Effective Field Theory Perspective On King Non-linearity

Gil Paz

Department of Physics and Astronomy,

Wayne State University,

Detroit, Michigan, USA

U.S. DEPARTMENT OF ENERGY

Office of Science

Based on

[Assi, Carey, Jäger, Lee, Paz, Perez, Zupan, arXiv:2512.03157]

Outline

- Introduction: King linearity and non-linearity

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- EFT Perspective On King Non-linearity

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- Phenomenological implications
- Conclusions

Introduction:

King linearity
and
King non-linearity

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   For hydrogen-like system

$$E_n^A = -\frac{1}{2n^2}(Z\alpha)^2 \frac{m_e m_A}{m_e + m_A} \approx -\frac{1}{2n^2} m_e (Z\alpha)^2 \left(1 - \frac{m_e}{m_A}\right)$$

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$$\Delta E_n = \frac{4\pi\alpha}{6}|\phi(0)|^2\langle r^2\rangle = \frac{4\pi\alpha}{6}\frac{(m_e Z\alpha)^3}{\pi n^3}\langle r^2\rangle$$

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- Two sources with a different electronic dependence and a different nuclear dependence

- Two sources $\Rightarrow$ linear relation between $\nu_i^{AA'}$'s $\Rightarrow$ King linearity
  [W. H. King, J. Opt. Soc. Am. **53** 638 (1963)]

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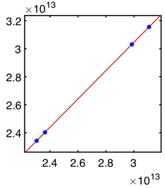

Image from [Counts et al., PRL **125** 123002 (2020)]

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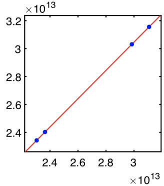

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- New interactions, e.g., coupling to e and n, can introduce 3rd source leading to non-linearity $V_\phi(r) = -(-1)^s y_e y_n (A-Z) e^{-m_\phi r}/4\pi r$ [Berengut, et al. Phys. Rev. Lett. **120**, 091801 (2018)]

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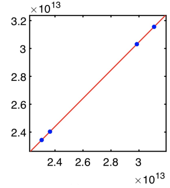

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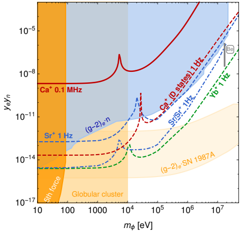

King non-linearity

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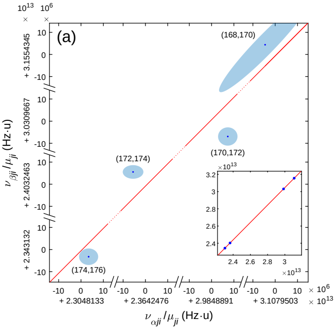

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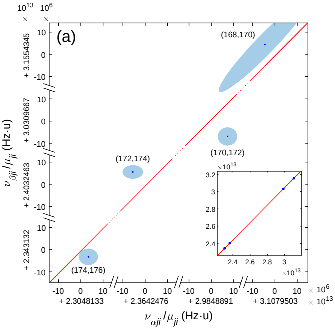

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- Is it new physics?
- Nuclear effects such as $\langle r^4 \rangle$ or $\langle r^2 \rangle^2$ also lead to non-linearity!

reminder: $\left[\langle r^k \rangle \equiv \int d^3r\, r^k\, \rho(r)\right]$

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- King non-linearity was observed with a large number of σ's
- Is it new physics?
- Are these all the possible nuclear effects?

EFT Perspective on King Non-linearity

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- We will use Non Relativistic QED (NRQED)

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- Step 3: Use the QM potentials to find $\Delta E = h\,\Delta\nu$

Step 1: Match SM onto NRQED

NRQED Lagrangian

- The NRQED Lagrangian up to dimension 8
  [Hill, Lee, GP, Solon, PRD **87** 053017 (2013)]

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$$\mathcal{L}_{\text{NRQED}} = \chi^\dagger \Big\{ iD_t + \frac{\boldsymbol{D}^2}{2M} + c_F g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2M} + c_D g \frac{[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{8M^2} + ic_S g \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E}-\boldsymbol{E}\times\boldsymbol{D})}{8M^2}$$

$$+\frac{\boldsymbol{D}^4}{8M^3} + c_{W1} g \frac{\{\boldsymbol{D}^2,\boldsymbol{\sigma}\cdot\boldsymbol{B}\}}{8M^3} - c_{W2} g \frac{D^i\boldsymbol{\sigma}\cdot\boldsymbol{B}D^i}{4M^3} + ic_M g \frac{\{\boldsymbol{D}^i,[\boldsymbol{\partial}\times\boldsymbol{B}]^i\}}{8M^3}$$

$$+c_{p'p} g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}\boldsymbol{B}\cdot\boldsymbol{D}+\boldsymbol{D}\cdot\boldsymbol{B}\boldsymbol{\sigma}\cdot\boldsymbol{D}}{8M^3} + c_{A1} g^2 \frac{\left(\boldsymbol{B}^2-\boldsymbol{E}^2\right)}{8M^3} - c_{A2} g^2 \frac{\boldsymbol{E}^2}{16M^3}$$

$$+c_{X1} g \frac{[\boldsymbol{D}^2,\boldsymbol{D}\cdot\boldsymbol{E}+\boldsymbol{E}\cdot\boldsymbol{D}]}{M^4} + c_{X2} g \frac{\{\boldsymbol{D}^2,[\boldsymbol{\partial}\cdot\boldsymbol{E}]\}}{M^4} + c_{X3} g \frac{[\boldsymbol{\partial}^2\boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4}$$

$$+ic_{X4} g^2 \frac{\{\boldsymbol{D}^i,[\boldsymbol{E}\times\boldsymbol{B}]^i\}}{M^4} + ic_{X5} g \frac{D^i\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E}-\boldsymbol{E}\times\boldsymbol{D})D^i}{M^4}$$

$$+ic_{X6} g \frac{\epsilon^{ijk}\sigma^i D^j[\boldsymbol{\partial}\cdot\boldsymbol{E}]D^k}{M^4} + c_{X7} g^2 \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} + c_{X8} g^2 \frac{[\boldsymbol{E}\cdot\boldsymbol{\partial}\boldsymbol{\sigma}\cdot\boldsymbol{B}]}{M^4}$$

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NRQED Lagrangian

- 1st simplification: King NL experiments are done with spin-zero nuclei

$$\mathcal{L}_{\text{NRQED}} = \chi^\dagger \Big\{ iD_t + \frac{\boldsymbol{D}^2}{2M} + c_F g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2M} + c_D g \frac{[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{8M^2} + ic_S g \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E}-\boldsymbol{E}\times\boldsymbol{D})}{8M^2}$$
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& + \frac{\boldsymbol{D}^4}{8M^3} + ic_M g \frac{\{\boldsymbol{D}^i, [\boldsymbol{\partial} \times \boldsymbol{B}]^i\}}{8M^3} \\
& + c_{A1} g^2 \frac{\left(\boldsymbol{B}^2 - \boldsymbol{E}^2\right)}{8M^3} - c_{A2} g^2 \frac{\boldsymbol{E}^2}{16M^3} \\
& + c_{X1} g \frac{[\boldsymbol{D}^2, \boldsymbol{D} \cdot \boldsymbol{E} + \boldsymbol{E} \cdot \boldsymbol{D}]}{M^4} + c_{X2} g \frac{\{\boldsymbol{D}^2, [\boldsymbol{\partial} \cdot \boldsymbol{E}]\}}{M^4} + c_{X3} g \frac{[\boldsymbol{\partial}^2 \boldsymbol{\partial} \cdot \boldsymbol{E}]}{M^4} \\
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& \Big\} \chi
\end{aligned}
$$

NRQED Lagrangian

- 2nd simplification: $1/r_N \ll M$, since $1/r_N \sim A^{-1/3}/r_p$ and $M \sim Am_p$

$$\mathcal{L}_{\text{NRQED}} = \chi^\dagger \Big\{ iD_t + c_D g \frac{[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{8M^2} + + c_{A1} g^2 \frac{\left(\boldsymbol{B}^2 - \boldsymbol{E}^2\right)}{8M^3} - c_{A2} g^2 \frac{\boldsymbol{E}^2}{16M^3} + c_{X3} g \frac{[\boldsymbol{\partial}^2\boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} \Big\}\chi$$

- Only 5 operators remain out of the 24

NRQED Lagrangian

- NRQED Lagrangian up to dim. 8 [Hill, Lee, GP, Solon, PRD **87** 053017 (2013)]

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Scalar NRQED Lagrangian in the $M \to \infty$ limit

- Changing $\chi \to S$, the scalar NRQED Lagrangian in the $M \to \infty$ limit takes the simple form

$$\mathcal{L}_S = S^\dagger \left\{ iD_t + \frac{c_{\langle r^2 \rangle}}{\Lambda_N^2} (\boldsymbol{\nabla} \cdot \boldsymbol{E}) + \frac{c_{E^2}}{\Lambda_N^3} \boldsymbol{E}^2 + \frac{c_{B^2}}{\Lambda_N^3} \boldsymbol{B}^2 + \frac{c_{\langle r^4 \rangle}}{\Lambda_N^4} \boldsymbol{\nabla}^2 (\boldsymbol{\nabla} \cdot \boldsymbol{E}) \right\} S$$

Scalar NRQED Lagrangian in the $M \to \infty$ limit

- Changing $\chi \to S$, the scalar NRQED Lagrangian in the $M \to \infty$ limit takes the simple form

$$\mathcal{L}_S = S^\dagger \left\{ iD_t + \frac{c_{\langle r^2 \rangle}}{\Lambda_N^2} (\boldsymbol{\nabla} \cdot \boldsymbol{E}) + \frac{c_{E^2}}{\Lambda_N^3} \boldsymbol{E}^2 + \frac{c_{B^2}}{\Lambda_N^3} \boldsymbol{B}^2 + \frac{c_{\langle r^4 \rangle}}{\Lambda_N^4} \boldsymbol{\nabla}^2 (\boldsymbol{\nabla} \cdot \boldsymbol{E}) \right\} S$$

- To determine the Wilson coefficients we perform matching and get

$$\mathcal{L}_S = S^\dagger \left\{ iD_t + eZ \frac{\langle r^2 \rangle}{6} (\boldsymbol{\nabla} \cdot \boldsymbol{E}) + \frac{\alpha_E}{2} \boldsymbol{E}^2 + \frac{\beta_M}{2} \boldsymbol{B}^2 + eZ \frac{\langle r^4 \rangle}{120} \boldsymbol{\nabla}^2 (\boldsymbol{\nabla} \cdot \boldsymbol{E}) \right\} S$$

- If the nuclear charge density is $\rho(r)$, $\langle r^k \rangle \equiv \int d^3r \, r^k \rho(r)$

- α_E is the electric polarizability and β_M the magnetic polarizability

Four-fields operators Lagrangian

- At dimension 8 we have many more operators
  Lorentz invariance implies relations between Wilson coefficients
  [Hill, Lee, GP, Solon, PRD **87** 053017 (2013)]

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\mathcal{L}_{S\psi_e} =& \frac{d_2 m_e}{\Lambda_N^3} S^\dagger S \psi_e^\dagger \psi_e + \frac{d_3}{\Lambda_N^4} S^\dagger D_+^i S \psi_e^\dagger D_+^i \psi_e + \frac{d_6}{m_e \Lambda_N^3} S^\dagger S \psi_e^\dagger (\boldsymbol{D}^2 + \overleftarrow{\boldsymbol{D}}^2) \psi_e \\
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where $D_\pm = D \pm \overleftarrow{D}$

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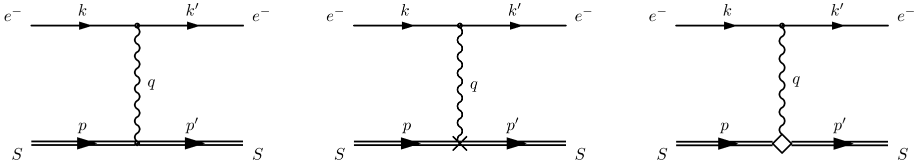
- We treat them as non-perturbative nuclear parameters

Step 2: Match NRQED onto QM

Example: $V_{\text{1-photon}}(\boldsymbol{r})$



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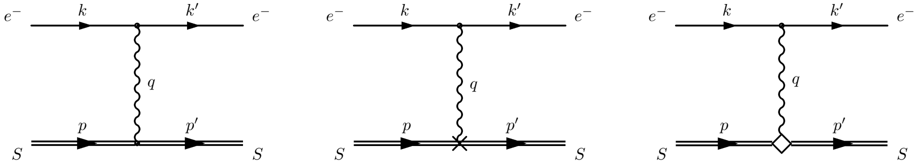

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$$i\mathcal{M}_Z = \frac{c}{q^2} \quad \Rightarrow \quad \tilde{V}_Z(\boldsymbol{q}) = \frac{-Ze^2}{\boldsymbol{q}^2} \quad \Rightarrow \quad V_Z(r) = -\frac{Z\alpha}{r},$$

$$i\mathcal{M}_{\langle r^2\rangle} = -\frac{c\langle r^2\rangle}{6q^2}\boldsymbol{q}^2 \quad \Rightarrow \quad \tilde{V}_{\langle r^2\rangle}(\boldsymbol{q}) = \frac{Ze^2}{6}\langle r^2\rangle \quad \Rightarrow \quad V_{\langle r^2\rangle}(\boldsymbol{r}) = \frac{4\pi Z\alpha}{6}\langle r^2\rangle\, \delta^3(\boldsymbol{r}),$$

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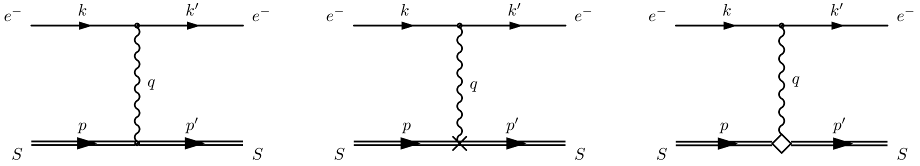

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- Coulomb potential, $V_Z(r)$, is solved exactly
  while $V_{\langle r^2\rangle}$ and $V_{\langle r^4\rangle}$ are treated as perturbations

Step 3: Use QM potentials to find ΔE

List of potentials

- Atomic scale: a nuclear scale: r_N. Classify potentials by $\epsilon = r_N/a$ This is a rough scaling powers of Z are also important

$$\text{Order } \epsilon^2 : \qquad V_{\langle r^2 \rangle}(\boldsymbol{r}) = \frac{4\pi Z\alpha}{6} \langle r^2 \rangle \, \delta^3(\boldsymbol{r}) \,,$$

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- Current level of experimental precision is ϵ^4

$\Rightarrow$ Use 1st and 2nd order PT for $V_{\langle r^2\rangle}$ and 1st order for all others

Phenomenological implications

List of potentials by interaction

- Use 1st and 2nd order PT for $V_{\langle r^2\rangle}$ and 1st order for all others

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King NL: Nuclear parameters

- Using the EFT results we can find the contributions to King NL

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2\pi\nu_i^{AA'} =& K_i\,\mu_{AA'} + F_i\left(\langle r_A^2\rangle_{\text{eff}} - \langle r_{A'}^2\rangle_{\text{eff}}\right) + G_i^{(2)}\left(\langle r_A^2\rangle^2 - \langle r_{A'}^2\rangle^2\right) \\
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- For $V(r) = Z_{\text{eff}}\,\alpha/r$ we have $\phi'_{n,0}(0) = -m_e Z_{\text{eff}}\,\alpha\,\phi_{n,0}(0)$ so that

$$F_{n,\ell} = \frac{4\pi Z\alpha}{6}\left|\phi_{n,\ell}(0)\right|^2\,\delta_{\ell 0},$$

$$G_{n,\ell}^{(2)} = \left(\frac{4\pi Z\alpha}{6}\right)^2 \tilde{G}_{Z_{\text{eff}}}^{n\text{-dep.}}(0,0,E_n)\,\left|\phi_{n,\ell}(0)\right|^2\,\delta_{\ell 0},$$

$$G_{n,\ell}^{(4)} = \begin{cases} \dfrac{4\pi Z\alpha}{20}\phi_{n,0}(0)\phi''_{n,0}(0), & \ell = 0,\\ \dfrac{4\pi Z\alpha}{80\pi}\left[R'_{n1}(0)\right]^2, & \ell = 1,\\ 0, & \ell > 1 \end{cases}$$

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- Recall that $G_i^{(2)}$, the electronic coefficient of $\langle r^2 \rangle^2$ depend on the **entire** spectrum of the Hamiltonian. Very hard to calculate.

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- Conclusion: current treatment of $\langle r^2 \rangle^2$ term needs to be reconsidered

Conclusions and outlook

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- Applied this to find a general expression for King NL and compare and test other more phenomenological approaches

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Thank you!