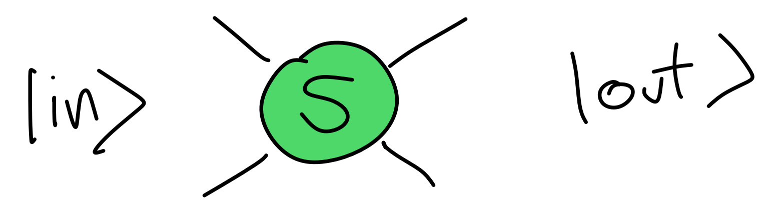
Non-perturbative scattering in higher dimensions

Amplitudes 2026, QMUL, London

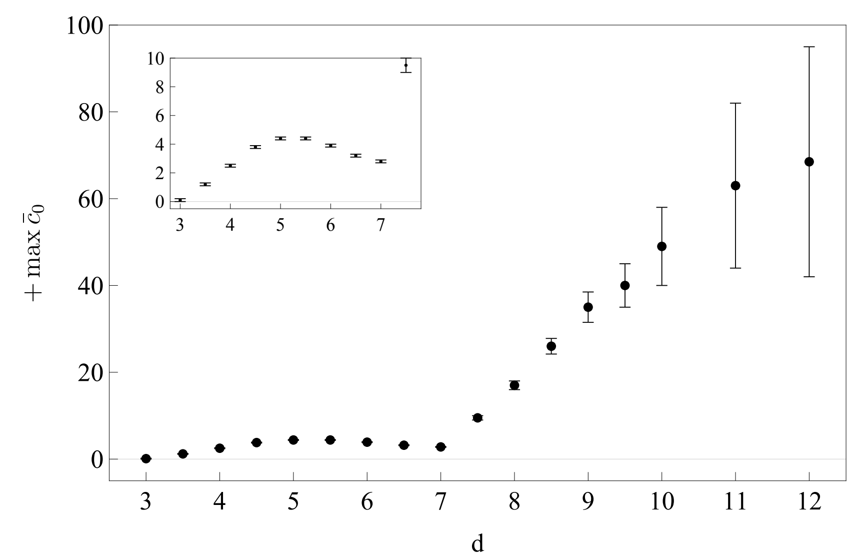
Piotr Tourkine, LAPTh, Annecy, 03/07/2026

Summary

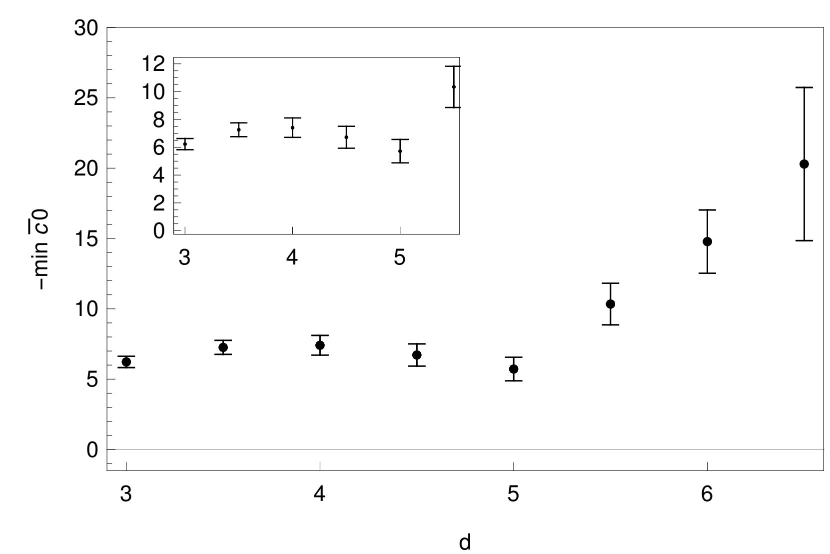


- I'll show some surprising results for "extremal couplings" (maximum interaction strengths) for scalar 2-to-2 scattering in massive QFTs as a function of spacetime dimension

  $d = 3, 3.5, 4, 4.5, \ldots, 12$

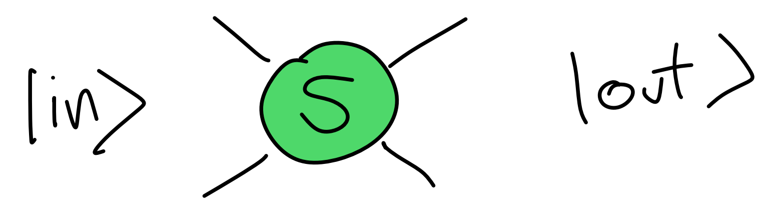


- In collaboration with two fantastic postdocs: **Mehmet A. Gumus, Simon Metayer**



[arXiv:2512.24474]
Tracking S-matrix bounds across dimensions
M. A. Gumus, S. Metayer, P. Tourkine

Summary



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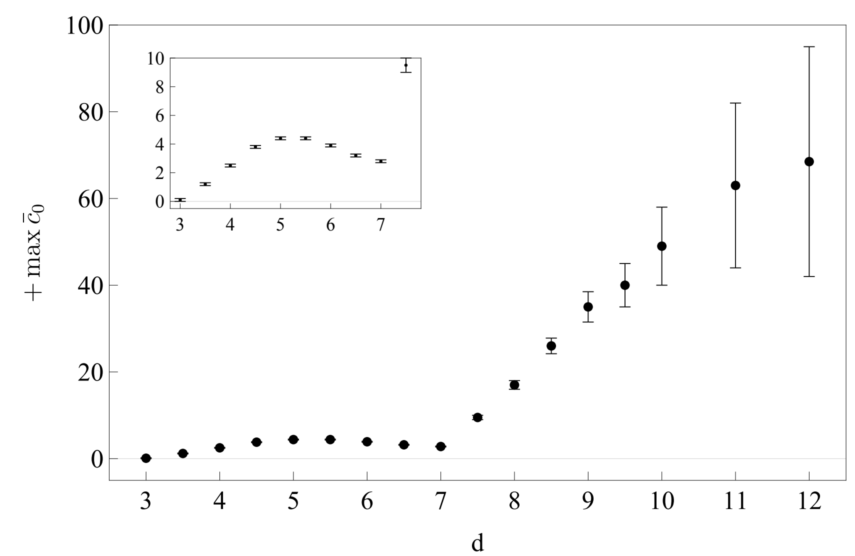

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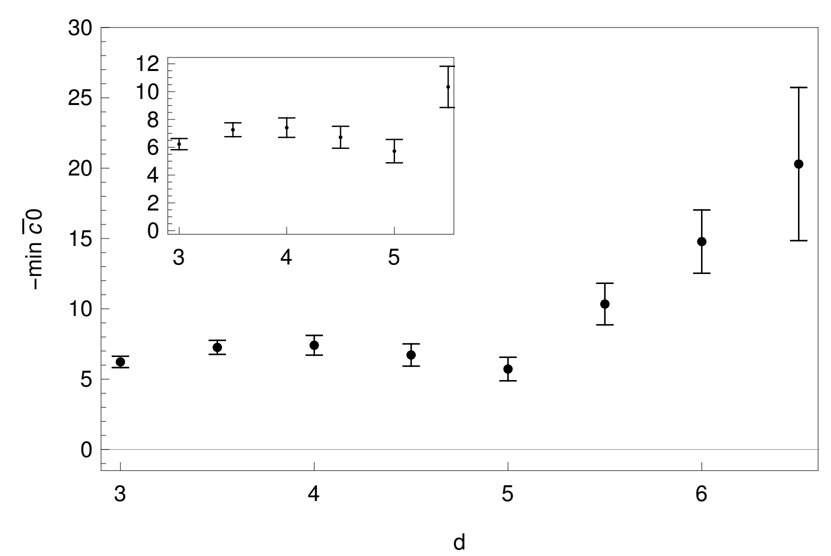


- Briefly open up on the status of a rigorous approach w/ **S. Zhiboedov** + **D. Leflot** and **M. A. Gumus** which could in principle give a way to understand these surprising results better

- Question to you: do you expect scattering to be **stronger** or **weaker** in higher dimensions?

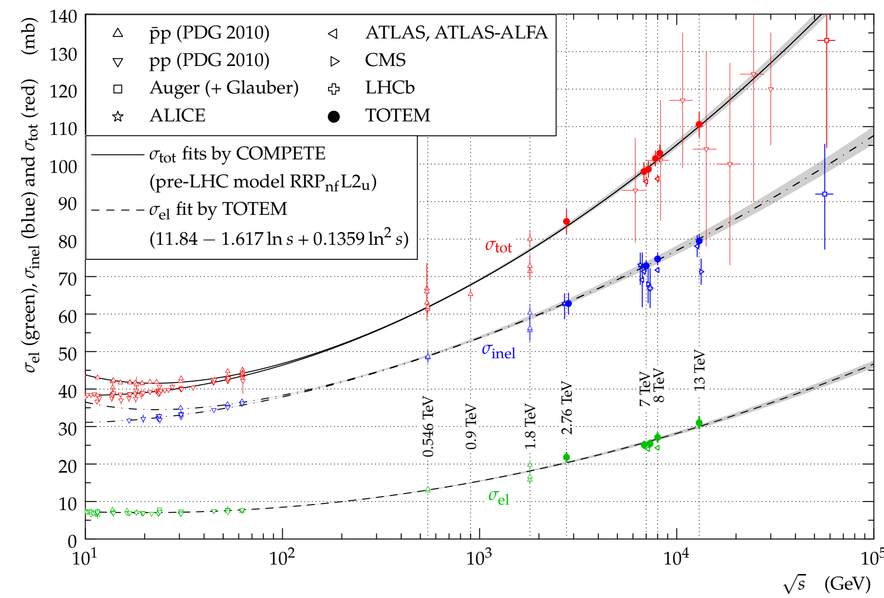
Plan

- What is non-perturbative S-matrix.
- What is non-perturbative S-matrix in higher dimensions?
- Bootstrap
- Results
- Conclusion & perspectives

S-matrix

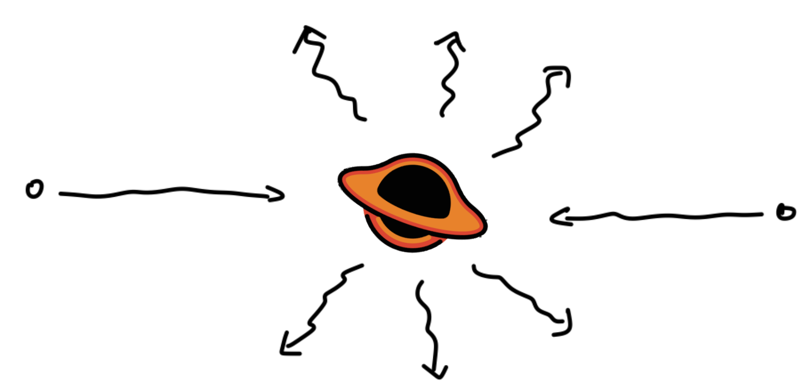
- Weakly coupled theories: amplitudes methods
- Strong coupling: standard perturbation theory not expected to converge. Integrability and large-N sometimes allow to connect weak to strong coupling.
- How to study generic strongly coupled, non-perturbative scattering?
- “analytic S-matrix methods” - first principle, bootstraps.
- **Regge theory.**

QCD: soft hadronic interactions



$pp, p\bar{p}$ cross-section

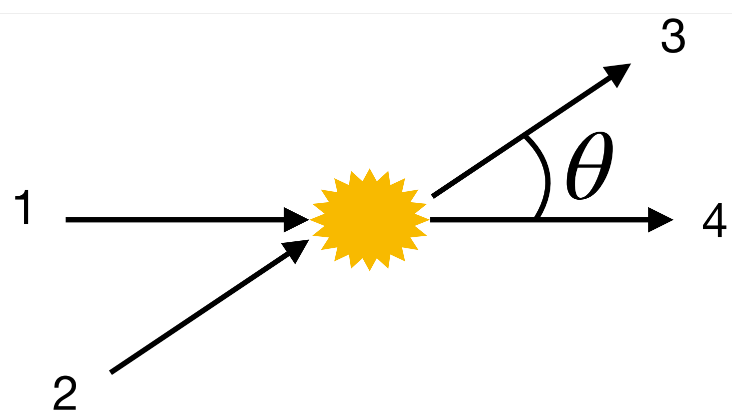
Quantum gravity and string theory



Where is M-theory ? Guerrieri, Murali, Penedones, Vieira

S-matrix

- Non-perturbative S-matrix here means: a function $T(s, t, u)$ that satisfies the axioms of:
  - **A**nalyticity, **C**rossing symmetry, **U**nitarity



$$s = (p1 + p2)^2$$
$$t = (p1 - p3)^2$$
$$u = (p1 - p4)^2$$

$$\cos(\theta) = 1 + \frac{2t}{s - 4m^2}$$
$$E^2_{c.o.m.} = s/4$$

$$s + t + u = 4m^2$$

Physical region for s-channel scattering:

$$s \geq 4m^2 \qquad t, u \leq 0$$

Similar for t- and u-channels

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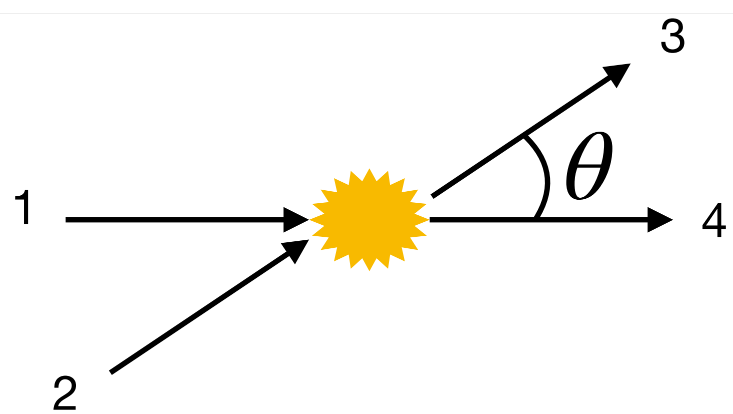

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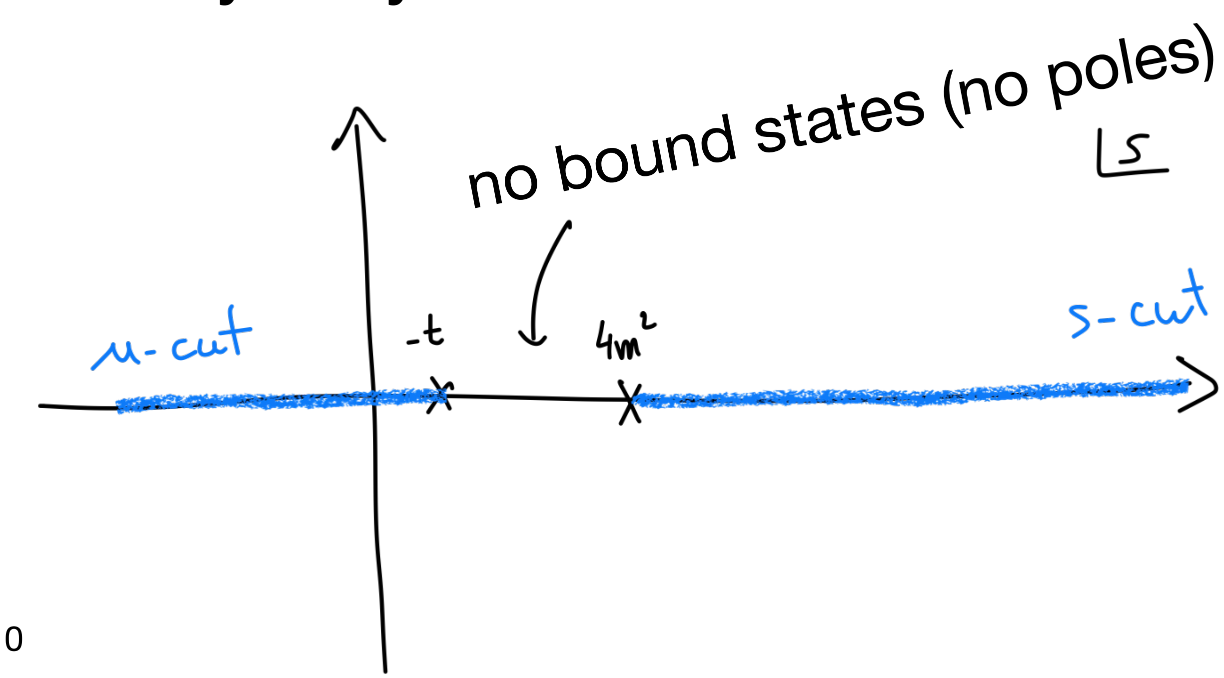
- Analyticity: reflects causality.
  Assumption (unproven): maximal analyticity



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- Crossing:

S-matrix

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  $$T(s, t) = T(t, s)$$

  very technical proof from first principles, Bros-Epstein-Glaser ‘60s

S-matrix

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- Unitarity:

S-matrix

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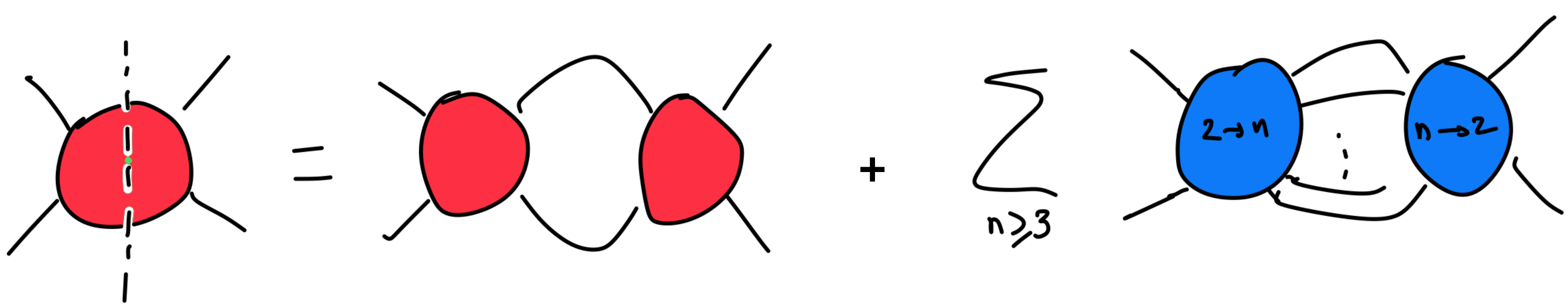
- Unitarity: 2-particle unitarity



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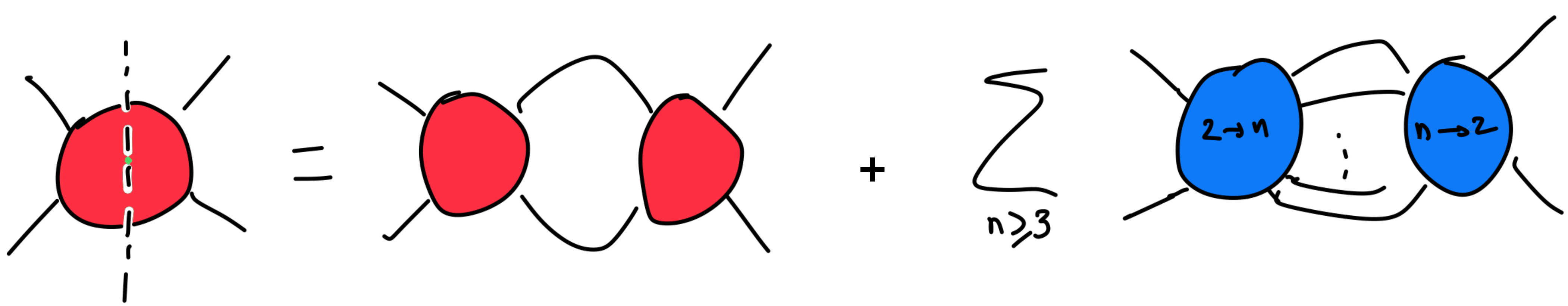

Tourkine Zhiboedov, +Leflot Gumus 2021 2023 2024 2026

S-matrix

- Non-perturbative S-matrix here means: a function $T(s, t)$ that satisfies the axioms of:
    - **A**nalyticity, **C**rossing symmetry, **Unitarity**
- Unitarity: Partial wave basis / decomposition into harmonics

  Yields a convex (semidefinite) problem, amenable to “easy” numerical solving

Paulos, Penedones, Toledo, van Rees, Vieira 2016, 2017

Partial wave unitarity

$$T(s,t) = \sum_{J=0}^{\infty} n_J^{(d)} f_J(s) P_J^{(d)}(\cos(\theta)) \qquad \cos(\theta) = 1 + \frac{2t}{s-4m^2}$$

$P_J^{(d)}$: Gegenbauer (orthogonal) polynomials

$n_J^{(d)}$: normalisation

$f_J(s)$: partial waves

$$T(s,t) \leftrightarrow f_J(s)$$

Partial wave unitarity

$$T(s,t) = \sum_{J=0}^{\infty} n_J^{(d)}\, f_J(s)\, P_J^{(d)}(\cos(\theta))$$

$$\cos(\theta) = 1 + \frac{2t}{s-4m^2}$$

the partial waves: $S_J(s) = 1 + i\frac{(s-4)^{(d-3)/2}}{\sqrt{s}} f_J(s)$ **diagonalise** unitarity:

- $|S_J(s)|^2 \leq 1, \qquad s \geq 4m^2$
- Straightforward to check by *partial wave projection*:

$$f_J(s) \propto \int_{-1}^{1} K_d(z)\, P_J^{(d)}(z)\, T(s,t(z))\, dz \qquad z = \cos(\theta)$$

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d appears analytically everywhere

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- Bound low energy observables
- Techniques from Elias-Miro, Guerrieri, Gumus 2022
- We will use bootstrap numerical methods in $d = 3 \to 12$
- See Chen Fitzpatrick Karateev 2022 for $d = 2.5 \to 4$
- Main difference with our study: inclusion of all terms allowed by unitarity (threshold divergences) gives a complete picture.

Microscopic picture?

- What is non-perturbative scattering?
- In $d = 4$: Yang-Mills g_{YM} is dimensionless : confinement in the IR, mass gap.
  Spectrum of resonances (glueballs), measured on lattice.

[arXiv:2312.00127] Phys.Rev.D **112** (2025) 094023
Constraining Glueball Couplings
A. L. Guerrieri, A. Hebbar, B. C. van Rees

Microscopic picture?

- What is non-perturbative scattering?
- In $d = 4$: Yang-Mills g_{YM} is dimensionless : confinement in the IR, mass gap.
  Spectrum of resonances (glueballs), measured on lattice.
- In $d = 5{,}6$, non-gravitational UV-complete QFTs exist: the superconformal field theories (5-brane worldvolumes)
- In $d \geq 7$, no non-gravitational solutions are known to exist (7d Yang-Mills discussed)

Itzhaki, Maldacena, Sonnenschein, Yankielowicz '98
and many many more

- "string universality"?

back to bootstrap

Observables

What are we bounding?

- *Low-energy data* at the crossing-symmetric point (pure convention)

$$s_0 = t_0 = u_0 = \frac{4}{3}m^2$$

- $T(s,t) = c_0 + c_2\left((s-s_0)^2 + (t-t_0)^2 + (u-u_0)^2\right) + \ldots$
- Main observables: *normalised couplings*
  - $\bar{c}_0 = \frac{N_d}{2}T(s_0,t_0),$ (~ quartic coupling), $\bar{c}_2 = \frac{N_d}{8}\frac{\partial^2}{\partial s^2}T\left(s,t_0\right)\Big|_{s=s_0}$
  - where $N_d \sim \dfrac{1}{\Gamma(d/2))} \ll 1$ at large d

Ansatz

The machinery

Ansatz

N : size of ansatz

J_{max} : spin cutoff

- $$T_N(s,t) = \sum_{i,j,k}^{N} \alpha_{i,j,k}\ \rho(s,\sigma_i)^i\, \rho(t,\sigma_j)^j\, \rho(u,\sigma_k)^k$$
  - basis functions: $s \to \rho(s,\sigma) = \dfrac{\sqrt{\sigma - 4m^2} - \sqrt{4m^2 - s}}{\sqrt{\sigma - 4m^2} + \sqrt{4m^2 - s}}$
  - Builds in **crossing** and **analytic** structure (cuts)
  - Optimize low-energy couplings over α 's
  - **Enforce** partial wave-**unitarity** $|S_J(s)|^2 \leq 1$ up to J_{max}

take double limit $N,\ J_{max}\ \to \infty$

Optimization

N : size of ansatz

J_{max} : spin cutoff

- **Remark: Clean semidefinite problem ($|S_J(s)|^2 \leq 1$)**

1. Use $\alpha^2 + \beta^2 \leq 1 \Leftrightarrow \det \begin{pmatrix} 1+\alpha & \beta \\ \beta & 1-\alpha \end{pmatrix} \geq 0$

2. Since $S_J(s) \sim \int T(s, t(z))\, P_J(z)\, dz \implies S_J(s) \sim$ linear in $\alpha_{i,j,k}$

Paulos, Penedones,
Toledo, van Rees, Vieira
2016, 2017

sdpb Public

A semidefinite program solver for the conformal bootstrap.

C++ ☆ 75 47

Simons Duffin, +Landry

One last (essential) piece

Threshold behaviour, $s \to 4m^2$

One last (essential) piece

Threshold behaviour, $s \to 4m^2$

- Essential part of our whole construction.
- Near the 2-part. threshold, the partial waves can be large — pure quantum effect.
- Unitarity dictates the worst possible behaviour:
- $f_0(s) \sim \dfrac{1}{(s-4m^2)^{(d-3)/2}} + \text{less singular} + \text{regular}$

[arXiv:2006.08221] JHEP **03** (2021)
An Analytical Toolkit for the S-matrix Bootstrap
M. Correia, A. Sever, A. Zhiboedov

Threshold behaviour

- In spirit of EFT, everything that is allowed by constraints should be included in ansatz
- For instance, in $d = 6$:
- $f_0(s) = \frac{c}{(s-4m^2)^{3/2}} + \frac{\#}{s-4m^2} + \frac{\#}{\sqrt{s-4m^2}} + cst + \#\sqrt{s-4m^2} + \dots$
- with log corrections in odd dimensions
- Only c is fixed by unitarity to 0 (regular) or 1 (singular), not the other coefficients
- Include in ansatz.
- *Modifying the ansatz to allow for these terms proved a very difficult task, absolutely essential to achieve numerical convergence.*

Strong/mild threshold

- **Strong** threshold if singularities with $\dfrac{1}{(s-4m^2)^n}$ with $n > 1$
- **Mild** threshold if singularities with $\dfrac{1}{(s-4m^2)^n}$ with $n \leq 1$

- *Crucial to explain the results.*

- Remark: any threshold with $n \geq 1$ implies a loss of positivity.
- *Positivity in higher dimensions becomes subtle*

Results

Supercomputer

Local cluster

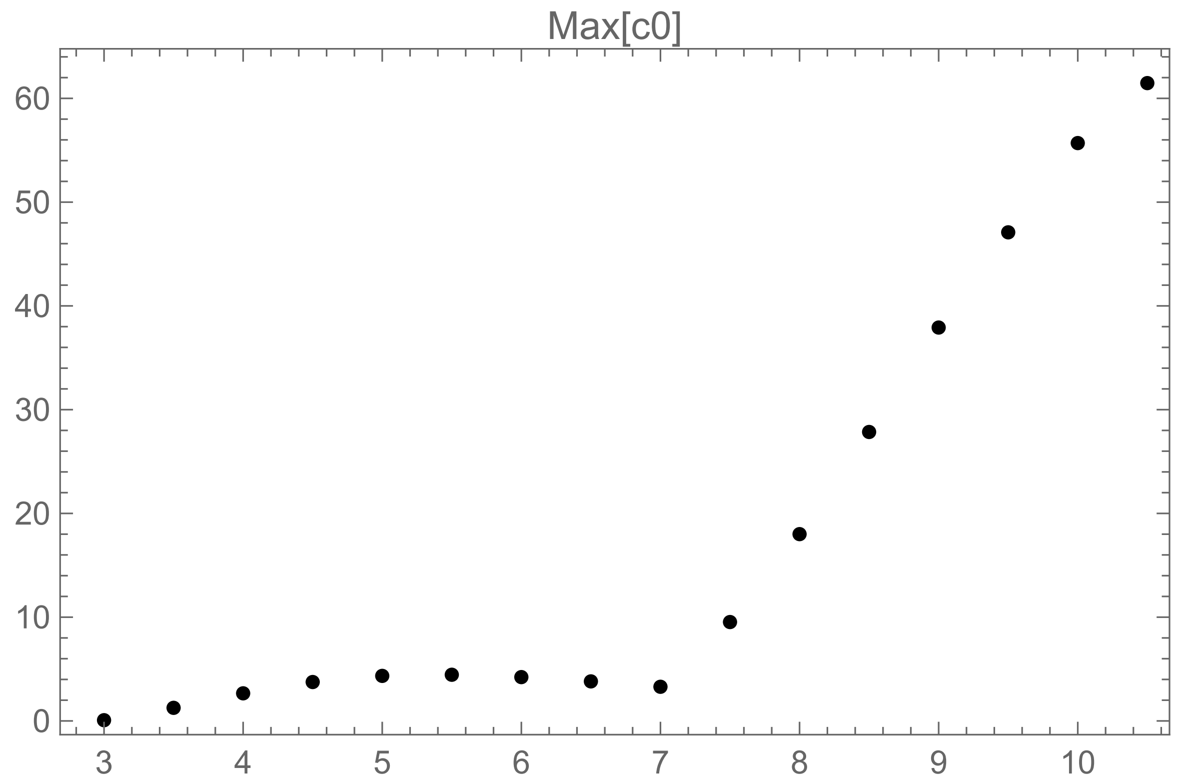
Results

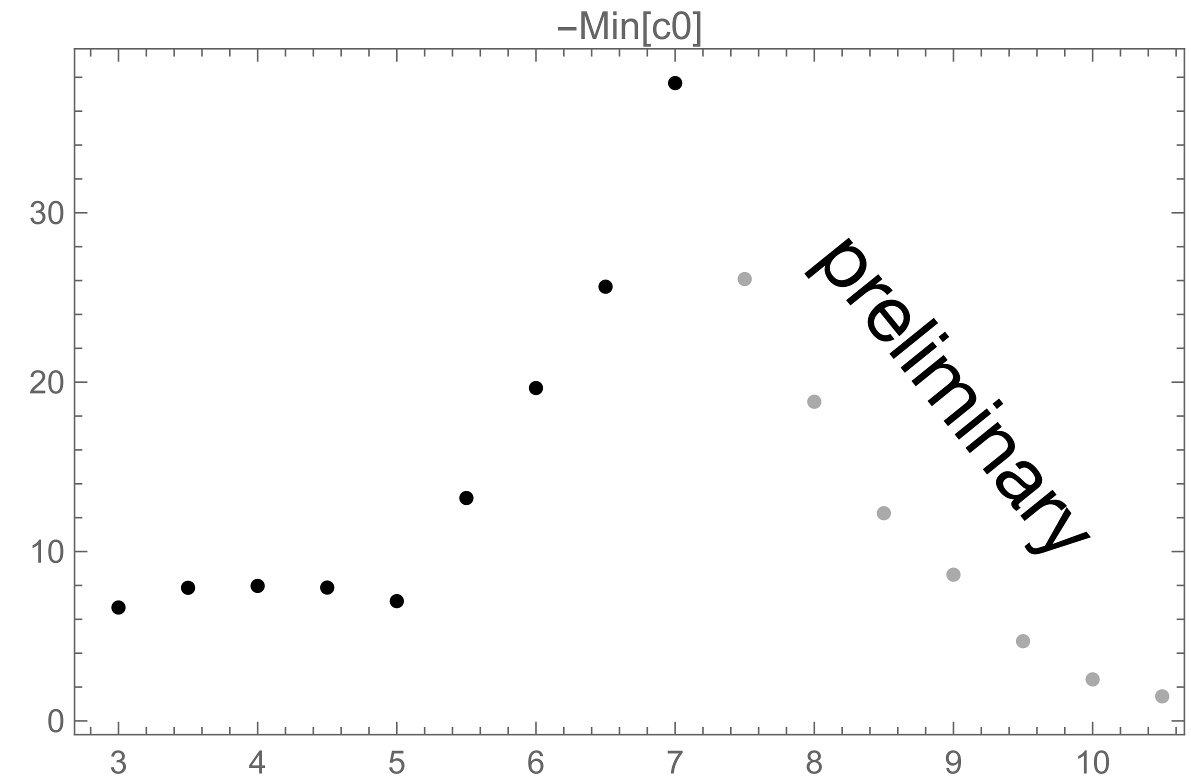
Develop an entire automatized pipeline to do robust and scalable numerics in all *d*, with SQL database

`SDPBWrapper` (in mathematica) to be made public eventually (Simon Metayer)

~ 100,000 CPU hours

Extremal bounds





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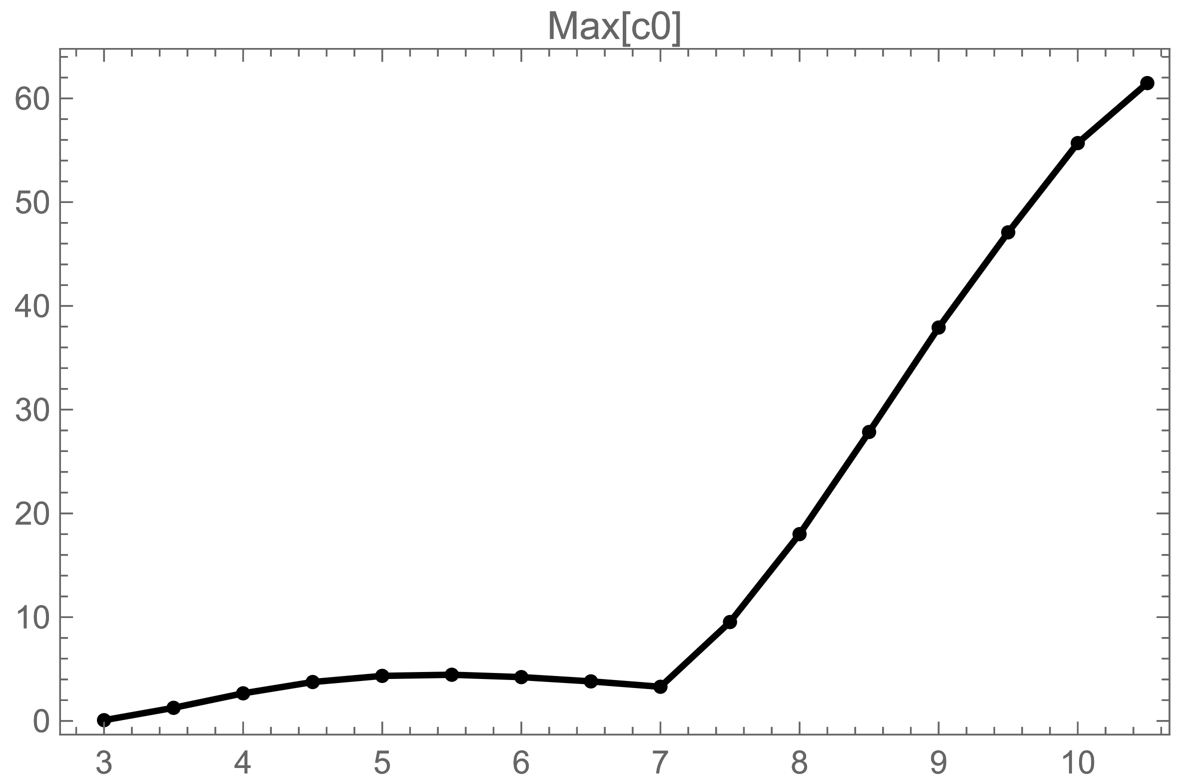

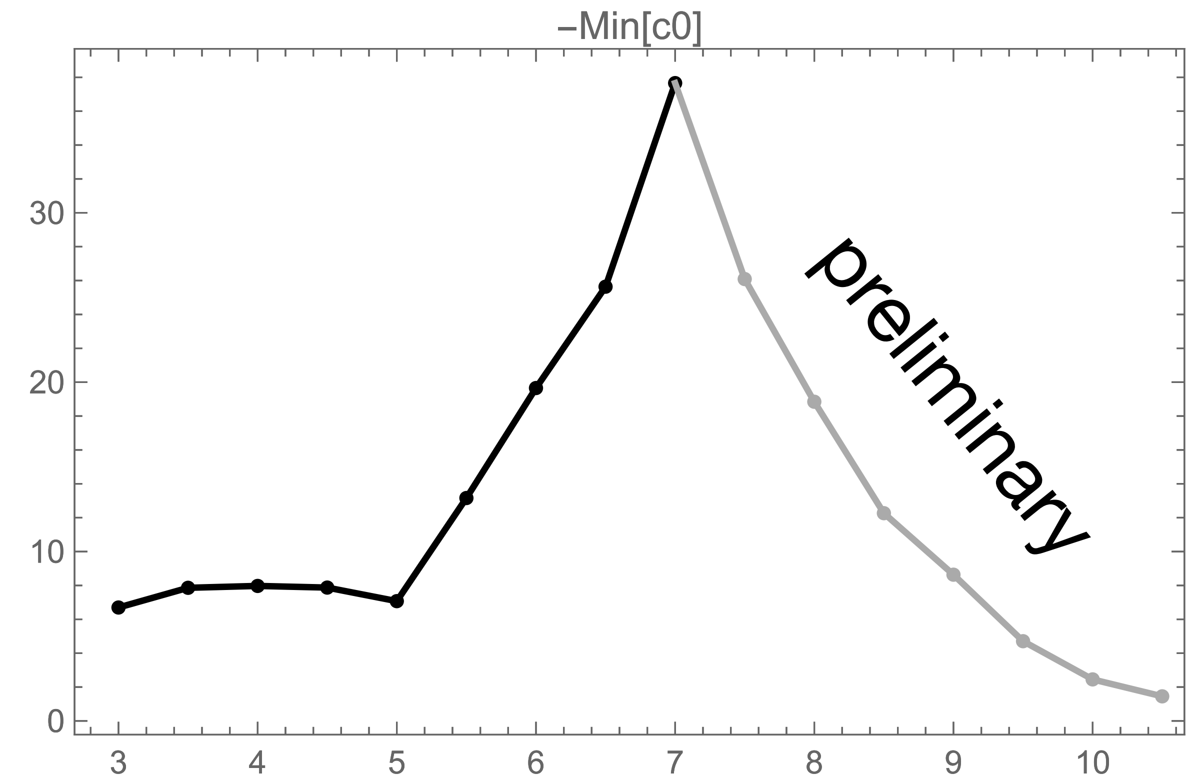

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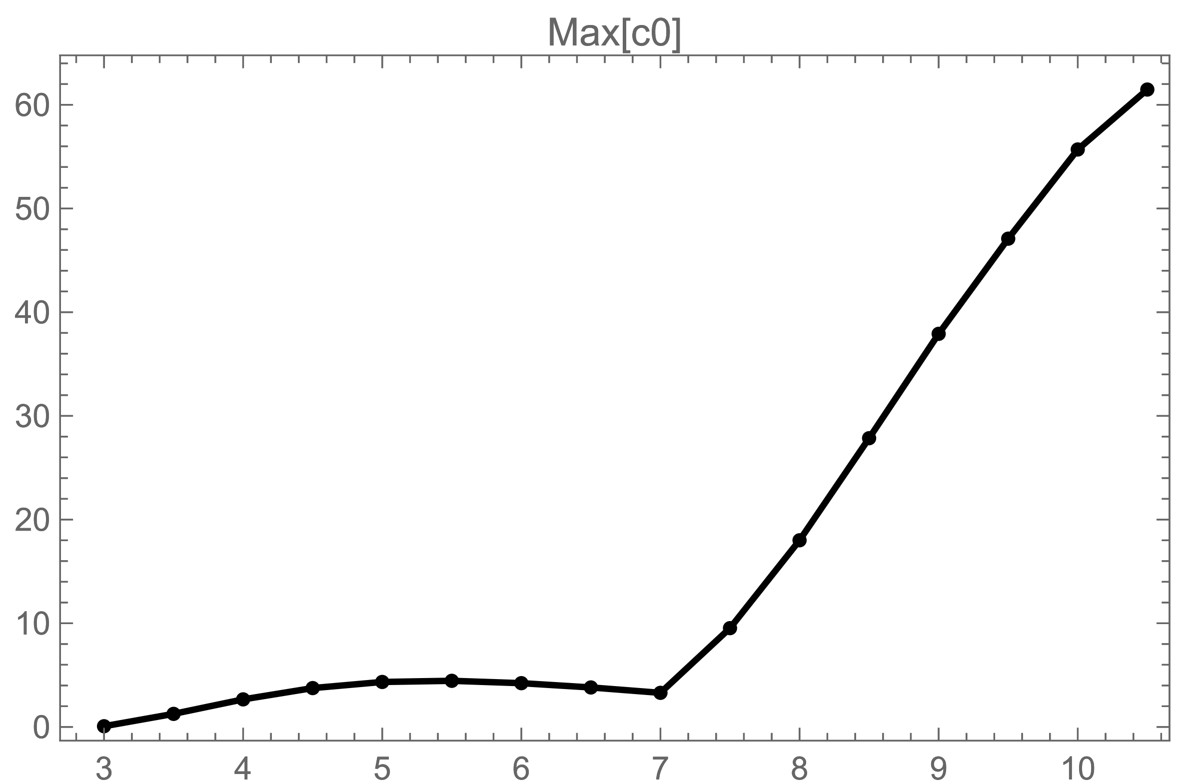
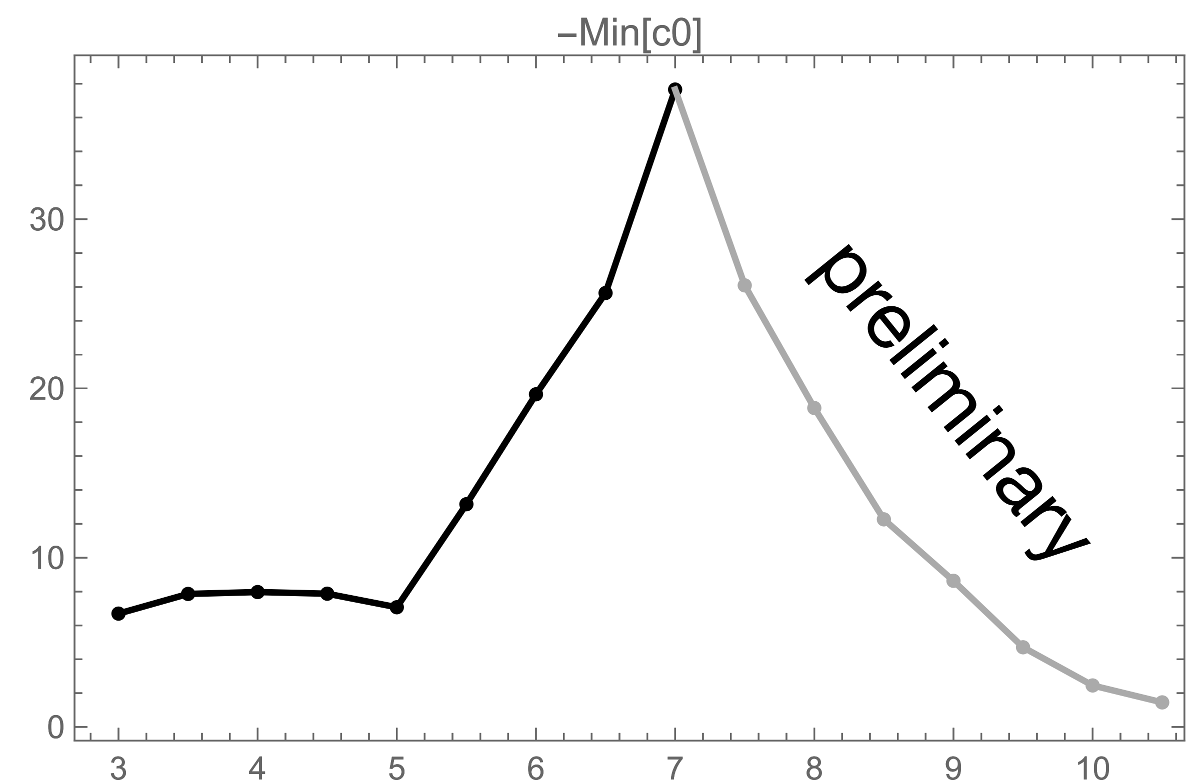
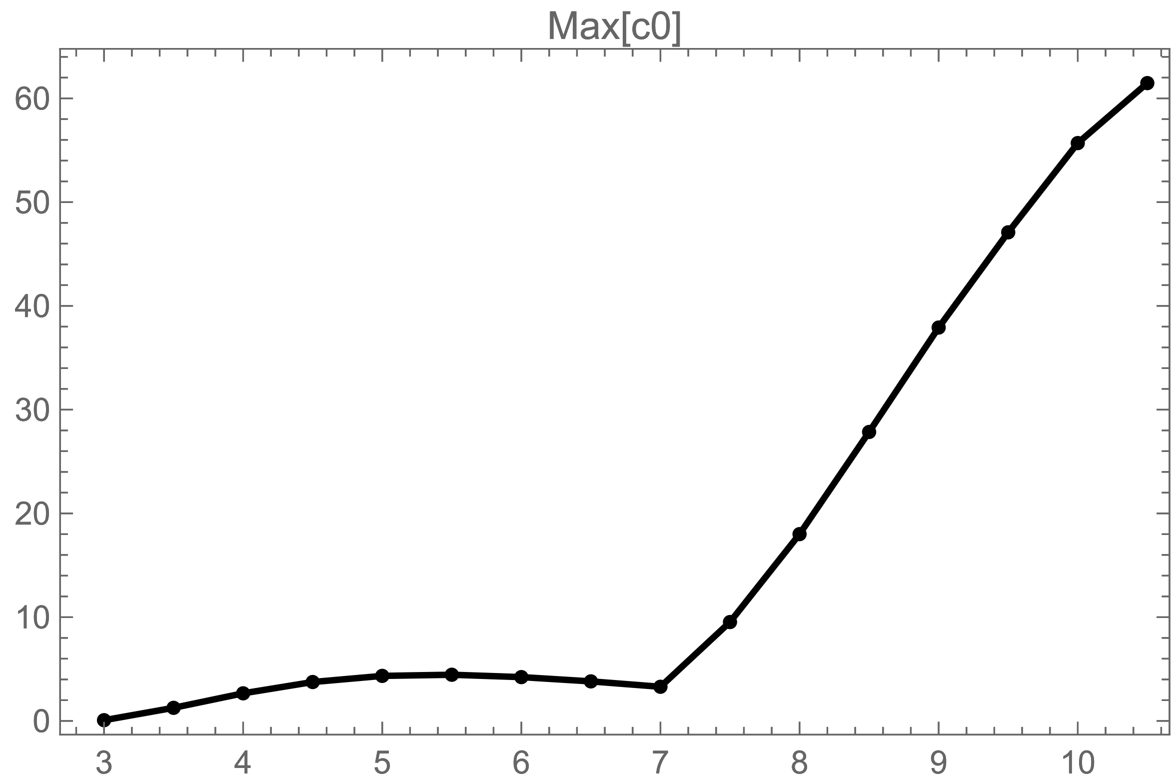


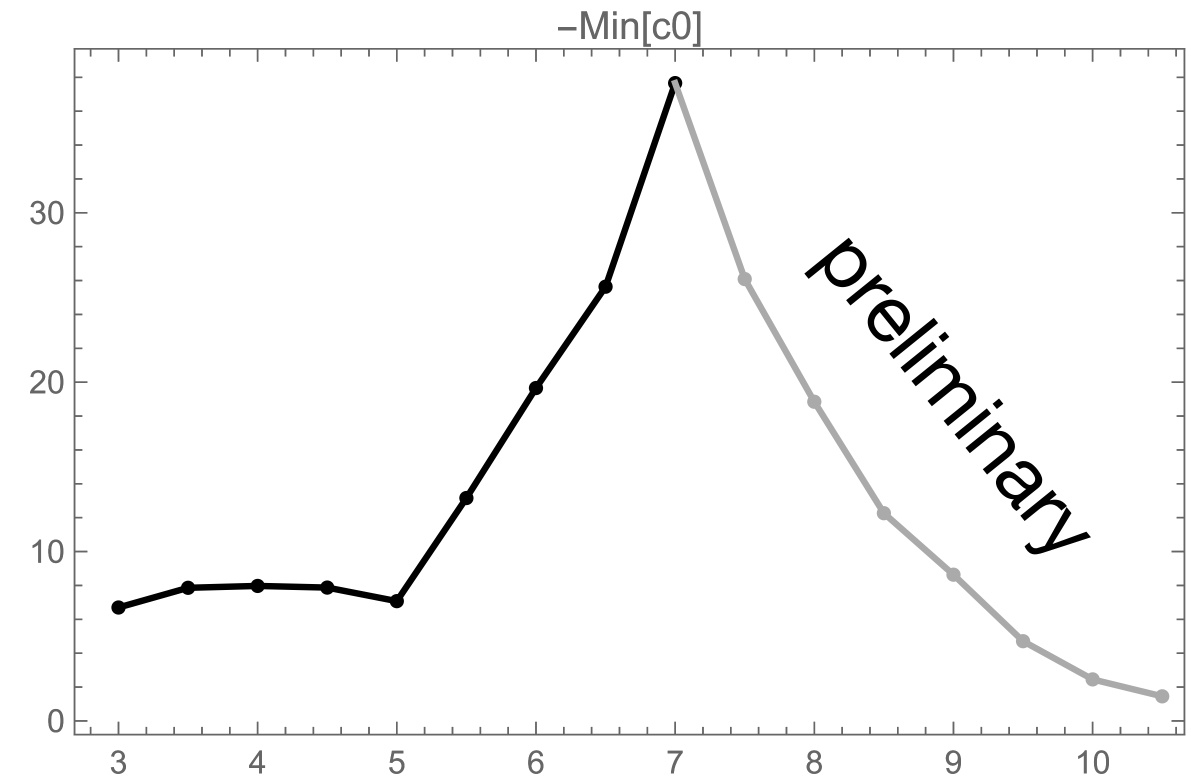


Sharp kinks at $d = 5$ and $d = 7$

How to explain?

Extremal bounds





Sharp kinks at $d = 5$ and $d = 7$

How to explain?

No microscopic origin, but clear amplitude structure

Organising principle for branches: threshold behaviour

- Repeat the analysis by maximizing / minimizing also c_2. Find similar kinks.

$$T(s,t) \sim \frac{1}{(s-4m^2)^n}$$

- Pattern:
  - before kink, mild threshold, even if strong is allowed by unitarity $\quad n \leq 1$
  - After kink: strong threshold $\quad n > 1$

e.g. d=6: $$f_0(s) = \frac{c}{(s-4m^2)^{3/2}} + \frac{\#}{s-4m^2} + \frac{\#}{\sqrt{s-4m^2}} + cst + \#\sqrt{s-4m^2} + \dots$$

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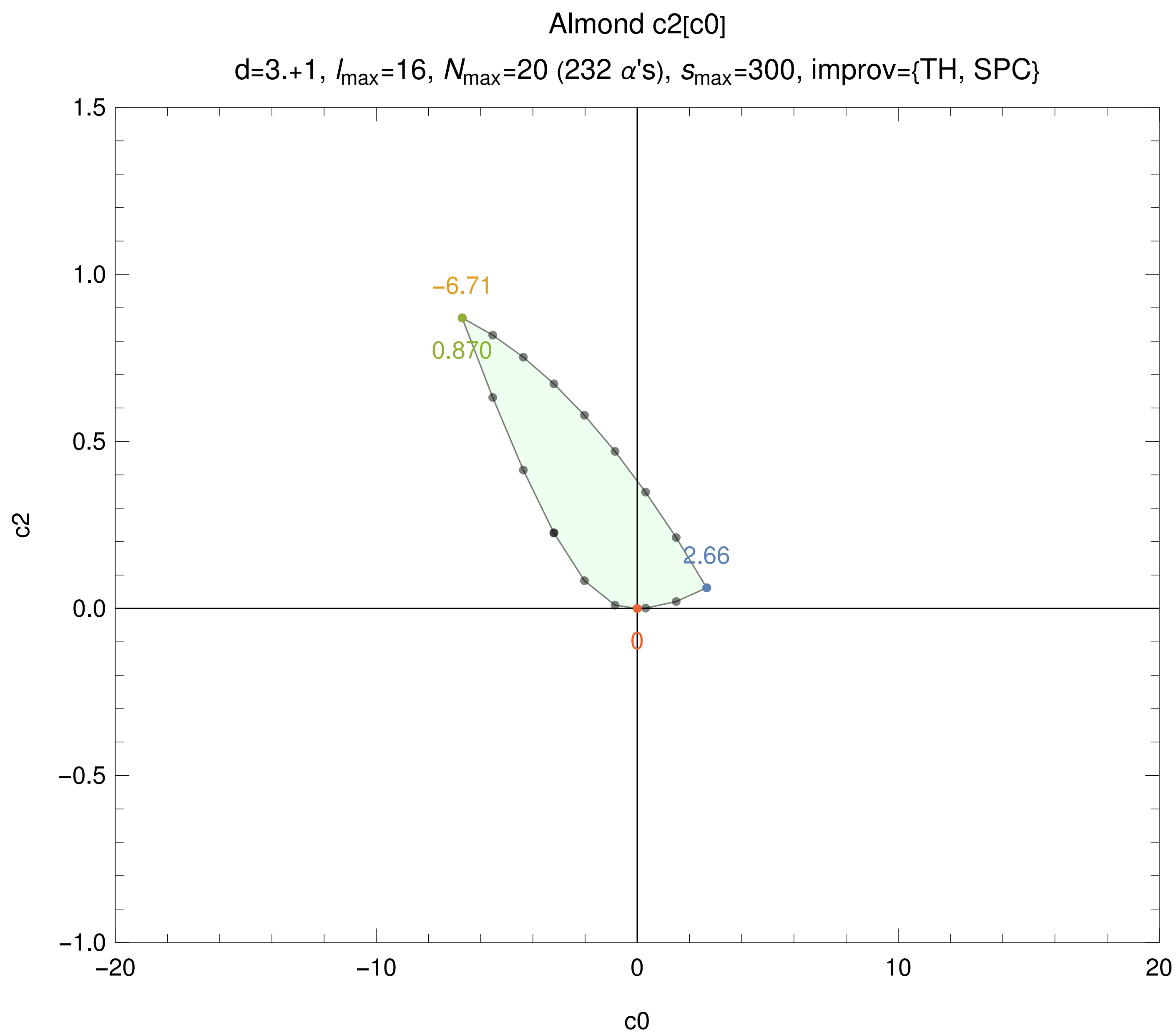
What is the meaning of this? Sign of stringiness? Quantum non-relativistic effects?

Extremal spaces
d=4

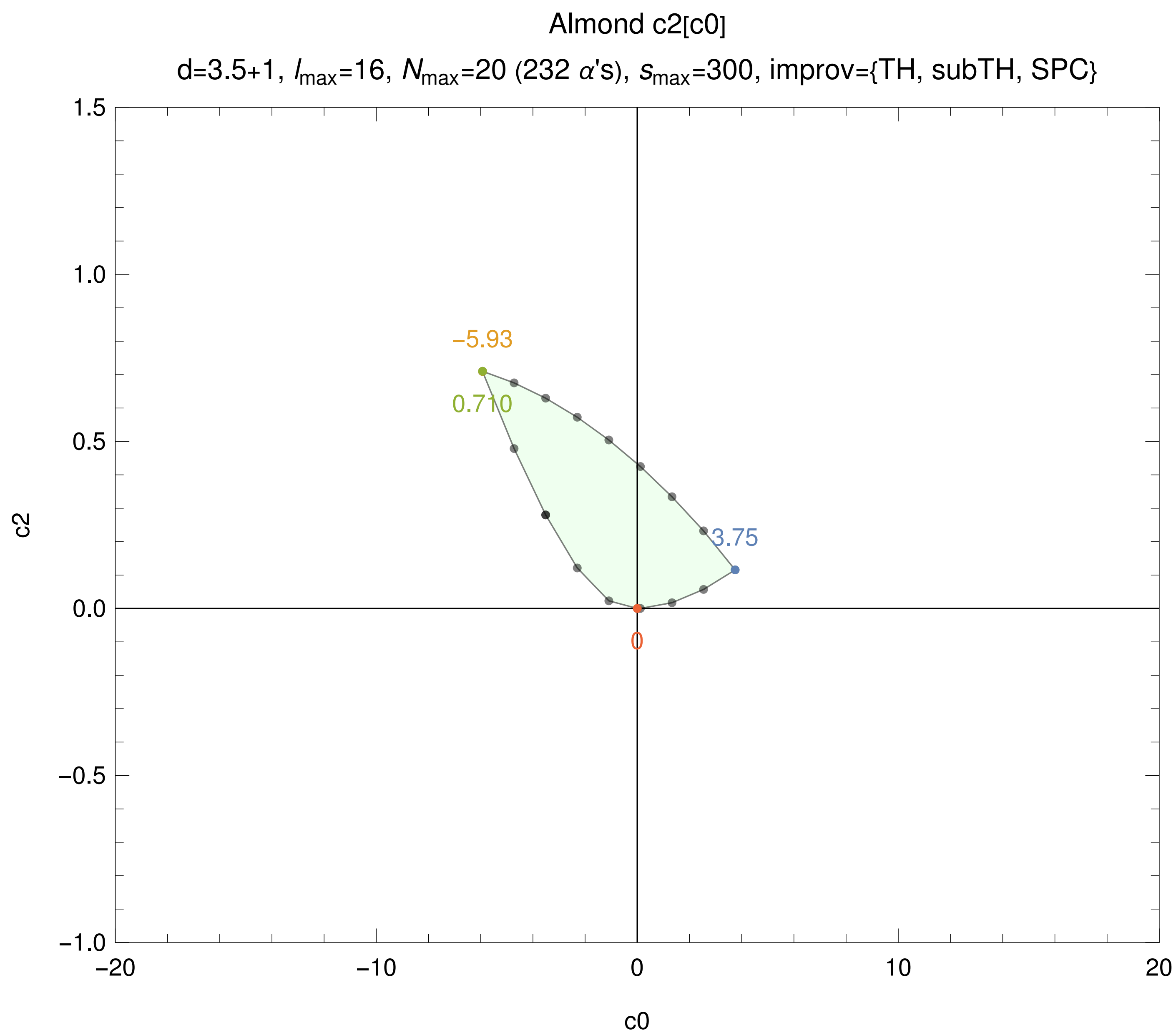
[arXiv:2210.01502] JHEP **05** (2023) 001
Bridging Positivity and S-matrix Bootstrap Bounds
J. Elias Miro, A. Guerrieri, M. A. Gumus



[arXiv:2506.04313]
Cross-Section Bootstrap: Unveiling the Froissart Amplitude
M. Correia, A. Georgoudis, A. L. Guerrieri

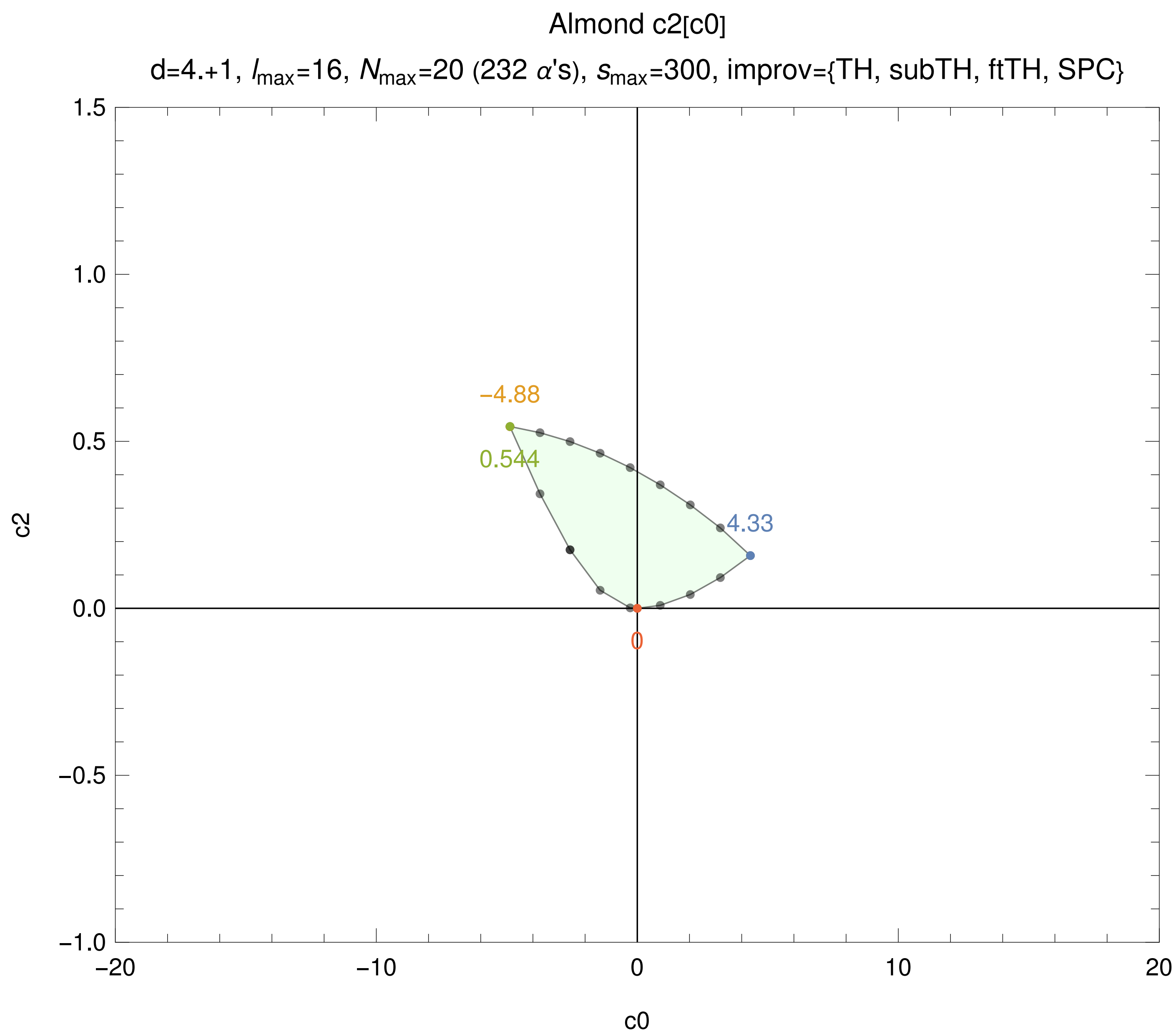
Extremal spaces

d=4.5



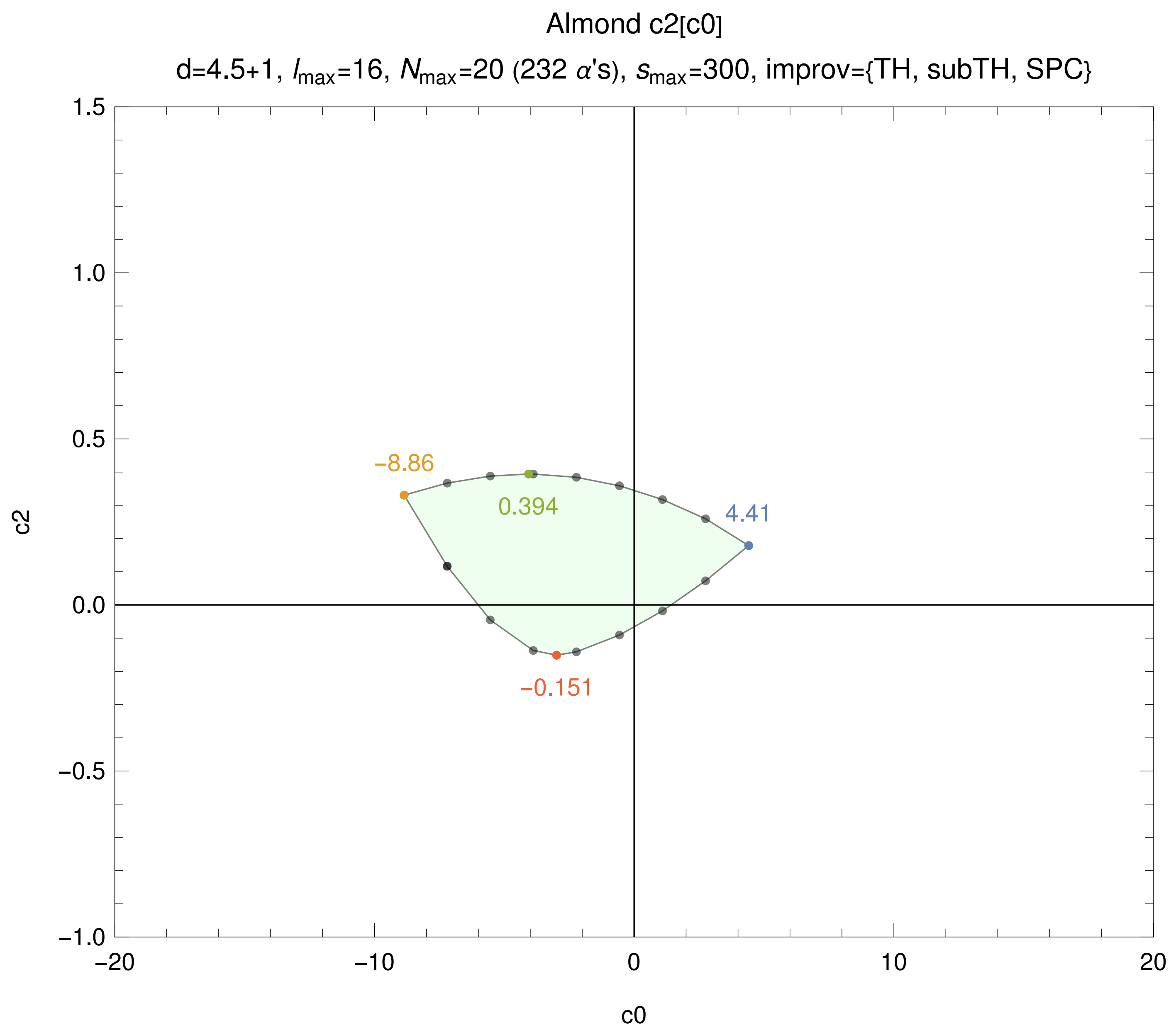
Extremal spaces

d=5



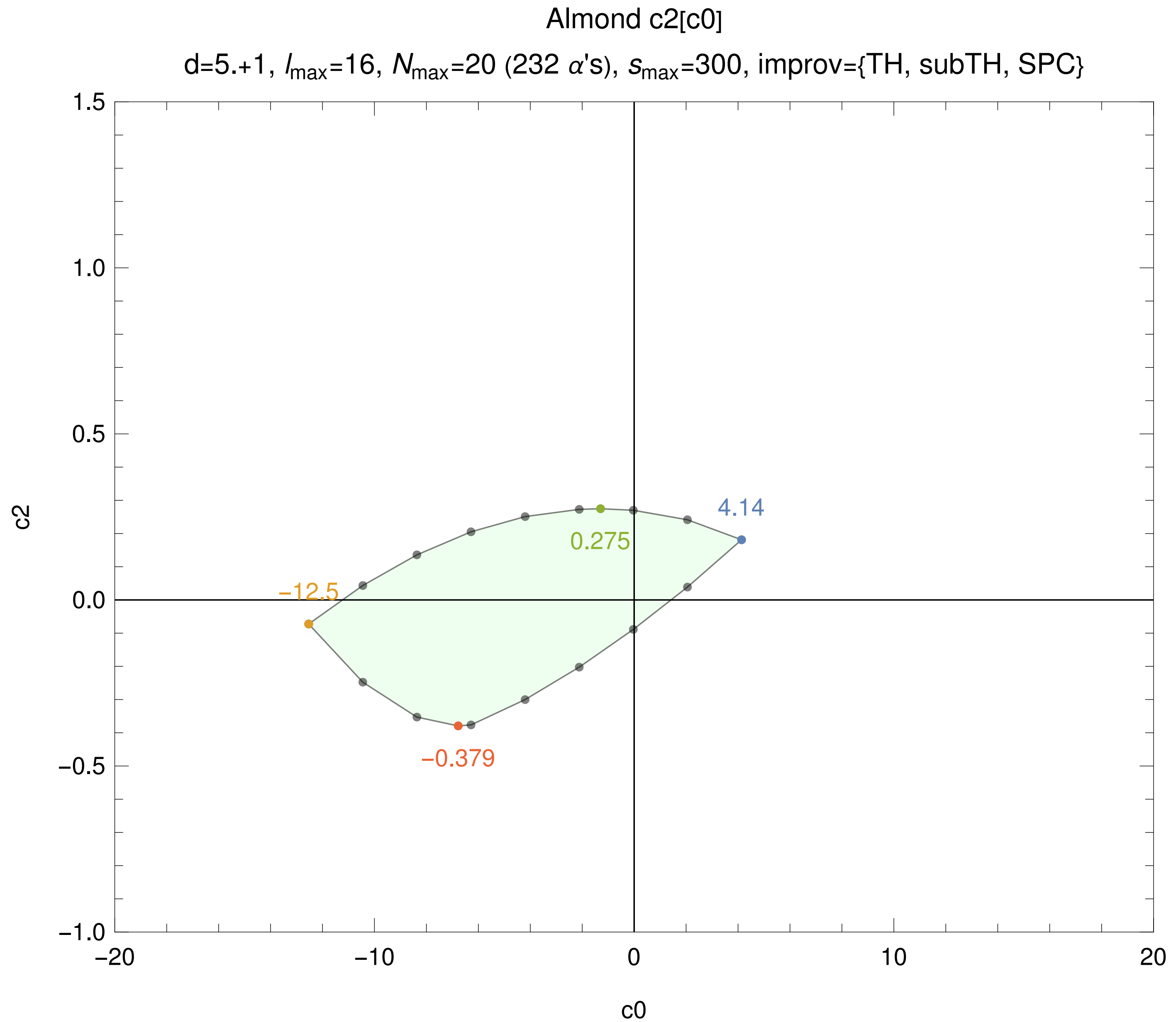
Extremal spaces

d=5.5



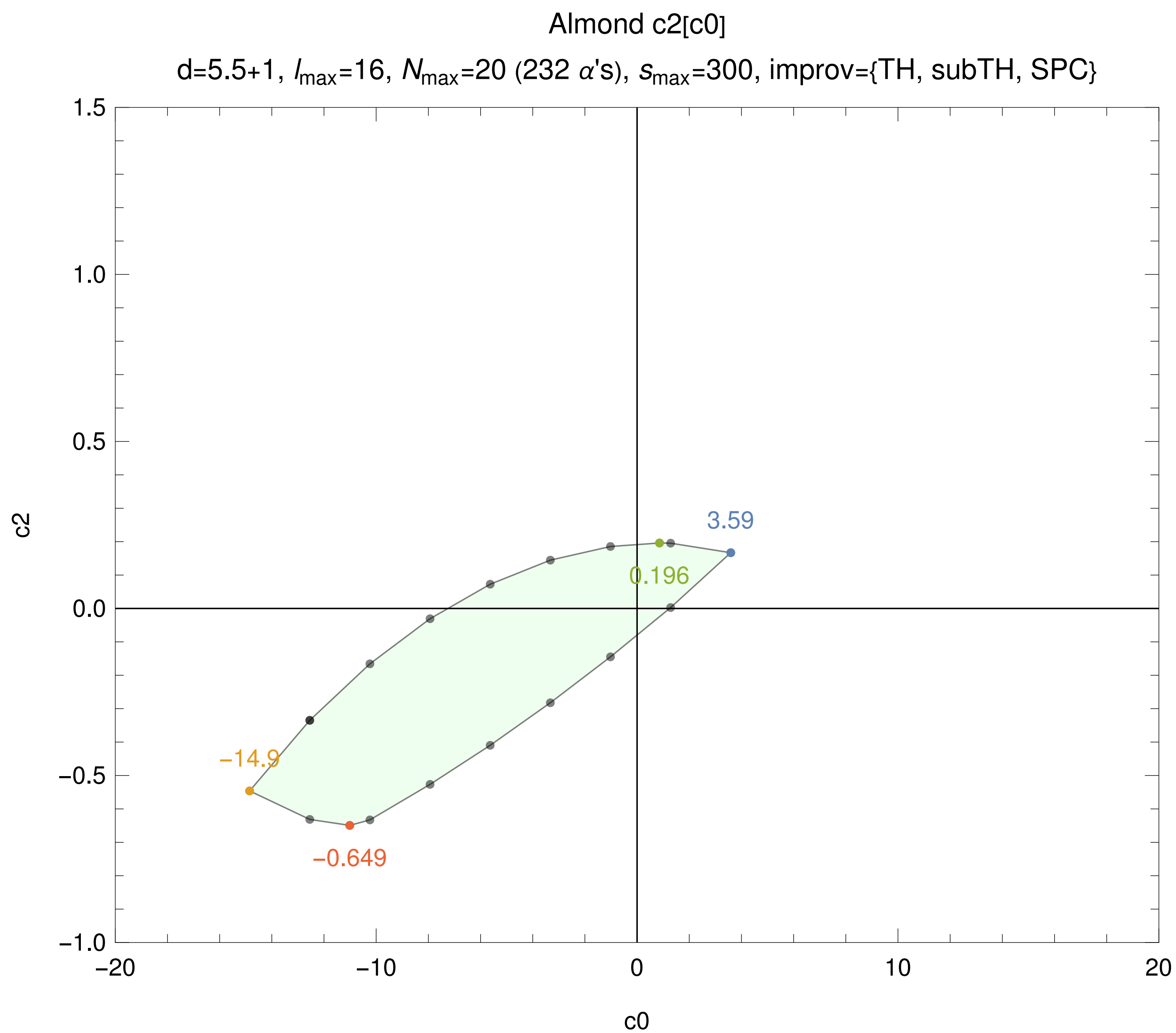
Extremal spaces

d=6



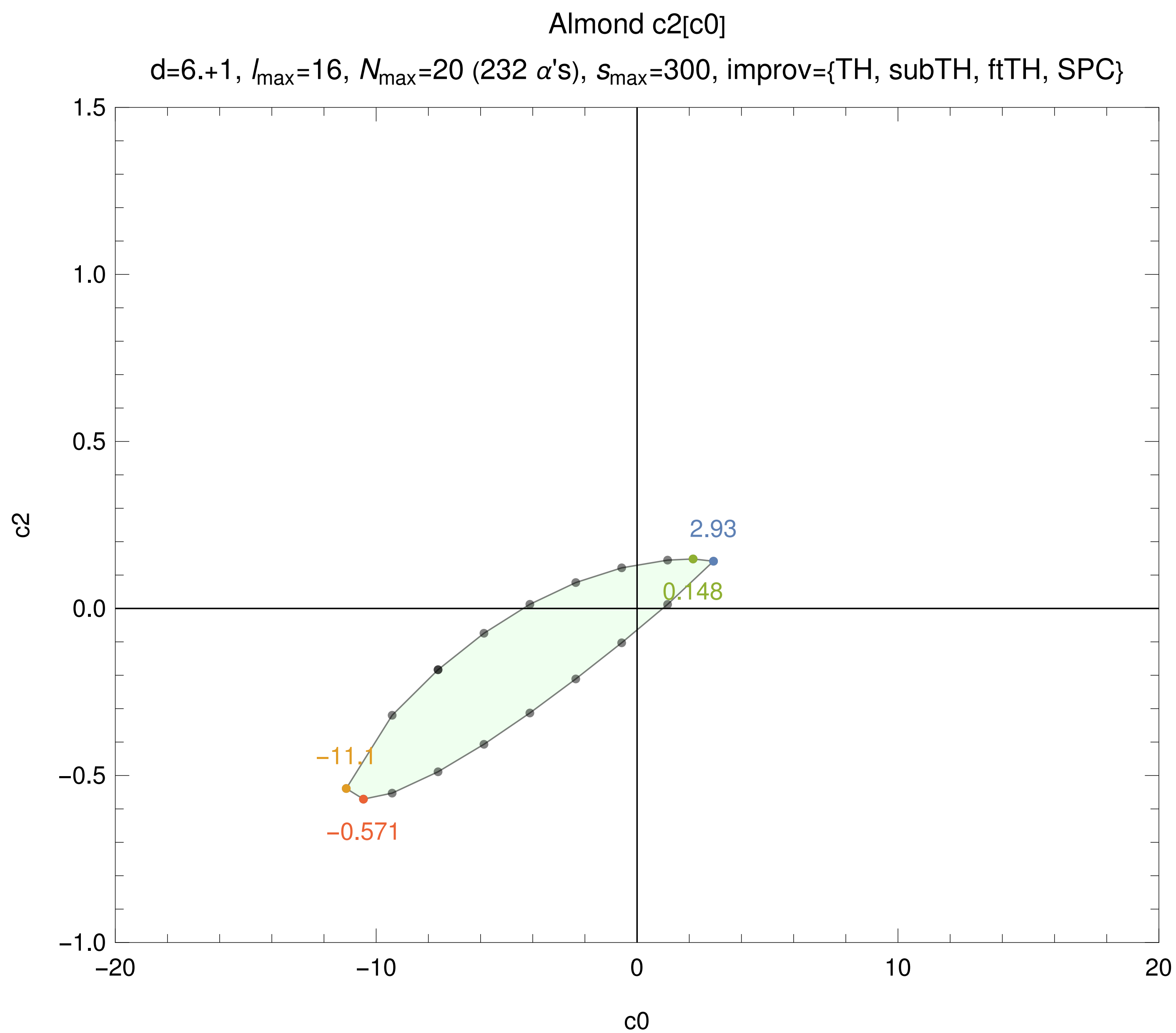
Extremal spaces

d=6.5



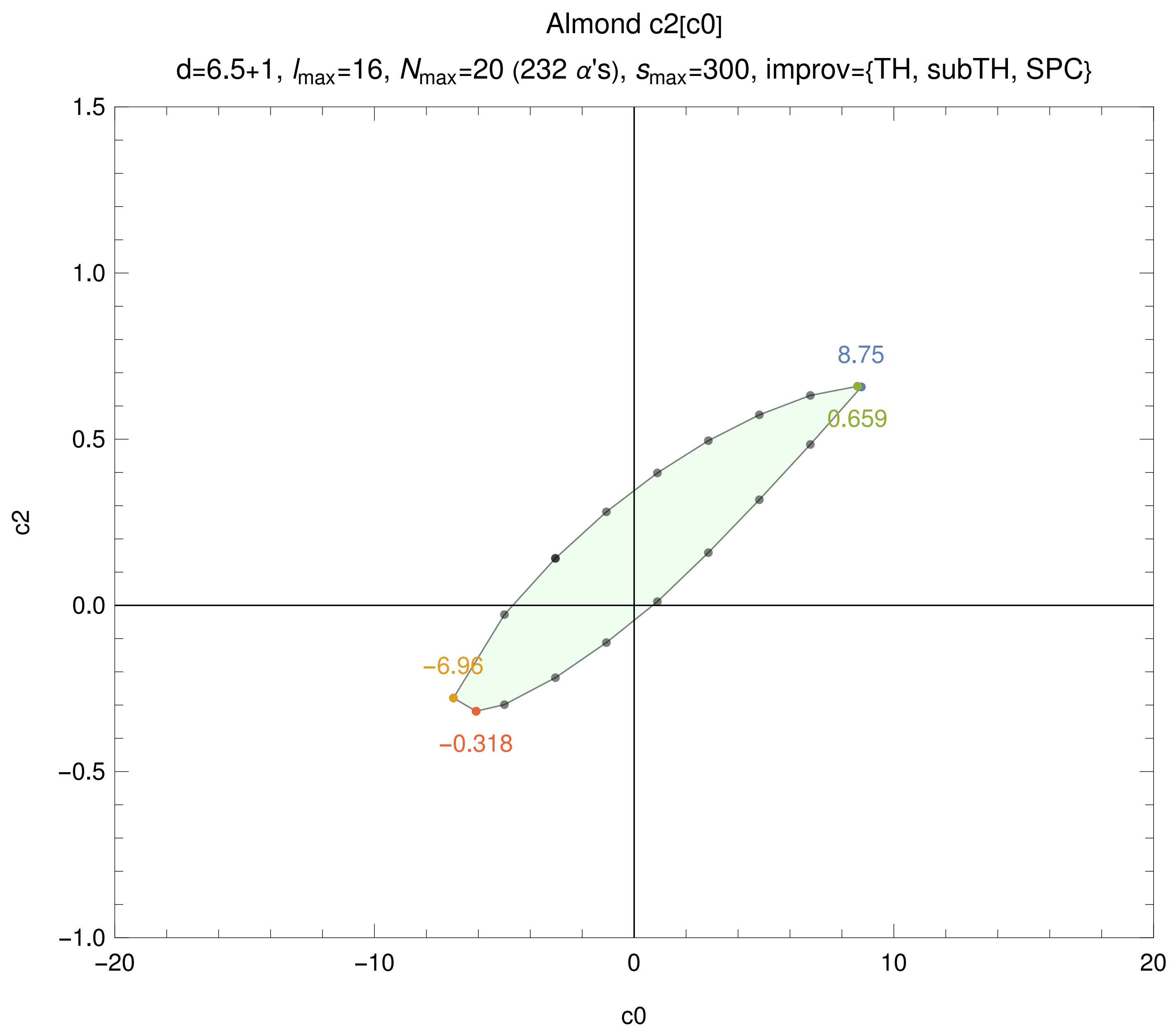
Extremal spaces

d=7



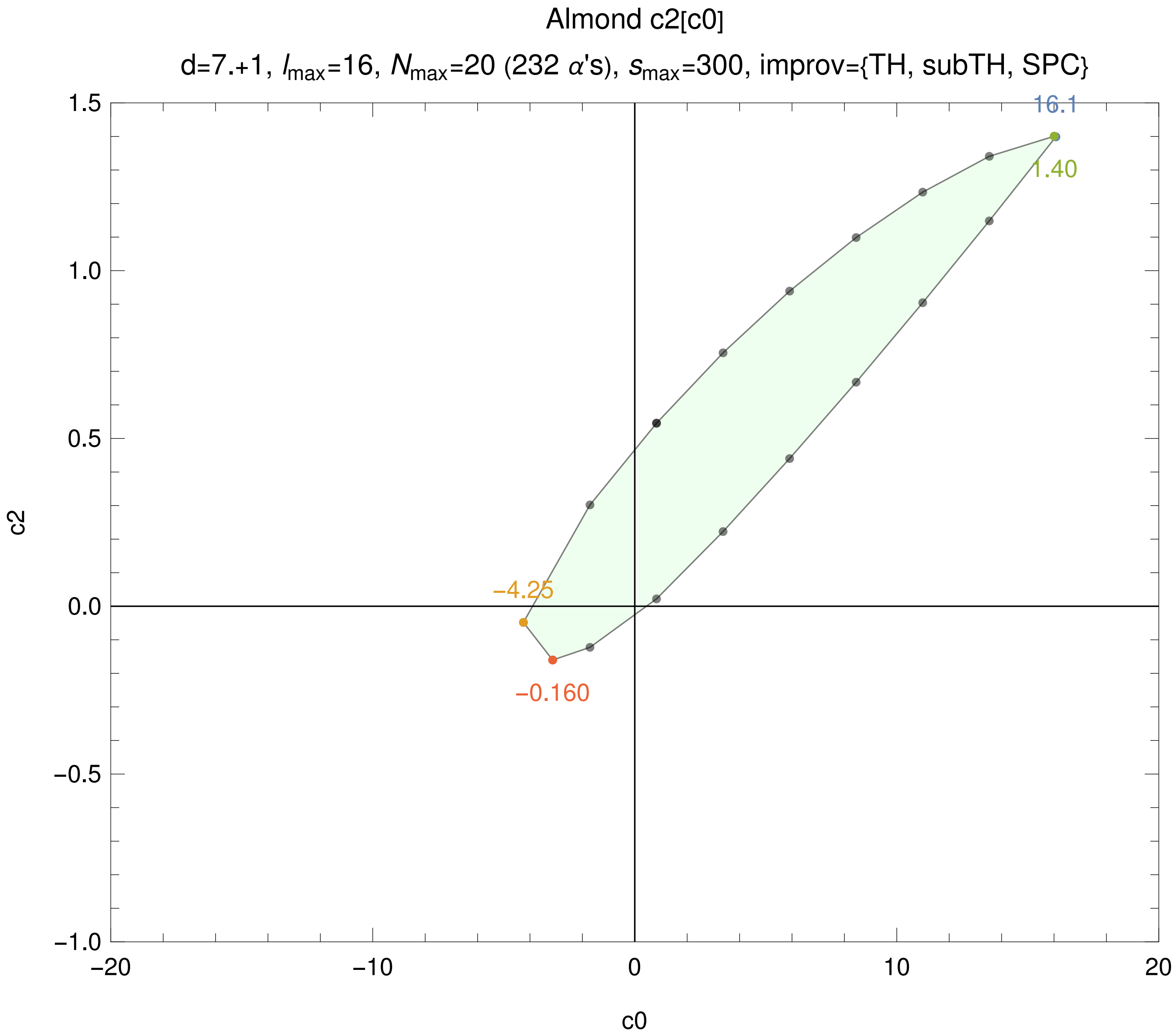
Extremal spaces

d=7.5



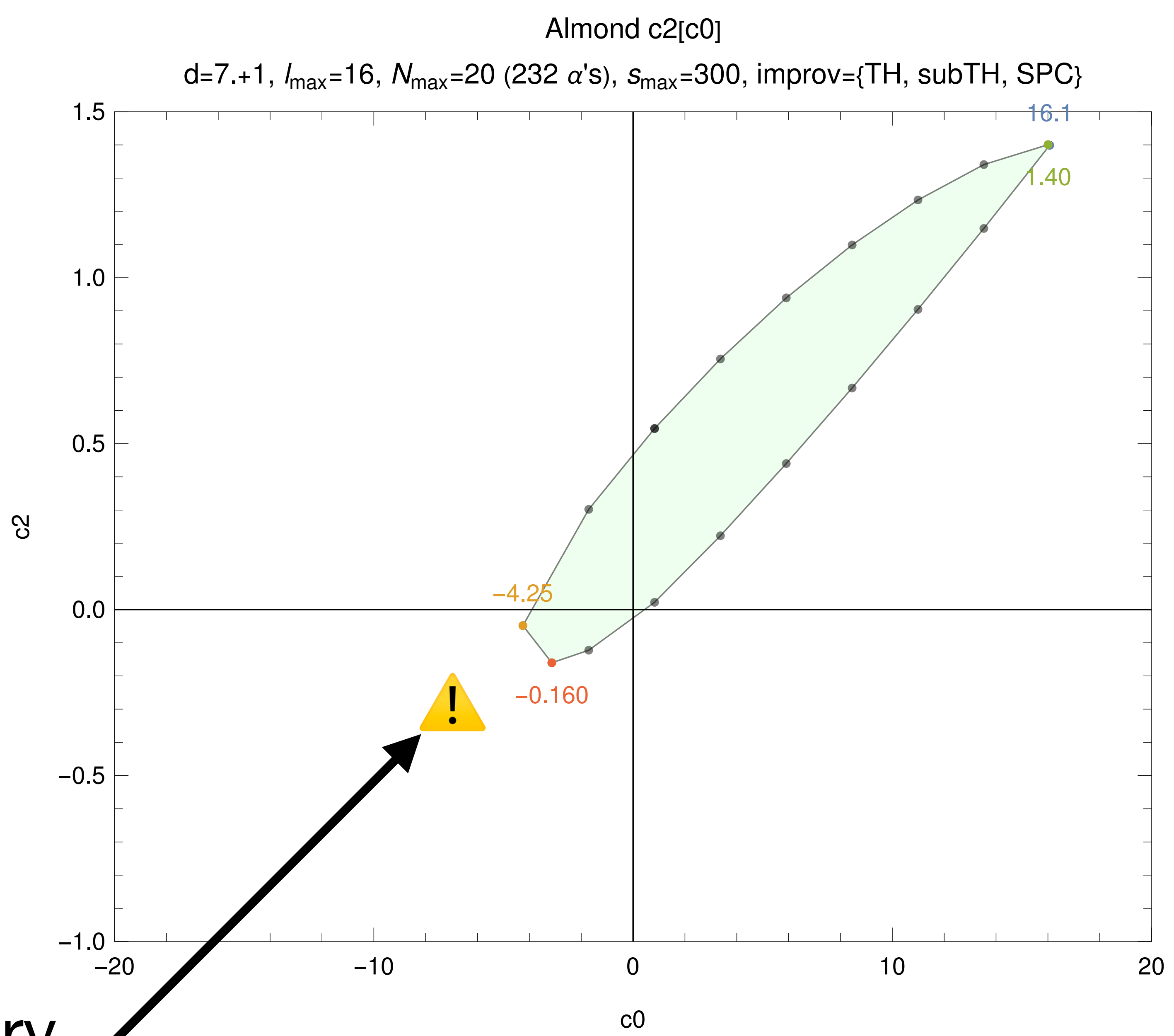
Extremal spaces

d=8



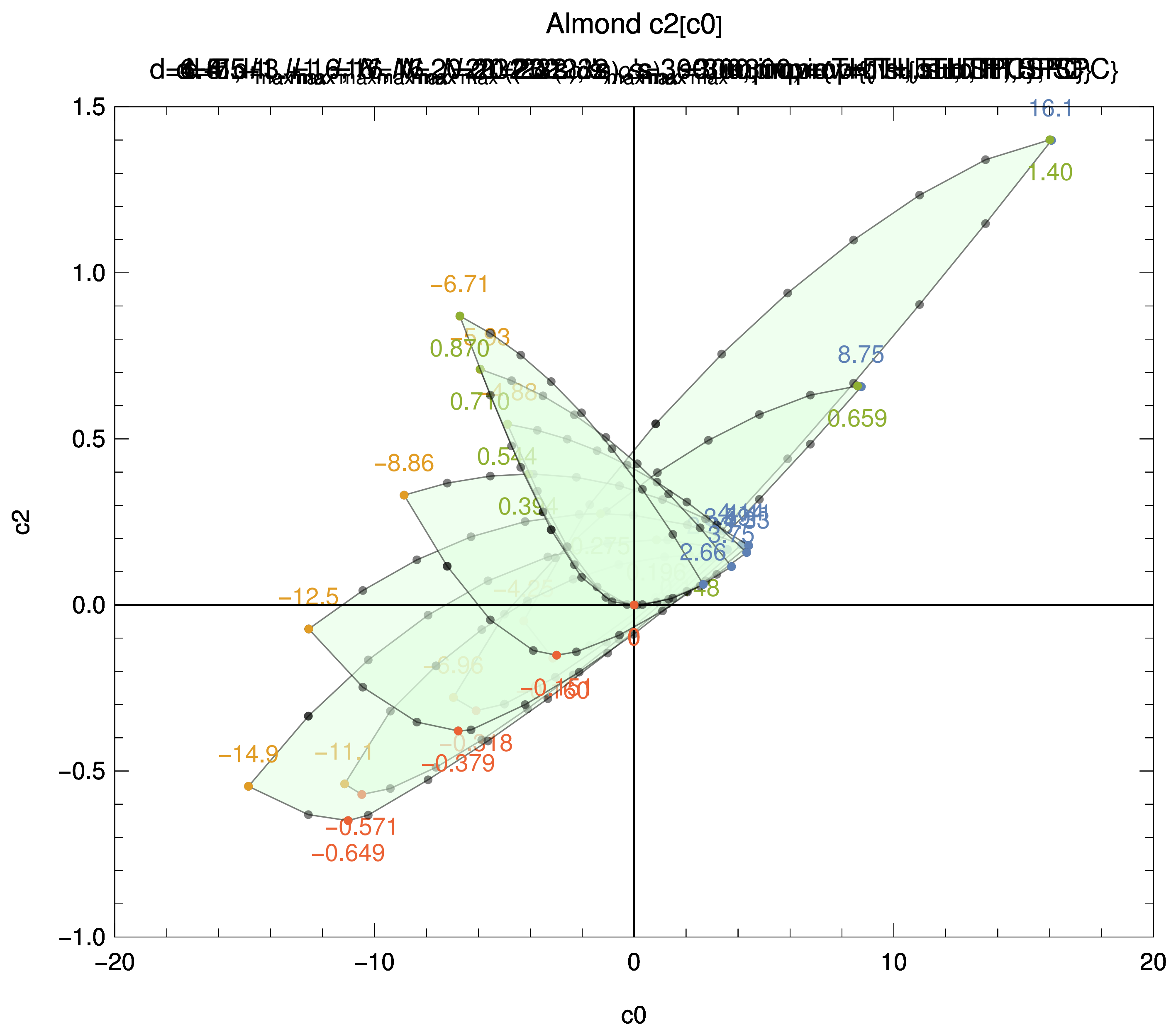
Extremal spaces

d=8



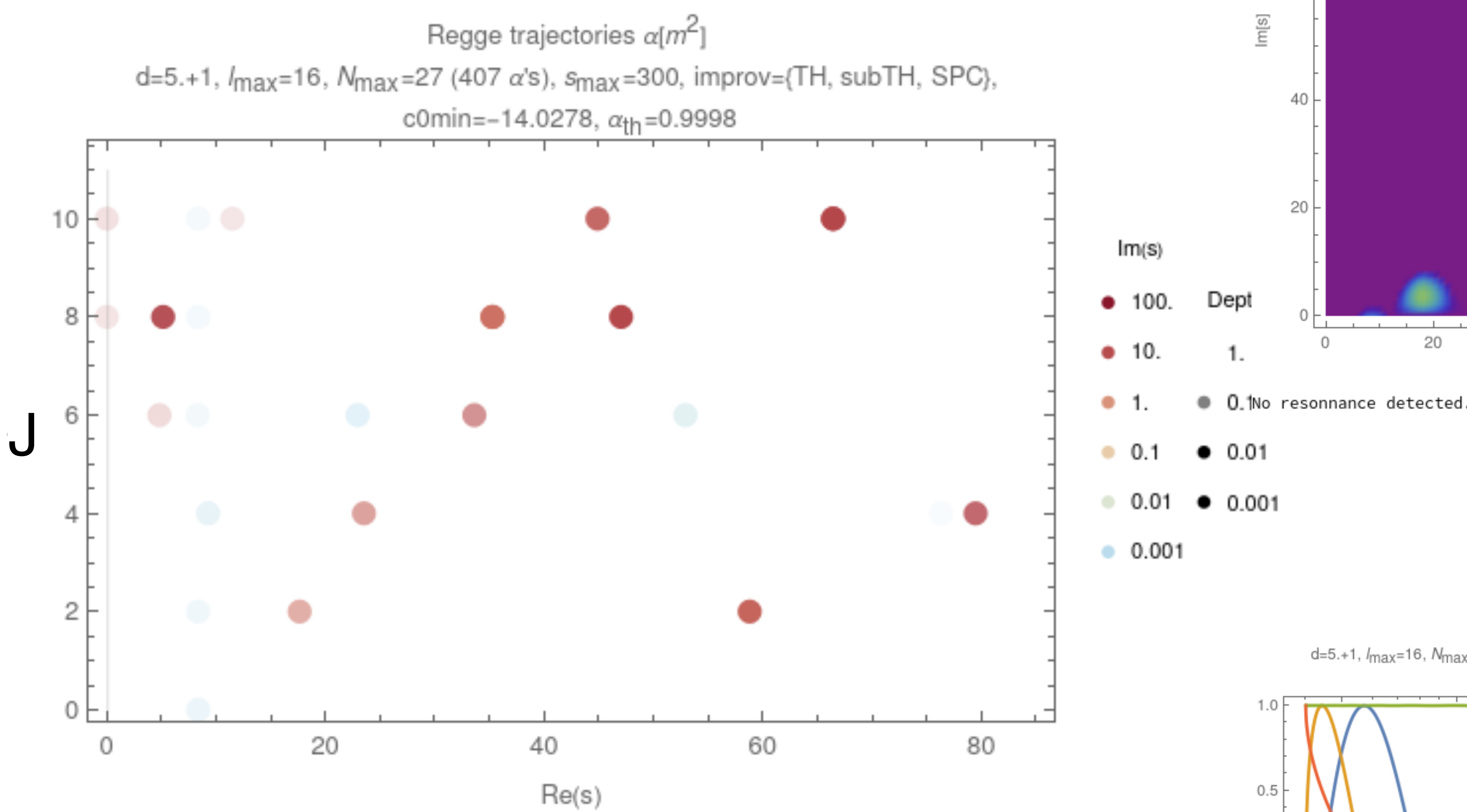
⚠️ preliminary region of slide 34

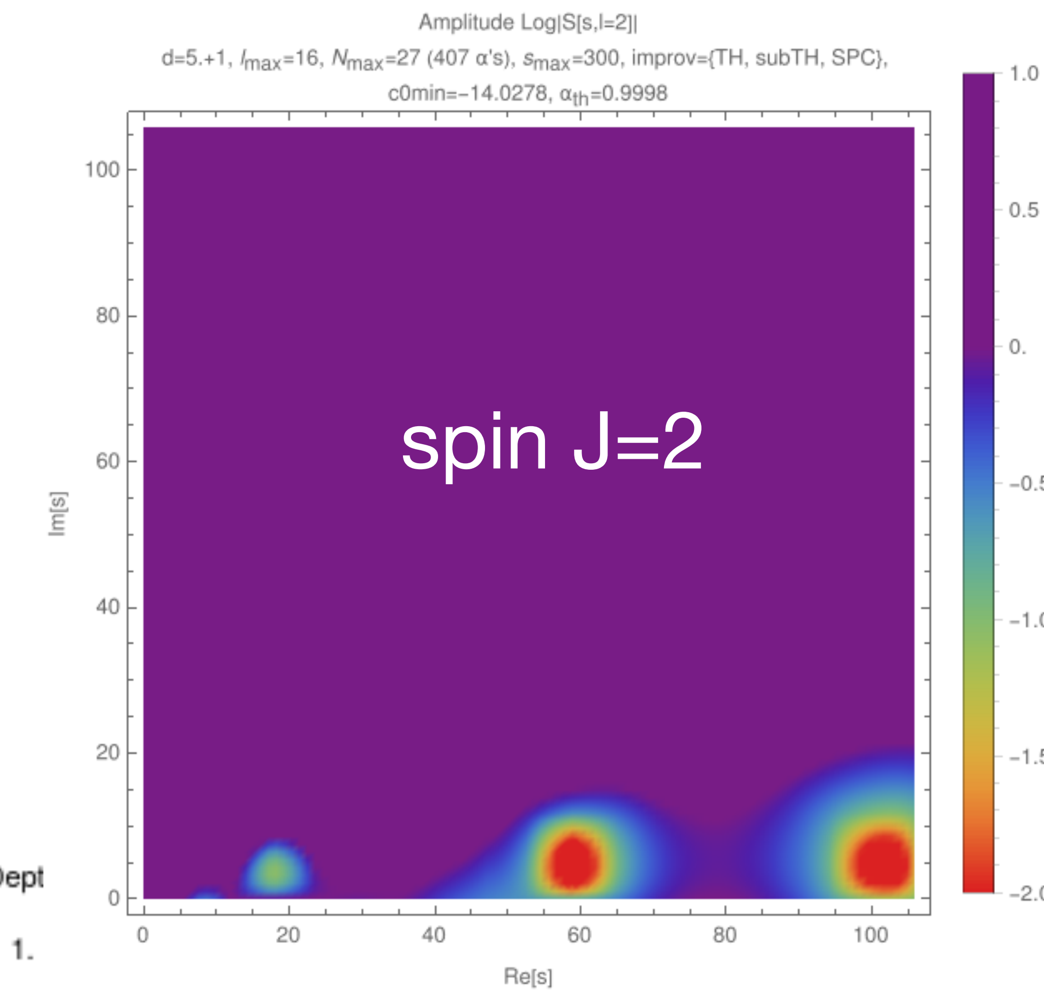
Extremal spaces



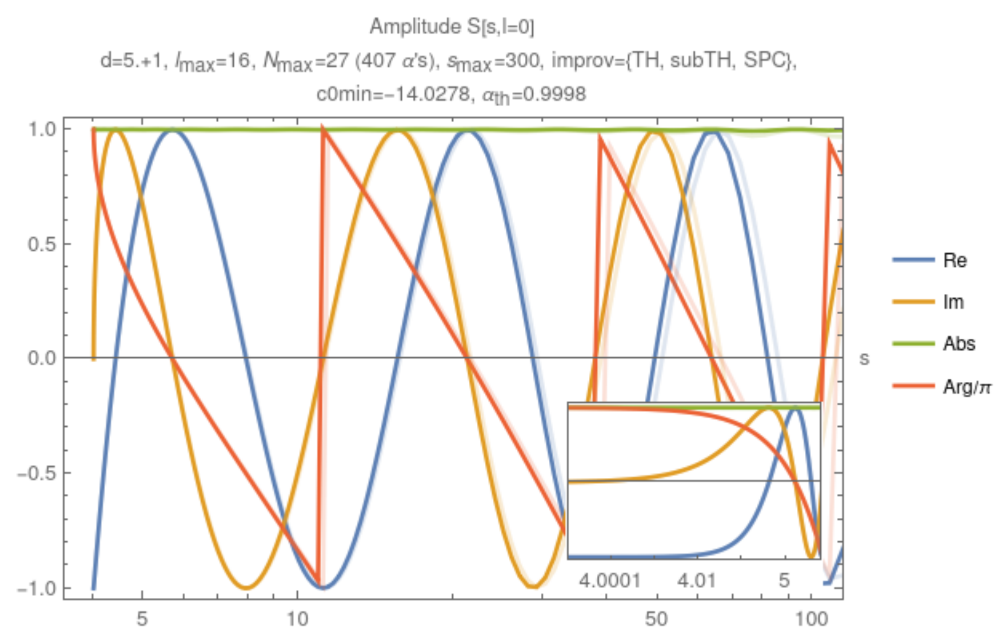
Amplitudes

d=6 min



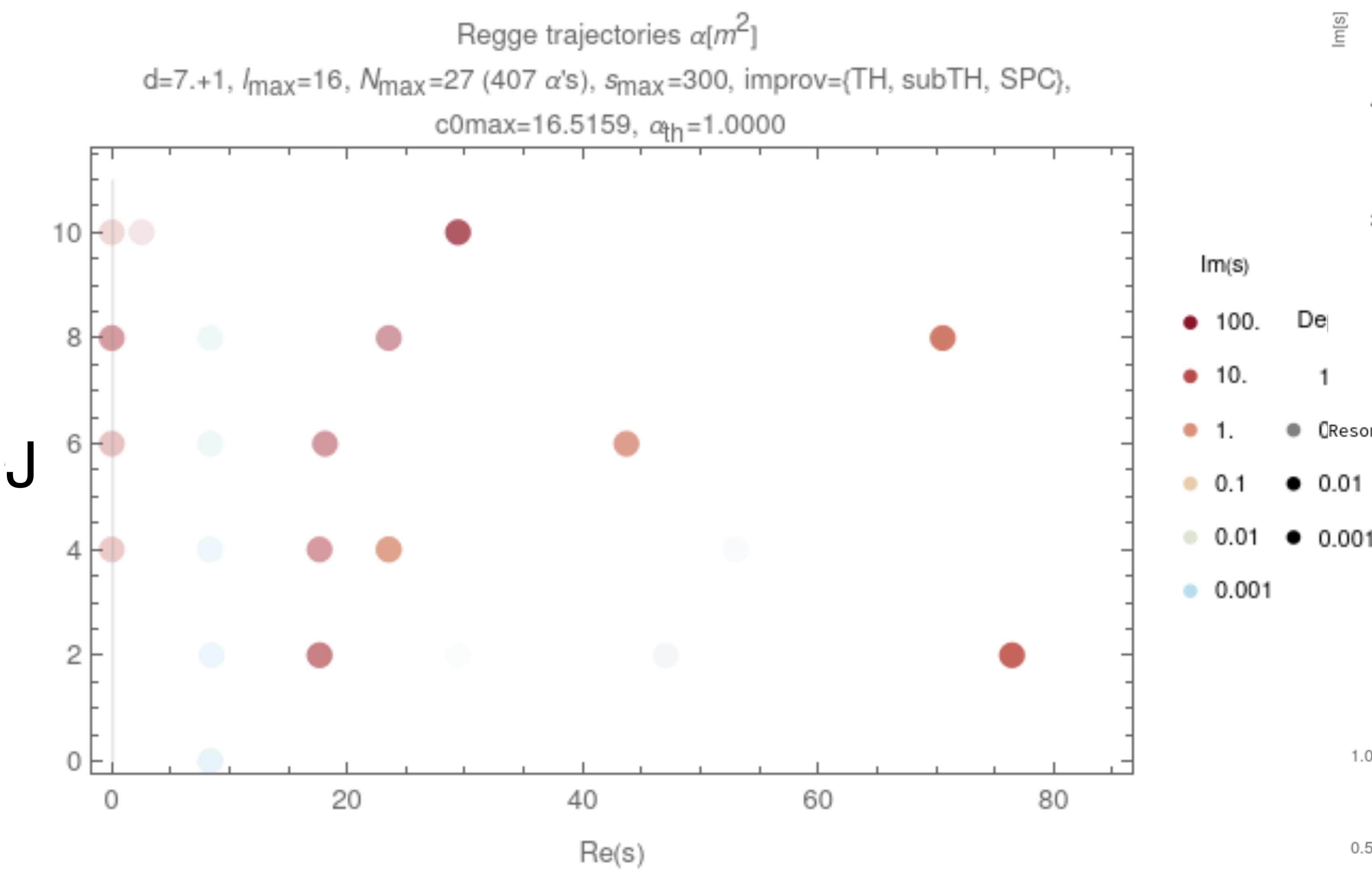


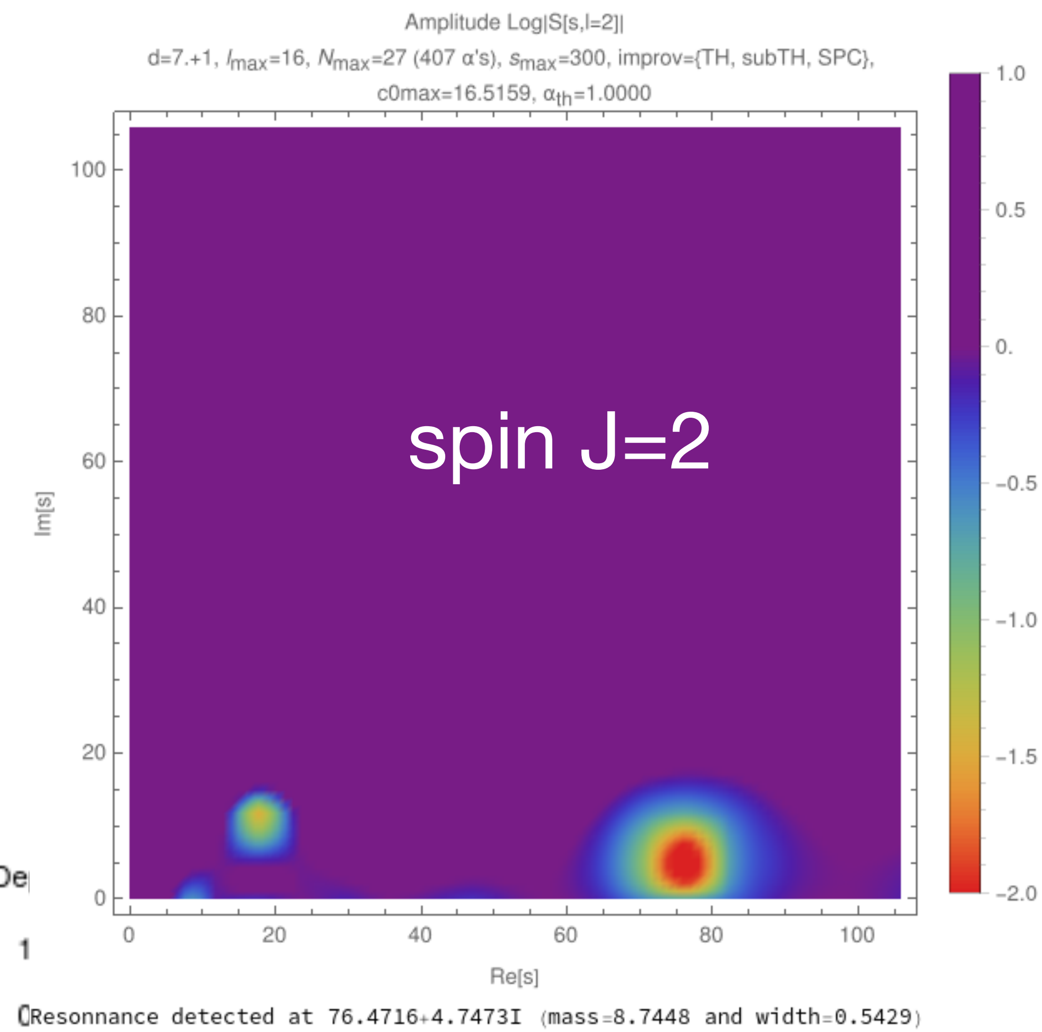
spin J=0



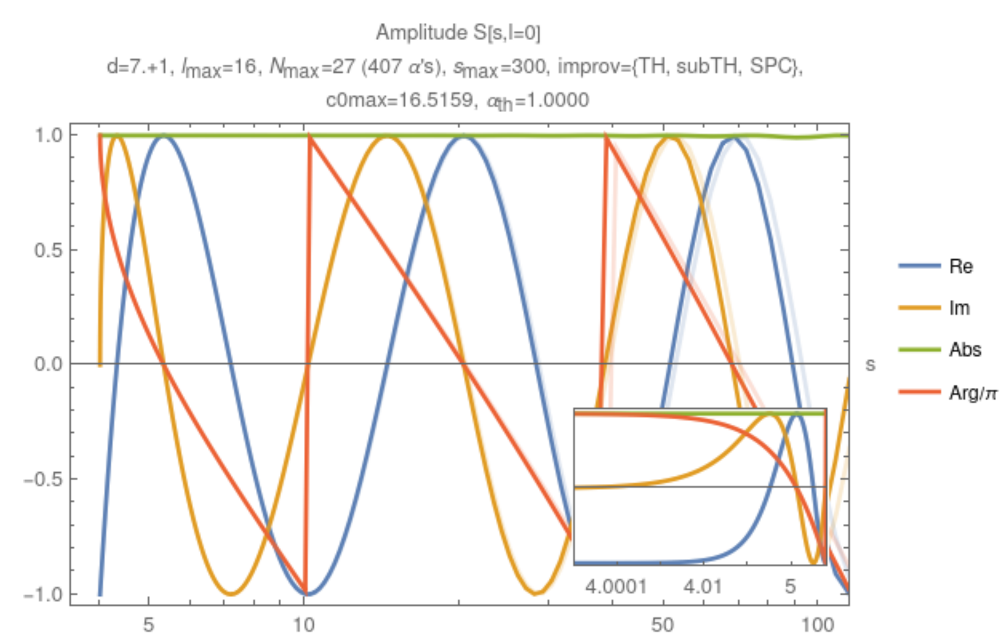
Amplitudes

d=8 max



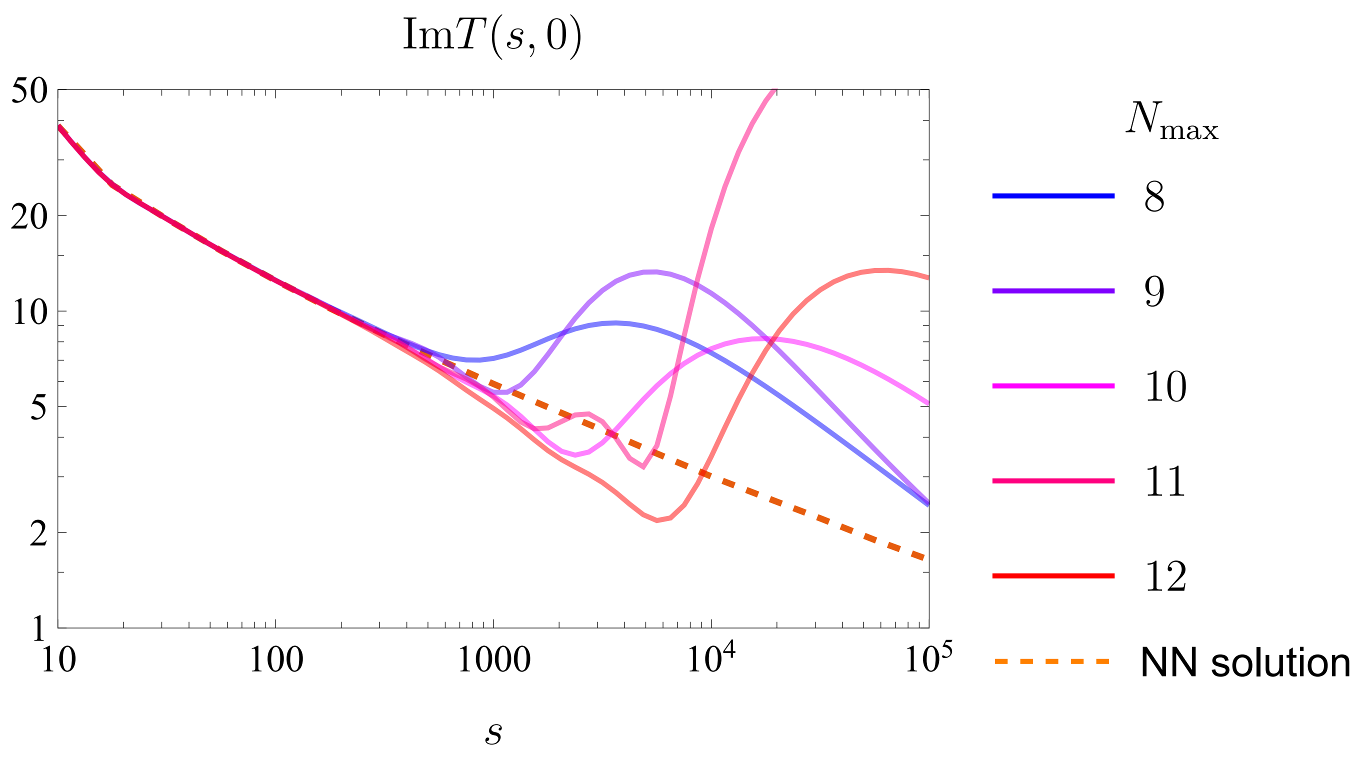


spin J=0



UV completion? Particle production?

- Subject of on-going work in progress using a completely different approach w/ S. Zhiboedov, +Gumus, Leflot
- Solve directly all spins $J \in \mathbb{C}, \mathrm{Re}(J) > 0$
- Generates realistic particle production
- UV complete amplitudes
- Numerical convergence more tedious



Conclusion

- Higher dimensional scattering is very rich.
- Clear, smooth structure in d with Regge/Froissart-like scattering and a new, not-yet-characterised type of scattering / UV completion in higher dimension.
- Microscopic realisation of IR gapped theories in d=5,6 ?
- Theories at the kink?
- Local QFT-like theory in $d = 7$?
- A lot of exciting possibilities to probe string universality and non-perturbative scattering in the clean arena of massive, gapped scalar scattering.

extras

Microscopic picture?

- What is non-perturbative scattering?
- In $d = 4$: Yang-Mills g_{YM} is dimensionless : confinement in the IR, mass gap.
  Spectrum of resonances (glueballs), measured on lattice.

[arXiv:2312.00127] Phys.Rev.D **112** (2025) 094023
Constraining Glueball Couplings
A. L. Guerrieri, A. Hebbar, B. C. van Rees

Microscopic picture?

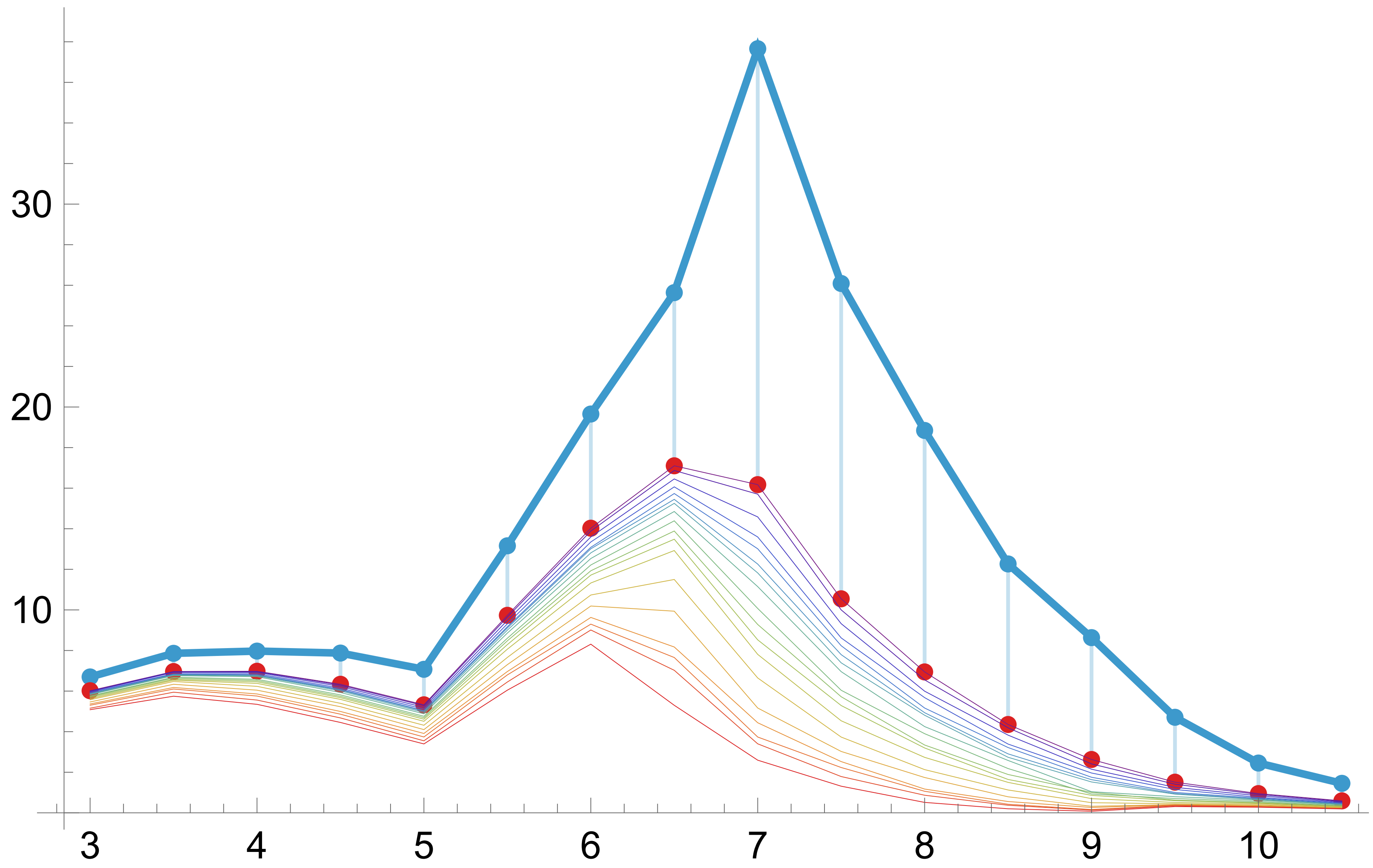
- What is non-perturbative scattering?
- In $d=4$: Yang-Mills g_{YM} is dimensionless : confinement in the IR, mass gap.
  Spectrum of resonances (glueballs), measured on lattice.
- In $d=5{,}6$, non-gravitational UV-complete QFTs exist: the superconformal field theories (5-brane worldvolumes)
- In $d \geq 7$, no non-gravitational solutions are known to exist (7d Yang-Mills discussed)

Itzhaki, Maldacena, Sonnenschein, Yankielowicz '98
and many many more

- "string universality"?

The Mount Everest of “maybe do not believe what you see”

This is not a kink (or is it?)



Threshold terms

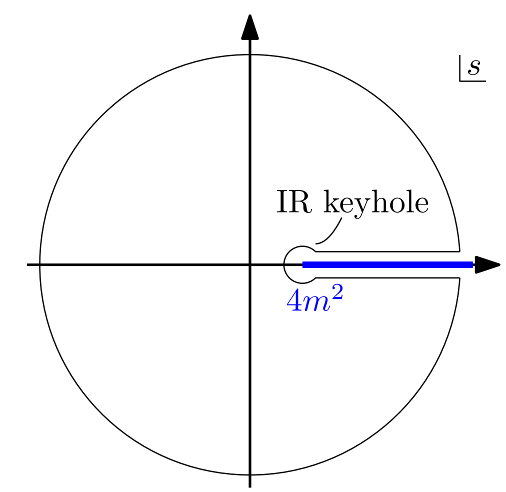
$$
\begin{aligned}
d = 3:&\quad \frac{\beta_0}{\log_s} + \frac{\beta_0'}{\log_s^2} \\
d = 3.5:&\quad \frac{\beta_0}{\sigma_s^{1/4}} \\
d = 4:&\quad \frac{\beta_0}{\sigma_s^{1/2}} \\
d = 4.5:&\quad \frac{\beta_0}{\sigma_s^{3/4}} + \frac{\beta_1}{\sigma_s^{1/2}} + \frac{\beta_2}{\sigma_s^{1/4}} \\
d = 5:&\quad \frac{\beta_0}{\sigma_s \log_s} + \frac{\beta_1}{\log_s} + \frac{\beta_0'}{\sigma_s \log_s^2} + \frac{\beta_1'}{\log_s^2} \\
d = 5.5:&\quad \frac{\beta_0}{\sigma_s^{5/4}} + \frac{\beta_1}{\sigma_s^{1/2}} \\
d = 6:&\quad \frac{\beta_0}{\sigma_s^{3/2}} + \frac{\beta_1}{\sigma_s} + \frac{\beta_2}{\sigma_s^{1/2}} \\
d = 6.5:&\quad \frac{\beta_0}{\sigma_s^{7/2}} + \frac{\beta_1}{\sigma_s^{3/2}} + \frac{\beta_2}{\sigma_s^{5/4}} + \frac{\beta_3}{\sigma_s} + \frac{\beta_4}{\sigma_s^{3/4}} + \frac{\beta_5}{\sigma_s^{1/2}} + \frac{\beta_6}{\sigma_s^{1/4}} \\
d = 7:&\quad \frac{\beta_0}{\sigma_s^2 \log_s} + \frac{\beta_1}{\sigma_s \log_s} + \frac{\beta_2}{\log_s} + \frac{\beta_0'}{\sigma_s^2 \log_s^2} + \frac{\beta_1'}{\sigma_s \log_s^2} + \frac{\beta_2'}{\log_s^2} \\
d = 7.5:&\quad \frac{\beta_0}{\sigma_s^{9/4}} + \frac{\beta_1}{\sigma_s^{3/2}} + \frac{\beta_2}{\sigma_s^{3/4}} \\
d = 8:&\quad \frac{\beta_0}{\sigma_s^{5/2}} + \frac{\beta_1}{\sigma_s^2} + \frac{\beta_2}{\sigma_s^{3/2}} + \frac{\beta_3}{\sigma_s} + \frac{\beta_4}{\sigma_s^{1/2}} \\
d = 8.5:&\quad \frac{\beta_0}{\sigma_s^{11/4}} + \frac{\beta_1}{\sigma_s^{5/2}} + \frac{\beta_2}{\sigma_s^{1/4}} + \frac{\beta_3}{\sigma_s^{9/4}} + \frac{\beta_4}{\sigma_s^2} + \frac{\beta_5}{\sigma_s^{7/4}} + \frac{\beta_6}{\sigma_s^{3/2}} + \frac{\beta_7}{\sigma_s^{5/4}} + \frac{\beta_8}{\sigma_s} + \frac{\beta_9}{\sigma_s^{3/4}} + \frac{\beta_{10}}{\sigma_s^{1/2}} \\
d = 9:&\quad \frac{\beta_0}{\sigma_s^3 \log_s} + \frac{\beta_1}{\sigma_s^2 \log_s} + \frac{\beta_2}{\sigma_s \log_s} + \frac{\beta_3}{\log_s} + \frac{\beta_0'}{\sigma_s^3 \log_s^2} + \frac{\beta_1'}{\sigma_s^2 \log_s^2} + \frac{\beta_2'}{\sigma_s \log_s^2} + \frac{\beta_3'}{\log_s^2} \\
d = 9.5:&\quad \frac{\beta_0}{\sigma_s^{13/4}} + \frac{\beta_1}{\sigma_s^{5/2}} + \frac{\beta_2}{\sigma_s^{7/4}} + \frac{\beta_3}{\sigma_s} + \frac{\beta_4}{\sigma_s^{1/4}} \\
d = 10:&\quad \frac{\beta_0}{\sigma_s^{7/2}} + \frac{\beta_1}{\sigma_s^3} + \frac{\beta_2}{\sigma_s^{5/2}} + \frac{\beta_3}{\sigma_s^2} + \frac{\beta_4}{\sigma_s^{3/2}} + \frac{\beta_5}{\sigma_s} + \frac{\beta_6}{\sigma_s^{1/2}} \\
d = 10.5:&\quad \frac{\beta_0}{\sigma_s^{15/4}} + \frac{\beta_1}{\sigma_s^{7/2}} + \frac{\beta_2}{\sigma_s^{5/4}} + \frac{\beta_3}{\sigma_s} + \frac{\beta_4}{\sigma_s^{3/4}} + \frac{\beta_5}{\sigma_s^{1/2}} + \frac{\beta_6}{\sigma_s^{1/4}} + \frac{\beta_7}{\sigma_s^{13/4}} + \frac{\beta_8}{\sigma_s^3} + \frac{\beta_9}{\sigma_s^{11/4}} + \frac{\beta_{10}}{\sigma_s^{5/2}} + \frac{\beta_{11}}{\sigma_s^{9/4}} + \frac{\beta_{12}}{\sigma_s^2} + \frac{\beta_{13}}{\sigma_s^{7/4}} + \frac{\beta_{14}}{\sigma_s^{3/2}} \\
d = 11:&\quad \frac{\beta_0}{\sigma_s^4 \log_s} + \frac{\beta_1}{\sigma_s^3 \log_s} + \frac{\beta_2}{\sigma_s^2 \log_s} + \frac{\beta_3}{\sigma_s \log_s} + \frac{\beta_4}{\log_s} + \frac{\beta_0'}{\sigma_s^4 \log_s^2} + \frac{\beta_1'}{\sigma_s^3 \log_s^2} + \frac{\beta_2'}{\sigma_s^2 \log_s^2} + \frac{\beta_3'}{\sigma_s \log_s^2} + \frac{\beta_4'}{\log_s^2} \\
d = 11.5:&\quad \frac{\beta_0}{\sigma_s^{17/4}} + \frac{\beta_1}{\sigma_s^{7/2}} + \frac{\beta_2}{\sigma_s^{11/4}} + \frac{\beta_3}{\sigma_s^2} + \frac{\beta_4}{\sigma_s^{5/4}} + \frac{\beta_5}{\sigma_s^{1/2}} \\
d = 12:&\quad \frac{\beta_0}{\sigma_s^{9/2}} + \frac{\beta_1}{\sigma_s^4} + \frac{\beta_2}{\sigma_s^{7/2}} + \frac{\beta_3}{\sigma_s^3} + \frac{\beta_4}{\sigma_s^{5/2}} + \frac{\beta_5}{\sigma_s^2} + \frac{\beta_6}{\sigma_s^{3/2}} + \frac{\beta_7}{\sigma_s} + \frac{\beta_8}{\sigma_s^{1/2}}
\end{aligned}
\tag{49}
$$

Positivity (loss)

$$A(s,t) = \frac{1}{2i\pi}\oint_s ds' K(s,s',t)\, A(s',t)\,, \quad K(s,s',t) = \frac{1}{s'-s}\left[\frac{s-s_0}{s'-s_0}\,\frac{s-4+s_0+t}{s'-4+s_0+t}\right]^{\frac{D-2}{2}}\left[\frac{s'-4}{s-4}\,\frac{s'+t}{s+t}\right]^{\frac{D-4}{2}}$$

$$c_2^+ \equiv c_2 = \int_4^\infty \frac{ds'}{\pi}\,\frac{\Im T(s',\frac{4}{3})}{(s'-\frac{4}{3})^3} \geq 0$$

$$c_4^+ \equiv c_2 - \frac{128}{9}c_4 = \int_4^\infty \frac{ds'}{\pi}\,\frac{(s'+\frac{4}{3})}{(s'-\frac{4}{3})^5}(s'-4)\Im T(s',\frac{4}{3}) \geq 0.$$



	$2<d<5$	$d<7$	$d<9$	$d<11$	$\cdots$	$d<(2\lambda+1)\to\infty$
c_2^+	c_2	✗	✗	✗	$\cdots$	✗
c_4^+	c_4	$c_2-\frac{128}{9}c_4$	✗	✗	$\cdots$	✗
c_6^+	c_6	$c_2-\frac{16384}{81}c_6$	$c_2-\frac{256}{9}c_4+\frac{16384}{81}c_6$	✗	$\cdots$	✗
c_8^+	c_8	$c_2-\frac{2097152}{729}c_8$	$c_2-\frac{64}{3}c_4+\frac{1048576}{729}c_8$	$c_2-\frac{128}{3}c_4+\frac{16384}{27}c_6-\frac{2097152}{729}c_8$	$\cdots$	✗
c_{10}^+	c_{10}	$c_2-\frac{268435456}{6561}c_{10}$	$c_2-\frac{512}{27}c_4+\frac{268435456}{19683}c_{10}$	$c_2-\frac{1024}{27}c_4+\frac{32768}{81}c_6-\frac{268435456}{19683}c_{10}$	$\cdots$	✗
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
c_{2n}^+	c_{2n}	$c_2-\left(\frac{128}{9}\right)^{n-1}c_{2n}$	$c_2-\frac{128}{9}\frac{n-1}{n-2}c_4+\left(\frac{128}{9}\right)^{n-1}\frac{1}{n-2}c_{2n}$	$c_2-\frac{256}{9}\frac{n-1}{n-2}c_4+\frac{16384}{81}\frac{n-1}{n-3}c_6-\left(\frac{128}{9}\right)^{n-1}\frac{2}{(n-3)(n-2)}c_{2n}$	$\cdots$	$\sum_{m=0}^{\lambda-2}(-1)^m\#\,c_{2m}+\#c_{2n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$c_{2n\to\infty}^+$	c_{2n}	$c_2-\left(\frac{128}{9}\right)^{n}c_{2n}$	$c_2-\frac{128}{9}c_4+\left(\frac{128}{9}\right)^{n}\frac{1}{n}c_{2n}$	$c_2-\frac{256}{9}c_4+\frac{16384}{81}c_6-\left(\frac{128}{9}\right)^{n}\frac{2}{n^2}c_{2n}$	$\cdots$	undefined