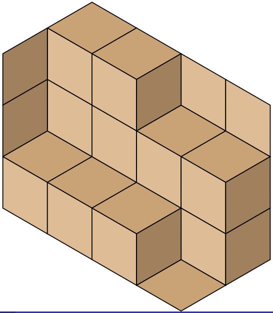
Conjectures and counterexamples for the amplituhedron.

Lauren K. Williams

Harvard

Based on



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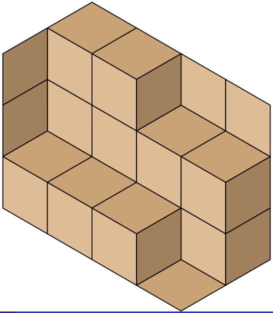

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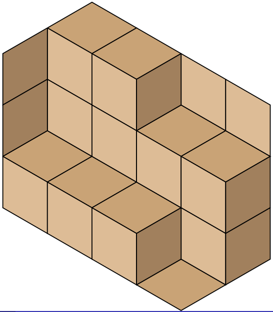
- *Signed Cells and G-gradings on Cluster Algebras*, joint with Alan Yan.



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- Facets of the amplituhedron and the sign flip conjecture

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- Generalized positive parts: a conjecture on amplituhedron tiles

The positive Grassmannian

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Plücker coordinates

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If we decompose it into pieces based on which Plucker coordinates are positive, we obtain the positroid cell decomposition

$$\mathrm{Gr}_{k,n}^{\geq 0} = \bigsqcup S_{\mathcal{M}}.$$

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Fix n, k, m with $k + m \leq n$ and a matrix Z in $\mathrm{Mat}^{>0}_{n,k+m}$, i.e. all maximal minors of Z are positive.

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Let Y in $\mathrm{Gr}_{k,k+m}$ be represented by a matrix with rows $y_1, \ldots, y_k$; and let $Z_1, \ldots, Z_n$ be the rows of Z.

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First surprise for $m \geq 6$: nonlinear facets

Theorem (AH-EZ-L-P-SB-T-W)

For even $m \geq 6$, n sufficiently large, and $k \neq 1$, the amplituhedron $\mathcal{A}_{n,k,m}$ has algebraic facets cut out by polynomials of degree

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The sign-flip conjecture is false in general: sign conditions on twistor coordinates cannot detect these nonlinear boundary components.

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Let $S \subset \mathrm{Gr}_{k,n}^{\geq 0}$ be a positroid cell.

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A tiling is a collection $\mathcal{C}$ of km-dimensional positroid cells such that $\widetilde{Z}$ is injective on each cell and every generic point of $\mathcal{A}_{n,k,m}(Z)$ has exactly one preimage in $\mathcal{C}$.

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- Parity duality relates tilings of $\mathcal{A}_{n,k,m}$ and $\mathcal{A}_{n,n-k-m,m}$ (Lam–Galashin).

$m = 6, k = 2$: conjectural "BCFW" tilings

We give a generalized BCFW-style recursive collection $\mathcal{C}(n, 2, 6)$ of positroid cells.

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Conjectural result: $\mathcal{C}(n, 2, 6)$ gives all-Z tilings of $\mathcal{A}_{n,2,6}$.

Contrast between low m and higher m

Conjecture (AH-EZ-L-P-SB-T-W)

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These are the first cases not covered by known families of tilings (or parity duals).

Relaxing the notion of tiling: r-fold tilings

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An *r-fold tiling* is a multiset of km-dimensional positroid cells whose images contain open km-dimensional sets and such that every generic point of $\mathcal{A}_{n,k,m}(Z)$ has exactly r preimages in the multiset.

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Relaxing the notion of tiling: r-fold tilings

By trying to generalize the BCFW recurrence for higher m, we found collections of cells which aren't quite tilings – but are close.

Definition

An *r-fold tiling* is a multiset of km-dimensional positroid cells whose images contain open km-dimensional sets and such that every generic point of $\mathcal{A}_{n,k,m}(Z)$ has exactly r preimages in the multiset.

New examples

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Magic Number Conjecture (Karp–Williams–Zhang)

Every tiling of $\mathcal{A}_{n,k,m}$ consists of

$$M\left(k, n-k-m, \frac{m}{2}\right) = \prod_{i=1}^{k} \prod_{j=1}^{n-k-m} \prod_{\ell=1}^{m/2} \frac{i+j+\ell-1}{i+j+\ell-2}$$

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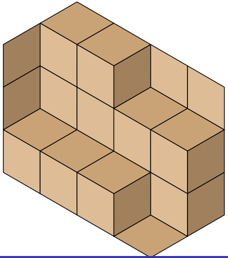
A plane partition can be viewed as a rhombic tiling of a hexagon.



Modified magic number conjecture

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In every r-fold tiling $\mathcal{T}$ of $\mathcal{A}_{n,k,m}$,

$$\frac{1}{r} \sum_{S \text{ in } \mathcal{T}} \iota_m(S) = M\left(k, n-k-m, \frac{m}{2}\right),$$

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Thus the "magic number" remains relevant if we keep track of intersection number when we count!

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Next ... brief discussion of signed seeds and amplituhedron tiles.

Cluster algebras from quivers

Let $\widetilde{Q}$ be an ice quiver: mutable vertices $1, \ldots, r$ and frozen vertices $r+1, \ldots, s$.

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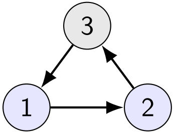

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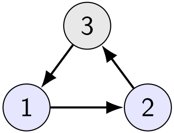

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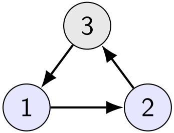

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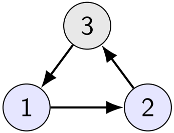

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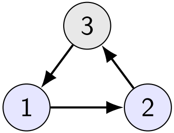

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The cluster algebra is the polynomial ring (over the polynomial ring in coefficients) generated by the cluster variables obtained by applying all possible sequences of mutations.

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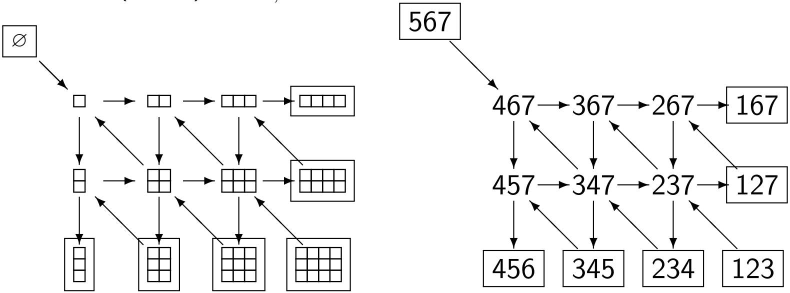

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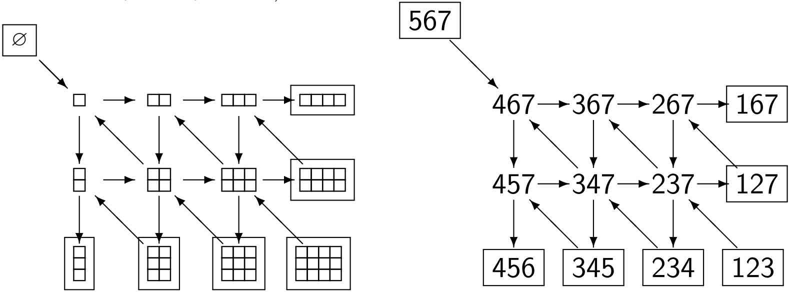

- The *rectangles seed* (encoded in two ways above for $Gr_{3,7}$) gives a cluster structure on $Gr_{K,N}$.

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- – BUT if we choose signs on initial cluster carefully so that they are "balanced" – i.e. both terms on the RHS of the exchange relation have the same sign – the answer is yes.

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If $\mathcal{X}$ is the set of all cluster variables, this gives a *grading* $\sigma : \mathcal{X} \to \{-1, 1\}$ on the set of cluster variables.

Signed seeds for cluster algebras: example

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$$\begin{tikzcd}
\boxed{123}\ + \arrow[dr] & & & & \\
& \overset{-}{124} \arrow[r] \arrow[d] & \overset{+}{125} \arrow[r] \arrow[d] & \overset{-}{126} \arrow[r] \arrow[d] & \overset{+}{\boxed{127}} \\
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E.g. there is a signed cell of $\mathrm{Gr}_{3,6}$ whose Plücker coords have signs below.

$$\begin{tikzcd}
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A signed cell is a kind of "generalized positive part."

Mapping twistor coordinates to Plücker coordinates

Recall: if Y in $\mathrm{Gr}_{k,k+m}$ is represented by a matrix with rows $y_1, \ldots, y_k$, and Z has rows $Z_1, \ldots, Z_n$, we define:

Twistor coordinate

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tiles in $\mathcal{A}_{n,k,m}(Z) \subset \mathrm{Gr}_{k,k+m}$ and signed seeds for $\mathrm{Gr}_{m,n}$.

Amplituhedron tiles are conjecturally signed cells

Fix n, k, m; Z in $\mathrm{Mat}_{n,k+m}^{>0}$; and let Z_S° be an open tile for $\mathcal{A}_{n,k,m}(Z)$.

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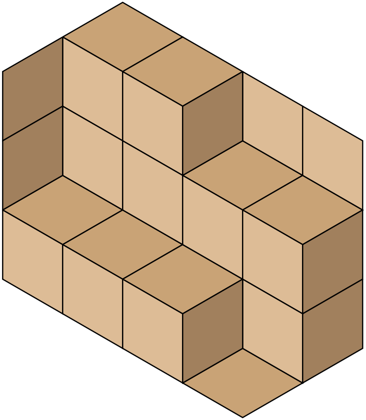
$m = 4$ BCFW tiles (EvanZohar-Lakrec-Parisi-ShermanBennett-Tessler-W).

Thank you



References

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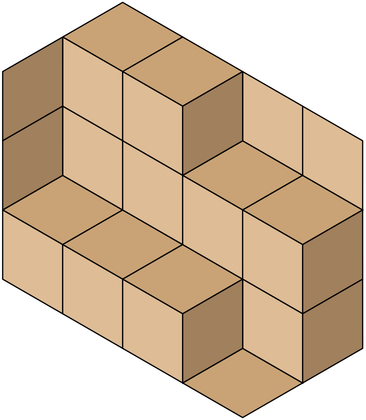

References

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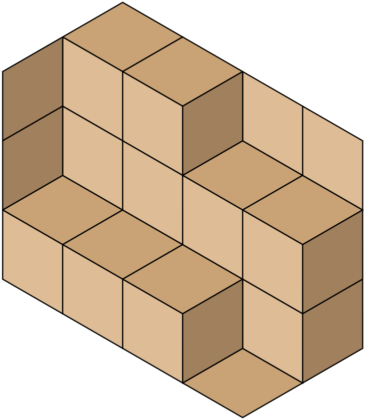

References

- *Higher m Amplituhedra*, in preparation, joint with Arkani-Hamed, Even-Zohar, Lakrec, Parisi, Sherman-Bennett, Tessler.
- *Signed Cells and G-gradings on Cluster Algebras*, in preparation, joint with Alan Yan.