LO})^2},$$

$$A_{4,\rm NNLO} = \frac{16}{3}\left[\frac{(\sigma_{\rm B,LO}\sigma_{\rm F,NNLO} - \sigma_{\rm F,LO}\sigma_{\rm B,NNLO})}{(\sigma_{\rm F,LO} + \sigma_{\rm B,LO})^2} - \frac{(\sigma_{\rm F,NLO} + \sigma_{\rm B,NLO})(\sigma_{\rm B,LO}\sigma_{\rm F,NLO} - \sigma_{\rm F,LO}\sigma_{\rm B,NLO})}{(\sigma_{\rm F,LO} + \sigma_{\rm B,LO})^3}\right].$$

	Scheme	$e^+e^- \to \mu^+\mu^-$ $\alpha(0)$	$\alpha(m_Z)$	G_μ	$e^+e^- \to u\bar{u}$ $\alpha(0)$	$\alpha(m_Z)$	G_μ	$e^+e^- \to d\bar{d}$ $\alpha(0)$	$\alpha(m_Z)$	G_μ
$\sqrt{s} =$ 150 GeV	LO	5.206	5.880	5.581	10.00	11.30	10.72	7.356	8.307	7.885
	+NLO	5.731	5.773	5.821	10.61	10.61	10.74	7.741	7.727	7.828
	+NNLO	5.783	5.789	5.817	10.67	10.69	10.73	7.750	7.751	7.782
240 GeV	LO	1.797	2.030	1.927	3.071	3.468	3.292	1.816	2.051	1.947
	+NLO	1.990	2.007	2.022	3.276	3.279	3.317	1.913	1.910	1.935
	+NNLO	2.010	2.012	2.021	3.293	3.296	3.308	1.911	1.910	1.917
1 TeV	LO	0.099	0.112	0.107	0.162	0.183	0.174	0.086	0.097	0.092
	+NLO	0.109	0.110	0.111	0.166	0.164	0.167	0.089	0.088	0.090
	+NNLO	0.110	0.110	0.110	0.165	0.165	0.165	0.088	0.088	0.088

Table: Partonic cross-sections in pb for different final states and in different renormalization schemes for the electroweak coupling.

Uncertainty from Higher Orders

Evaluate potential impacts of NNLO bosonic corrections through two methods: 1) Assume that the ratio NNLO/NLO is similar to NLO/LO for $N_f = 0$.

2) Assume that the ratio for $N_f = 0$ and $N_f = 1$ is similar at NLO and NNLO

Additionally, evaluate N3LO corrections as

$$\frac{\sigma^2_{\mathrm{NNLO},N_f=2} + \sigma^2_{\mathrm{NNLO},N_f=1}}{\sqrt{\sigma^2_{\mathrm{NLO},N_f=1} + \sigma^2_{\mathrm{NLO},N_f=0}}}$$

Get overall estimate as

$$\delta\sigma_{\mathrm{perturb.,EW}} = \sqrt{\left(\max\left\{\delta_a\sigma_{\mathrm{NNLO},N_f=0}, \delta_b\sigma_{\mathrm{NNLO},N_f=0}\right\}\right)^2 + \delta\sigma^2_{\mathrm{N3LO}}}$$

Finally, get estimate of uncertainty from scheme dependence

Uncertainties (cont.)

$\sqrt{s}$	[fb]	$e^+e^- \to \mu^+\mu^-$	$e^+e^- \to u\bar{u}$	$e^+e^- \to d\bar{d}$
150 GeV	$\sigma_{\mathrm{NNLO,ferm}}$	5816.7	10729.5	7782.4
	$\bar{m}_{u,d,s}$ uncertainty	0.15	0.15	2.0
	$\delta_{\mathrm{perturb.,EW}}$	8.8	32.0	70.0
	scheme dep.	27.7	43.7	31.7
	max. of prev. two rows	27.7 (0.48%)	43.7 (0.40%)	70.0 (0.90%)
240 GeV	$\sigma_{\mathrm{NNLO,ferm}}$	2021.3	3308.1	1916.8
	$\bar{m}_{u,d,s}$ uncertainty	0.04	0.06	0.6
	$\delta_{\mathrm{perturb.,EW}}$	4.1	18.7	25.7
	scheme dep.	9.3	12.4	7.3
	max. of prev. two rows	9.3 (0.46%)	18.7 (0.57%)	25.7 (1.3%)
1 TeV	$\sigma_{\mathrm{NNLO,ferm}}$	110.44	164.87	88.06
	$\bar{m}_{u,d,s}$ uncertainty	0.003	0.006	0.04
	$\delta_{\mathrm{perturb.,EW}}$	0.90	5.0	3.1
	scheme dep.	0.36	0.17	0.16
	max. of prev. two rows	0.90 (0.8%)	5.0 (3.0%)	3.1 (3.5%)

Table: Evaluation of uncertainties from light quark masses and missing higher orders, using the G_μ scheme as reference.