

Dispersion relation and unitarization of electroweak amplitudes

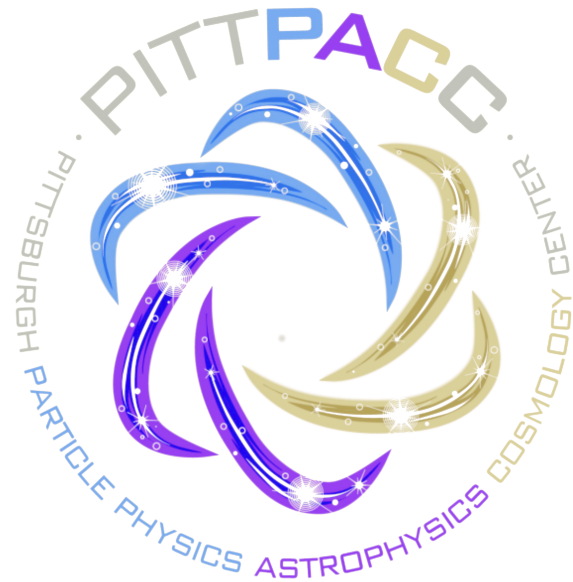


University of Chicago

Chi Shu

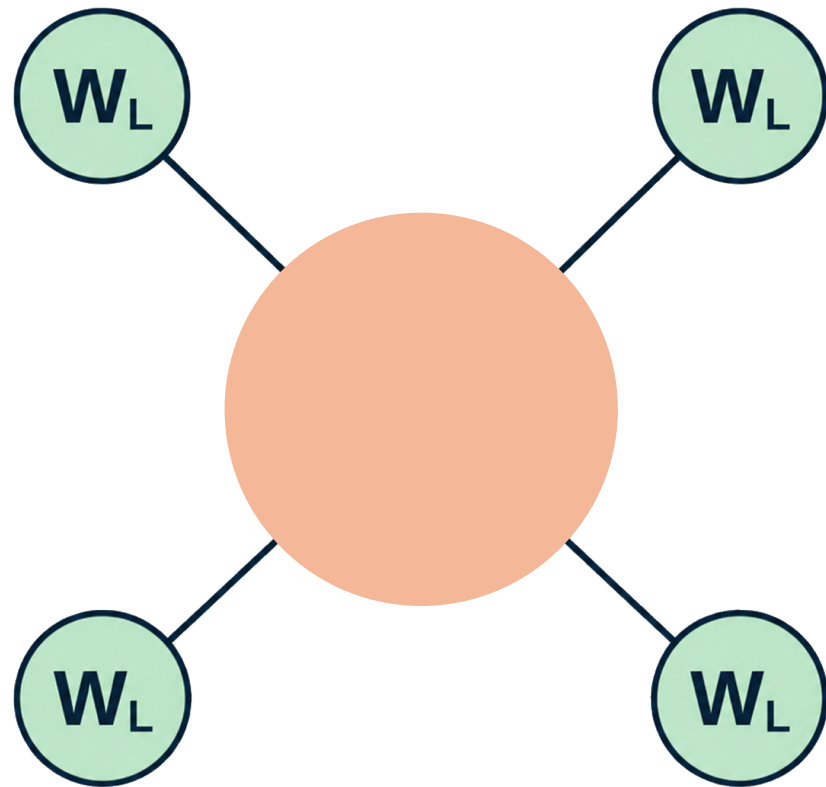
Pheno 2026

In collaboration with Zhen Liu, Lian-Tao Wang and Yohei Ema
Work in progress



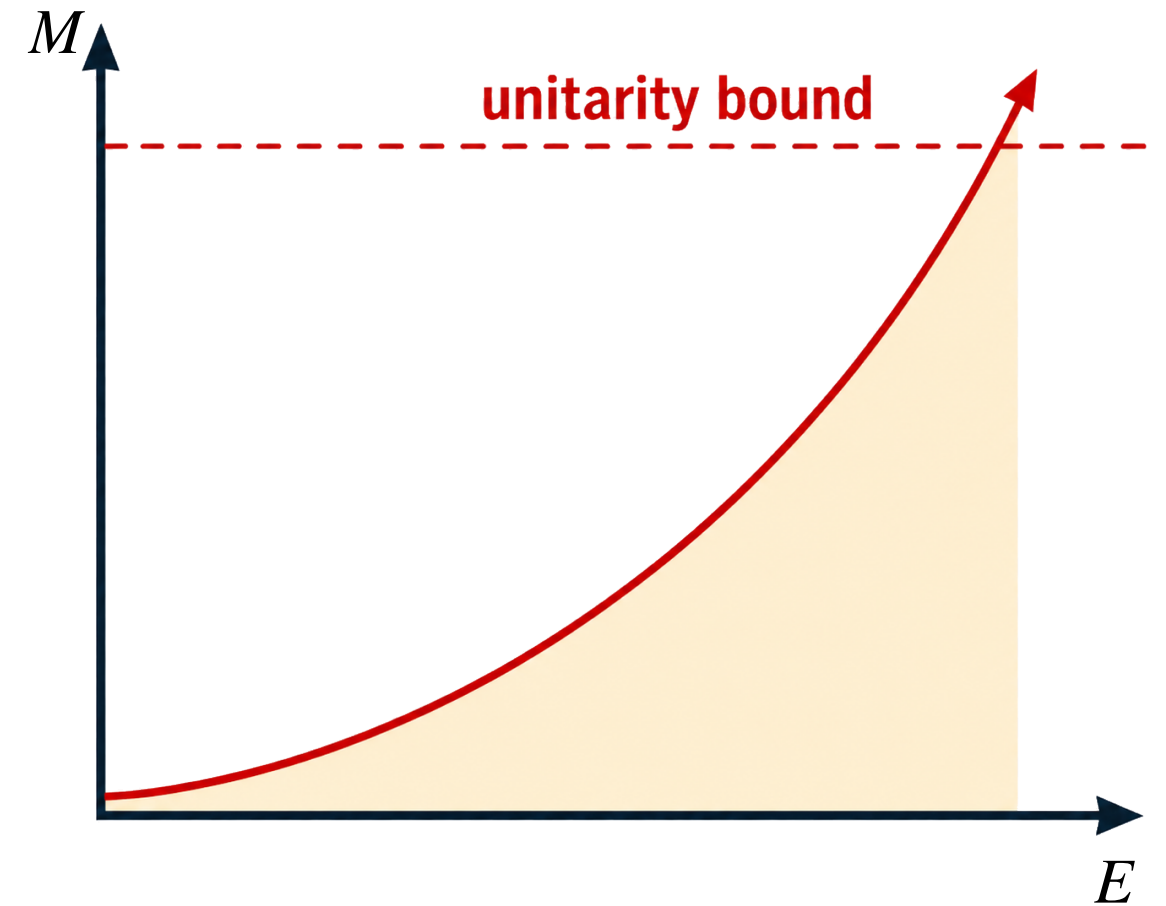
The example for unitarity fails

Consider gauge boson exchange only

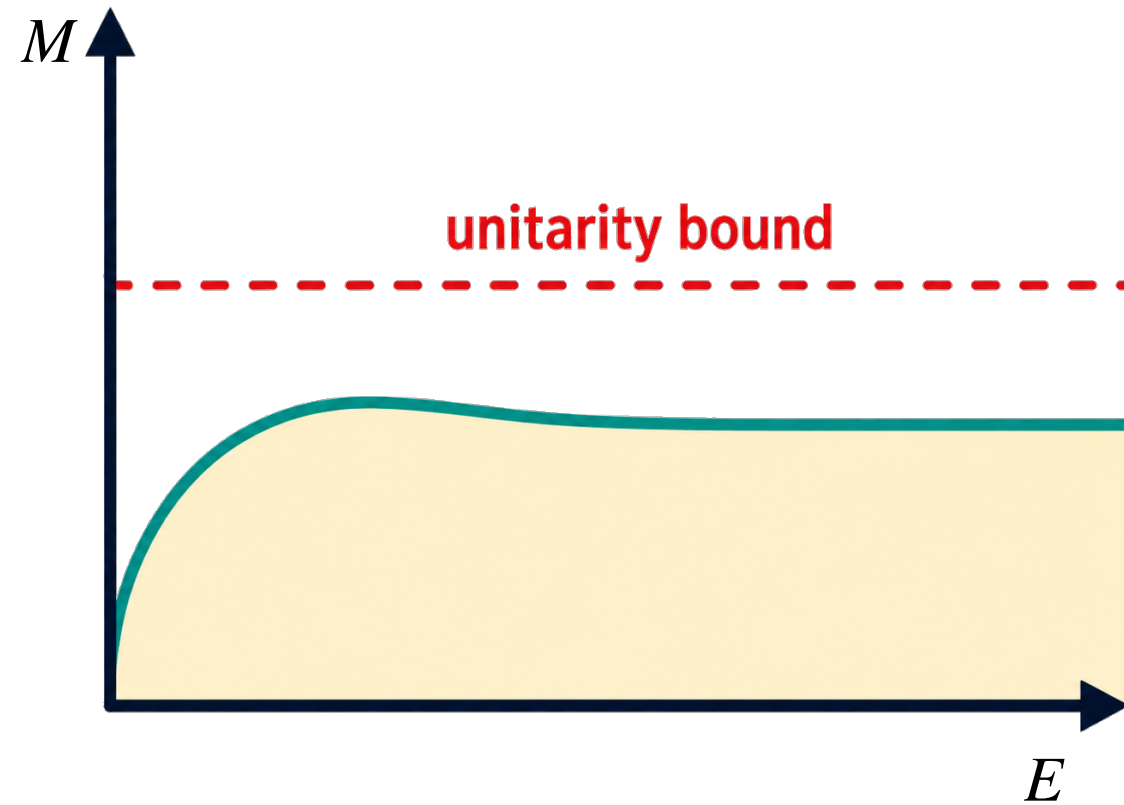
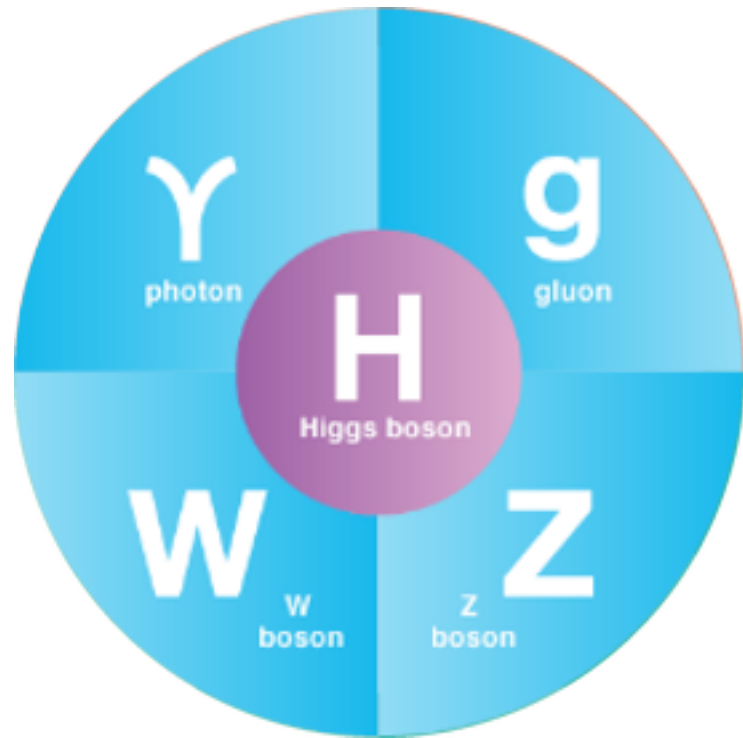


No Higgs
Cancellation

Unitarity
Fails



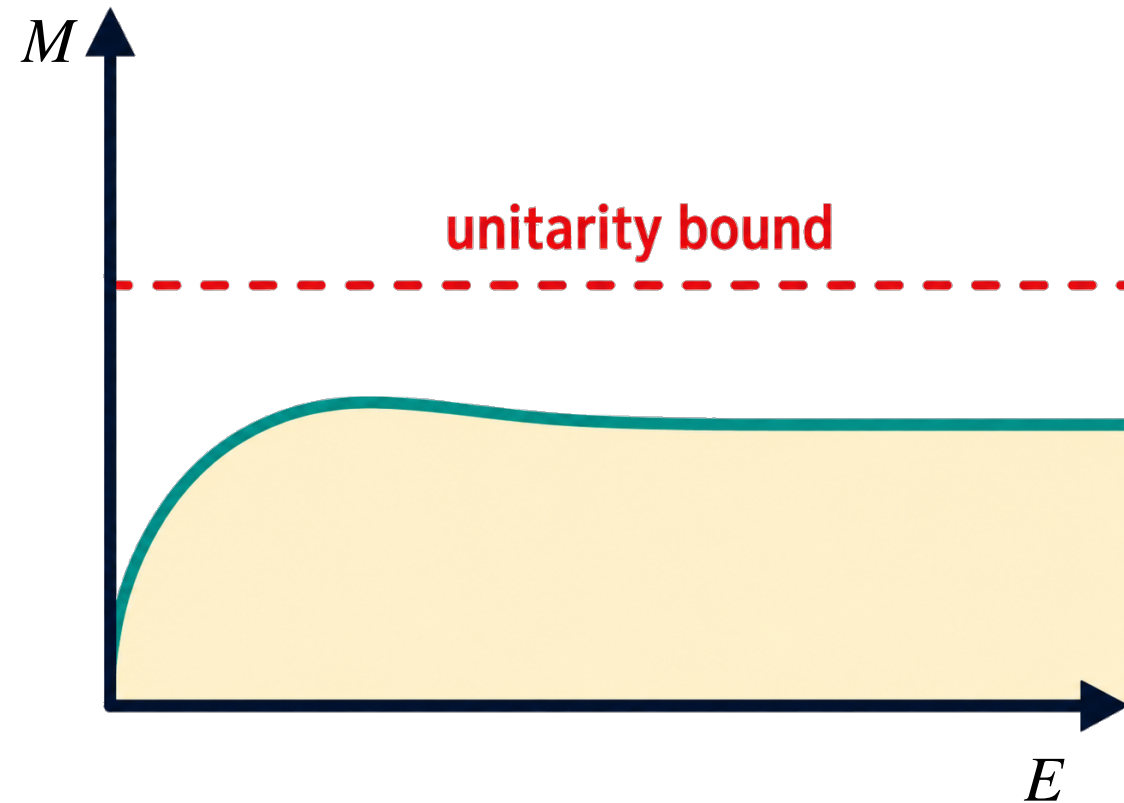
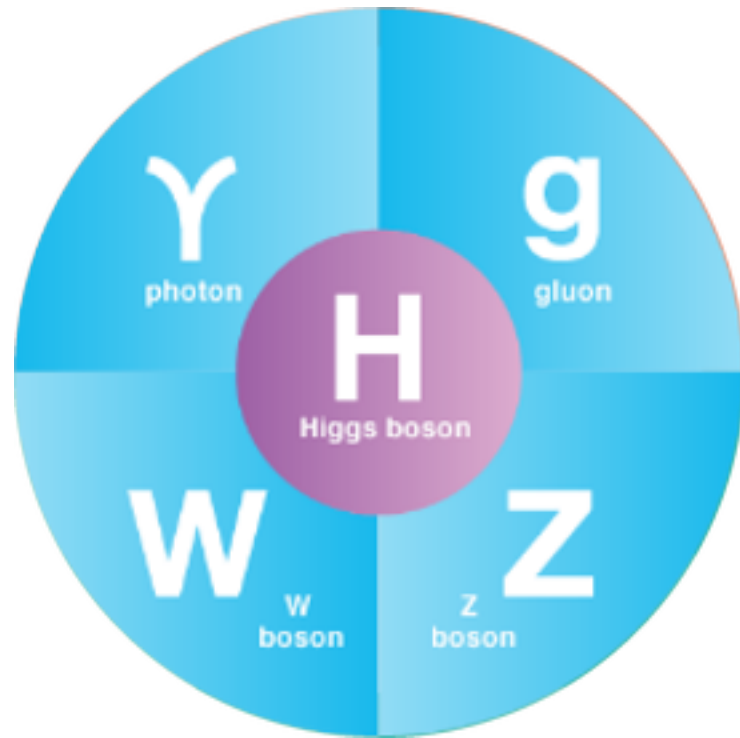
The Higgs solution



$$m_h \leq \sqrt{\frac{16\pi}{3}} v \approx 1 \text{ TeV}.$$

Lee–Quigg–Thacker bound

The possible solution



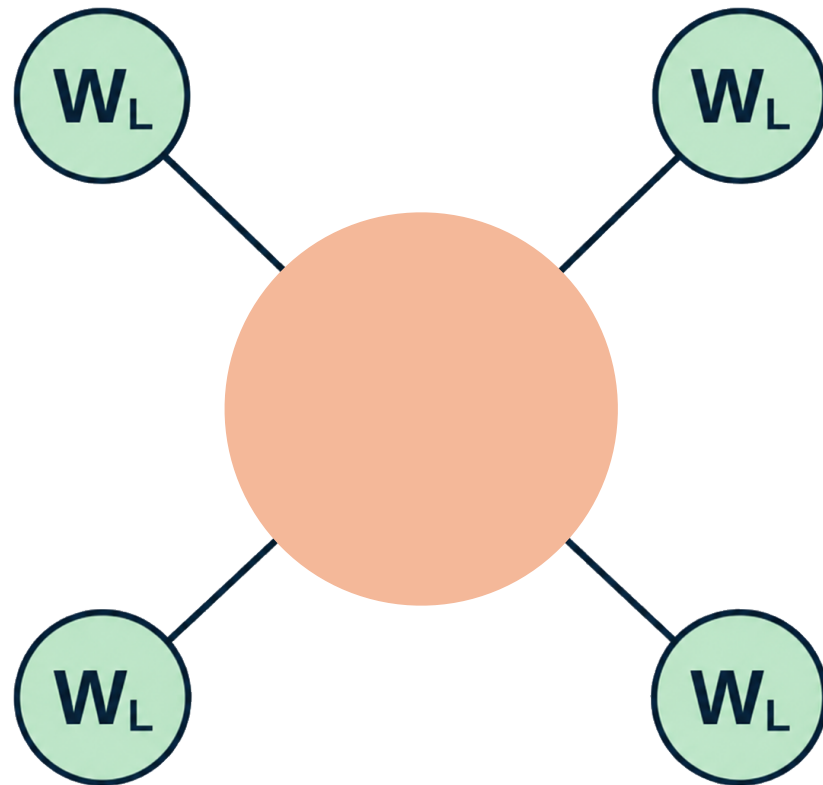
$$m_h \leq \sqrt{\frac{16\pi}{3}} \frac{1}{v} \approx 1 \text{ TeV}.$$

Lee–Quigg–Thacker bound

- KK tower model
- Veneziano-type Amplitude

Question:
How can we construct UV spectral densities that unitarize the amplitude?

Setup: IR Matching and UV Softness



IR matching: reproduce the low-energy longitudinal W scattering amplitude.

$$A_4^{\text{IR}}(s, x) = \frac{g^2}{m_W^2}(s + u) + \mathcal{O}(1) = \frac{g^2}{2m_W^2}s(1 + x) + \dots, \quad x = \cos \theta$$

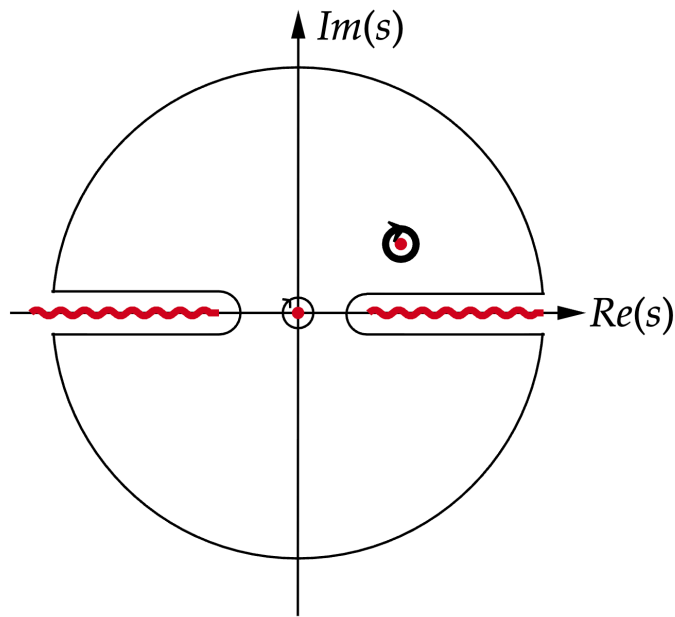
UV softness: require the full amplitude to grow slower than s^k at fixed angle.

$$\lim_{s \rightarrow \infty} \frac{A_4(s, x)}{s^k} = 0, \quad k \geq 1$$

Sum rules for discrete mass spectra

Cauchy integral

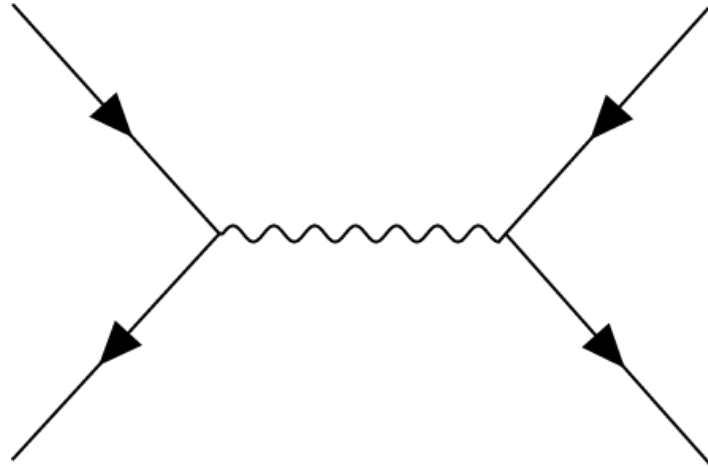
$$0 = -\frac{1}{2\pi i} \int_{\mathcal{C}} ds \frac{A_4(s, x)}{s^{k+1}} = \text{Res}_{s=0} \frac{A_4(s, x)}{s^{k+1}} + \sum_{\text{UV poles}} \text{Res} \frac{A_4(s, x)}{s^{k+1}}$$



Crossing symmetry

$$\begin{aligned} 0 &= \sum_{s=0, m^2} \text{Res} \left[\frac{A_4}{s^{k+1}} \right] + \sum_{u=m^2} \text{Res}_s \left[\frac{A_4}{s^{k+1}} \right] \\ &= \text{Res}_{s=0} \left[\frac{A_4(s, x)}{s^{k+1}} \right] + \frac{1}{m^{2(k+1)}} \left[\text{Res}_{s=m^2} A_4 + \left(-\frac{1-x}{2} \right)^k \text{Res}_{u=m^2} A_4 \right]. \end{aligned}$$

four-point amplitude factorization



$$\lim_{s \rightarrow m^2} A_4 = - \sum_{\lambda} A_3^{(\lambda)}(p_1, p_2, p_I) \frac{1}{s - m^2} A_3^{(\bar{\lambda})}(p_3, p_4, -p_I),$$

This can be written as Legendre polynomial¹

$$\lim_{s \rightarrow m^2} A_4 = - \frac{g_0^2}{s - m^2} P_S \left(1 + \frac{2u}{m^2} \right),$$

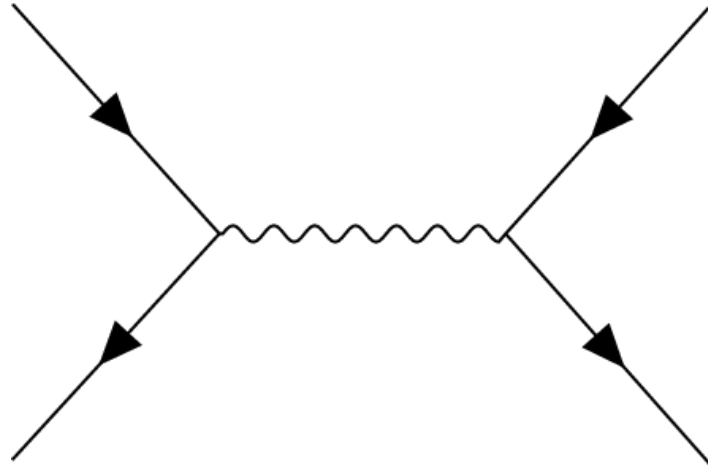
We focus on $k = 1$, $\lim_{s \rightarrow \infty} \frac{A_4(s, x)}{s} = 0$, leading to

$$0 = \text{Res}_{s=0} \left[\frac{A_4(s, x)}{s^{k+1}} \right] + \frac{1}{m^{2(k+1)}} \left[\text{Res}_{s=m^2} A_4 + \left(-\frac{1-x}{2} \right)^k \text{Res}_{u=m^2} A_4 \right].$$

$$0 = \frac{g^2}{2m_W^2} (1+x) - \frac{g_0^2}{m^4} \left[P_S(x) - \frac{1-x}{2} P_S \left(1 - \frac{4}{1-x} \right) \right].$$

1: Arxiv: 1906.11687

Example



$$0 = \frac{g^2}{2m_W^2}(1+x) - \frac{g_0^2}{m^4} \left[P_S(x) - \frac{1-x}{2} P_S \left(1 - \frac{4}{1-x} \right) \right].$$

We focus on $S = 0$ scalar exchange, Higgs-like

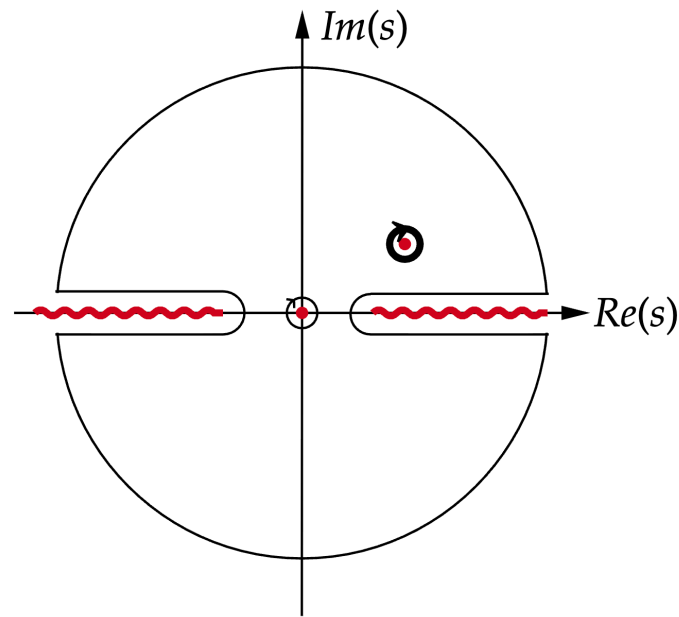
$$0 = \frac{1+x}{2} \left[\frac{g^2}{m_W^2} - \frac{g_0^2}{m^4} \right],$$

We focus on $S = 1$ vector exchange, Z' -like

$$0 = \frac{1+x}{2} \left[\frac{g^2}{m_W^2} - \frac{3g_0^2}{m^4} \right].$$

Sum rules for Continuum mass spectra

Cauchy integral



$$0 = \text{Res}_{s=0} \frac{A_4}{s^{k+1}} - \frac{1}{2\pi i} \left(\int_{s-\text{ch}} ds \frac{2i \text{Im} A_4(s, x)}{s^{k+1}} - \int_{t, u-\text{ch}} ds \frac{2i \text{Im} A_4(s, x)}{s^{k+1}} \right)$$

The imaginary part of the amplitude is given by the spectral density as

$$\text{Im}_s A_4(s) = \sum_J (2J + 1) \rho_J(s) P_J \left(1 + \frac{2u}{s} \right), \quad \rho_J < 2$$

$$\text{Res}_{s=0} \left[\frac{A_4(s)}{s^{k+1}} \right] = \frac{1}{\pi} \sum_J \left[\int_{m_0^2}^{\infty} ds \frac{\rho_J(s)}{s^{k+1}} \right] (2J + 1) \left[P_J(x) + \left(-\frac{1-x}{2} \right)^k P_J \left(1 - \frac{4}{1-x} \right) \right]$$

matching equation

$$\text{Res}_{s=0} \left[\frac{A_4(s)}{s^{k+1}} \right] = \frac{1}{\pi} \sum_J \left[\int_{m_0^2}^{\infty} ds \frac{\rho_J^{(0)}(s)}{s^{k+1}} \right] (2J + 1) \left[P_J(x) + \left(-\frac{1-x}{2} \right)^k P_J \left(1 - \frac{4}{1-x} \right) \right]$$

We can define the "coupling" constants as

$$g_{k;J} = \frac{1}{\pi} \left[\int_{m_0^2}^{\infty} ds \frac{\rho_J(s)}{s^{k+1}} \right] (2J + 1)$$

$$A_4^{\text{IR}}(s, x) = \frac{g^2}{2m_W^2} s(1+x) + \dots$$

$$\frac{g^2}{2m_W^2} (1+x) = \sum_J [g_{1;J}]^2 \left(P_J(x) - \frac{1-x}{2} P_J \left(1 - \frac{4}{1-x} \right) \right)$$

$$\frac{g^2}{2m_W^2}(1+x) = \sum_J [g_{1;J}]^2 \left(P_J(x) - \frac{1-x}{2} P_J \left(1 - \frac{4}{1-x} \right) \right)$$

Note that, $J = 0$ gives L.H.S. $\propto (1+x)$, the trivial solution

$$\frac{g^2}{2m_W^2}(1+x) = [g_{1;0}]^2 \left(P_0(x) - \frac{1-x}{2} P_0 \left(1 - \frac{4}{1-x} \right) \right) = [g_{1;0}]^2 \frac{1+x}{2}$$

This can be viewed as particular solution. We want to find the homogeneous solution

Let us suppose a function $H(z)$ satisfies $H(x) - \frac{1-x}{2} H \left(1 - \frac{4}{1-x} \right) = 0$.

Infinite spin support gives non-unique solutions

$$\frac{g^2}{2m_W^2}(1+x) = \sum_J [g_{1;J}]^2 \left(P_J(x) - \frac{1-x}{2} P_J \left(1 - \frac{4}{1-x} \right) \right) \quad H(x) - \frac{1-x}{2} H \left(1 - \frac{4}{1-x} \right) = 0.$$

One set solution $H(z) = 2 - \frac{4}{3-z} = 2 - 4 \sum_{J=0}^{\infty} (2J+1) Q_J(3) P_J(z).$

Find coefficient $[g_{1;J}]^2$ to satisfy the equation. Give 1 set solution here

$$[g_{1;0}]^2 = a_0 = \frac{g^2}{m_W^2} - \lambda(2 - 2 \ln 2),$$

$$[g_{1;1}]^2 = a_1 = \lambda(18 \ln 2 - 12) > 0,$$

$$[g_{1;J}]^2 = b_J = 4\lambda(2J+1)Q_J(3), \quad J \geq 2,$$

$$g_{k;J} = \frac{1}{\pi} \left[\int_{m_0^2}^{\infty} ds \frac{\rho_J(s)}{s^{k+1}} \right] (2J+1)$$

A particularly simple choice is $\rho_J(s) = \frac{\pi [g_{k;J}]^2}{(2J+1)\Delta} \Theta(M_+^2 - s) \Theta(s - M_-^2)$

$$A_4(s, x) = \frac{1}{\pi} \sum_J (2J+1) \int \frac{ds'}{s'} \left[\frac{s}{s'-s} P_J(x) + \frac{u}{s'-u} P_J \left(1 - \frac{4}{1-x} \right) \right] \rho_J^{(0)}(s'),$$

$$A_4(s, x) = \Psi(s) \left[\frac{g^2}{m_W^2} - \lambda \frac{2(\cos \theta - 1)}{\cos \theta - 3} \right] + \Psi(u) \left[\frac{g^2}{m_W^2} - \lambda \frac{4}{3 - \cos \theta} \right].$$

$$\Psi(s) = \frac{1}{\Delta} \left[\log \frac{M_+^2 - s}{M_-^2 - s} - \log \frac{M_+^2}{M_-^2} \right].$$



Thanks

