Algebraic underpinnings of genus-one string amplitudes

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Work with: D. Dorigoni, M. Doroudiani, J. Drewitt, M. Hidding, N. Matthes, F. Porkert, O. Schlotterer, L. Schneps, B. Verbeek [2209.06772] [2403.14816] [2406.05099] [2508.02800]

Context and plan

- String amplitudes a clean laboratory for special functions appearing in scattering amplitudes:
  $\Rightarrow$ Multiple polylogarithms, elliptic functions, ...
- Have very rich algebraic structures that control many of the transcendental numbers appearing in them
  $\Rightarrow$ zeta generators, coactions, derivation algebras, ...
- Developing story going up in genus of the world sheet

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(thanks to Bastien Duboeuf/chatGPT)

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Today: status for genus one (torus)

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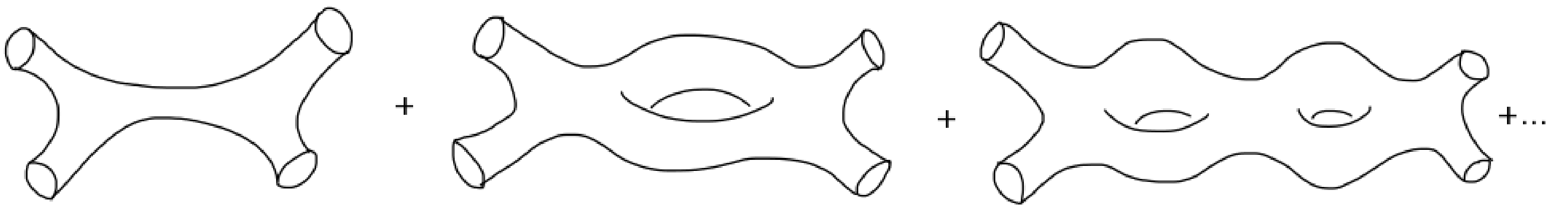
String amplitudes deconstructed

Perturbative string expansion=world-sheet genus expansion



Weighted by $\sim g_s^{2(h-1)}$ with string coupling g_s and genus h

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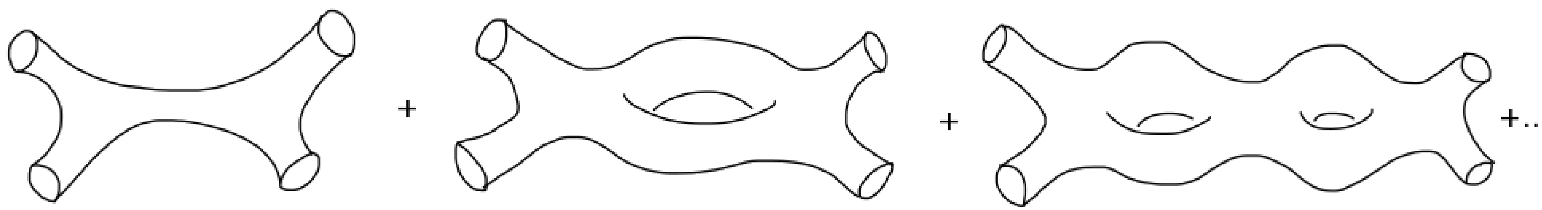
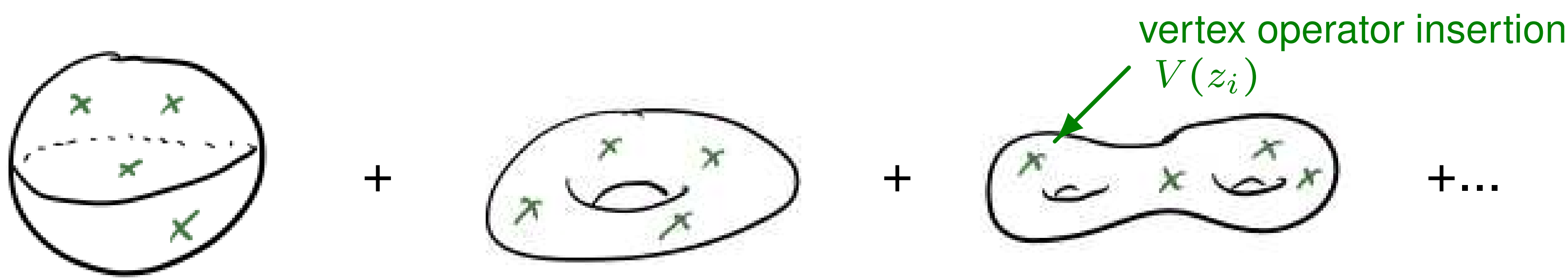


Weighted by $\sim g_{\mathrm{s}}^{2(h-1)}$ with string coupling g_{s} and genus h

Conformal invariance: shrink external states to punctures



Amplitude given by moduli space integral of CFT correlator

String amplitudes deconstructed

n-point amplitude at genus h

$$\mathcal{A}_h(1,\dots,n) = \int\limits_{\mathcal{M}_{h,n}} d\mu_{h,n} \langle V(z_1)\cdots V(z_n)\rangle_\Sigma = \int\limits_{\mathcal{M}_h} d\mu_h \int\limits_{\Sigma^n} d^{2n}z \langle V(z_1)\cdots V(z_n)\rangle_\Sigma$$

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shape of world-sheet Σ

- subtleties from boundaries of $\mathcal{M}_{h,n}$ and gauge fixing
- configuration space integral must be integrable on $\mathcal{M}_h$
  $\rightarrow$ modularity
- difficult to integrate CFT correlator for $h > 0$
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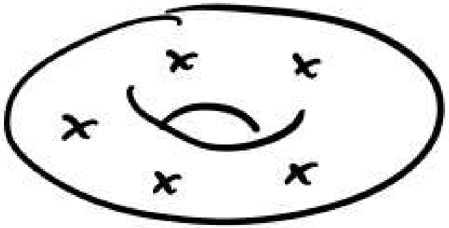
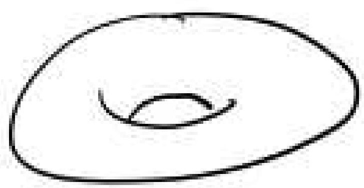
Focus on types of functions for configuration space integrals

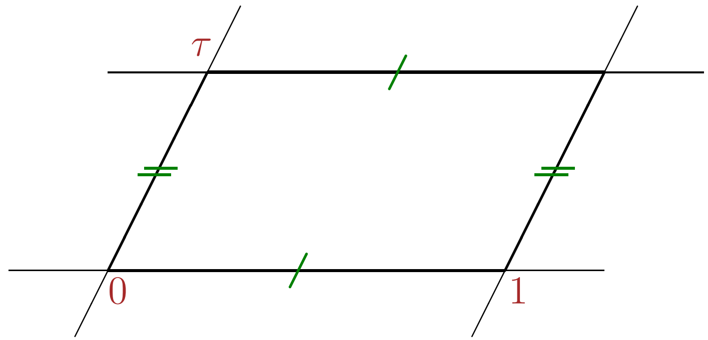
Genus one

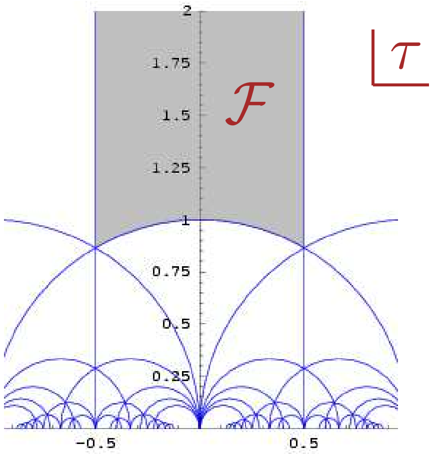


Genus one geometry

A torus can be described by identifying the opposite sides of a parellelogramm $\Sigma = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$







space of inequivalent tori

$\mathcal{M}_{1,1}$=upper half-plane / $SL(2,\mathbb{Z})$

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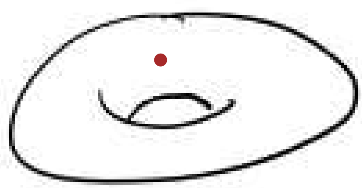

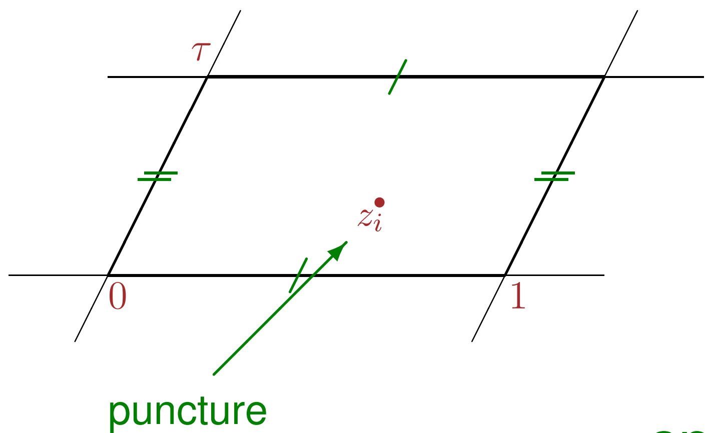

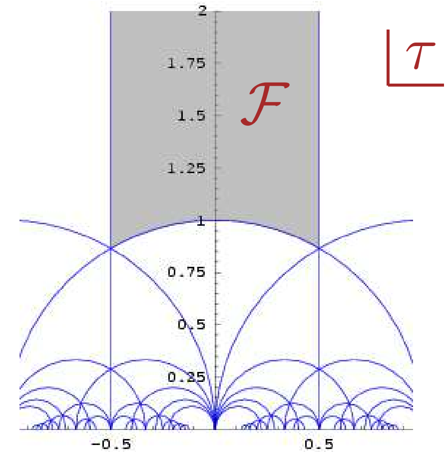

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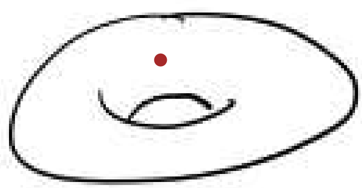

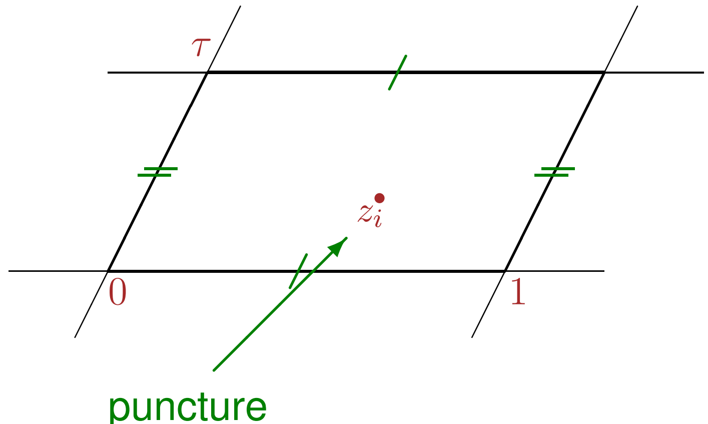

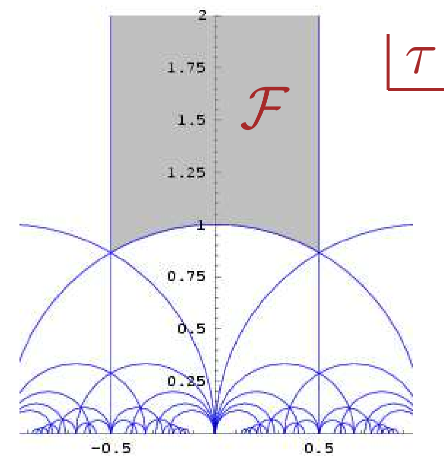

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Configuration space integral

$$\int_{\Sigma^n} d^{2n}z \langle V(z_1)\cdots V(z_n)\rangle_\tau \to \int_{\Sigma^{n-1}} \prod_{i=2}^{n} \frac{d^2 z_i}{\operatorname{Im}\tau} \underbrace{e^{\sum_{1\leq i<j\leq n} s_{ij} G(z_i - z_j;\tau)}}_{\text{Koba–Nielsen factor (universal)}} \cdot (\ldots)$$

'insertions' (depend on states)

Mandelstam variables $s_{ij} = \alpha' k_i \cdot k_j$

Genus one configuration space integrals

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with Green function ($p = m\tau + n$, $z = u\tau + v$, $\langle p, z\rangle = nu - mv$)

$$G(z;\tau) = -\log\left|\frac{\theta_1(z;\tau)}{\theta_1'(0;\tau)}\right|^2 + \frac{2\pi(\operatorname{Im} z)^2}{\operatorname{Im}\tau} = \frac{\operatorname{Im}\tau}{\pi}\sum_{p\neq 0}\frac{e^{2\pi i\langle p,z\rangle}}{|p|^2}$$

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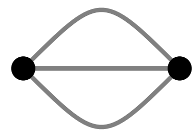
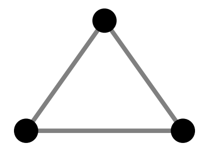
results in modular graph forms [D'Hoker, Green Vanhove]:

- Nested lattice sums giving non-holomorphic functions transforming well under $SL(2,\mathbb{Z})$: $\tau \mapsto \frac{a\tau+b}{c\tau+d}$
- Can be depicted graphically as Feynman diagrams of a scalar 2D QFT with discrete momenta

Differential equations

Modular graph forms satisfy (unexpected) relations, e.g.

 $=$  $+\ \zeta_3$ [Zagier]

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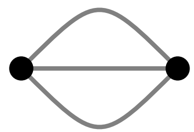
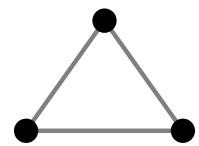
 $=$  $+\ \zeta_3$ [Zagier]

To study them systematically, organise MGFs by looking at their differential equations. Variant of Maass operator [D'Hoker Green]

$$\nabla = 2i(\operatorname{Im}\tau)^2\partial_\tau \qquad \text{key:} \quad \nabla\frac{(\operatorname{Im}\tau)^\alpha}{p^\alpha\bar{p}^\beta} = \alpha\frac{(\operatorname{Im}\tau)^{\alpha+1}}{p^{\alpha+1}\bar{p}^{\beta-1}}$$

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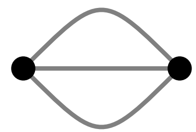
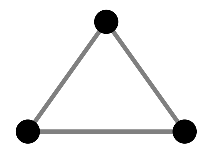
 $=$  $+\ \zeta_3$ [Zagier]

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$\Rightarrow$ ∇^m leads eventually to holomorphic dependence. Can be expressed through holomorphic Eisenstein series

$$\mathrm{G}_k(\tau) = \sum_{p\neq 0}\frac{1}{p^k} = 2\zeta_k + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n>0}\sigma_{k-1}(n)q^n \qquad (q = e^{2\pi i\tau})$$

divisor sum

Iterated Eisenstein integrals

Repeated application of ∇ leads to expression involving G_k

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$$\partial_\tau \mathbb{I}(\epsilon_k;\tau) = -\mathbb{I}(\epsilon_k;\tau)\mathbb{A}(\epsilon_k;\tau)$$

with connection

$$\mathbb{A}(\epsilon_k;\tau) = \sum_{k\geq 4}\sum_{j=0}^{k-2}(-1)^j \frac{(k-1)(2\pi i)^{1+j-k}}{j!}\tau^j \mathrm{G}_k(\tau)\epsilon_k^{(j)} d\tau$$

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[+ regularisation]

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S-cocycle

$$U_S^{-1}\epsilon_k^{(j)} U_S = \#\epsilon_k^{(k-2-j)}$$

S-cocycle records period integrals $\int_0^{i\infty}$ called multiple modular values [Brown] and consists of

- multiple zeta values $\zeta_{s_1,\ldots,s_\ell} = \sum_{0<n_1<\cdots<n_\ell} n_1^{-s_1}\cdots n_\ell^{-s_\ell}$
- L-values and other transcendental numbers associated with $SL(2,\mathbb{Z})$ cusp forms (e.g. Ramanujan discriminant)

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drop out in string theory
Pollack relations
$[\epsilon_4,\epsilon_{10}]-3[\epsilon_6,\epsilon_8]=0$

Equivariant iterated Eisenstein series

Similar to single-valued polylogs can make $\mathbb{I}$ transform correctly under $SL(2,\mathbb{Z})$

- Linear combination of $\mathbb{I}$ with its complex conjugate and $\mathbb{Q}[\mathrm{MZV}]$ coefficients
- Occurrence of coefficients can also be summarised algebraically [Brown][Dorigoni, Doroudiani, Drewitt, Hidding, AK Schlotterer, Schneps, Verbeek]

$$\mathbb{I}^{\mathrm{eqv}}(\epsilon_k;\tau) = (\mathsf{sv}\,\mathbb{M}_z)^{-1}\overline{\mathbb{I}(\epsilon_k;\tau)}^T(\mathsf{sv}\,\mathbb{M}_\sigma)\mathbb{I}(\epsilon_k;\tau)$$

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$\mathsf{sv}\,\mathbb{M}_\bullet$ is a derivation-valued generating series of single-valued MZVs (uses f-alphabet [Brown])

$$\mathsf{sv}\,\mathbb{M}_\sigma = 1 + 2\sum_{w\in 2\mathbb{N}+1}\zeta_w\sigma_w + 2\sum_{w_1,w_2\in 2\mathbb{N}+1}\zeta_{w_1}\zeta_{w_2}\sigma_{w_1}\sigma_{w_2} + \ldots$$

Zeta generators

Derivations σ_w in sv $\mathbb{M}_\sigma$ are called zeta generators and normalise 'geometric algebra' $\mathfrak{u} = \langle \epsilon_k^{(j)} \rangle$ (mod Pollack), i.e.

$$\left[\sigma_w, \epsilon_k^{(j)}\right] \subset \mathfrak{u}$$

$\left[\begin{array}{c}\text{Hain}\\ \text{Matsumoto}\end{array}\right]$

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Decomposition (non-unique) [Brown] [Dorigoni, Doroudiani, Drewitt, Hidding AK, Schlotterer, Schneps, Verbeek]

$$\sigma_w = z_w + \sigma_w^{\text{geo}}, \qquad \sigma_w^{\text{geo}} = -\frac{\epsilon_{w+1}^{(w-1)}}{(w-1)!} + \ldots$$

arithmetic part $\notin \mathfrak{u}$ geometric part $\in \mathfrak{u}$

Generating series $\mathbb{M}_z$ uses only arithmetic part z_w of σ_w

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Single-valued version more canonical

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but not $SL(2,\mathbb{Z})$ covariant! No similar distinction for genus zero.

Many structures...

Genus-one string integrals lead to iterated Eisenstein integrals

- modular and single-valued properties
- Tsunogai derivation algebra $\epsilon_k^{(j)}$ with Pollack relations (associated with $\text{Lie}[a,b]$ from once-punctured torus)
- zeta generators σ_w: control occurrence of MZVs
- canonical choices and implementations up to triply iterated integrals [Dorigoni, Doroudiani, Drewitt, Hidding AK, Schlotterer, Schneps, Verbeek]

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- canonical choices and implementations up to triply iterated integrals [Dorigoni, Doroudiani, Drewitt, Hidding AK, Schlotterer, Schneps, Verbeek]
- studies of coaction on iterated Eisenstein integrals [AK, Porkert Schlotterer]; symbols for elliptic polylogs [Broedel, Duhr, Dulat Penante, Tancredi] [Forum Hippel] [Wilhelm Zhang]

What is this good for?

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- elliptic curves show up in QFT amplitudes [many talks][snowmass 2203.07088], often via congruence subgroups
  $\Rightarrow$ extend above structure [Duhr Lippert] [Claasen, Doroudiani, Duhr, AK Lippert, Schlotterer, Sohnle;w.i.p.]
- explicit instances of functions spaces in QFT amplitudes involve matrix representations of abstract generators
- fast numerical evaluations

Generalisations and outlook

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- keep (some) punctures on torus unintegrated: elliptic multiple polylogarithms [Brown Levin] [Broedel, Duhr Dulat, Tancredi] [Broedel, Mafra Matthes, Schlotterer] [Enriquez Zerbini] and elliptic modular graph forms [D'Hoker, AK Schlotterer] [Schlotterer Sohnle, Tao]

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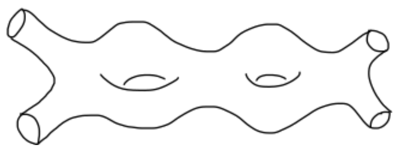

- Riemann surfaces of higher genus. Different but related flat connections [Enriquez] [D'Hoker, Hidding Schlotterer] [Baune, Broedel Im, Lisitsyn, Zerbini] and iterated integrals. Generalisation of derivation/braid algebra [Enriquez]. Many open questions...

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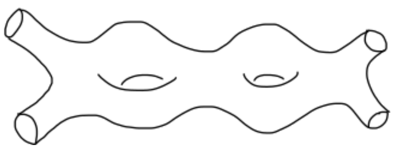

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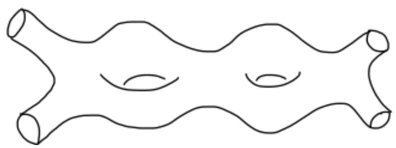

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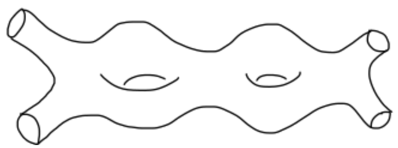

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Thank you for your attention!

Extra slides

Examples

Expansion of equivariant iterated Eisenstein integrals slightly more convenient in 'modular frame' [DDDHKSSV] with kernels ($y = \pi \, \mathrm{Im}\, \tau$)

$$\omega_+\left[\begin{smallmatrix} j \\ k \end{smallmatrix}; \tau, \tau_1\right] = \left(\frac{\tau-\tau_1}{4y}\right)^{k-2-j} (\bar{\tau}-\tau_1)^j \mathrm{G}_k(\tau_1) \frac{\mathrm{d}\tau_1}{2\pi i}$$

$$\omega_-\left[\begin{smallmatrix} j \\ k \end{smallmatrix}; \tau, \tau_1\right] = -\left(\frac{\tau-\bar{\tau}_1}{4y}\right)^{k-2-j} (\bar{\tau}-\bar{\tau}_1)^j \overline{\mathrm{G}_k(\tau_1)} \frac{\mathrm{d}\bar{\tau}_1}{2\pi i}$$

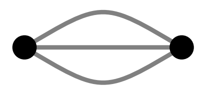
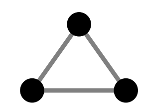
Corresponding equivariant generating series can be related to $\mathbb{I}(\epsilon_k; \tau)$ easily. Expansion coefficients in $\epsilon_k^{(j)}$ are denoted by

$$\beta^{\mathrm{eqv}}\left[\begin{smallmatrix} j_1 \ldots j_\ell \\ k_1 \ldots k_\ell \end{smallmatrix}; \tau\right]$$

Examples (II)

$$\beta^{\mathrm{eqv}}\left[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau\right] = \beta_+\left[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau\right] + \beta_-\left[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau\right] - \frac{2\zeta_{k-1}}{(k-1)(4y)^{k-2-j}}$$

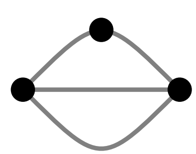
For $j = \frac{k-2}{2}$ this is proportional to non-holomorphic Eisenstein series $\mathrm{E}_{k/2}$.

Can rederive Zagier's relation  $=$  $+\zeta_3$ easily in this language.

Another example

$$\beta^{\mathrm{eqv}}\left[\begin{smallmatrix} 2 & 0 \\ 4 & 4 \end{smallmatrix}\right] = \beta_+\left[\begin{smallmatrix} 2 & 0 \\ 4 & 4 \end{smallmatrix}\right] + \beta_+\left[\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\right]\beta_-\left[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right] + \beta_-\left[\begin{smallmatrix} 0 & 2 \\ 4 & 4 \end{smallmatrix}\right]$$
$$- \frac{2\zeta_3}{3}\left(\beta_+\left[\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\right] + \frac{\beta_-\left[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right]}{(4y)^2}\right) - \frac{\pi^2\tau\bar{\tau}\zeta_3}{1080y} - \frac{5\zeta_5}{216y} + \frac{\zeta_3^2}{72y^2}$$

MGF connection

 $= -18\beta^{\mathrm{eqv}}\left[\begin{smallmatrix} 2 & 0 \\ 4 & 4 \end{smallmatrix}\right] - 126\beta^{\mathrm{eqv}}\left[\begin{smallmatrix} 3 \\ 8 \end{smallmatrix}\right]$

Coaction

In order to describe coaction one needs to introduce motivic and de Rham versions of the iterated integrals. For de Rham no full construction (absence of de Rham projection), so proceed formally [AK, Porkert Schlotterer]

$$\Delta \mathbb{I}^{\mathfrak{m}} = (\mathbb{M}_{\sigma}^{\mathfrak{dr}})^{-1} \mathbb{I}^{\mathfrak{m}} \mathbb{M}_{\sigma}^{\mathfrak{dr}} \otimes \mathbb{I}^{\mathfrak{dr}}$$

Was checked to satisfy many consistency requirements

- shuffle compatibility
- τ derivatives
- limiting values for multiple modular values

Multiple modular values

Numbers (MMVs) can be obtained from iterated Eisenstein integrals in the regularised limit $\tau \to 0$ [Brown]

$$\mathfrak{m}\left[\begin{smallmatrix} j_1 & j_2 & \ldots & j_\ell \\ k_1 & k_2 & \ldots & k_\ell \end{smallmatrix}\right] = \int_0^{i\infty} \tau_\ell^{j_\ell} \mathrm{G}_{k_\ell}(\tau_\ell)\, \mathrm{d}\tau_\ell \int_{\tau_\ell}^{i\infty} \cdots \int_{\tau_2}^{i\infty} \tau_1^{j_1} \mathrm{G}_{k_1}(\tau_1)\, \mathrm{d}\tau_1$$

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Reduce to MZVs and other periods (including L-values of cusp forms), e.g.

$$\mathfrak{m}\Big[\begin{smallmatrix} j \\ k \end{smallmatrix}\Big] = \int_0^{i\infty} \tau_1^{j} \mathrm{G}_{k}(\tau_1)\, \mathrm{d}\tau_1 = \begin{cases} -\dfrac{2\pi i \zeta_{k-1}}{k-1} & \text{for } j=0 \\ \dfrac{2(-1)^{j+1} j! (2\pi i)^{k-1-j}}{(k-1)!} \zeta_{j+1}\zeta_{j+2-k} & \text{for } 0 < j \le k-2 \end{cases}$$

$$\mathfrak{m}\Big[\begin{smallmatrix} 0 & 0 & 0 \\ 4 & 4 & 8 \end{smallmatrix}\Big] = -\frac{i\pi^{13}\zeta_3}{22325625} + \frac{16 i\pi^9 \zeta_7}{416745} + \frac{4 i \pi^7 \zeta_9}{19845} - \frac{11 i \pi^5 \zeta_{11}}{189} + \frac{365983 i \pi^3 \zeta_{13}}{677180} - \frac{4 i \pi^3 \zeta_{3,3,7}}{441} + \frac{8 i \pi^3 \zeta_{3,5,5}}{735} - \frac{352 i \pi^{14} \Lambda(\Delta_{12}, 13)}{48974625}$$

Multiple modular values (II)

Multiple modular values are recorded in the S-cocycle (failure of modularity) of generating series $\mathbb{I}(\epsilon_k;\tau)$ of iterated Eisenstein integrals

$$\mathbb{I}(\epsilon_k;-\tfrac{1}{\tau}) = \mathbb{S}(\epsilon_k)U_S^{-1}\mathbb{I}(\epsilon_k;\tau)U_S$$

via

$$\mathbb{S} = 1 + \sum_{k\geq 4}\sum_{j=0}^{k-2}(-1)^j\frac{k-1}{j!}(2\pi i)^{1+j-k}\mathfrak{m}\big[\begin{smallmatrix} j \\ k \end{smallmatrix}\big]\,\epsilon_k^{(j)} + \ldots$$

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Coaction suggests a simplified form

$$\mathbb{S}^{\mathfrak{m}}(\epsilon_k) = (\mathbb{M}_\sigma^{\mathfrak{m}})^{-1}\,\mathbb{X}_S^{\mathfrak{m}}(\epsilon_k)U_S^{-1}\mathbb{M}_\sigma^{\mathfrak{m}}U_S$$

with $\mathbb{X}_S^{\mathfrak{m}}$ only a $\mathbb{Q}[\pi^2]$ series and S-transformation on ϵ_k:

$$U_S^{-1}\epsilon_k^{(j)}U_S = (-1)^j(2\pi i)^{k-2-2j}\frac{j!}{(k-j-2)!}\epsilon_k^{(k-j-2)}$$

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consistent with [Brown]
huge computational advantage

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Tsunogai derivation algebra

On $\mathsf{Lie}[a,b]$ (free Lie algebra) define the derivation ϵ_{2m} by

$$\epsilon_{2m}(a) = \mathrm{ad}_a^{2m}(b)\,, \quad \epsilon_{2m}([a,b]) = 0 \qquad \text{for } m \in \mathbb{Z}_{\geq 0}$$

implying $\mathrm{ad}_{\epsilon_0}^{k-1}\epsilon_k = 0$ and

$$\epsilon_0(b) = 0\,, \quad \epsilon_{2m}(b) = \sum_{j=0}^{m-1} (-1)^j \left[\mathrm{ad}_a^j(b), \mathrm{ad}_a^{2m-1-j}(b)\right]\,, \quad m \geq 1\,.$$

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Generate a derivation algebra $\mathfrak{u}$ (geometric) [Tsunogai] [Hain Matsumoto]

From every holom. $SL(2,\mathbb{Z})$ cusp form get relations [Pollack]

$$[\epsilon_{10}, \epsilon_4] - 3[\epsilon_8, \epsilon_6] = 0 \qquad \text{(period poly of Ramanujan } \Delta\text{)}$$

$$\ldots$$

Matrix representations of ϵ_k

In string integrals get representations of Tsunogai derivations [Mafra Schlotterer] [Gerken, AK Schlotterer] as $(n-1)! \times (n-1)!$ matrices of operators.

Explicit form for $n = 3$

$$R_{\eta_2,\eta_3}(\epsilon_0) = \frac{1}{\eta_{23}^2}\begin{pmatrix} s_{12} & -s_{13} \\ -s_{12} & s_{13} \end{pmatrix} + \frac{1}{\eta_2^2}\begin{pmatrix} 0 & 0 \\ s_{12} & s_{12}+s_{23} \end{pmatrix} + \frac{1}{\eta_3^2}\begin{pmatrix} s_{13}+s_{23} & s_{13} \\ 0 & 0 \end{pmatrix}$$
$$- \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big(\frac{1}{2}s_{12}\partial_{\eta_2}^2 + \frac{1}{2}s_{13}\partial_{\eta_3}^2 + \frac{1}{2}s_{23}(\partial_{\eta_2}-\partial_{\eta_3})^2 + 2\pi i(\bar\eta_2\partial_{\eta_2}+\bar\eta_3\partial_{\eta_3})\Big)$$

$$R_{\eta_2,\eta_3}(\epsilon_k) = \eta_{23}^{k-2}\begin{pmatrix} s_{12} & -s_{13} \\ -s_{12} & s_{13} \end{pmatrix} + \eta_2^{k-2}\begin{pmatrix} 0 & 0 \\ s_{12} & s_{12}+s_{23} \end{pmatrix} + \eta_3^{k-2}\begin{pmatrix} s_{13}+s_{23} & s_{13} \\ 0 & 0 \end{pmatrix}$$

Operators can be generated easily for higher n

In QFT get matrix representations and truncation for finite k

Zeta generators

Genus zero for odd $w \geq 3$ and use canonical polynomial

$$[M_w, e_0] = 0\,, \qquad [M_w, e_1] = \big[e_1, g_w(e_0, e_1)\big]$$

from Drinfeld associator

Genus one similar, but use as auxiliary object the free Lie algebra $\mathrm{Lie}[a, b]$ from fund. group of once-punctured torus. First Tsunogai derivations ϵ_{2i}:

$$\epsilon_{2i}(a) = \mathrm{ad}_a^{2i}(b) \qquad \epsilon_{2i}([a, b]) = 0$$

$$\Longrightarrow \quad \epsilon_0(b) = 0 \qquad \epsilon_{2i}(b) = \sum_{j=0}^{i-1}(-1)^j \big[\mathrm{ad}_a^j(b), \mathrm{ad}_a^{2i-1-j}(b)\big]$$

Zeta generators

$$\sigma_w(s_{12}) = 0 \qquad \sigma_w(s_{01}) = \big[s_{01}, g_w(s_{12}, -s_{01})\big]$$

$$s_{01} = -b - \sum_{n\geq 1} \frac{\mathrm{B}_n}{n!}\mathrm{ad}_a^n(b) \qquad s_{12} = [b, a]$$

from degenerating KZB connection

Zeta generators (II)

Generating series

$$\mathsf{sv}\,\mathbb{M}_\sigma = \sum_{\ell=0}^{\infty} \sum_{\substack{i_1,i_2,\ldots,i_\ell \\ \in 2\mathbb{N}+1}} \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_\ell}\,\rho^{-1}\Big(\mathrm{sv}\big(f_{i_1}f_{i_2}\ldots f_{i_\ell}\big)\Big)$$

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Generating series

ρ : MZV $\to$ f-alphabet

non-commutative letters f_i

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single-valued map

where the f-alphabet [Brown] is a conjectured isomorphic description of multiple zeta values (proven for motivic).

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where the f-alphabet [Brown] is a conjectured isomorphic description of multiple zeta values (proven for motivic). Isomorphism ρ not unique, as many rational parameters as independent irreducible MZVs. Preferred choice described in [2406.05099]. Also

shuffle product

$$\text{sv}(f_{i_1}f_{i_2}\cdots f_{i_\ell}) = \sum_{j=0}^{\ell} f_{i_j}\cdots f_{i_2}f_{i_1} \sqcup\!\sqcup\, f_{i_{j+1}}\cdots f_{i_\ell}$$

[Schnetz]

Examples of zeta generators

First two zeta generators (extending [Brown])

$$\sigma_3 = z_3 - \frac{1}{2}\epsilon_4^{(2)} + \frac{1}{480}[\epsilon_4, \epsilon_4^{(1)}] - \frac{1}{120\,960}[\epsilon_4, \epsilon_6^{(1)}] + \frac{1}{30\,240}[\epsilon_4^{(1)}, \epsilon_6] +$$

$$\sigma_5 = z_5 - \frac{1}{24}\epsilon_6^{(4)} - \frac{5}{48}[\epsilon_4^{(1)}, \epsilon_4^{(2)}]$$

$$+ \frac{1}{5760}\big([\epsilon_4^{(0)}, \epsilon_6^{(3)}] - [\epsilon_4^{(1)}, \epsilon_6^{(2)}] + [\epsilon_4^{(2)}, \epsilon_6^{(1)}]\big) + \dots$$

with lowest arithmetic parts [Hain Matsumoto]

projector to an $\mathfrak{sl}_2$ rep

$$[z_w, \epsilon_k] = \frac{\mathrm{B}_{w+k-1}\, k!}{\mathrm{B}_k\, (w+k-1)!} t^{w+1}(\epsilon_{w+1}, \epsilon_{w+k-1}) + \dots ,$$

Single-valued iter. Eisenstein integrals

Construction of [Brown] rewritten in [DDDHKSSV] as

$$\mathrm{sv}\,\mathbb{I}(\epsilon_k;\tau) = (\mathrm{sv}\,\mathbb{M}_\sigma)^{-1}\overline{\mathbb{I}(\epsilon_k;\tau)^T}(\mathrm{sv}\mathbb{M}_\sigma)\mathbb{I}(\epsilon_k;\tau)$$

Expansion of coefficients are not modular forms under $SL(2,\mathbb{Z})$ in general!