

Progress at the 5th post-Minkowskian Order in Gravitational-Wave Physics

Zvi Bern

July 3, 2026

Amplitudes 2026

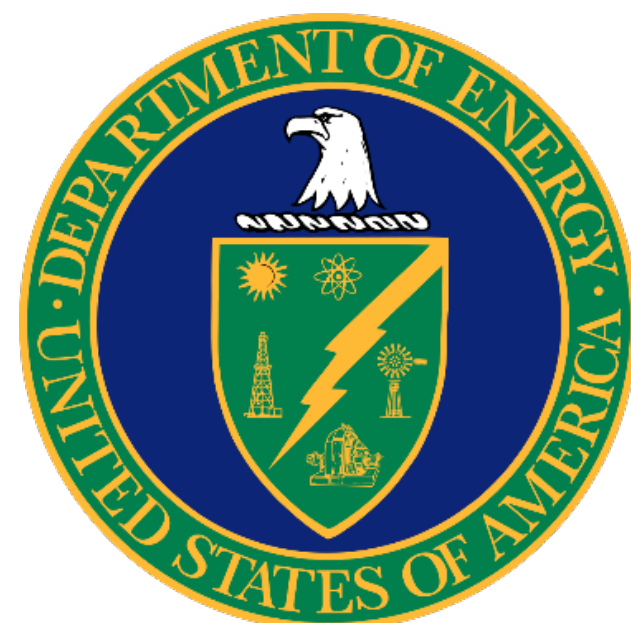
Queen Mary University of London



ZB, E. Herrmann, R. Roiban, M. Ruf, V.A. Smirnov, A.V. Smirnov, M. Zeng:
arXiv:2305.08981, arXiv:2406.01554; arXiv:2509.17412.

ZB, E. Herrmann, R. Roiban, M. Ruf, A.V. Smirnov, S. Smith, M. Zeng: arXiv:2512.15383.

ZB, Jackman, Mansfield, Ruf: ArXiv:2603.15283.



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Direct contributors to this talk.



Avery Jackman
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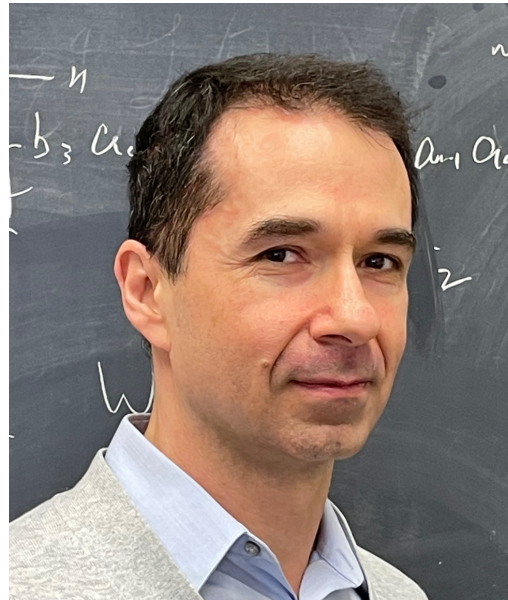
Enrico Herrmann
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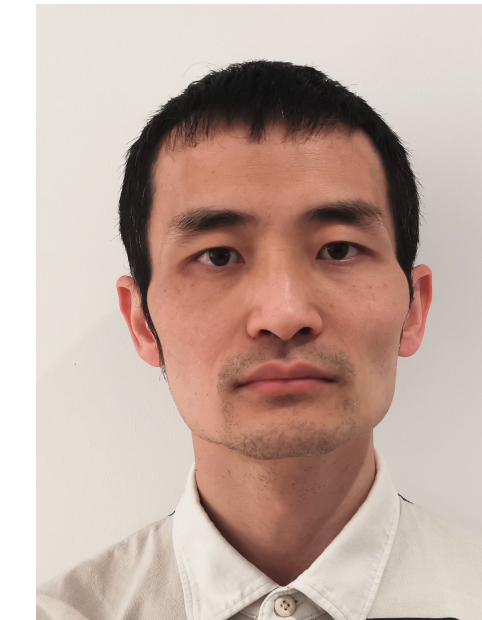
Volodya Smirnov
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Mao Zeng
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Also, many contributors to earlier work.

Outline

1. Summary of the problem and our approach to PM.

See the review from David Kosower and the 5PM talk from Jakobsen.

Also see talks from Bohnenblust, Brown, Chen, Isabella, O'Connell, Zhou

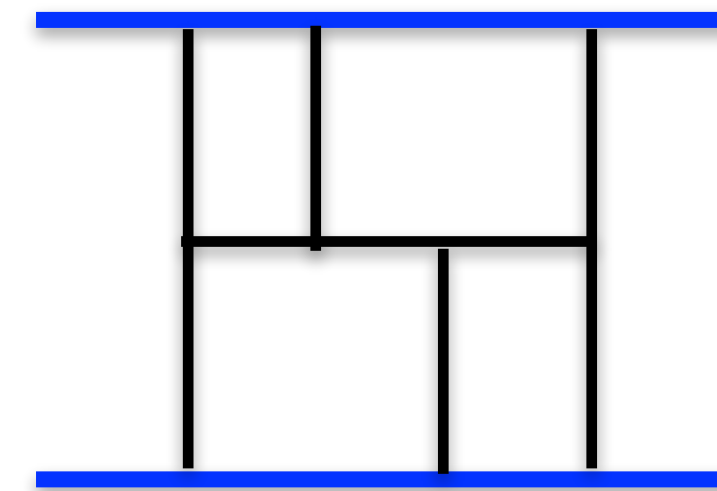
2. Brief review of basics:

- Quantum scattering amplitudes and gravity.
- Unitarity as a tool.
- Double copy.

2. Applications to state-of-the-art precision gravitational-wave computations.

3. Update on $O(G^5)$

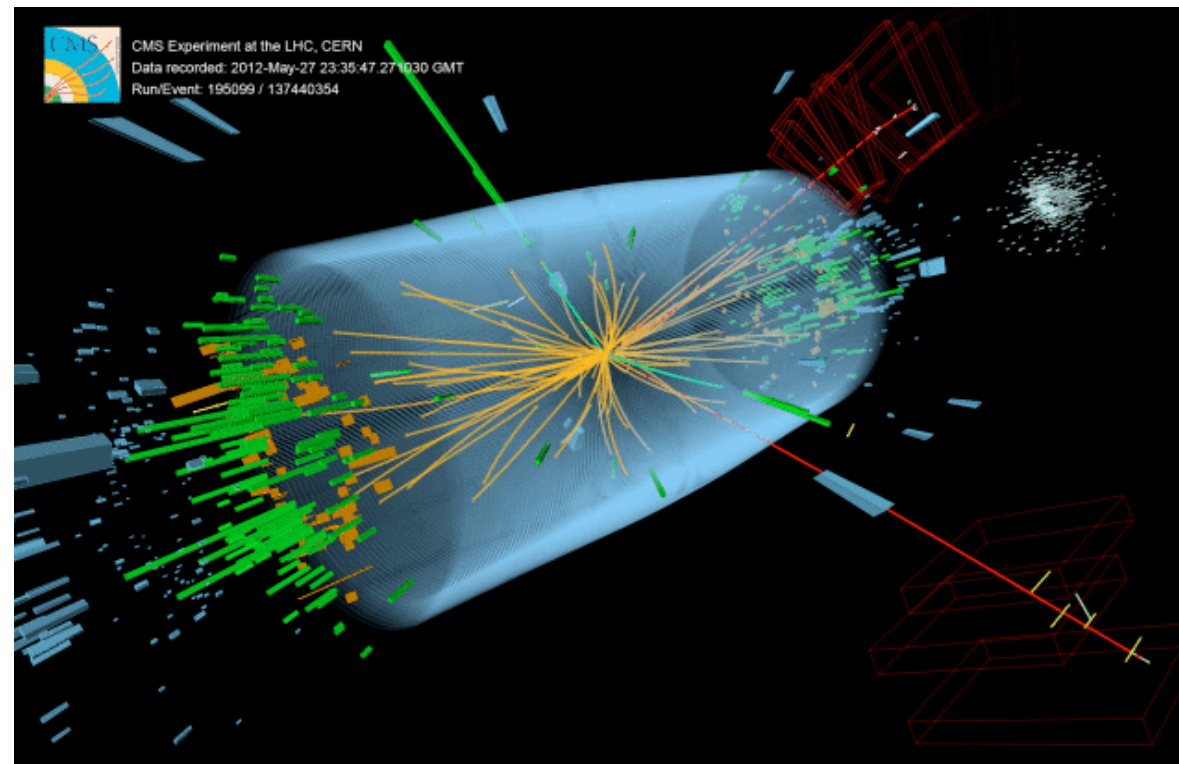
4. Conclusions and outlook.



Scattering Amplitudes and Gravitational Waves

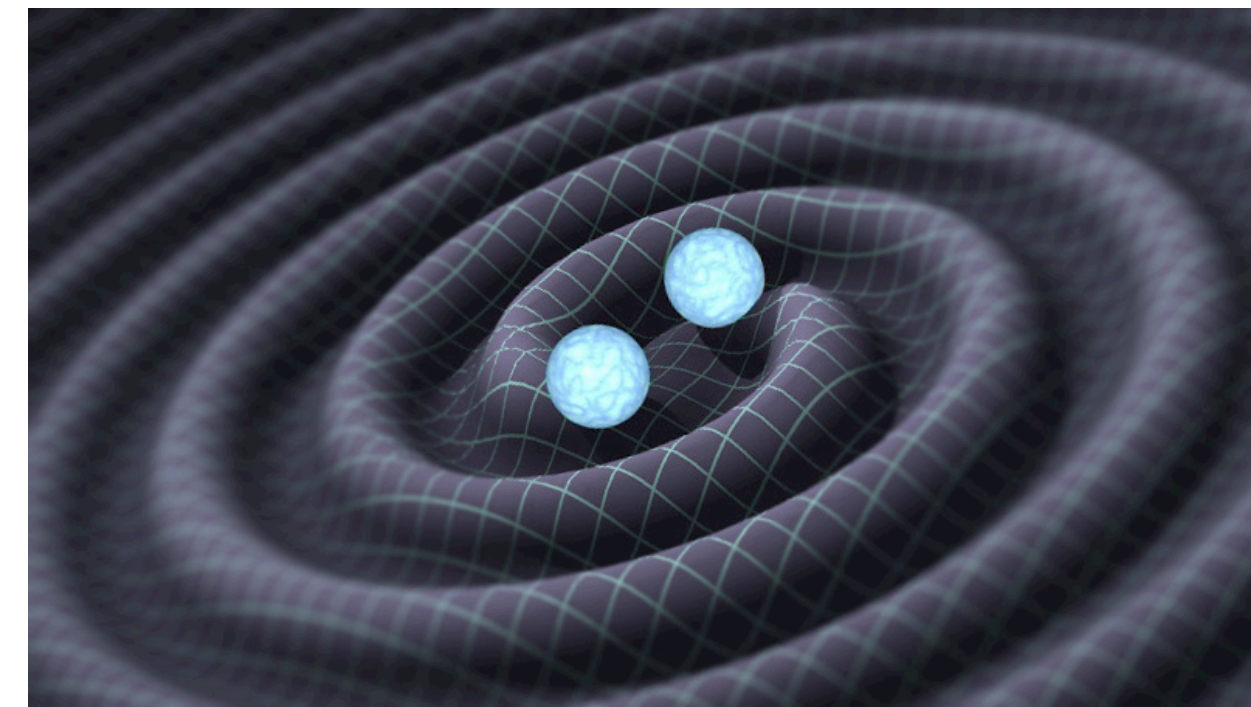
What does particle physics have to do with classical dynamics of astrophysical objects?

unbounded trajectory



**gauge theories, QCD, electroweak
quantum field theory**

bound orbit



**General Relativity
classical physics**

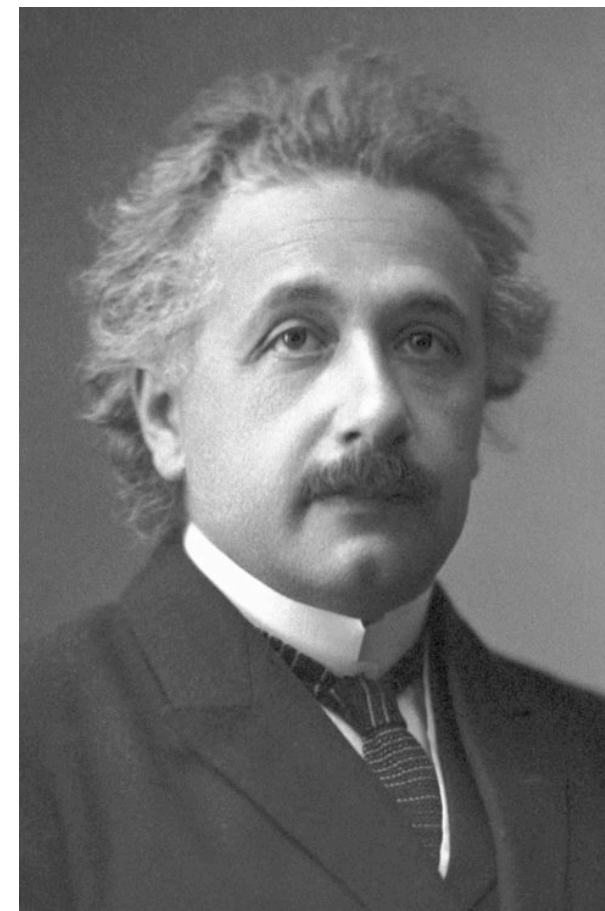
Effective field theory ideas: Black holes and neutron stars are point particles as far as long-wavelength radiation is concerned.

Iwasaki (1971); **Goldberger, Rothstein (2006)**; Neill and Rothstein; Vaidya, Foffa, Porto, Rothstein, Sturani; Kol; Bjerrum-Bohr, Donoghue, Holstein, Plante, Pierre Vanhove; Levi, Steinhoff; Vines, etc

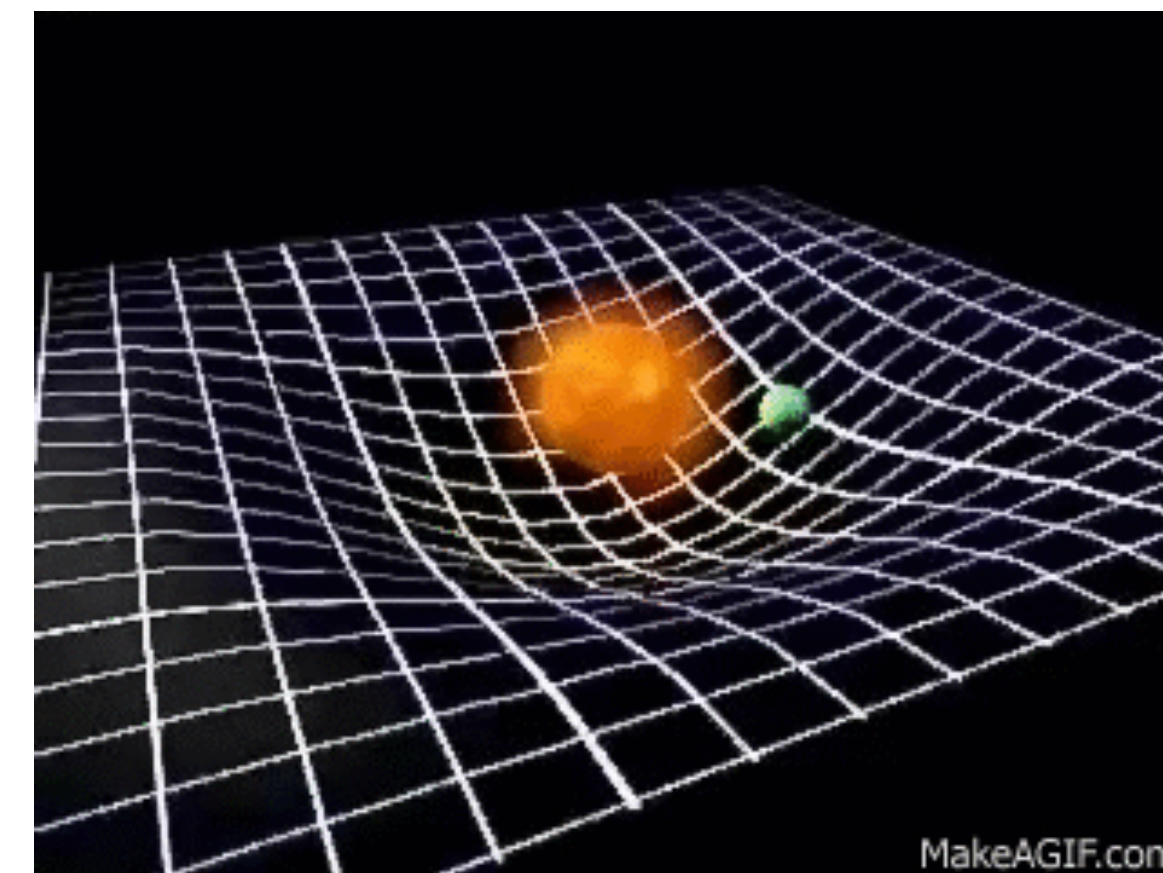
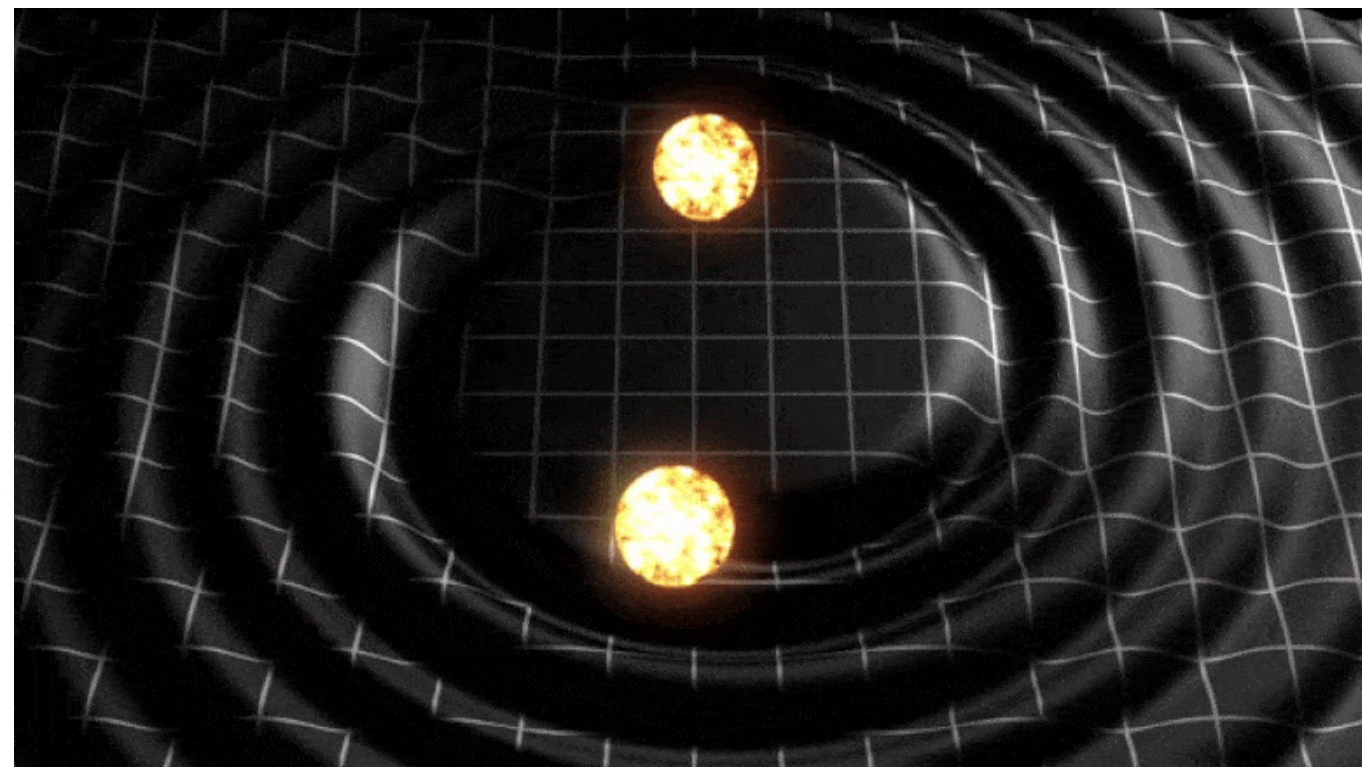
General Relativity

Usually, people start from Einstein's field equations:

$$\overset{\text{curvature}}{\nearrow} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \nwarrow \text{energy-momentum tensor}$$



Geometry
Curvature of space time

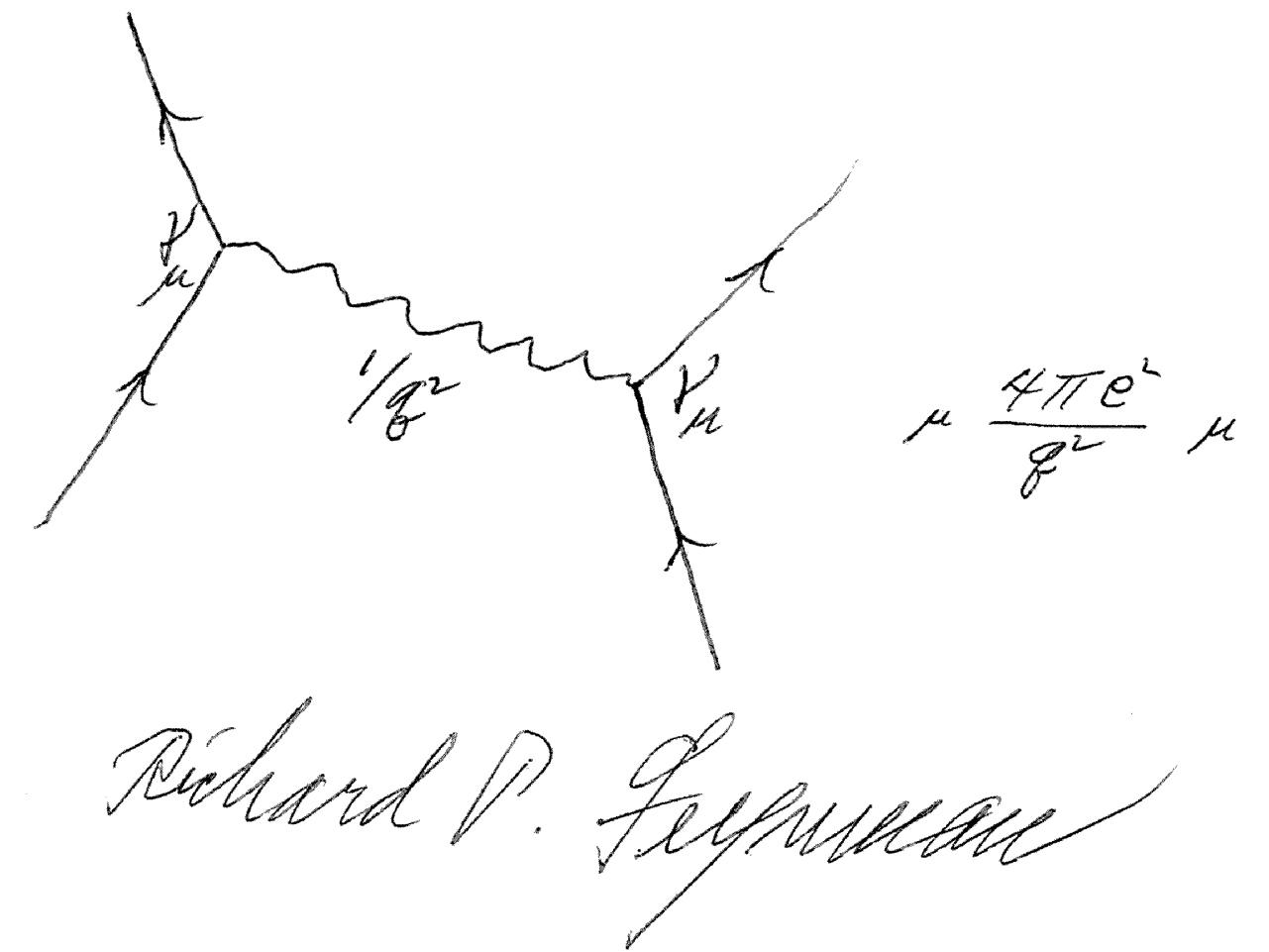


Amplitudes Approach to General Relativity

Amplitudes approach does *not* start from Einstein's field equations.

$$\cancel{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}}$$

Feynman: Einstein's gravity follows if we assume existence of massless spin-2 particle



Our approach: Assume massless spin-2 gravitons exist

Not suitable for all problems but effective for certain problems.

Various ideas to make it useful:

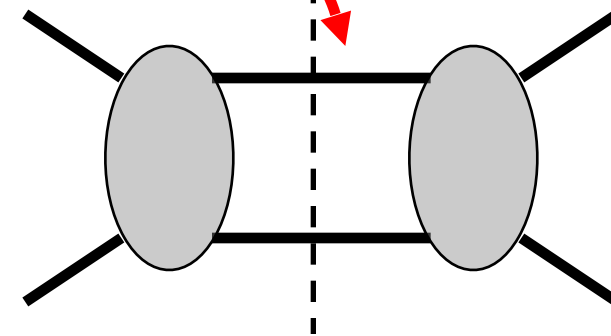
- **Double copy:** Convert spin 1 scattering to spin 2, gravity = (gauge theory)²
- **Unitarity Methods:** Higher orders from lower orders

Basic Tool: Generalized Unitarity Method

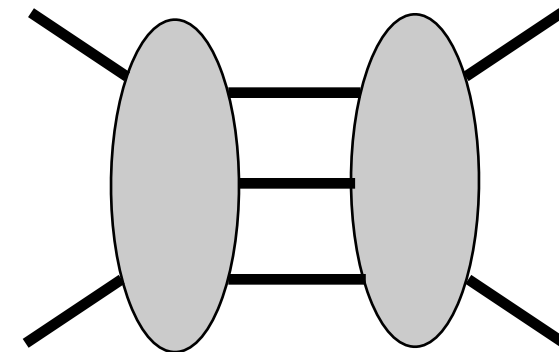
No Feynman rules; no need for virtual particles.

$$E^2 = \vec{p}^2 + m^2 \leftarrow \text{on-shell}$$

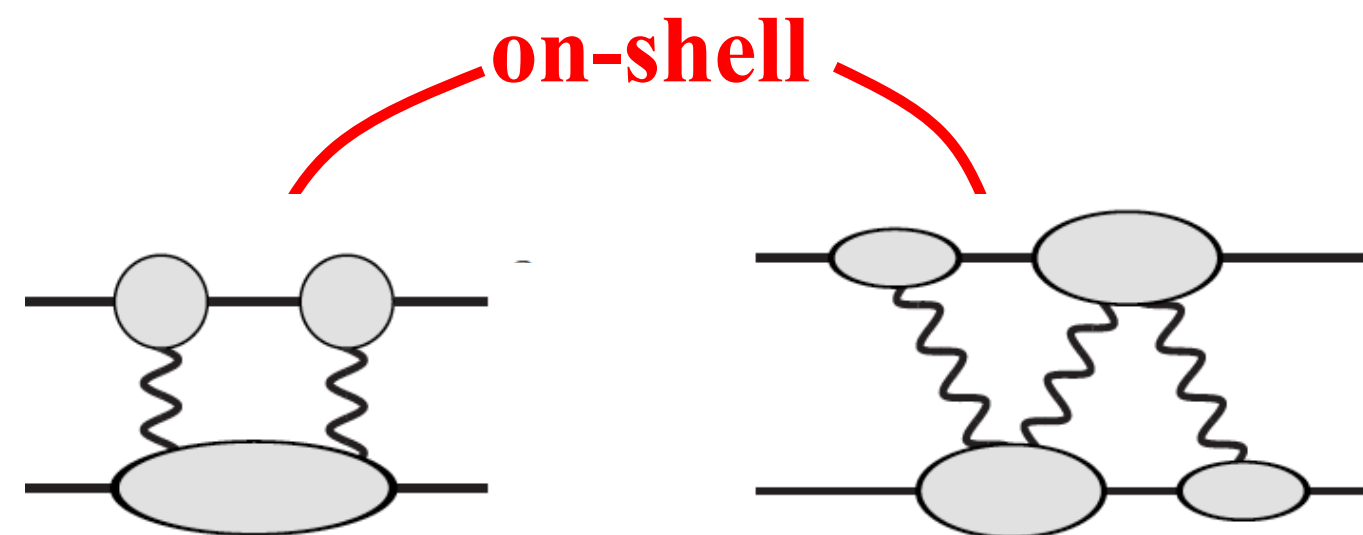
Two-particle cut:



Three-particle cut:



Generalized unitarity as a practical tool for loops.



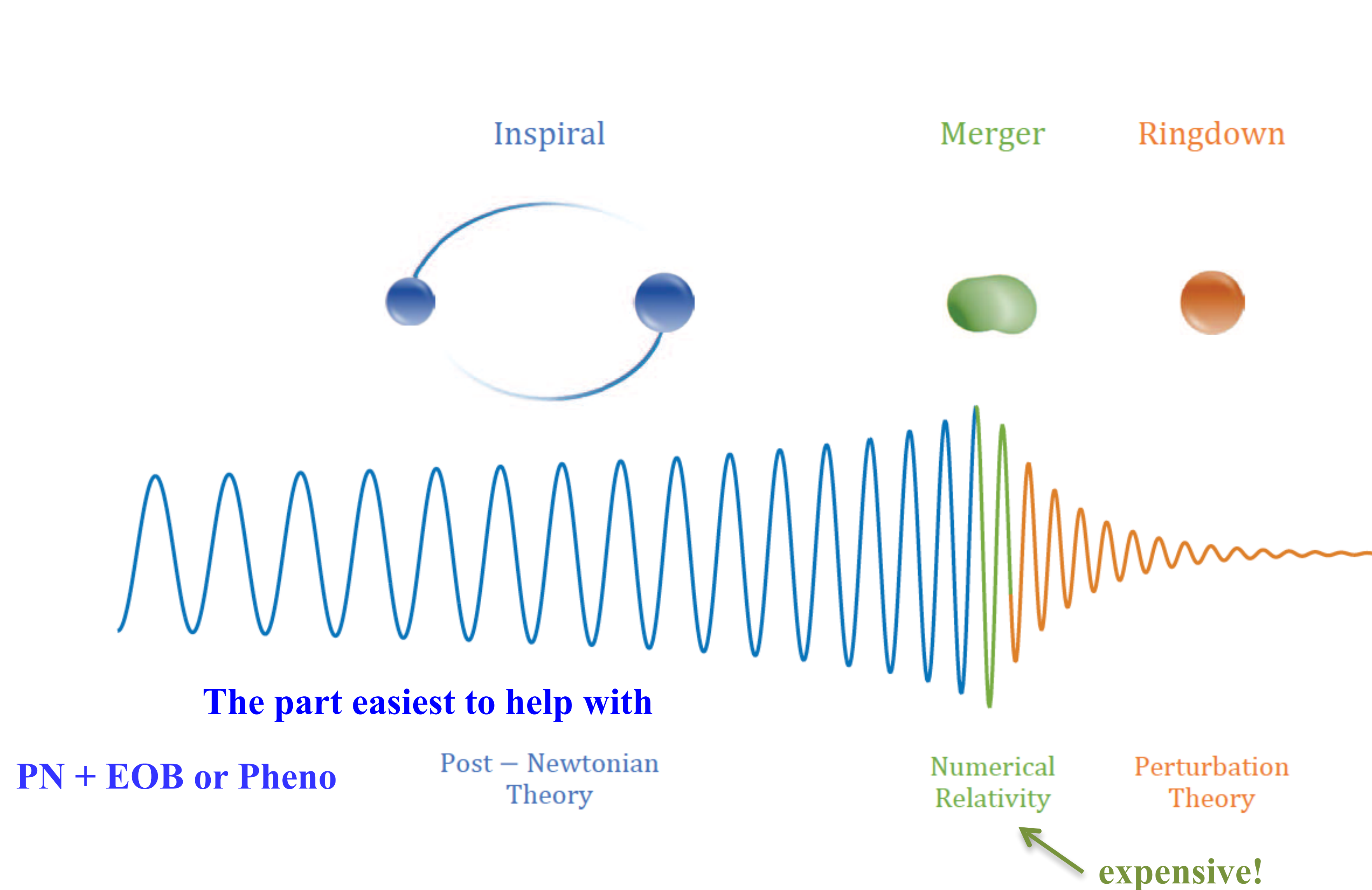
ZB, Dixon, Dunbar and Kosower (1994)

- Systematic assembly of complete loop amplitudes from tree amplitudes.
- Works for any number of particles or loops.

ZB, Dixon and Kosower;
ZB, Morgan;
Britto, Cachazo, Feng;
Ossala, Papadopoulos, Pittau;
Ellis, Kunszt, Melnikov;
Forde; Badger;
ZB, Carrasco, Johansson, Kosower
and many others

Designed to match results one would obtain from Feynman diagrams.
Idea used in the “NLO revolution” in QCD collider physics.
Ideas used here for gravity.

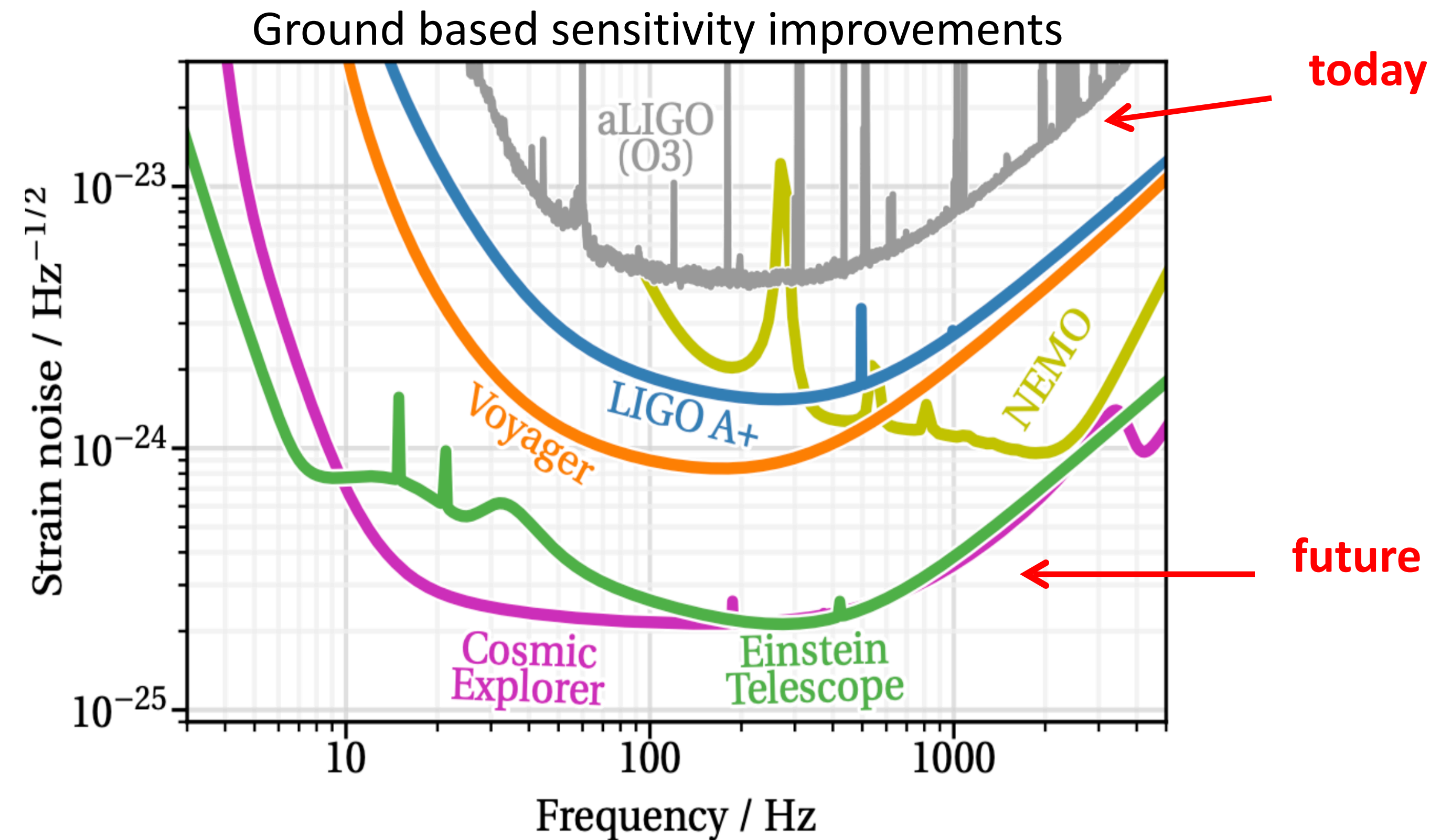
Goal: High Precision Waveforms



Small errors accumulate. Need for high precision.

From Antelis and Moreno, arXiv:1610.03567

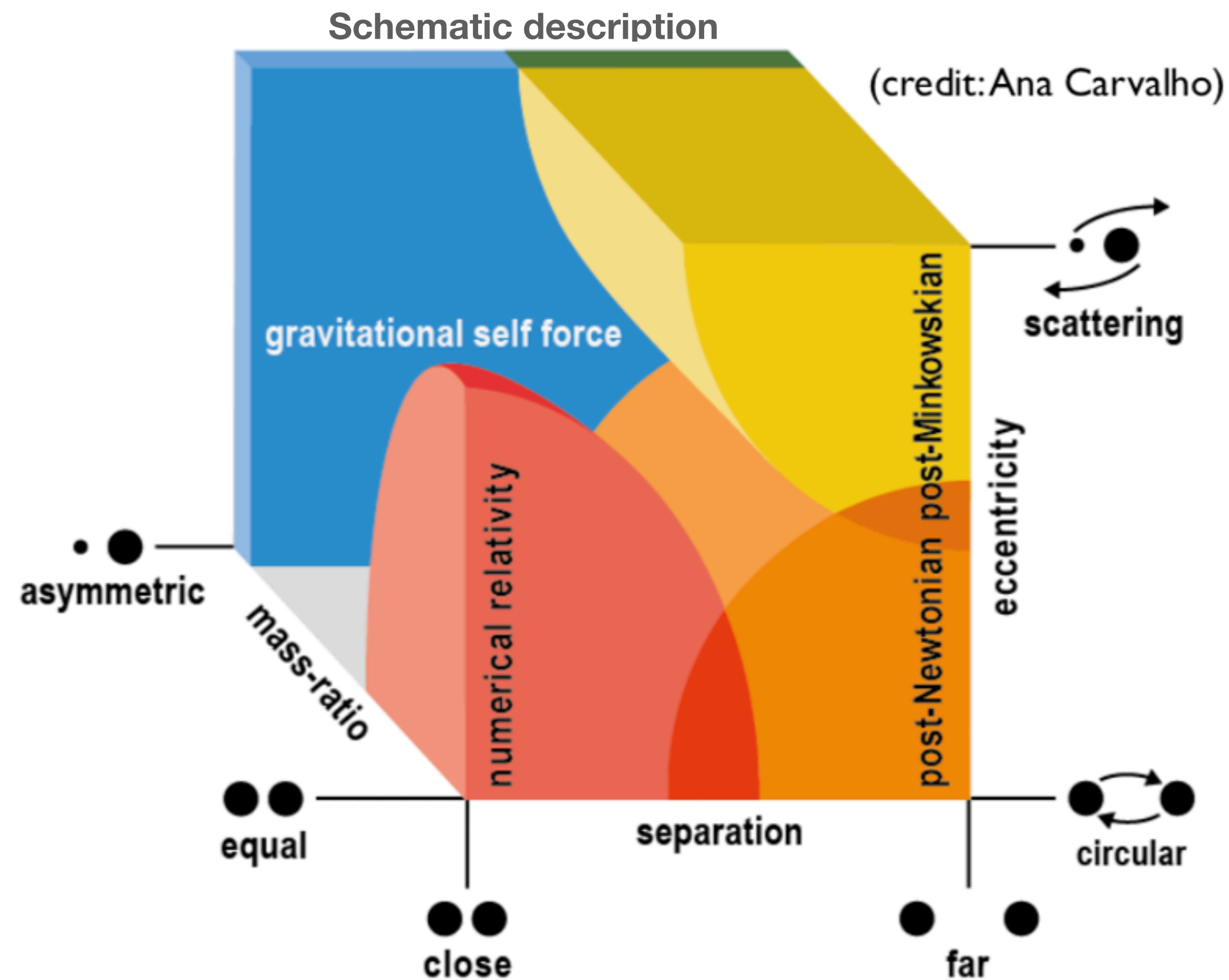
Future Detectors



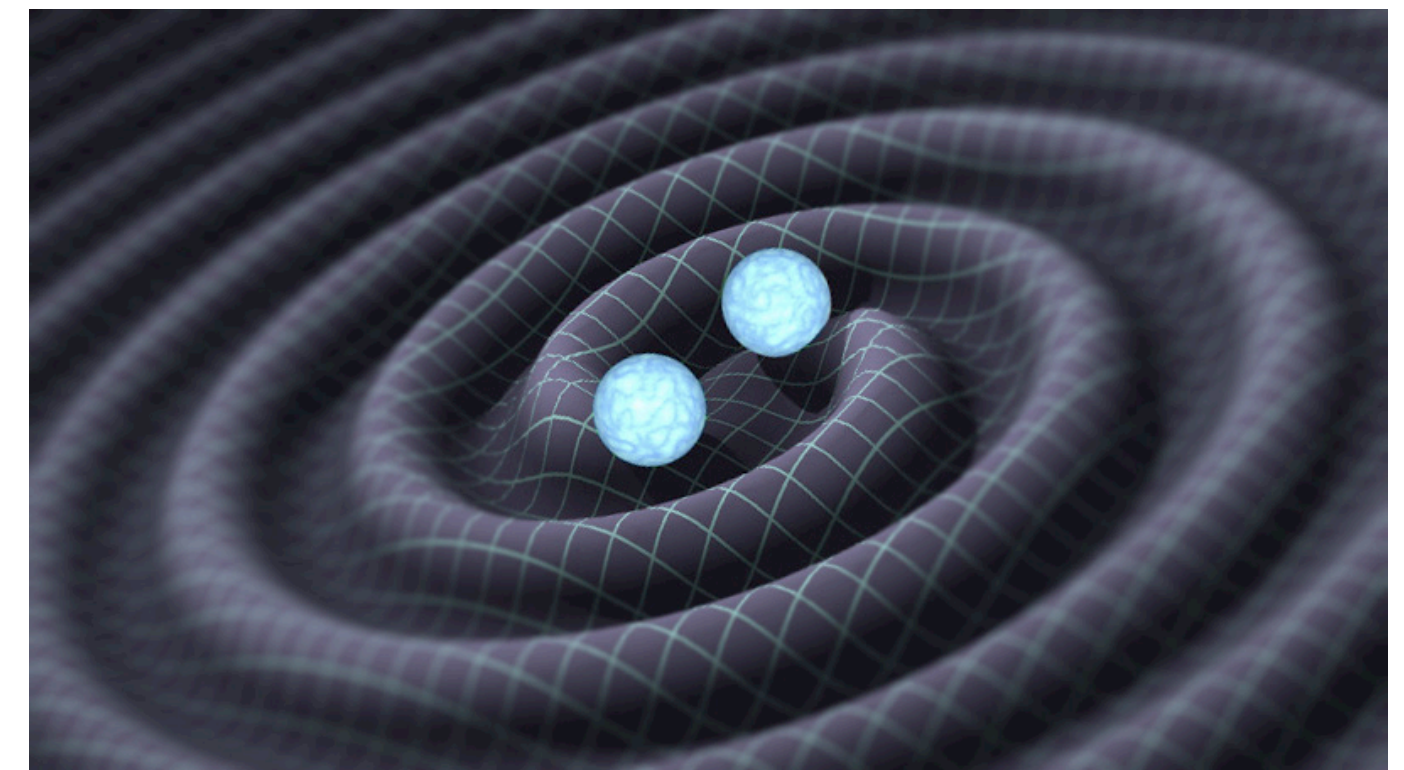
<https://cosmicexplorer.org/sensitivity.html>

- Factor of 100 increase in sensitivity is coming.
- Also space-based detectors at much lower frequency.
- Highly nontrivial theory challenge. Two more orders in perturbation theory.

Post-Minkowskian (PM) Approach



Amplitudes fits naturally
with post-Minkowskian approach



Different approximations are valid in different regions.

PM is an expansion in G

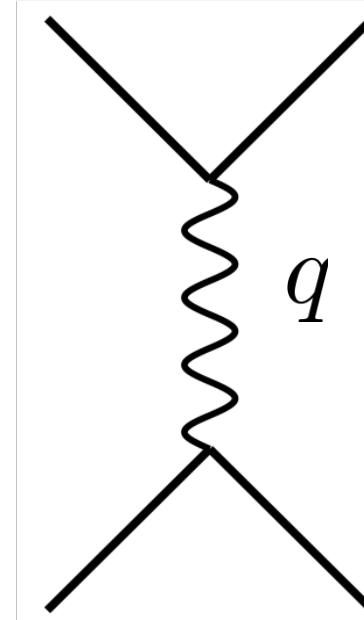
Information needs to be combined into a unified (EOB) waveform model.

2 Body Potentials and Amplitudes

Tree-level: Fourier transform gives classical potential.

$$V(r) \sim \int \frac{d^3 q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} A^{\text{tree}}(\mathbf{q})$$

$$H(\mathbf{p}, \mathbf{r}) = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2} + V(\mathbf{p}, \mathbf{r})$$



**Newtonian potential follows
follows from tree amplitude**

$$V(\mathbf{p}, \mathbf{r}) = \sum_{i=1}^{\infty} c_i(\mathbf{p}^2) \left(\frac{G_N}{|\mathbf{r}|} \right)^i$$

$$c_1(\mathbf{p}^2) = \frac{\nu^2 m^2}{\gamma^2 \xi} (1 - 2\sigma^2)$$

Beyond 1 loop less obvious:

- **Loops have classical pieces.**
- **$1/\hbar^L$ scaling of at L loop.**
- **Double counting and iteration.**

$$e^{iS_{\text{classical}}/\hbar}$$



**Piece of loops are classical: Our task is to efficiently extract these pieces.
Also dissipative pieces are encoded.**

General Relativity: Unitarity + Double Copy

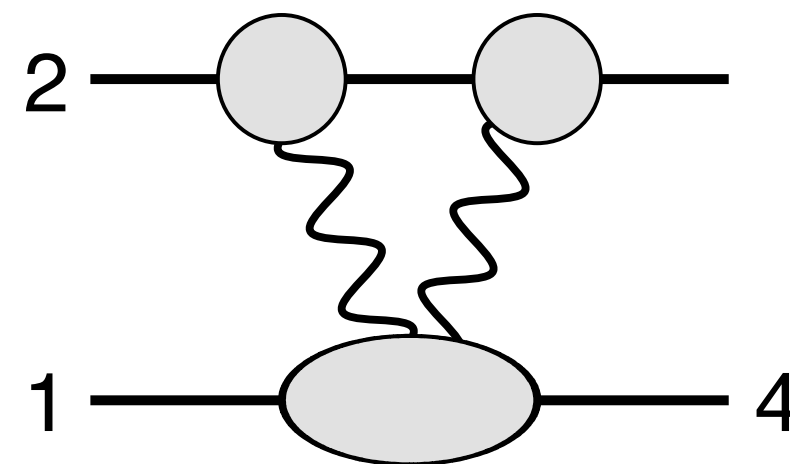
- Long-range force: Two matter lines separated by graviton propagators.
- Classical physics: 1 matter line per loop is cut (on-shell).

$$E^2 = \vec{p}^2 + m^2$$

Only independent unitarity cut for $O(G^2)$.

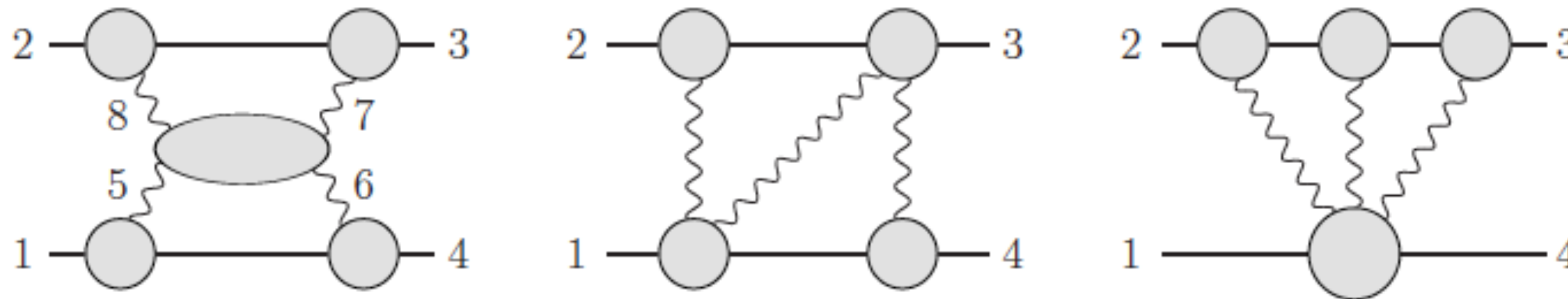
G is Newton's constant

2 black hole scattering



**Treat exposed lines on-shell (long range).
Pieces we want are simple!**

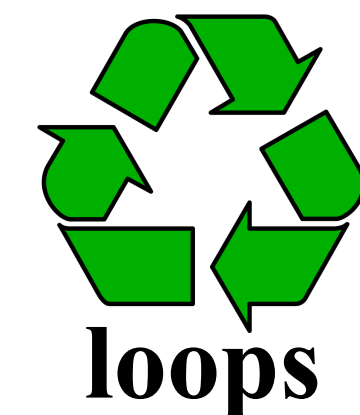
Independent generalized unitarity cuts for $O(G^3)$.



**Amplitude tools fit well for
extracting the pieces we want.**



gravity



loops

Methods for Extracting Classical Physics

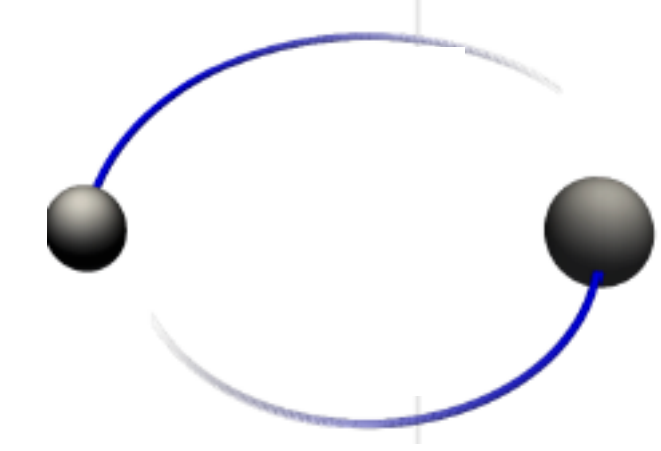
There are now multiple alternative ways to extract classical physics.

- **EFT matching to 2 body Hamiltonian** Cheng, Solon, Rothstein;
ZB, Cheung, Roiban, Shen, Zeng
- **Map to EOB** Bini, Damour, Geralico
- **Calculate physical observables** Kosower, Maybee, O'Connell (KMOC)
- **Eikonal phase** Amati, Ciafaloni, Veneziano;
Di Vecchia, Heissenberg, Russo, Veneziano
- **Amplitude radial-action relation** ZB, Parra-Martinez, Roiban, Ruf,
Shen, Solon, Zeng
- **Exponential representation** Damgaard, Plante, Vanhove;
Bjerrum-Bohr, Plante, Vanhove
- **Heavy mass field theory** Brandhuber, Chen, Travaglini, Wen
Damgaard, Haddad, Helset
- **Worldline formalisms** Goldberger, Rothstein; Levi, Steinhoff; Dlapa, Kälin, Liu, Porto;
Jakobson, Mogul, Plefka, Steinhoff; Edison, Levi; etc
- **Magnusian approach** Kim, Patil, Scheopner, Steinhoff; Guo, J. Kim, J. Kim, S. Kim, Lee, Li;
Gonzo, Mogull

These days, radial action and KMOC are our preferred methods.

Our First Paper: Conservative 3PM Hamiltonian

ZB, Cheung, Roiban, Shen, Solon, Zeng (2019)



The 3PM two body Hamiltonian:

$$H(\mathbf{p}, \mathbf{r}) = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2} + V(\mathbf{p}, \mathbf{r})$$

$$V(\mathbf{p}, \mathbf{r}) = \sum_{i=1}^3 c_i(\mathbf{p}^2) \left(\frac{G}{|\mathbf{r}|} \right)^i,$$

Newton in here

$$\begin{aligned} c_1 &= \frac{\nu^2 m^2}{\gamma^2 \xi} (1 - 2\sigma^2), & c_2 &= \frac{\nu^2 m^3}{\gamma^2 \xi} \left[\frac{3}{4} (1 - 5\sigma^2) - \frac{4\nu\sigma (1 - 2\sigma^2)}{\gamma\xi} - \frac{\nu^2 (1 - \xi) (1 - 2\sigma^2)^2}{2\gamma^3 \xi^2} \right], \\ c_3 &= \frac{\nu^2 m^4}{\gamma^2 \xi} \left[\frac{1}{12} (3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3) - \frac{4\nu (3 + 12\sigma^2 - 4\sigma^4) \operatorname{arcsinh} \sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2 - 1}} \right. \\ &\quad - \frac{3\nu\gamma (1 - 2\sigma^2) (1 - 5\sigma^2)}{2(1 + \gamma)(1 + \sigma)} - \frac{3\nu\sigma (7 - 20\sigma^2)}{2\gamma\xi} - \frac{\nu^2 (3 + 8\gamma - 3\xi - 15\sigma^2 - 80\gamma\sigma^2 + 15\xi\sigma^2) (1 - 2\sigma^2)}{4\gamma^3 \xi^2} \\ &\quad \left. + \frac{2\nu^3 (3 - 4\xi)\sigma (1 - 2\sigma^2)^2}{\gamma^4 \xi^3} + \frac{\nu^4 (1 - 2\xi) (1 - 2\sigma^2)^3}{2\gamma^6 \xi^4} \right], \end{aligned}$$

$$\begin{aligned} m &= m_A + m_B, & \mu &= m_A m_B / m, & \nu &= \mu / m, & \gamma &= E / m, \\ \xi &= E_1 E_2 / E^2, & E &= E_1 + E_2, & \sigma &= \mathbf{p}_1 \cdot \mathbf{p}_2 / m_1 m_2, \end{aligned}$$

Proved to gravitational-wave community that scattering-amplitude methods can solve problems of interest.

How do we know it is right?

Original checks:

- Compared to 4PN Hamiltonians (after canonical transformation)
- In test mass limit, $m_1 \ll m_2$, matches Schwarzschild Hamiltonian

Damour, Jaranowski, Schäfer; Bernard, Blanchet, Bohé, Faye, Marsat

Results met with initial skepticism.

Damour, arXiv:1912.02139v1

Subsequent calculations confirm our 3PM result:

1. Subsequent papers confirm our result in 6PN overlap.

Blümlein, Maier, Marquard, Schäfer; Bini, Damour, Geralico

2. Calculations reproducing 3PM result.

Cheung and Solon; Kälin, Liu, Porto;

3. Adding (non-conservative) real radiation clarifies a puzzle at high energies.

Di Vecchia, Heissenberg, Russo, Veneziano; Damour

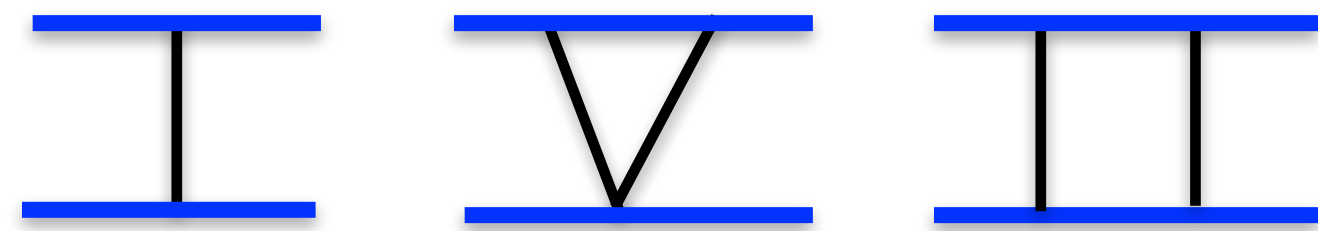
3PM results have passed highly nontrivial checks and careful scrutiny.

Generally, cross checks, especially with PN are critical as we go on.

New Challenges and Theoretical Issues at Each Order

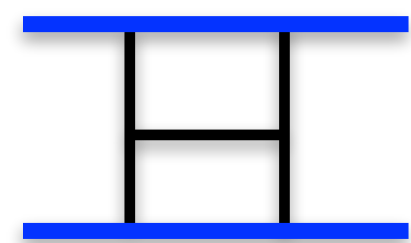
Moving up in orders of PM new effects and features encountered:

1PM and 2PM: Fixed by geodesic motion, 0SF.



$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2$$

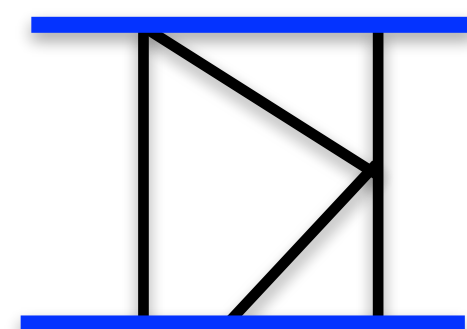
3PM: Interesting structure in high energy limit. 1SF, m_1/m_2 , beyond geodesic motion



$$\log(E^2/m_1 m_2)$$

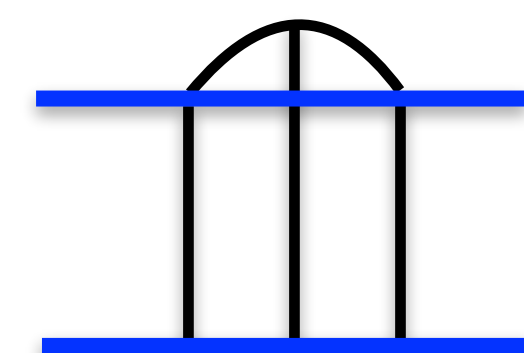
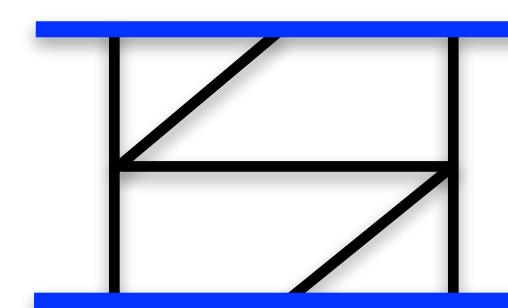
Cancels against real radiation

4PM: Tail effect, nontrivial analytic continuations, elliptic integrals, *noncancellation* of poor high-energy behavior. Nonlocal in time “tail” effect.

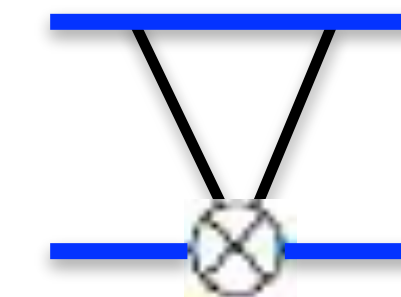
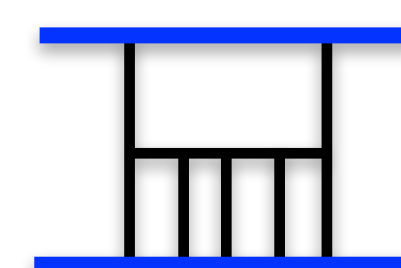


$$\sim K^2 \left(\frac{\sigma-1}{\sigma+1} \right)$$

5PM: 2SF, **Tough IBPs and DEs**, CY in master integral basis.

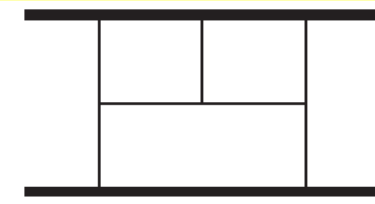


6PM: New features surely await.
Distinguish BHs from neutron stars, tidal effects



Complete Conservative Contribution $O(G^4)$

ZB, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng



test particle

1st self force

Iteration. No need to compute

$O(G^4)$ amplitude

$$\mathcal{M}_4^{\text{cons}} = G^4 M^7 \nu^2 |q| \pi^2 \left[\mathcal{M}_4^{\text{p}} + \nu \left(4\mathcal{M}_4^{\text{t}} \log\left(\frac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}} \right) \right] + \underbrace{\int_{\ell} \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_{\ell} \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_{\ell} \frac{\tilde{I}_{r,1} \tilde{I}_{r,3}}{Z_1} + \int_{\ell} \frac{\tilde{I}_{r,2}^2}{Z_1}}_{\text{tail effect}}$$

$$D = 4 - 2\epsilon$$

$$\mathcal{M}_4^{\text{t}} = r_1 + r_2 \log\left(\frac{\sigma+1}{2}\right) + r_3 \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}, \quad \mathcal{M}_4^{\text{p}} = -\frac{35(1-18\sigma^2+33\sigma^4)}{8(\sigma^2-1)}$$

$$\mathcal{M}_4^{\pi^2} = r_4 \pi^2 + r_5 K\left(\frac{\sigma-1}{\sigma+1}\right) E\left(\frac{\sigma-1}{\sigma+1}\right) + r_6 K^2\left(\frac{\sigma-1}{\sigma+1}\right) + r_7 E^2\left(\frac{\sigma-1}{\sigma+1}\right), \quad \leftarrow \text{elliptic integrals}$$

$$\mathcal{M}_4^{\text{rem}} = r_8 + r_9 \log\left(\frac{\sigma+1}{2}\right) + r_{10} \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11} \log(\sigma) + r_{12} \log^2\left(\frac{\sigma+1}{2}\right) + r_{13} \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \log\left(\frac{\sigma+1}{2}\right) + r_{14} \frac{\text{arccosh}^2(\sigma)}{\sigma^2-1} \\ + r_{15} \text{Li}_2\left(\frac{1-\sigma}{2}\right) + r_{16} \text{Li}_2\left(\frac{1-\sigma}{1+\sigma}\right) + r_{17} \frac{1}{\sqrt{\sigma^2-1}} \left[\text{Li}_2\left(-\sqrt{\frac{\sigma-1}{\sigma+1}}\right) - \text{Li}_2\left(\sqrt{\frac{\sigma-1}{\sigma+1}}\right) \right].$$

$$\nu = m_1 m_2 / (m_1 + m_2)^2$$

$$\sigma = p_1 \cdot p_2 / m_1 m_2,$$

r_i rational coefficients

This is *complete* conservative contribution.

$$\mathcal{M}_4^{\text{radgrav,f}} = \frac{12044}{75} p_\infty^2 + \frac{212077}{3675} p_\infty^4 + \frac{115917979}{793800} p_\infty^6 - \frac{9823091209}{76839840} p_\infty^8 + \frac{115240251793703}{1038874636800} p_\infty^{10} + \dots$$

First 3 terms match 6PN results of Bini, Damour, Geralico!

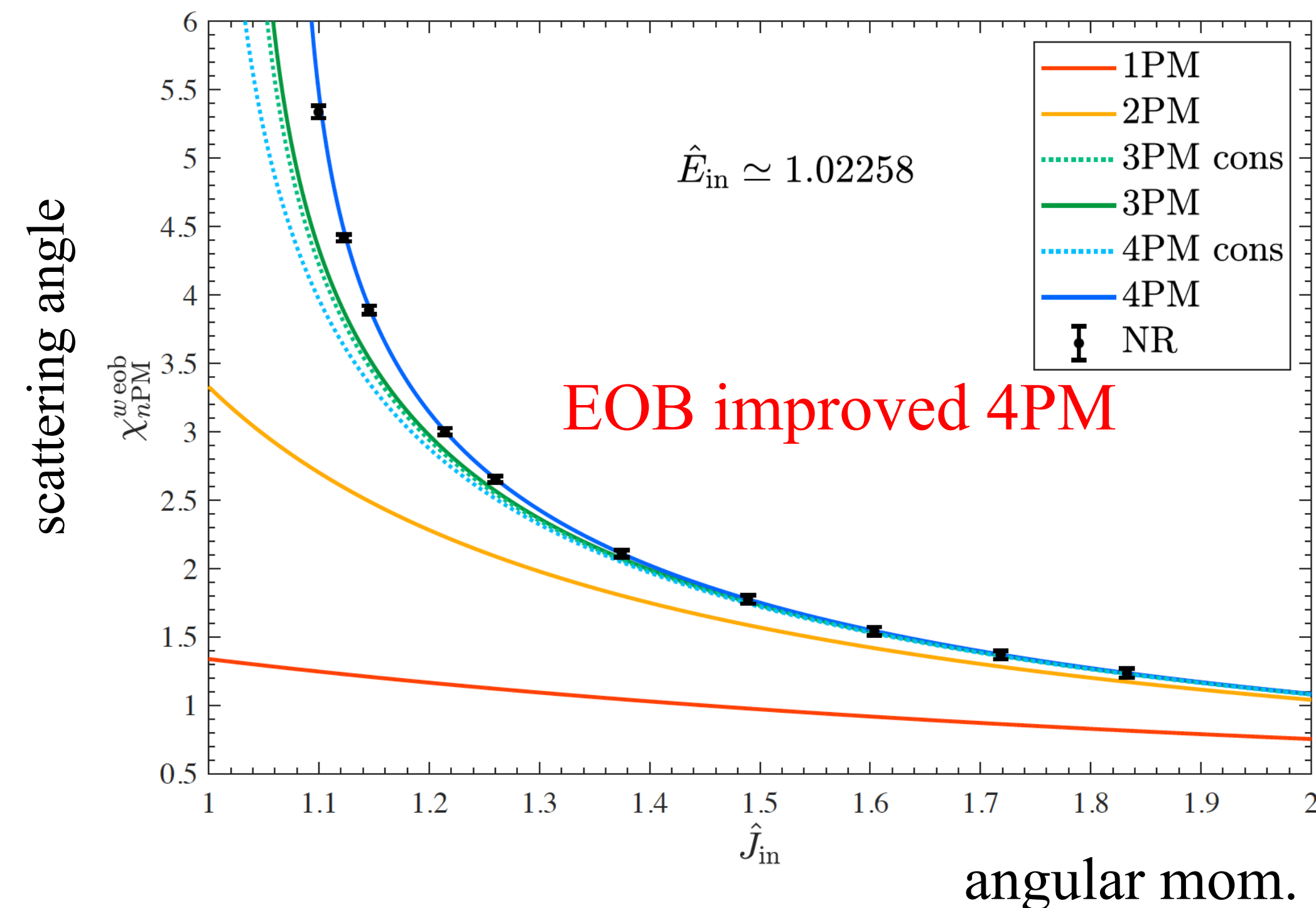
Nontrivial analytic continuation of certain time nonlocal term terms.

Partial patch.

Khalil, Buonanno, Steinhoff, Vines; Dlapa, Kalin, Liu, Porto; Damour, Retegno

Comparison with Numerical Relativity

Khalil, Buonanno, Vines, Steinhoff; Damour and Retegno



Plot uses:

4PM Conservative: ZB, Parra-Martinez, Roiban, Ruf. Shen, Solon, Zeng

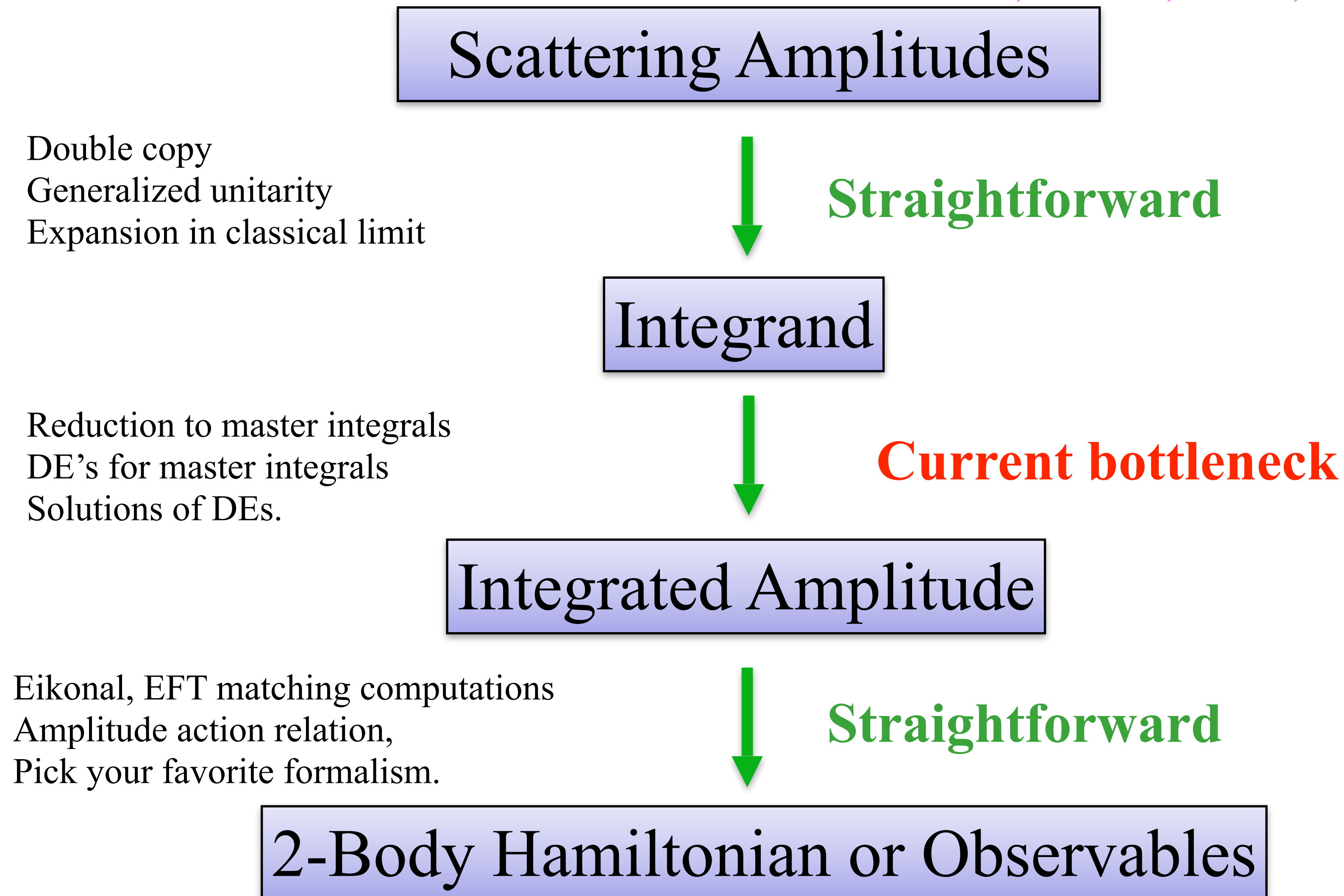
4PM Conservative + Dissipative: Manohar, Shen and Ridgeway; Dlapa, Kalen, Lui, Neef, Porto; Damgaard, Hansen, Planté, Vanhove.

NR: Damour, Guercilena, Hinder, Hopper, Nagar, Rezzolla;

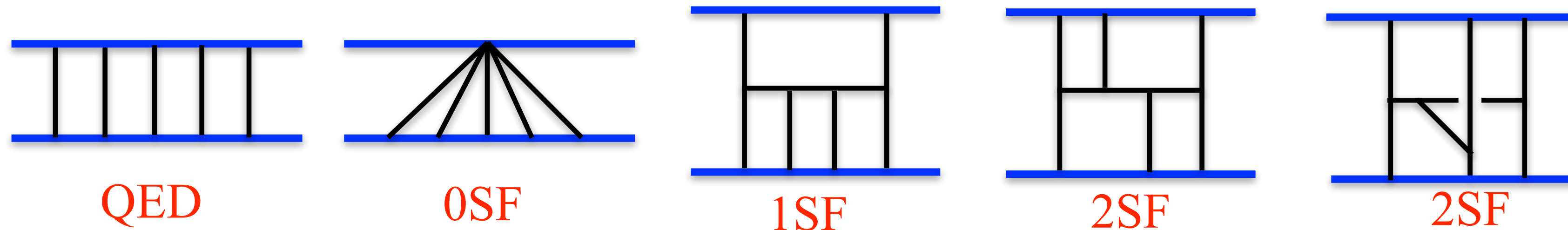
- **Good agreement with numerical relativity.**
- **Motivated us to go on to next order.**

5PM Pipeline, $O(G^5)$

ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov; Zeng;
Driesse, Jakobsen, Klemm, Mogull, Nega Plefka, Sauer, Usovitsch



Deal With 5PM Integration in Stages



Stages:

- 1. QED warmup. Potential mode contributions. Done.**
ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng
- 2. 1SF Conservative + Dissipative Done.**
Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch
- 3. $N = 8$ sugra (lower tensor rank) Both 1SF and 2SF potential done.**
ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng
- 4. GR 2SF “Conservative”. Done**
ZB, Herrmann, Roiban, Ruf, Smirnov, Smith, Zeng;
Driesse, Jakobsen, Mogul, Nega, Plefka, Sauer, Usovitch
- 5. 2SF dissipative effects. To be done**

$$M_{5\text{PM}} = M_{5\text{PM}}^{0\text{SF}} + \nu M_{5\text{PM}}^{1\text{SF}} + \nu^2 M_{\text{PM}}^{2\text{SF}}$$

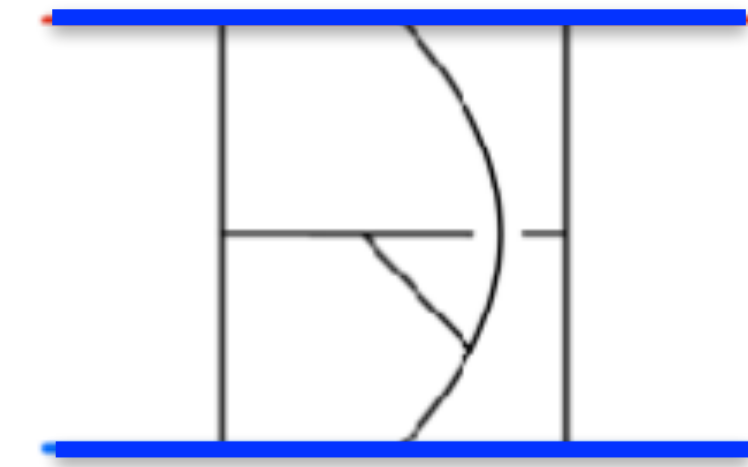
$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

Learn from each stage to push forward the next one.

Bottleneck: Integral Evaluation

Primary bottleneck is integrals:

22 indices: 13 propagator and 9 irreducible scalar products (ISPs)



$$\int \frac{d^4Dk [u_2 \cdot k_1]^{a-14} [u_2 \cdot k_4]^{a-15} [u_1 \cdot k_2]^{a-16} [u_1 \cdot k_3]^{a-17} [k_1 \cdot q]^{a-18} [k_2 \cdot q]^{a-19} [k_1 \cdot k_2]^{a-20} [k_1 \cdot k_4]^{a-21} [k_2 \cdot k_3]^{a-22}}{[-2u_2 \cdot k_2]^{a_1} [-2u_2 \cdot k_{123}]^{a_2} [2u_1 \cdot k_{234}]^{a_3} [2u_1 \cdot k_{1234}]^{a_4} [k_1^2]^{a_5} [k_2^2]^{a_6} [k_3^2]^{a_7} [k_{13}^2]^{a_8} [k_4^2]^{a_9} [k_{34}^2]^{a_{10}} [k_{234}^2]^{a_{11}} [(k_{123} - q)^2]^{a_{12}} [(k_{1234} - q)^2]^{a_{13}}}$$

Very similar to wordline integrals. KMOC version gives identical integrand as WLQFT

Capatti and Zeng

Encounter up to 8 numerator powers and 4 doubled propagators.

ZB, Herrmann, Roiban, Ruf, Smirnov, Smith, Zeng;

Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch

The fundamental issue isn't the millions of integrals, it's the complexity.

Public codes cannot deal with this.

Basic Tool Is IBP

Talks from Feng and Zhang

In QCD/Amplitudes advanced technology for loop integrals which we import.

1. IBP greatly simplified in classical limit.

Chetyrkin, Tkachov; Laporta; Henn; Henn and Smirnov
Beneke and Smirnov; Parra-Martinez, Ruf, Zeng

2. Choose master integrals carefully to simplify IBPs.

A. Smirnov and V. Smirnov; Usovitsch

3. Use finite prime fields and reconstruction for toughest IBPs.

Manteuffel, Schabinger; Peraro

Kotikov, ZB, Dixon and Kosower; Gehrmann, Remiddi

IBP:

$$0 = \int \prod_i^L \frac{d^D \ell_i}{(2\pi)^D} \frac{\partial}{\partial \ell_i^\mu} \frac{N^\mu(\ell_k, p_M)}{Z_1 \dots Z_n}$$

**Solve linear relations for master integrals,
usually the simplest ones.**

Our basic tools:

1. *Highly modified* private version of FIRE.

Smirnov, Chuharev; Lee; Smirnov, Zeng

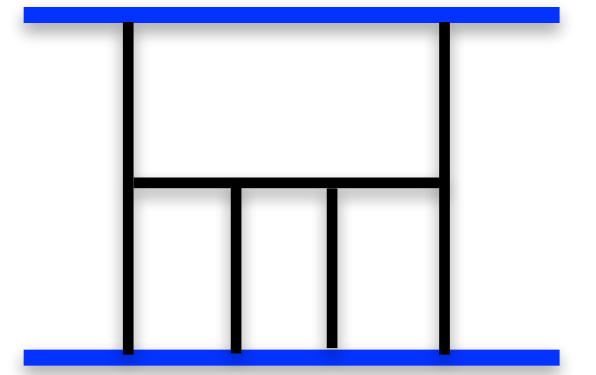
2. Finite fields and reconstruction.

Von Manteuffel and a Schabinger; Peraro

KIRA has also been upgraded.

Maierhoefer, Usovitsch, Uwer;
Klappert, Lange, Maierhöfer, Usovitsch

All integrals reduced to “Master integrals”



Finite Field Methods and Supercomputers

Question: Given a large linear system (million unknowns) whose solution is rational functions over multiple variables, how to solve analytically?

Answer: Finite field reconstruction.

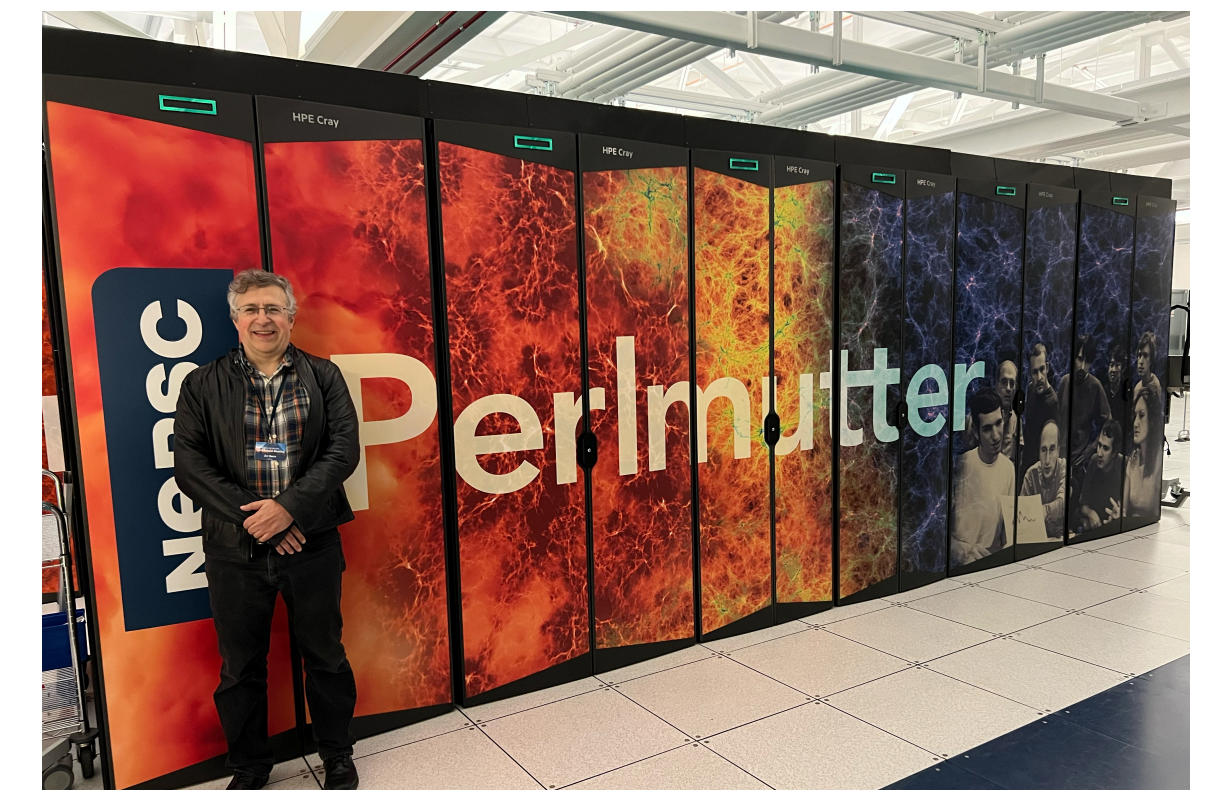
A. von Manteuffel and R. Schabinger; T. Peraro

- Problem reduced to numerical evaluation on a prime field.
- Rational coefficients of master integrals fully reconstructed.

Trivially parallelizable.

Scale of the calculation requires supercomputers.

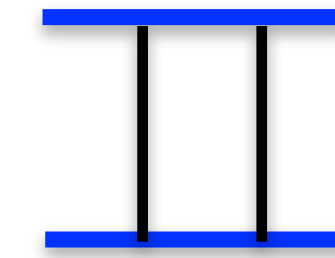
Supercomputers critical but better algorithms even more critical



DEs for Integrals.

Kotikov; ZB, Dixon, Kosower; Gehrmann, Remiddi; Henn

IBP can also be used to generate differential equations:



$$\begin{aligned}\frac{\partial}{\partial y} I[1,1,1,1] &= \frac{(u_2 - y u_1)}{1 - y^2} \cdot \frac{\partial}{\partial u_1} I[1,1,1,1] \\ &\stackrel{\text{Lorenz-invariance}}{=} \frac{y}{1 - y^2} I[1,1,1,1] + \frac{1}{1 - y^2} I[1,1,2,0] \\ &\stackrel{\text{IBP}}{=} \frac{y}{1 - y^2} I[1,1,1,1] + \frac{3 - d}{1 - y^2} I[1,1,0,0]\end{aligned}$$

$$y = u_1 \cdot u_2$$

The DEs can be solved either analytically or by series expansion.

Structure of DEs

1st order coupled ODE

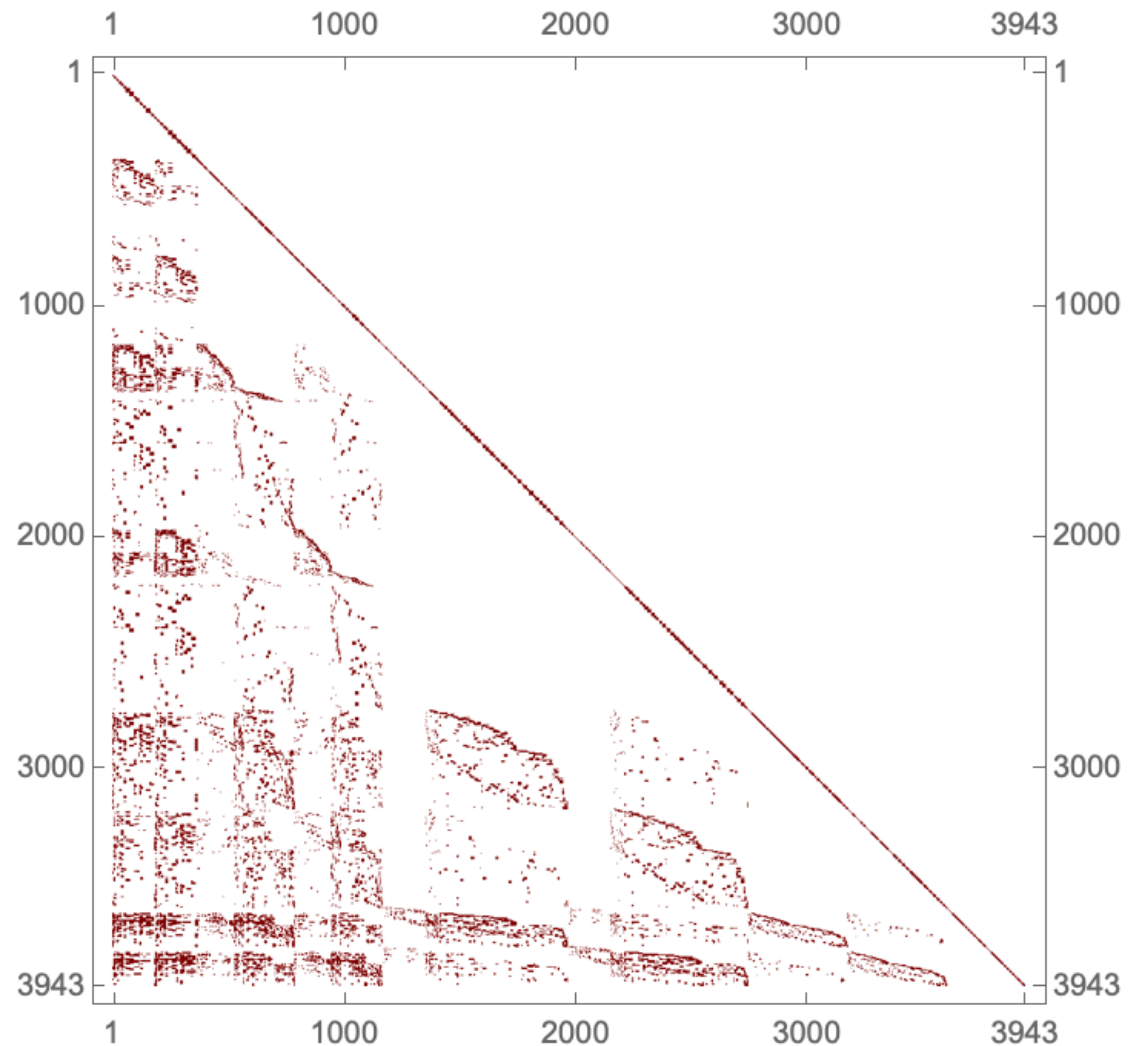
$$\frac{\partial}{\partial y} \begin{bmatrix} I_1 \\ \vdots \\ I_m \end{bmatrix} = M(y, \epsilon) \begin{bmatrix} I_1 \\ \vdots \\ I_m \end{bmatrix}$$

$m \times m$ matrix with rational entries

$$D = 4 - 2\epsilon$$

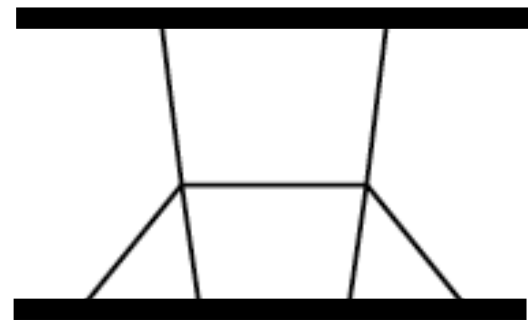
It takes considerable effort to clean up M .
Want “canonical form”

A 5 PM example of the DE matrix



Nontrivial Special Functions

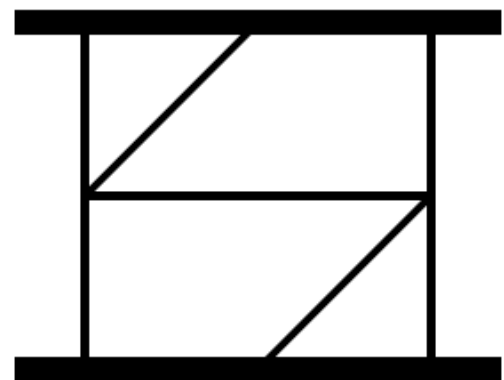
Elliptic, Heun and Calabi-Yau integrals appear.



At 1 SF, Elliptic integrals.

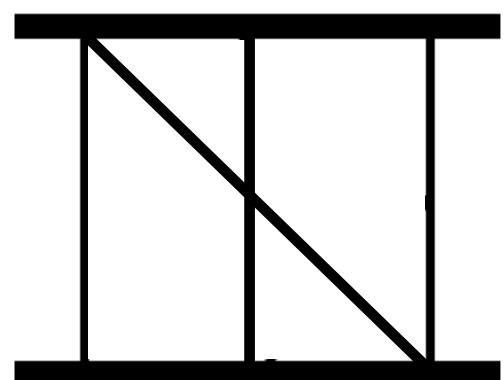
Driesse, Jakobsen, Klemm, Mogull, Nega Plefka, Sauer, Usovitsch
ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Smirnov, Zeng.

$$\Gamma_{i_1, \dots, i_n}(x) = \int_1^x dx_n k_{i_n}(x_n) \cdots \int_1^{x_2} dx_1 k_{i_1}(x_1)$$



At 2 SF, Calabi-Yau 3-fold integrals.

Complicated IBP and DEs



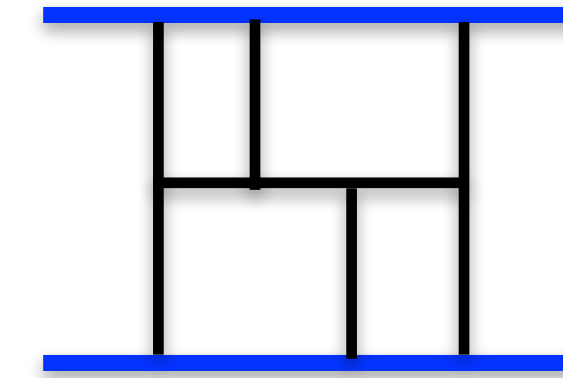
At 2 SF, Heun type integral.

Frellesvig, Morales, Wilhelm; Klemm, Nega, Sauer, Plefka;
Driesse, Jakobsen, Klemm, Mogull, Nega, Pelfka, Sauer, Uovi
Frellesvig, Morales, Pogel, Weinzierl, Wilhelm;
ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng;
ZB, Herrmann, Roiban, Ruf, Smirnov, Smith, Zeng;

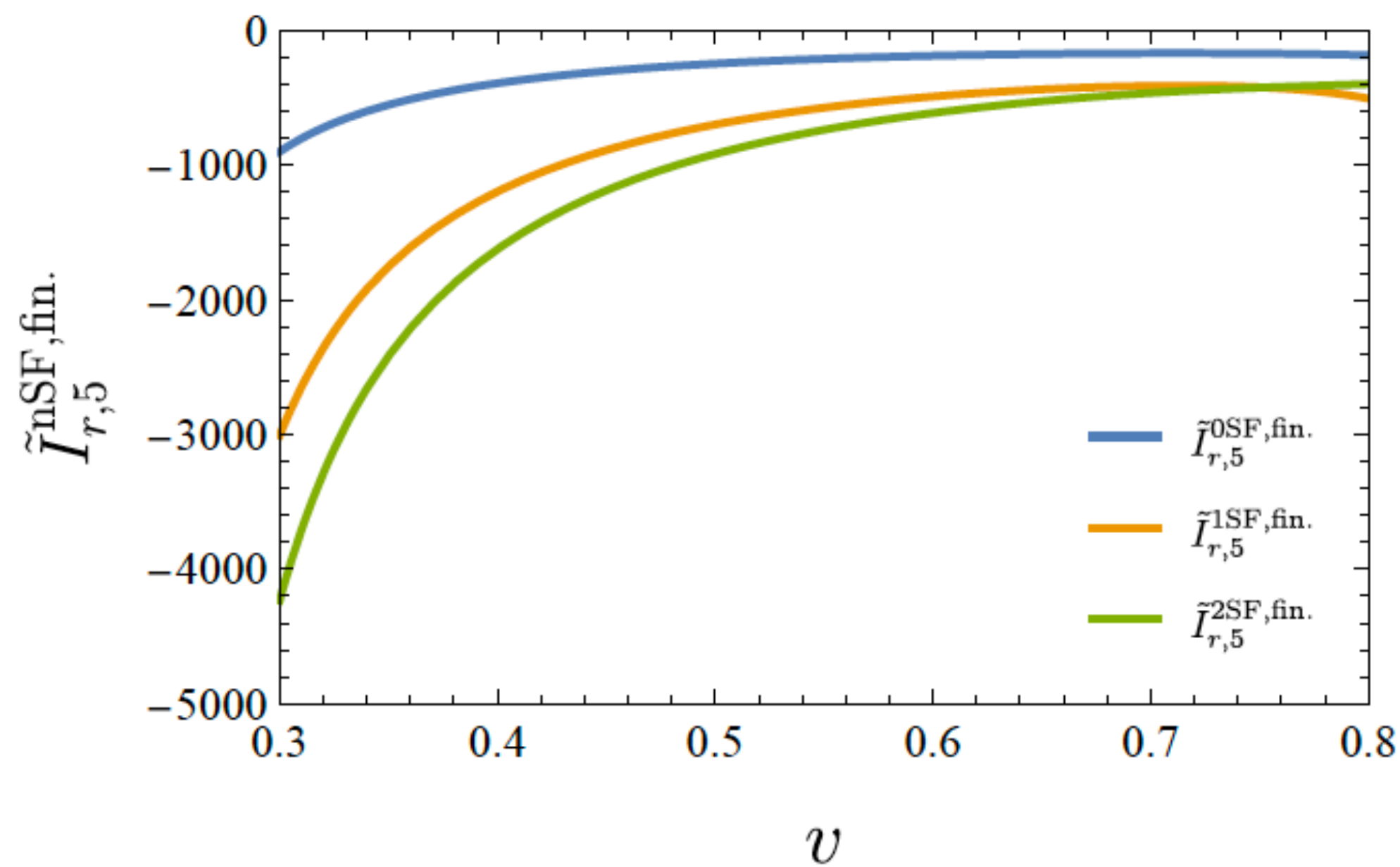
5PM GR Radial Action (2SF Potential)

ZB, Herrmann, Roiban, Ruf, Smirnov, Smith, Zeng

$$\tilde{I}_{r,5} = m_1^2 m_2^2 (m_1^4 + m_2^4) \tilde{I}_{r,5}^{0\text{SF}} + m_1^3 m_2^3 (m_1^2 + m_2^2) \tilde{I}_{r,5}^{1\text{SF}} + m_1^4 m_2^4 \tilde{I}_{r,5}^{2\text{SF}}$$



- While Calabi-Yau drops out they complicate the solution.
- We know it drops out because the boundary condition from DE drops out.
- Easier to use series solution



Scattering angle: $\chi = -\frac{\partial I_r}{\partial J}$

$$\begin{aligned} \tilde{I}_{r,5}^{2\text{SF,fin.}} = & -\frac{3}{320\bar{\sigma}^4} + \frac{31}{160\bar{\sigma}^3} - \frac{3543}{640\bar{\sigma}^2} - \frac{142651}{1920\bar{\sigma}} - \frac{22066709}{230400} + \dots \\ & - \pi^2 \left(\frac{41}{64\bar{\sigma}} + \frac{9523}{1152} + \frac{44005\bar{\sigma}}{2304} + \dots \right) \end{aligned}$$

22 terms explicitly given in paper

See Jakobsen's talk representing Berlin group
analytic expressions + radiative part

The way Elliptic and Calabi-Yau integrals drop out seems mysterious.

Can we get to 6PM?

More immediate question: Can we banish supercomputers at 5PM?

- **Very fast progress though 4 PM (~1 day modest cluster, public codes).**
 - No Integration bottleneck. Many order of magnitude simpler than 5PM.
 - FIRE6 or KIRA2 available for collider physics scattering amplitudes
- **Progress slowed at 5PM because integrals beyond the ability of public IBP codes.**
 - Upgraded IBP programs: FIRE7 and KIRA3, revamped algorithms, better seeding.

FIRE7: Smirnov, Zeng; KIRA3: Lang, Usovitsch, Wu

Progress feeds back to collider physics!

Improving integration active area.

Liu, Mitov; Von Hippel, Wilhelm; de la Cruz, Kosower Shih; epsilon collaboration;
Berman, Charton, Luna, Wilhelm, Zeng; Smith and Zeng; Vergu;
Bree, Gasparotto, Matijašić, Mazloumi, Melnichenko, Pögel, Teschke, Wang, Weinzierl, Wu, Xu; etc

We heard multiple talks: Feng, Vergu, Zhang, Zoia

We need new ideas



Example of New Approach

ZB, Jackman, Mansfield, Ruf

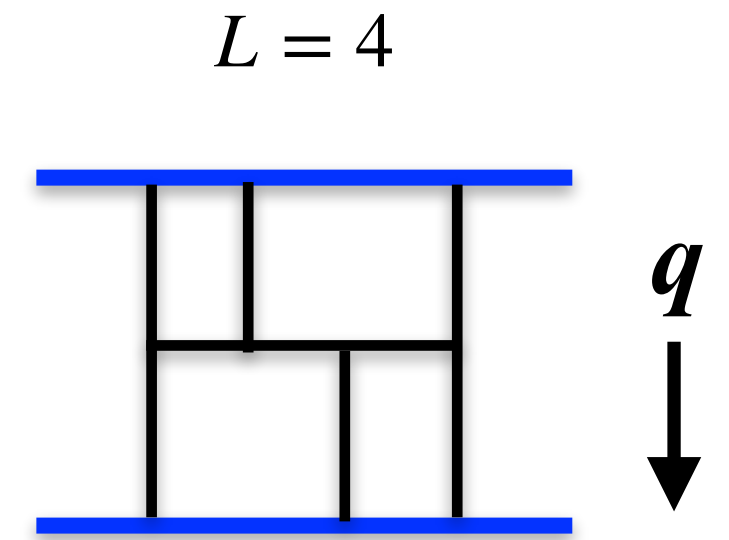
**Why does elliptic cancel and CY cancel? Complete mystery via IBP.
There should be a better way to do the integrals!**

In the conservative sector the scattering amplitudes behave as

$$\mathcal{M}_{\text{cl}} \sim \begin{cases} (-q^2)^{\frac{L}{2}-1} \ln(-q^2), & L \text{ even} \\ (-q^2)^{\frac{L}{2}-1}, & L \text{ odd.} \end{cases}$$

q is the momentum transfer

$$V(r) \sim \frac{1}{r^{L+1}}$$



In dim. reg, the only source of $\ln(-q^2)$ are divergences

divergence:

 $\frac{1}{\epsilon} (-q^2)^{-L\epsilon} = \left[\frac{1}{\epsilon} - L \ln(-q^2) + \mathcal{O}(\epsilon) \right] \quad D = 4 - 2\epsilon$

- Diverge structure is completely ignored in the standard IBP approach.
- IBP can scramble IR and UV and introduces spurious divergences.

Key observation: Singular regions generally much easier to calculate than nonsingular regions

A Simpler Integration Strategy?

Test of the idea:

1) Reproduced 2 loop (3PM) conservative results. IBP methods not challenging here.

2) Can we explain missing Calabi-Yau and Elliptic integrals at 5PM?

Two loop (3PM) example:

$$\text{Diagram} = \int \frac{d^D \ell_1 d^D \ell_2}{(i\pi^{D/2})^2} \frac{\delta(2u_1 \cdot \ell_1) \delta(2u_2 \cdot \ell_2) \mathcal{N}}{\ell_1^2 \ell_2^2 (\ell_1 - q)^2 (\ell_2 - q)^2 (\ell_1 + \ell_2 - q)^2}$$

UV region

$$\ell_1, \ell_2 \gg q$$

$$\begin{array}{c} q \\ \downarrow \end{array} \text{Diagram} \Big|_{\text{UV}} = \text{Diagram} \Big|_{\text{UV}} = \frac{i\pi}{\epsilon} \frac{\text{arccosh}(\sigma)(i\pi - \text{arccosh}(\sigma))}{2\sqrt{\sigma^2 - 1}}$$

UV divergence

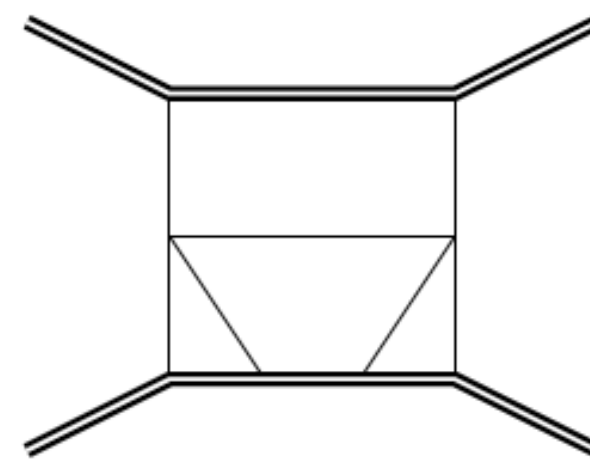
$$(\ell_1 + q)^2 \rightarrow \ell_1^2$$

- In UV region integral simplifies into a cusp integral.
- Integrals simpler.
- Reproduced 3PM results

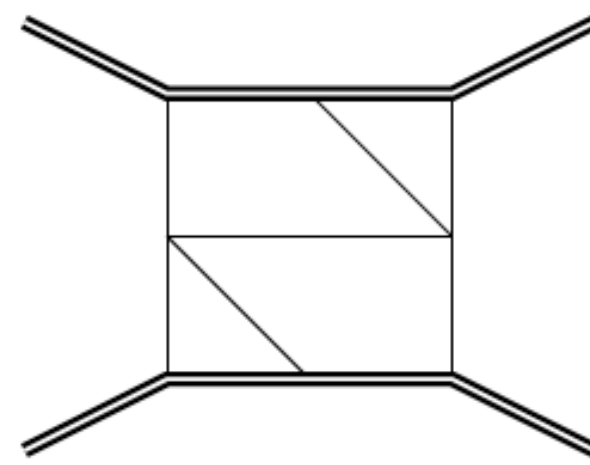
Source of Calabi-Yau, Elliptic and Heun

Integration facts we exploit:

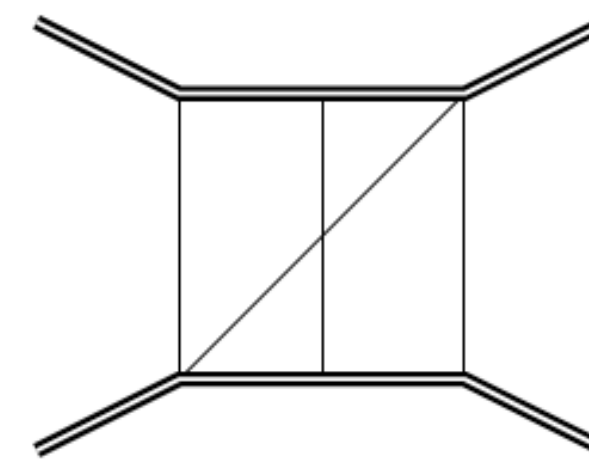
- The special functions beyond GPLs are controlled by only 3 sources:



Elliptic



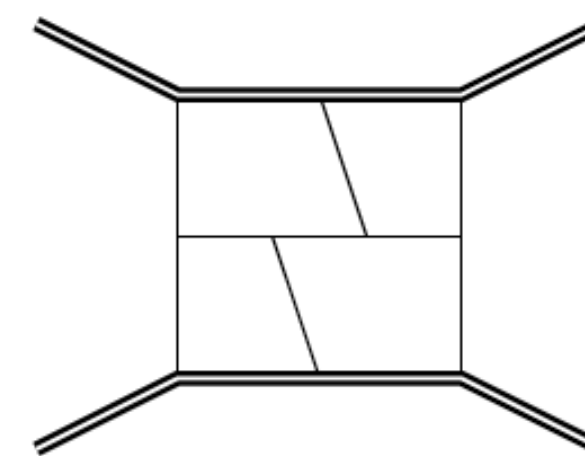
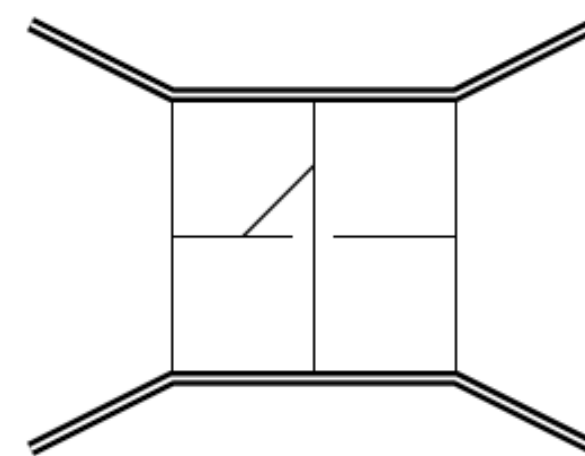
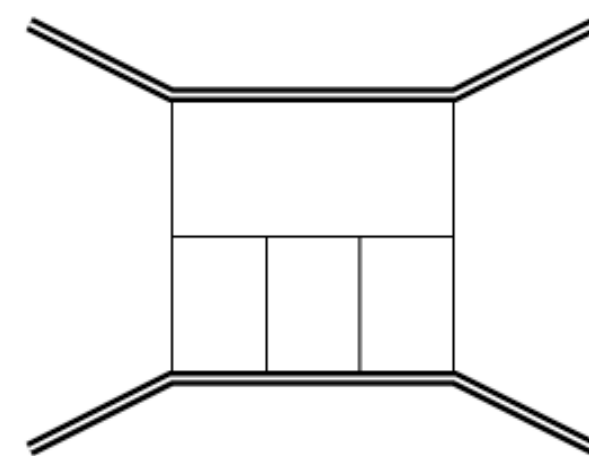
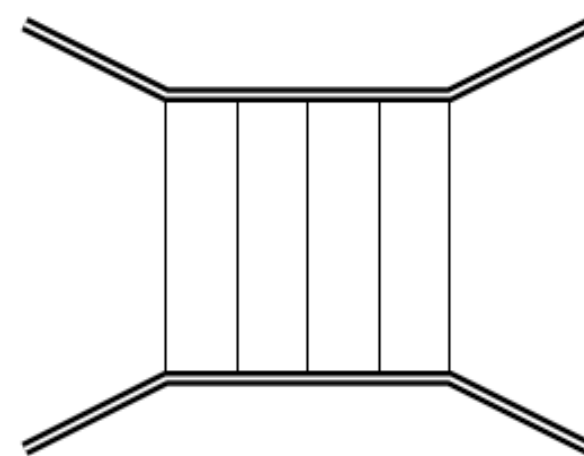
Calabi-Yau



Heun

These are the diagrams to focus on

- Above diagrams have only UV divergences, no IR
- Maximal cut analysis shows above diagrams must be present for functions beyond GPLS



Task: Determine what happens to the Elliptic, CY and Heun in the UV limit.

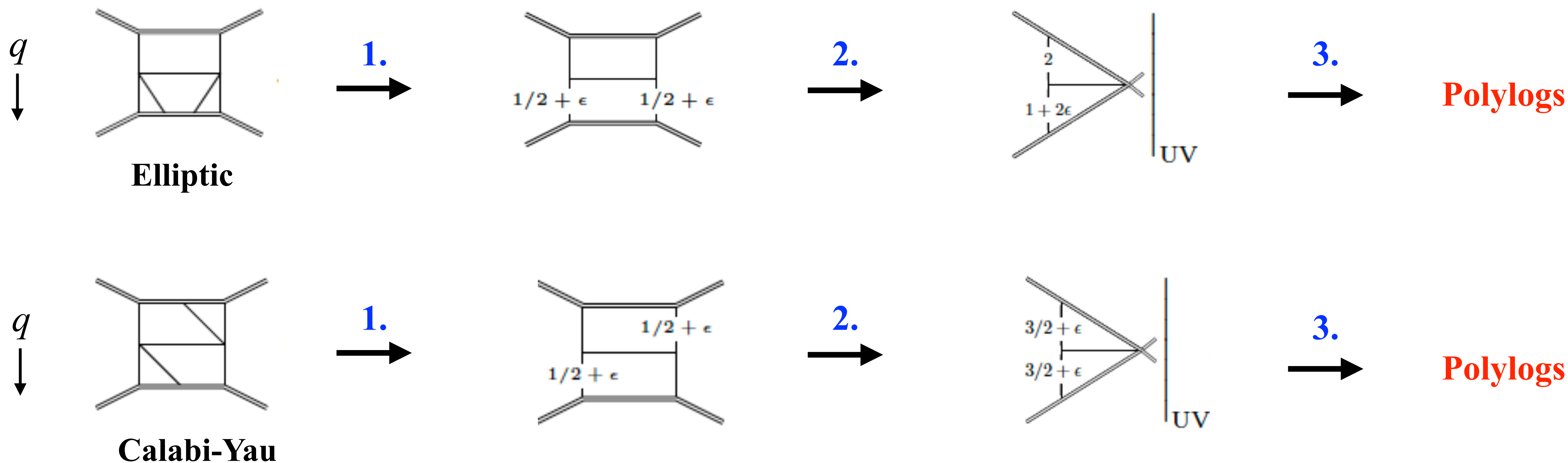
Integrals simplify greatly in the UV

1. Integrate out triangle subdiagrams.

$$D = 4 - 2\epsilon$$

2. Take UV limit: $\ell_i \gg q$. $(\ell_1 + q)^2 \rightarrow \ell_1^2$

3. Integrate simpler UV integral (See Appendix of arXiv:2603.15383)



Elliptic and CY are not present in the conservative part. (Heun remains).

It will be interesting to see how far these ideas can be pushed.

Many Aspects not Discussed Here

I only discussed spinless, no tidal, no absorption. Other aspects:

See David Kosower's review talk

- **Waveforms** Cristofoli, Gonzo, Kosower, O'Connell; Herrmann, Parra-Martinez, Ruf, Zeng; Di Vecchia, Heissenberg, Russo, Veneziano; Herderschee, Roiban, Teng; Georgoudis, Heissenberg, Vazquez-Holm; Brandhuber, Brown, Chen, De Angelis, Gowdy. Travaglini; De Angelis, Gonzo, Novichkov, Alaverdian, ZB, Kosmopoulos, Luna, RR, Scheopner, Teng; Bohnenblust, Ita, Kraus, Schlenk; Brunello, De Angelis, Kosower; etc
- **Finite-size effects** Cheung and Solon; Haddad and Helset; Kälin, Liu, Porto; Cheung, Shah, Solon; ZB, Parra-Martinez, Roiban, Sawyer, Shen; etc
- **Compton amplitudes** Ivanov, Li, Parra-Martinez, Zhou; M. Correia and G. Isabella; Caron-Huot, Correia, Isabella, Solon; Kosmopoulos, Perrone, and Solon; Correia, Gopakka, Isabella, Wolz; Brunello, Meo, Smith; Bautista, Driesse, Haddad, Jakobsen; etc
- **Spin** Vaidya; Porto; Levi, Steinhoff; Guevara, Ochirov, Vines; Chung, Huang, Kim, Lee; ZB, Luna, Roiban, Shen, Zeng; Mogull, Plefka, Steinhoff; Kosmopoulos, Luna; Febres Cordero, Kraus, Lin, Ruf, Zeng; Aoude Haddad, Helset; ZB, Kosmopoulos, Luna, Roiban, Scheopner, Teng, Vines; Cangemi, Chiodaroli, Johansson, Ochirov, Pichini, Skvortsov; Gatica; Luna, Moynihan, O'Connell, Ross; Akpinar, Febres Cordero, Kraus, Ruf, Zeng; Bautista, Guevara, Kavanagh, Vines; etc.
- **Absorption** Jones and Ruf, Chen, Hsieh, Y-t Huang, J-W Kim; Bautista, Guevara, Kavanagh, Vines; etc
- **Renormalization Effects** Goldberger and Rothstein; Goldberger, Ross; Porto, Galley, Leibovich; Levi; Barack, ZB, Herrmann, Long, Parra-Martinez, Roiban, Ruf, Shen, Solon, Teng, Ivanov, Li, Parra-Martinez, Zhou; Caron-Huot, Correia, Isabella, Solon, etc
- **The standard quantities of interest for the inspiral phase of binary black holes can be computed in amplitudes approach. A lot of work remains!**

Summary

- Pushing state of the art: real progress on 5PM . “Conservative” part under control.
See also talk from Jakobsen
- Progress is slower now that we are beyond what standard IBP tools can do.
- New ideas and methods for tackling integrals
 - Better IBP seeding.
 - UV, IR analysis is one new idea, at least for certain classes of integrals.
 - Elliptic and CY integral cancellations explained.
- 5PM radiative, dissipative contributions under study.
- Longer term goal is 6PM. First need to banish supercomputers from 5PM.

5PM is coming under control. Further improvements and results are coming.

Amplitudes

**Congratulations to all the speakers for the wonderful talks.
Future of the Amplitudes is in good hands!**

Amplitudes 2026 at Queen Mary: 225+ participants



“amplitudes 1993 reunion”: 4 participants



Thank You to the Local Organizers



**Andreas
Brandhuber**



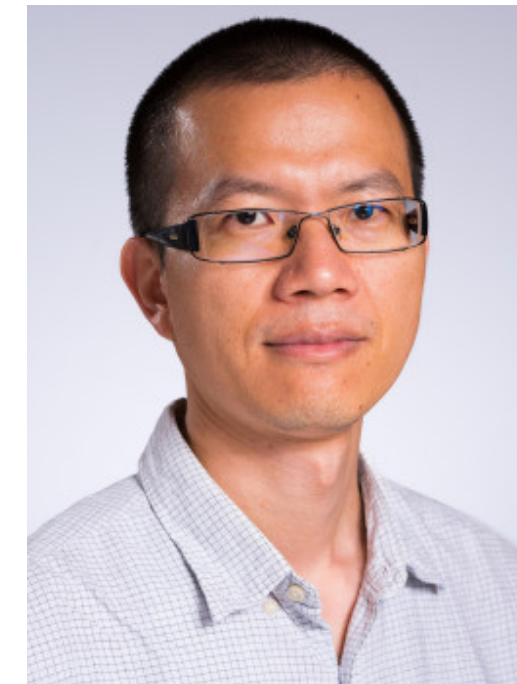
**Ricardo
Monteiro**



**Mary
Thomas**



**Gabriele
Travaglini**



**Congkao
Wen**



**Chris
White**

Thank you for a wonderful conference!