

Strings from Almost Nothing

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Queen Mary University of London

The background is a dark, textured blue-grey. It is filled with numerous small, multi-colored dots (orange, yellow, blue, white) scattered across the left side. On the right side, there are several thick, flowing, multi-colored lines (orange, yellow, blue) that swirl and loop. In the upper right, there are thin, light-colored curved lines. A horizontal band of semi-transparent blue-grey is centered across the image, containing the word 'Introduction' in white serif font.

Introduction

The amplitudes bootstrap

- Many successes of string theory: supplying quantum gravity amplitudes that are well behaved in the UV (along with other achievements like BH microstate counting, unification of forces, etc.)
- Open question: Can string theory can be *bootstrapped*? Lots of effort in investigating this question, especially in the last couple years!

Cheung, **GR**, Sciotti, Tarquini [2508.09246];

Cheung, **GR** [2505.05553, 2308.03833, 2302.12263, 2210.12163];

Arkani-Hamed, Figueiredo, **GR** [2412.20639]; Basile, **GR**, Staudt [2603.04485];

Arkani-Hamed, Cheung, Figueiredo, **GR** [2312.07652];

Cheung, Hillman, **GR** [2408.03362, 2406.02665];

Maldacena, **GR** [2207.06426]; **GR**, Huang [2203.00696];

Caron-Huot, Komargodski, Sever, Zhiboedov [1607.04253]; Häring, Zhiboedov [2311.13631];

Albert, Knop, Rastelli [2406.12959]; Boisvert, Knop, Rastelli [2606.09980];

Evang, Herderschee, Morales [2601.11705]

- Many theories we know and love can indeed be constructed by simply imposing nice constraints on the S-matrix (GR, gauge theory, ...).
- What is the math question to which string amplitudes are the answer? Are strings the only game in town?

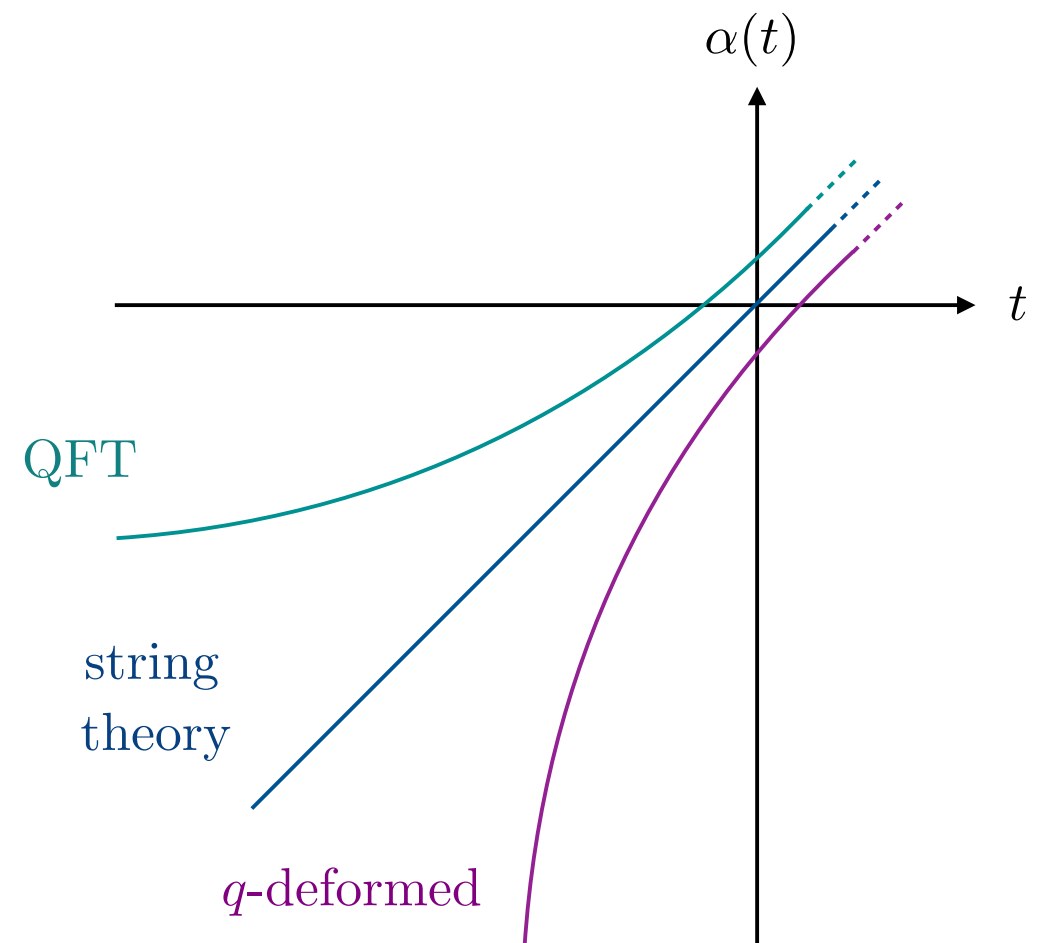
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Plan of Proof

String amplitudes

- Define unknown spectra of masses squared and spins: $\mu(n)$ and $J(n)$
- The level n is an integer, and $\mu(n)$ is strictly monotonic, without loss of generality.
- Define a “Regge trajectory” $\alpha(t)$, which just describes the large- s behavior of the amplitude, also without loss of generality:

$$A(s, t) \stackrel{s \rightarrow \infty}{\sim} f(t)(-s)^{\alpha(t)}$$

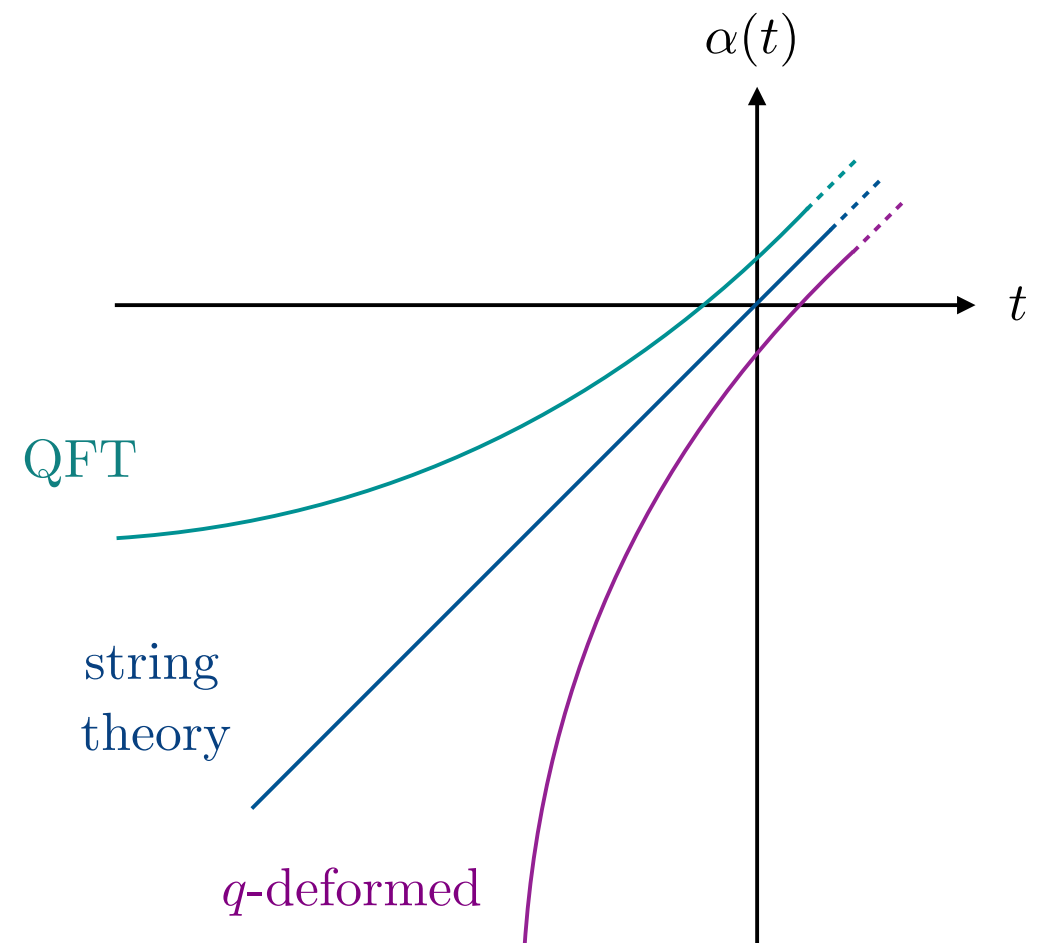


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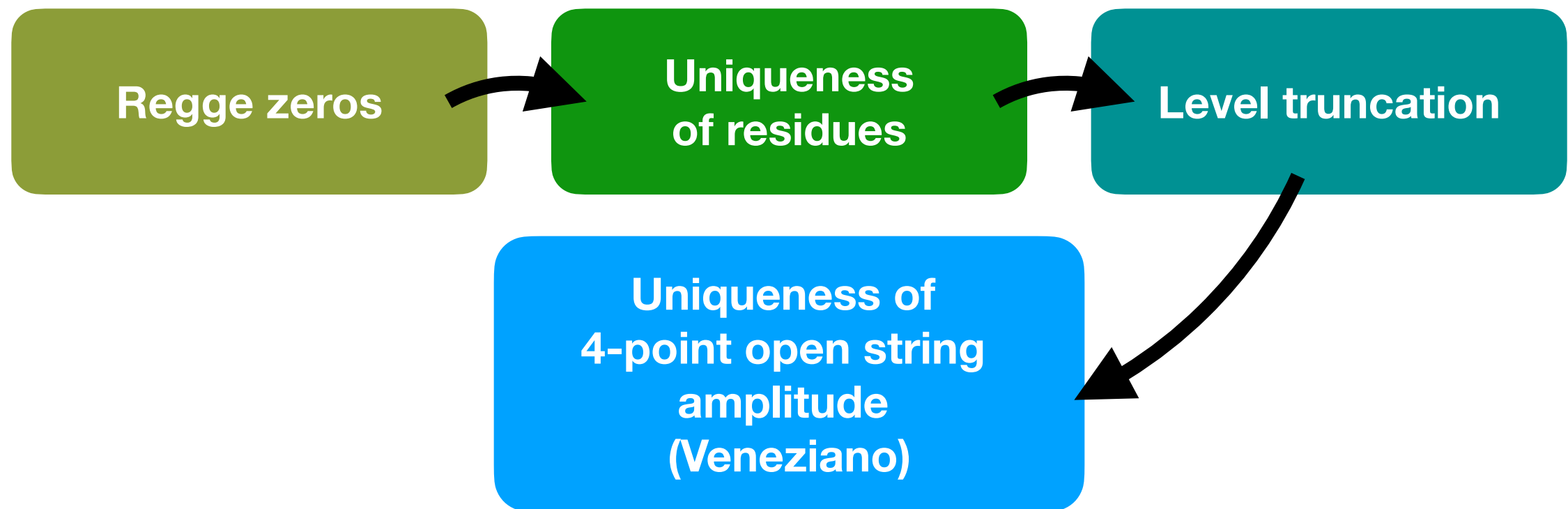
$$A(s, t) \stackrel{s \rightarrow \infty}{\sim} f(t)(-s)^{\alpha(t)}$$

- **Regge zeros:**
We will show that all amplitudes that are:
 - Lorentz invariant
 - Crossing-symmetric
 - Tree-level
 - Unitaryhave residues that vanish when $\alpha(t)$ crosses certain negative integers.



String amplitudes

- General plan of proof:



- Generalizations:

**Uniqueness of
4-point closed string
amplitude
(Virasoro-Shapiro)**

**Uniqueness of
5-point open string
amplitude**

String amplitudes

- Just two assumptions:

String amplitudes

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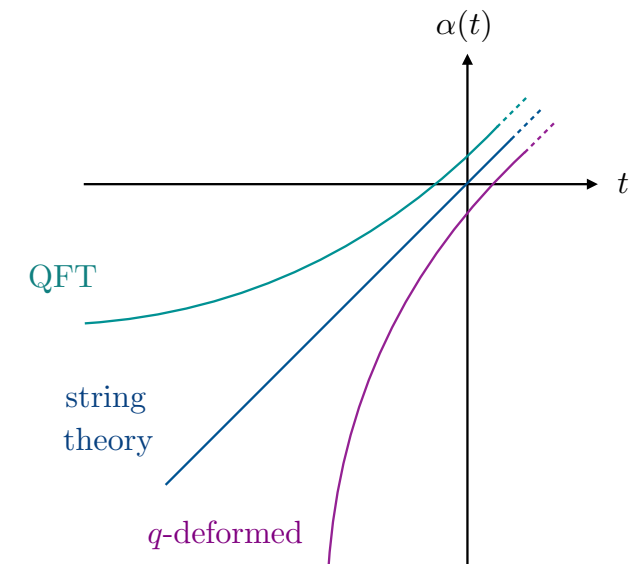
i) *Ultrasoftness.*

The Regge trajectory $\alpha(t)$ is increasing and bijective for $t < 0$.

Exhibited by string theory, but not QFT
(where softness is bounded [Cerulus, Martin \(1964\)](#)),

nor by q -deformations, cf. [Coon \(1969\)](#); **GR**, Maldacena [2207.06426]

Motivation: superpolynomial boundedness [Häring, Zhiboedov \[2311.13631\]](#)



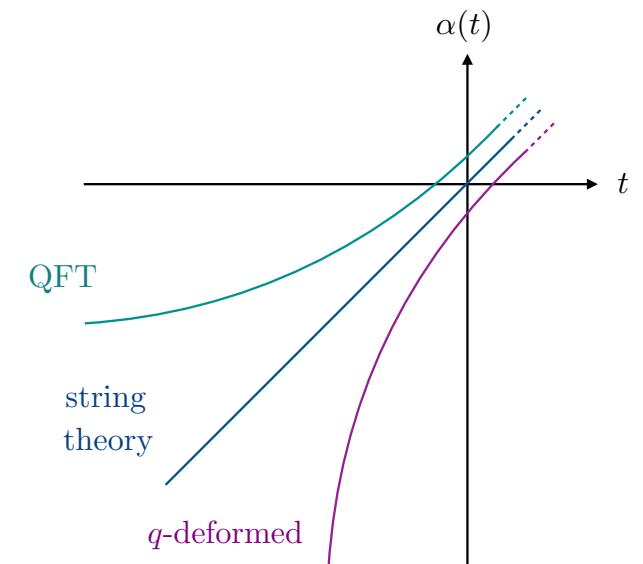
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ii) Minimal Zeros.

Every zero of a residue is a Regge zero
mandated by self consistency.

This is a minimality condition: residues could have extra zeros without obvious physical inconsistency, but these go beyond what is required.

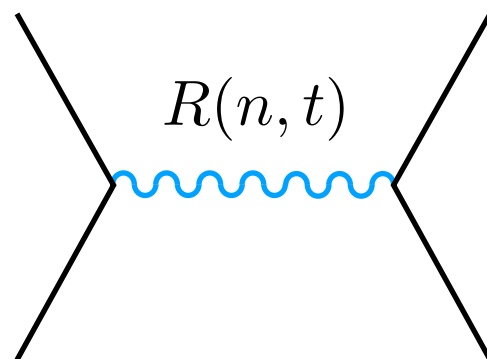
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Regge Zeros

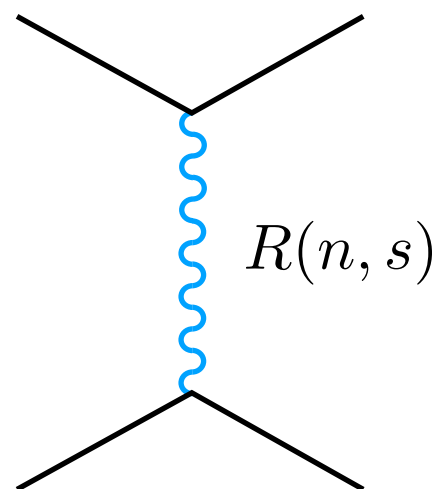
Four-point planar scattering

- Amplitude $A(s, t)$, crossing symmetric $A(s, t) = A(t, s)$
- External states: identical color-ordered scalars
- Regge trajectory $A(s, t) \stackrel{s \rightarrow \infty}{\sim} f(t)(-s)^{\alpha(t)}$

- Residues:



s -channel



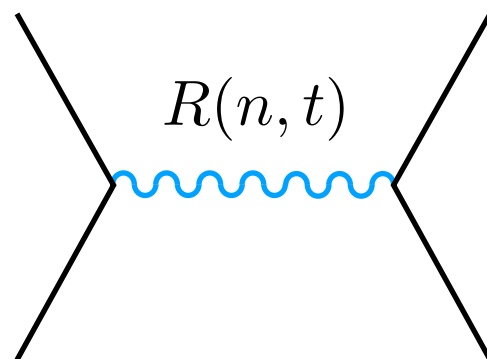
t -channel

$$\begin{aligned} R(n, t) &= \lim_{s \rightarrow \mu(n)} (\mu(n) - s) A(s, t) \\ &= \text{polynomial of degree } J(n) \end{aligned}$$

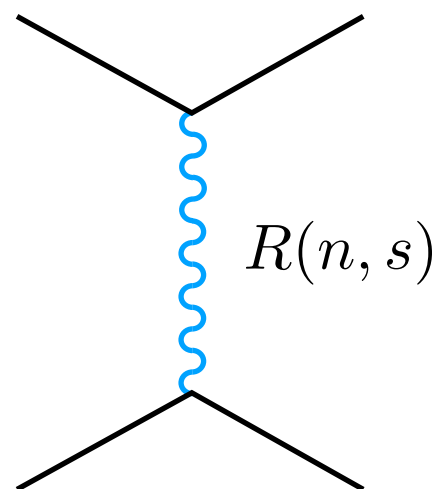
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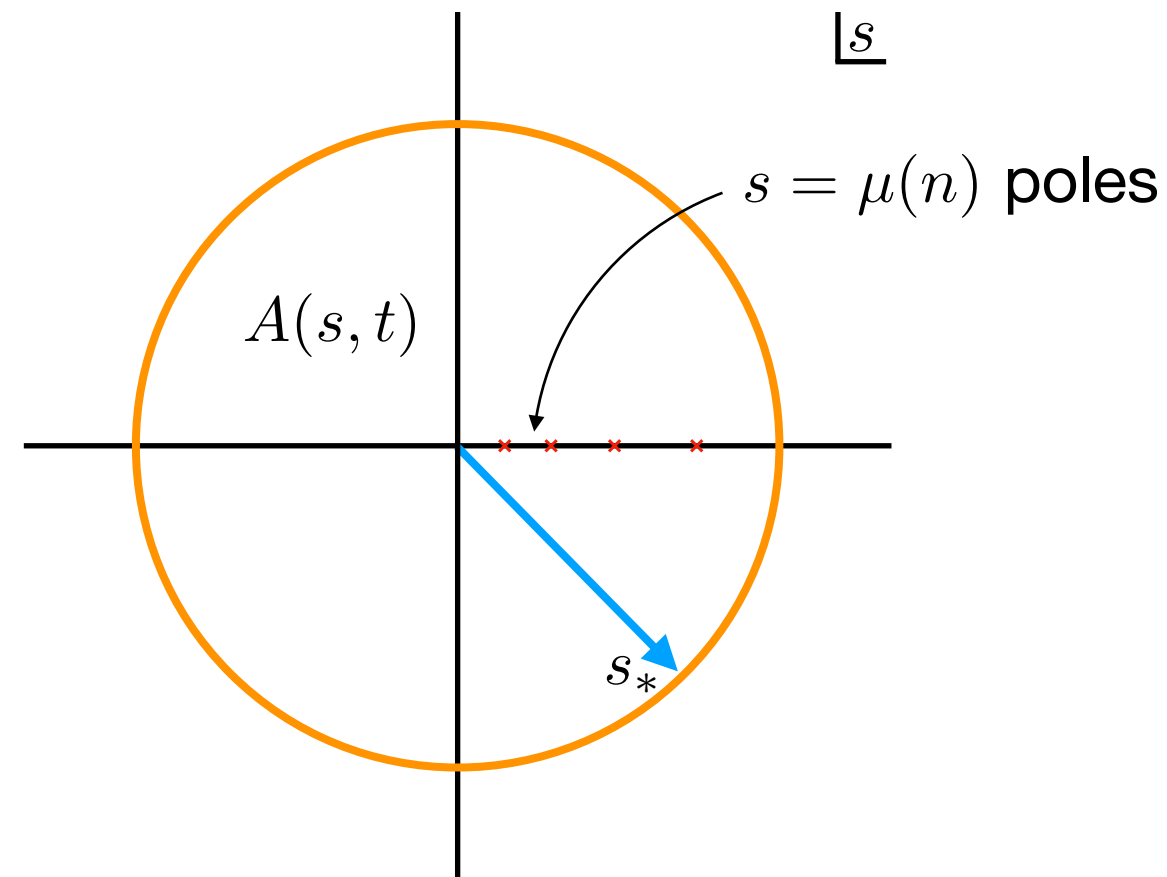
- Mass-spin relation (holds by definition of the Regge trajectory):
 $A(s, t) \sim t^{\alpha(s)}$ at large t , with poles at $s = \mu(n)$ with residues of degree $J(n)$

$$\implies \alpha(\mu(n)) = J(n) \text{ for all integers } n \geq 0.$$

Big arc integral

- Compute a contour integral around a big circle of radius s_* :

$$\oint_{\gamma} ds A(s, t)$$



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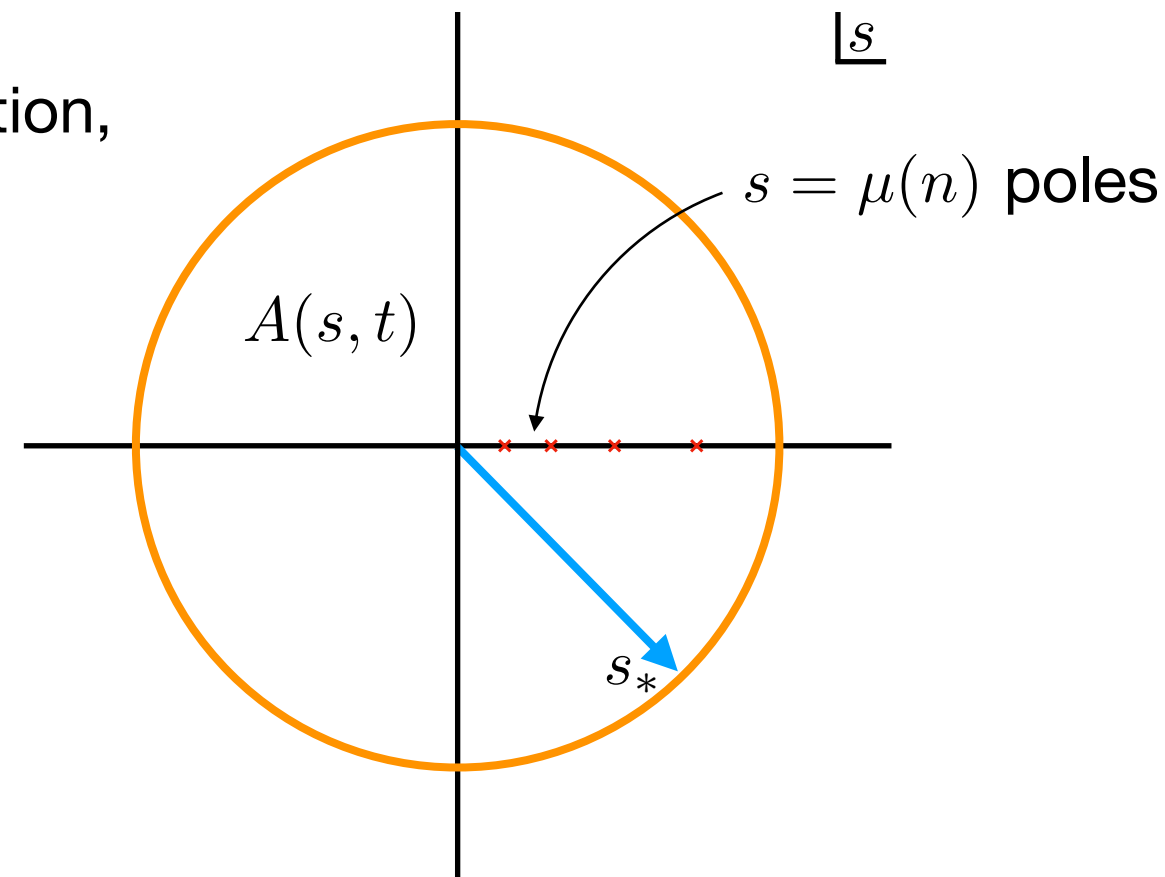
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- By direct calculation from Regge limit definition,

$$A(s, t) \stackrel{s \rightarrow \infty}{\sim} f(t)(-s)^{\alpha(t)}$$

we have

$$\oint^{s_*} ds A(s, t) \sim f(t) s_*^{\alpha(t)+1} \frac{\sin \pi \alpha(t)}{\alpha(t) + 1}$$



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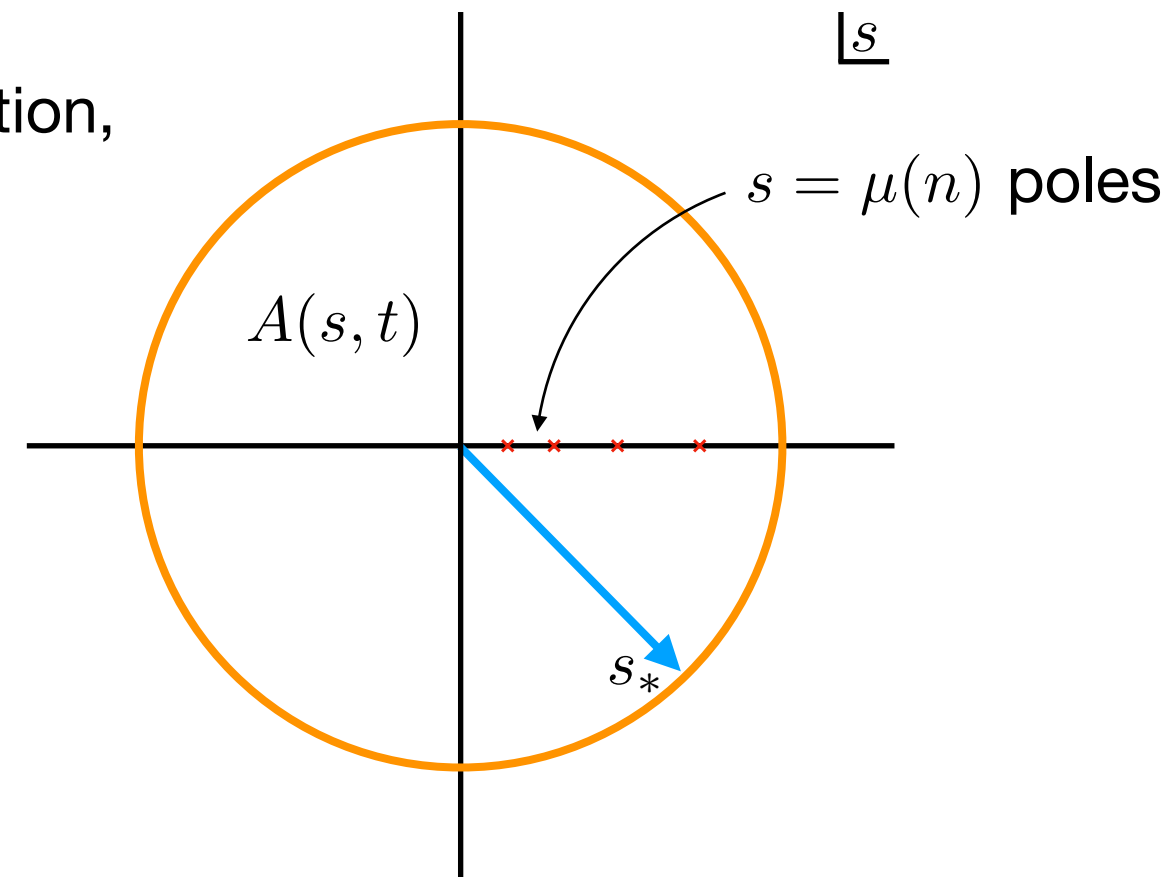
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Note that for integer $\alpha(t)$, this integral is *independent* of s_* :

$$\alpha(t) = -1 \quad \implies \quad \oint^{s_*} ds (-s)^{\alpha(t)} = 1$$

$$\alpha(t) \in \mathbb{Z} \neq -1 \quad \implies \quad \oint^{s_*} ds (-s)^{\alpha(t)} = 0$$



Big arc integral

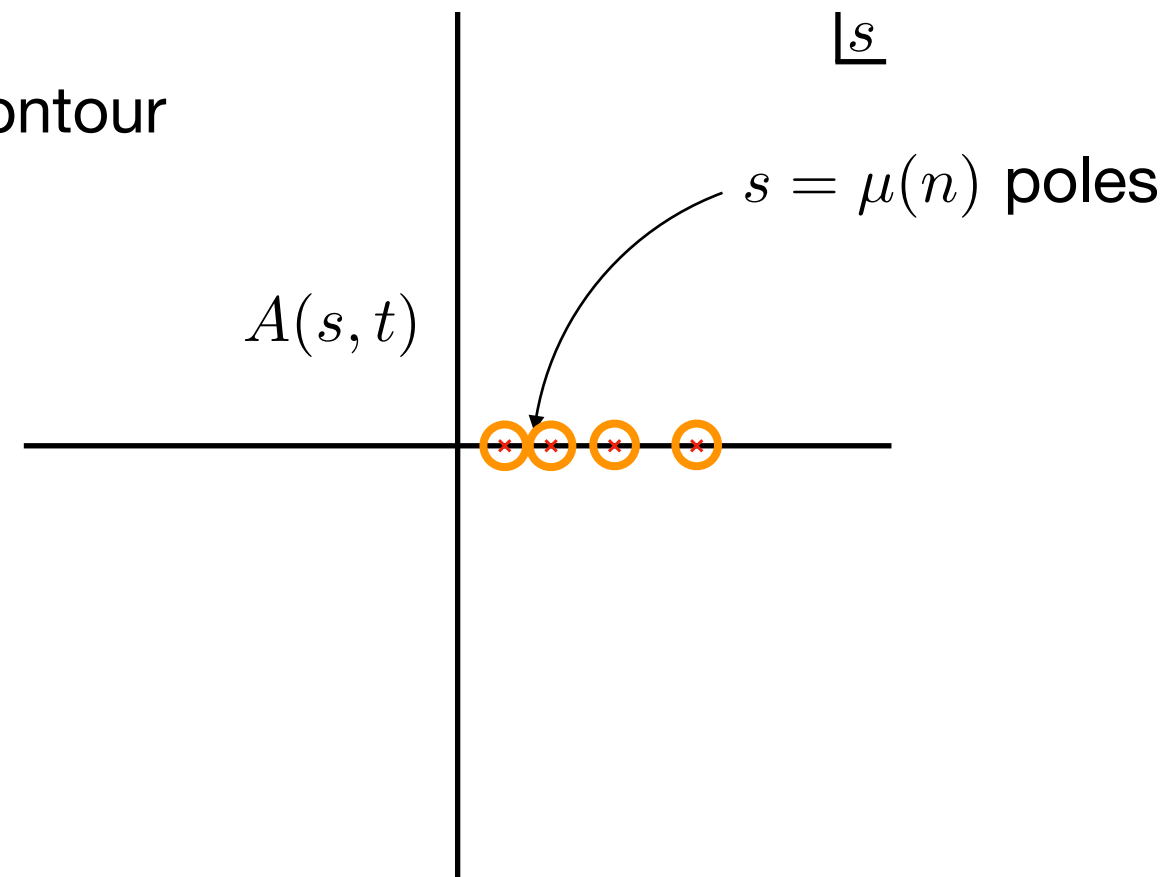
- Compute a contour integral around a big circle of radius s_* :

$$\oint_{|s|=s_*} ds A(s, t)$$

- By Cauchy's theorem, we can deform the contour inward to encircle each pole:

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where $s_* \sim \mu(n_*)$



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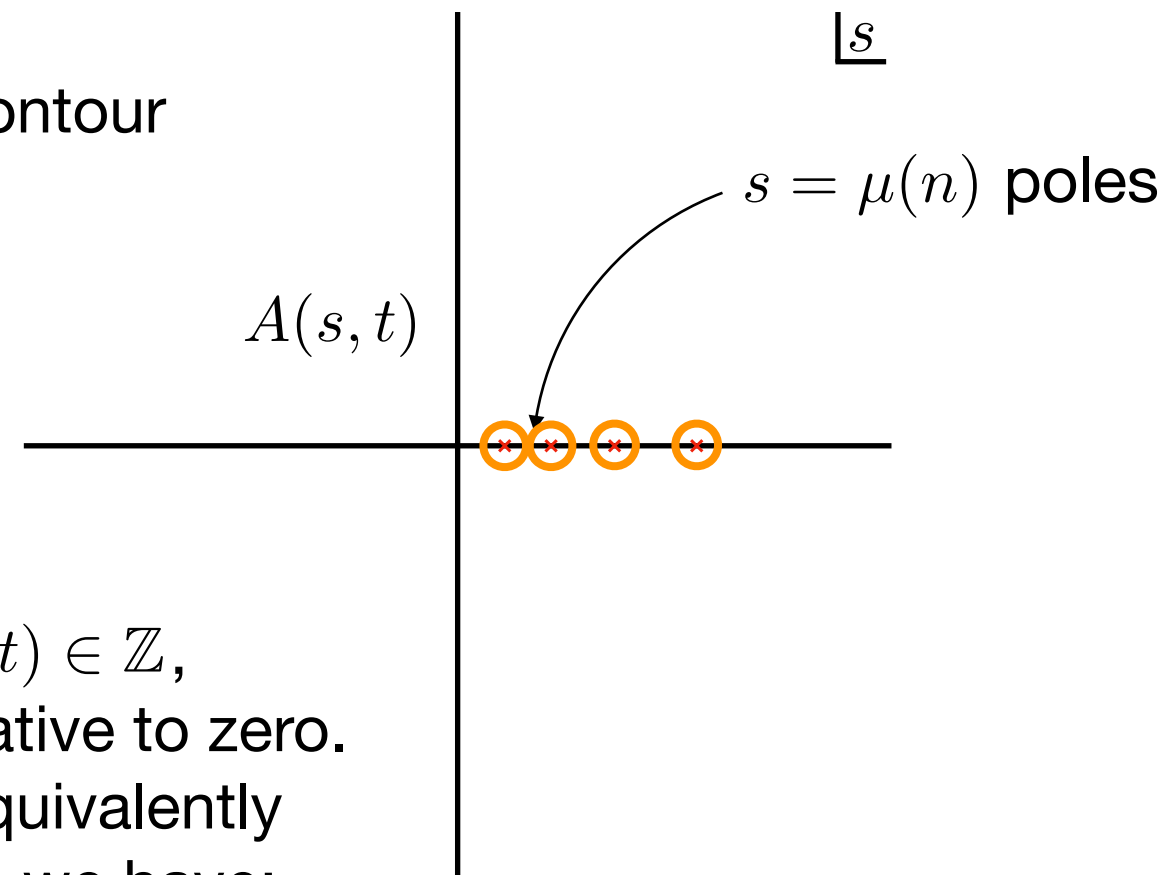
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- Since this must be n_* -independent when $\alpha(t) \in \mathbb{Z}$, we can differentiate by n_* and set the derivative to zero. Making the approximation $\sum_n \rightarrow \int dn$, or equivalently subtracting two concentric big arc integrals, we have:

$$R(n, t) \sim f(t) \mu'(n) \mu(n)^{\alpha(t)} \sin \pi \alpha(t)$$

The approximation holds when $|t| \ll \mu(n)$ and $\mu'(n) \ll \mu(n)$.



Regge zeros

- The factor of $\sin \pi \alpha(t)$ implies the *Regge zeros*:

$$R(n, t) = 0 \text{ for integer } \alpha(t)$$

- Two caveats:

1. Since $f(t)$ has poles at $t = \mu(n')$ for any integer $n' \geq 0$, this can cancel against the $\sin \pi \alpha(t)$. A genuine Regge zero requires integer $\alpha(t)$ for t off resonance, i.e., $\neq \mu(n')$.

2. Unitarity $\implies R(n, t) = \sum_{\ell=0}^{\infty} a_{n,\ell} G_{\ell}^{(D)}\left(1 + \frac{2t}{\mu(n)}\right)$ for $a_{n,\ell} \geq 0$



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$$\implies R(n, t) > R(n, 0) > 0 \text{ for } t > 0$$

Regge zeros

Summary:

In the Regge limit of $|t| \ll \mu(n)$, we have

$$R(n, t) = 0$$

on the Regge zeros, defined to be where

$$\alpha(t) + r = 0$$

for integer r , for which the real roots satisfy $t < 0$ and $t \neq \mu(n')$ for any integer $n' \geq 0$.

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Required by consistency for *any* unitary, tree-level, color-ordered, four-point amplitude.

Regge zeros nest: A Regge zero of $R(n, t)$ is also a Regge zero of $R(n', t)$ for any $n' > n$.

Regge zeros: examples

Softer behavior $\implies \alpha(t)$ crosses more negative integers
 $\implies$ more Regge zeros among the residues

Let's see how this works in various examples.

- Veneziano amplitude:

$$A_{\text{str}}(s, t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)} \underset{s \rightarrow \infty}{\sim} s^t \quad \text{ultrasoft}$$

Expected Regge zeros at $t + r = 0$ are indeed present in the residues:

$$R_{\text{str}} = \frac{1}{n!} \prod_{r=1}^n (t + r) \sim n^t \quad \checkmark$$

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- Deformed worldsheet amplitudes of [Gross \(1969\)](#). Example:

$$A(s, t) = \int_0^1 u^{-s-1} (1-u)^{-t-1} e^{gu(1-u)} du = A_{\text{str}}(s, t) \times {}_2F_2 \left[\begin{matrix} -s, -t \\ -\frac{s+t}{2}, \frac{1-s-t}{2} \end{matrix} ; \frac{g}{4} \right]$$

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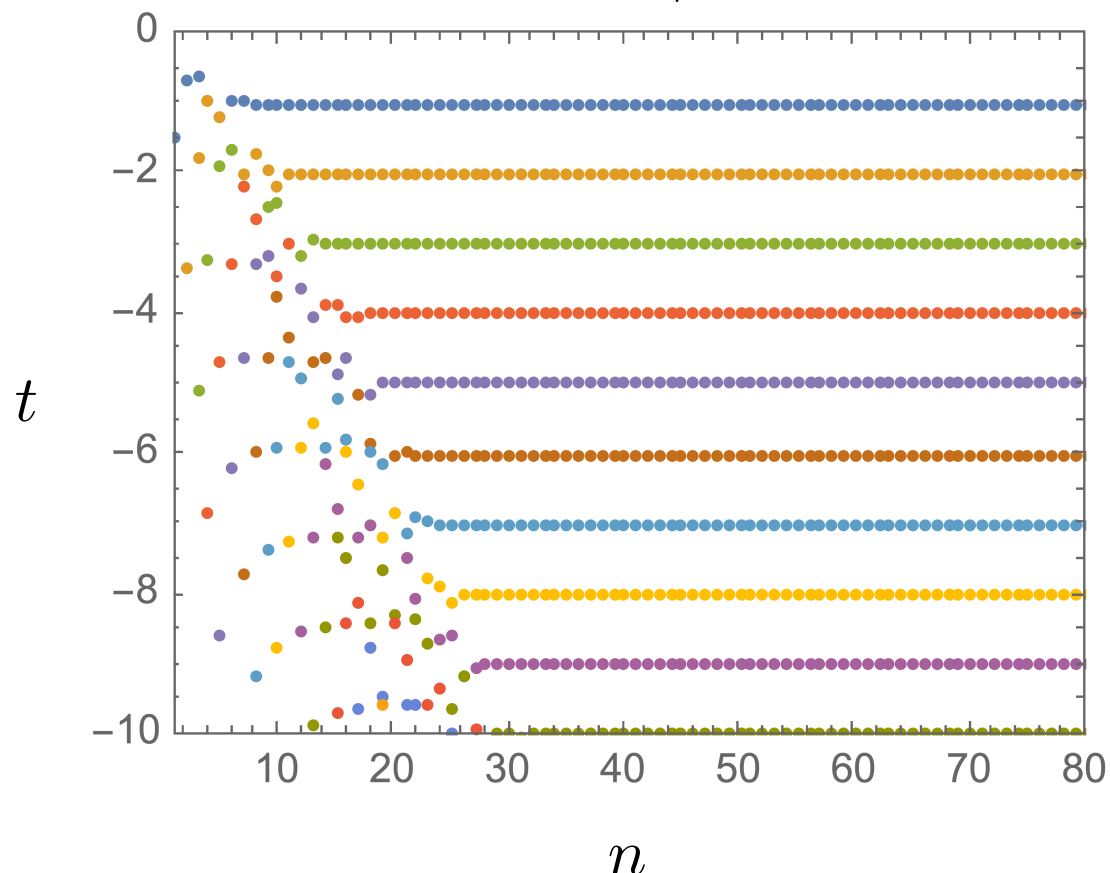
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$$g = 1/2$$

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- Three-parameter amplitudes of [Häring, Zhiboedov \[2311.13631\]](#):

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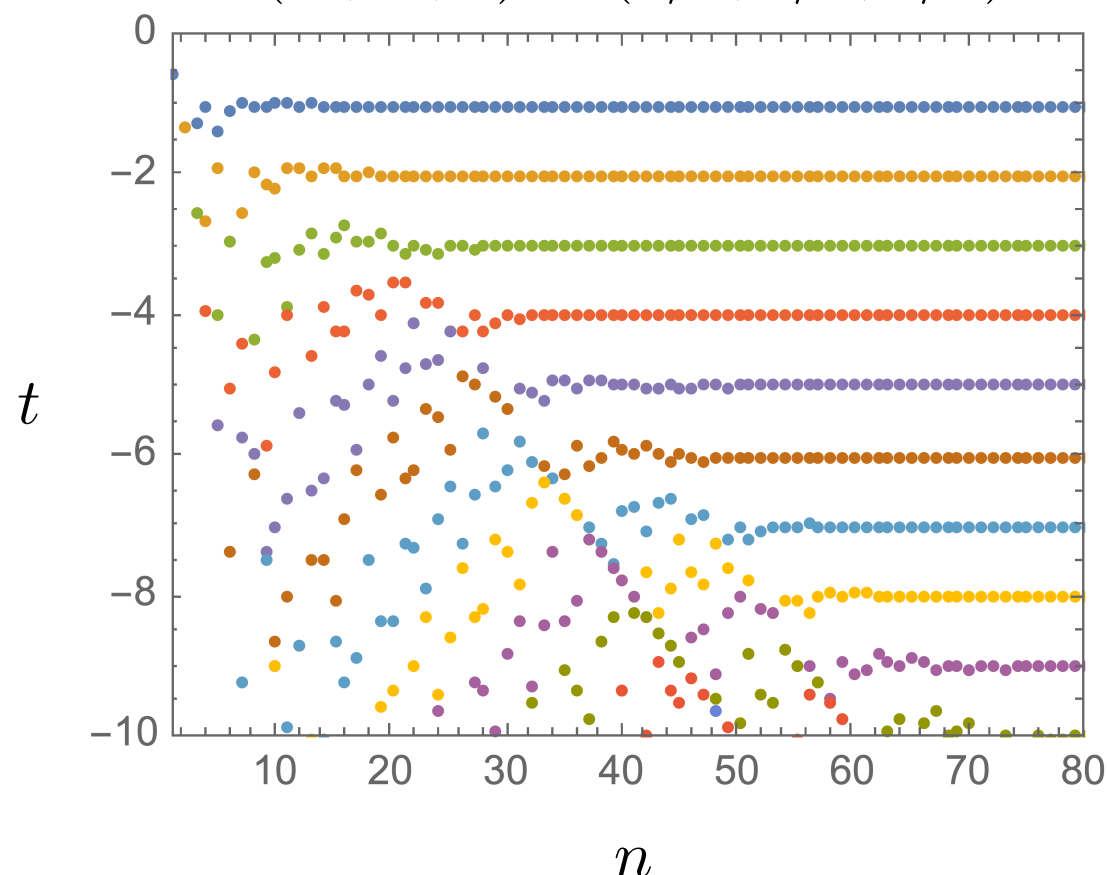
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$$(c_0, c_1, \lambda) = (1/3, 1/4, 1/4)$$



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Uniqueness of Strings

Asymptotic trajectories

So far, we have just used our ultrasoftness assumption *i*) to find an infinite collection of Regge zeros.

Let us now use assumption *ii*), i.e., minimality: The *only* zeros of the residues are the Regge zeros.

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Regge zeros nest: zeros of $R(n, t)$ are zeros of $R(n', t)$ for $n' > n$.
(*level truncation* in Cheung, Hillman, **GR** [2406.02665])

Let us use this fact to bound the type of function $\alpha(t)$ can be. We start with our result from the big arc integral:

$$R(n, t) \sim f(t) \mu'(n) \mu(n)^{\alpha(t)} \sin \pi \alpha(t)$$

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Let us use this fact to bound the type of function $\alpha(t)$ can be. We start with our result from the big arc integral, and take its log derivative in n :

$$\alpha(t) \frac{\mu'(n)}{\mu(n)} + \frac{\mu''(n)}{\mu'(n)} \sim \frac{d \log R(n, t)}{dn} \sim \log \frac{R(n, t)}{R(n-1, t)}$$

so $\alpha(t) \sim \log P(n, t)$, where $P(n, t) = R(n, t)/R(n-1, t)$ is *polynomial* in t .

Write $P(n, t) \stackrel{\text{large } n}{\sim} P(t) + \dots$.

Asymptotic trajectories

Either $P(t)$ is constant or it's not.

Non-constant $P(t)$: $P(t) \stackrel{\text{large } t}{\sim} t^k$ for some $k > 0$

$\implies$ logarithmic Regge trajectory $\alpha(t) \sim k \log t$

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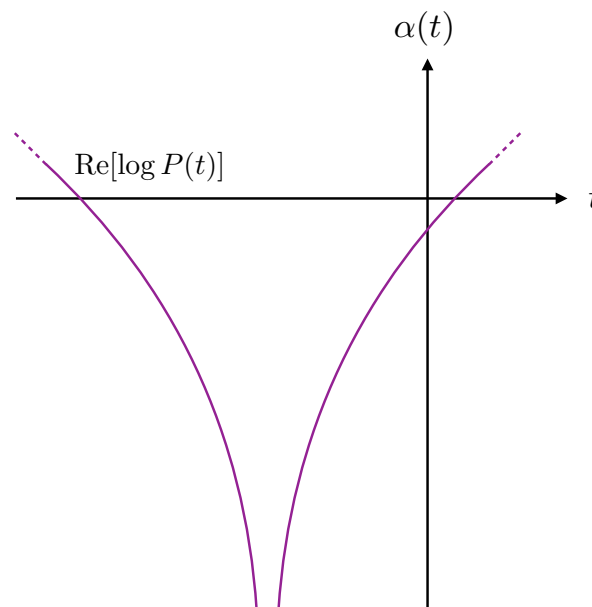
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From the mass-spin relation we found earlier, $\alpha(\mu(n)) = J(n)$, we have

$$\mu(n) \sim e^{J(n)/k}$$

This is precisely the q -deformed spectrum of the Coon amplitude [Coon \(1969\)](#), see also [Maldacena, GR \[2207.06426\]](#). This Regge trajectory is not ultrasoft for $t < 0$, violating our assumption $i)$, so we drop it:



Asymptotic trajectories

Either $P(t)$ is constant or it's not.

Constant $P(t)$: t -dependence is subleading in n , but still polynomial since $P(n, t)$ is polynomial for all n , by minimality *ii*).

$\implies \alpha(t) = \log P(n, t)$ is polynomial in t .

Minimality

- Our assumption *ii*) enforces minimality: the only zeros of the residues are the Regge zeros.

$\implies$ Residues are products over factors of the form $\alpha(t) + r$.
Since unitarity implies all Regge zeros are at $t < 0$, bijectivity implies that, on the Regge zero, $r = -\alpha(t) > -\alpha(0)$.

We can shift $\alpha(t)$ so that the minimal value is at $r = 1$.

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- The residue must therefore be of the form:

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$N(n) = \frac{J(n)}{h \deg(\alpha)}$

multiplicity of zeros

normalization

Minimality

- Comparing our residue,

$$R(n, t) = c(n) \prod_{r=1}^{N(n)} (\alpha(t) + r)^h$$

to that of the string,

$$R_{\text{str}}(n, t) = \frac{1}{n!} \prod_{r=1}^n (t + r) \sim n^t$$

we see that $R(n, t) \sim R_{\text{str}}(N(n), \alpha(t))^h \sim N(n)^{h\alpha(t)}$ at large n .

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then implies $\mu(n) \sim N(n)^h \sim J(n)^h$.

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then implies $\mu(n) \sim N(n)^h \sim J(n)^h$.

- From our mass-spin relation we derived earlier, $\alpha(\mu(n)) = J(n)$, we find the Regge trajectory, $\alpha(t) \sim t^{1/h}$.
- Since we argued the Regge trajectory must be polynomial, we find $h = 1$.

Linear spectrum and Regge trajectory

- We have thus arrived at the fundamental result:

$$\mu(n) \sim J(n) \quad \text{and} \quad \alpha(t) \sim t$$

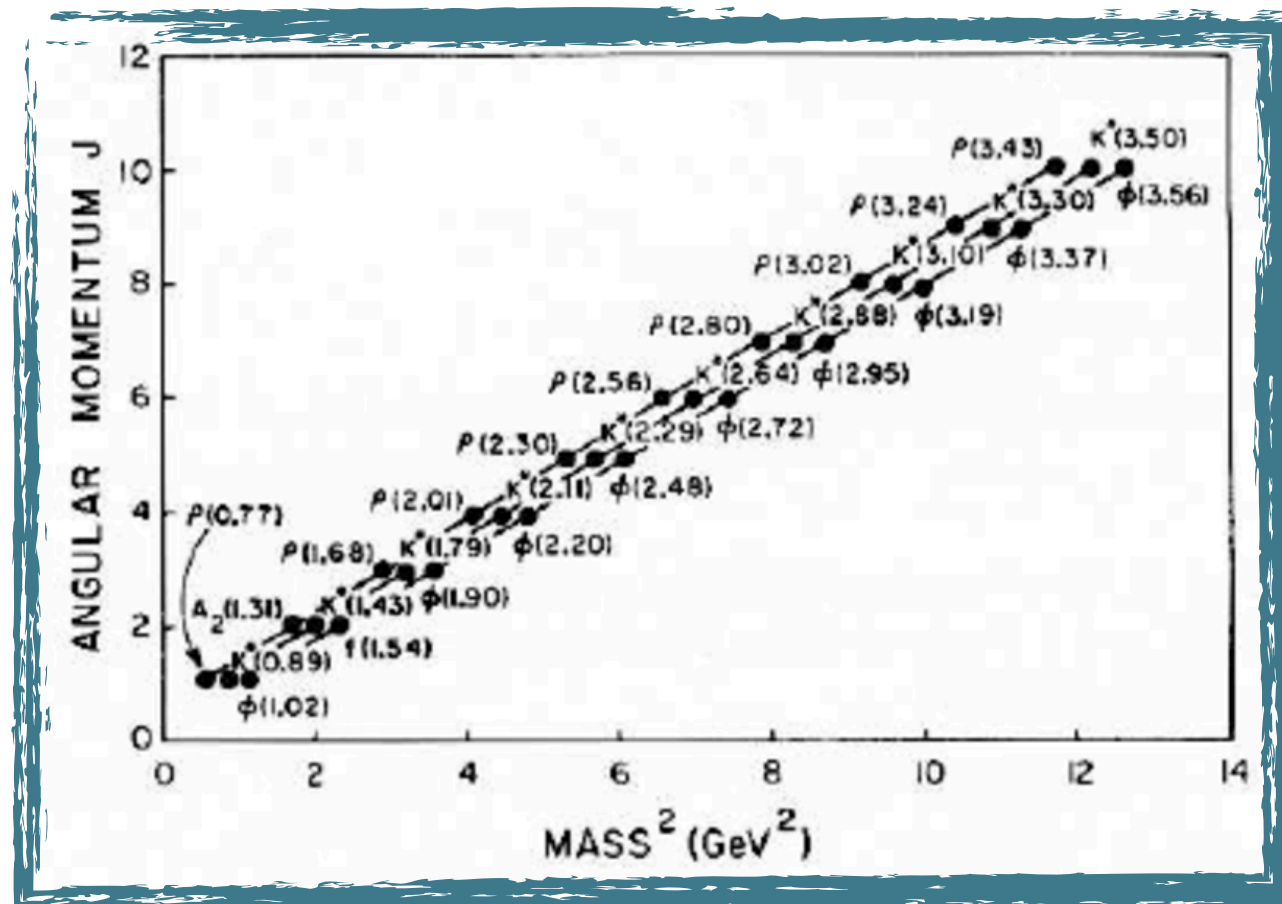
- Further, we can fix $J(n)$ as a function of n via subtracted dispersion relations, finding $J(n) \sim n$.

Linear spectrum and Regge trajectory

- We have thus arrived at the fundamental result:

$$\mu(n) \sim J(n) \sim n \quad \text{and} \quad \alpha(t) \sim t$$

- We have *derived*—from very sparse assumptions—the precise spectrum of the Chew-Frautschi plot that was the historical reason for the development of string theory:



From residues to strings

- Using just our two assumptions, we have arrived at a very constrained structure for the amplitude:

$$A(s, t) = \sum_{n=0}^{\infty} \frac{c(n) \prod_{r=1}^n (t + r)}{n - s}$$

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$$A(s, t) = \sum_{n=0}^{\infty} \frac{c(n) \prod_{r=1}^n (t + r)}{n - s}$$

- *Level truncation*: At special values of t , the amplitude becomes a rational function of s .

$$A(s, -k) = \sum_{n=0}^{k-1} \frac{c(n) \prod_{r=1}^n (r - k)}{n - s}$$

From residues to strings

- Using just our two assumptions, we have arrived at a very constrained structure for the amplitude:

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- Level truncation*: At special values of t , the amplitude becomes a rational function of s .

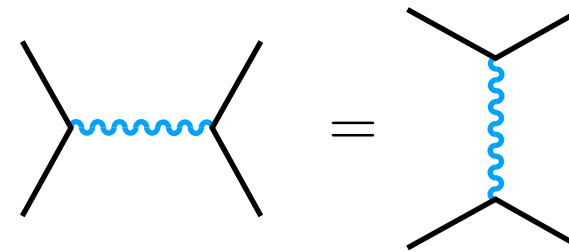
$$A(s, -k) = \sum_{n=0}^{k-1} \frac{c(n) \prod_{r=1}^n (r - k)}{n - s}$$

- In [Cheung, Hillman, GR \[2406.02665\]](#), we showed that level truncation and superpolynomial softness uniquely fix the string amplitude. Let's see how this works.

From residues to strings

Crossing symmetry:

$$A(s, t) = A(t, s)$$



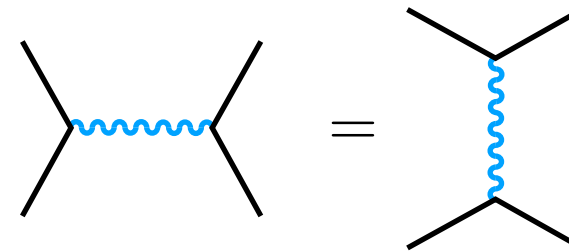
$$\Rightarrow A(-k_1, -k_2) = A(-k_2, -k_1)$$

$$\Rightarrow \sum_{n=0}^{k_2-1} \frac{c(n) \prod_{r=1}^n (r - k_2)}{n + k_1} = \sum_{n=0}^{k_1-1} \frac{c(n) \prod_{r=1}^n (r - k_1)}{n + k_2}$$

From residues to strings

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$\Rightarrow$ overconstrained algebraic system for the $c(n)$, with the unique solution $c(n) = 1/n!$

Uniqueness of strings

Thus, our simple assumptions of *i*) ultrasoftness and *ii*) minimal zeros ***uniquely*** give us a single tree-level amplitude for four-point planar scattering:

$$\begin{aligned} A(s, t) &= \sum_{n=0}^{\infty} \frac{\frac{1}{n!} \prod_{r=1}^n (t + r)}{n - s} \\ &= \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s - t)} \end{aligned}$$

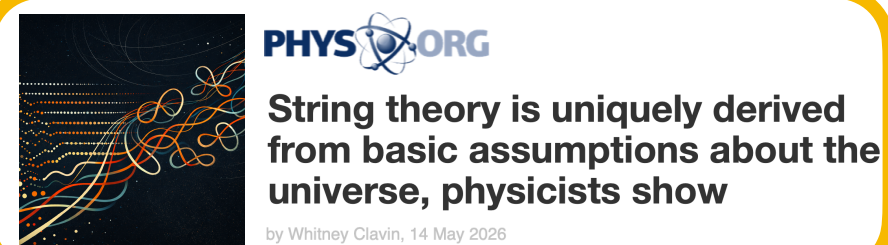
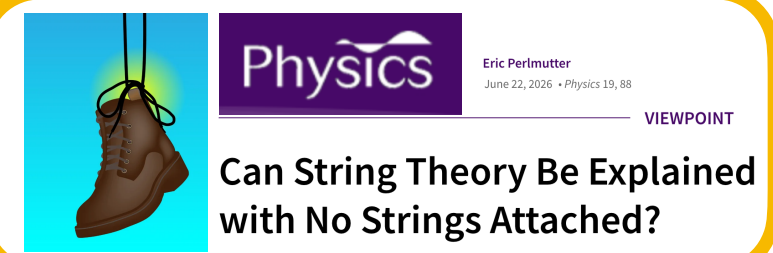
This is precisely the Veneziano amplitude of open string theory.

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What about non-planar scattering?

- We can similarly bootstrap string amplitudes beyond the planar limit, e.g., gravitational amplitudes.
- Consider the four-point amplitude for massless identical scalars. Fully Bose symmetric in s, t, u . Poles at $s, t, u = \mu(n)$, $s + t + u = 0$.
- Again take the mass, spin, and Regge trajectory as arbitrary. Assume only ultrasoftness and minimality as before.
- Implementing an analogous argument—involving the big arc integral, level truncation, etc.—yields...

Uniqueness of strings

...*uniquely* the following tree-level amplitude for four-point scattering:

$$\begin{aligned} A(s, t) &= \sum_{n=0}^{\infty} \left(\frac{1}{n-s} + \frac{1}{n-u} \right) \frac{1}{(n!)^2} \prod_{r=1}^{n-1} (t+r)^2 \\ &= - \frac{\Gamma(-s)\Gamma(-t)\Gamma(-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \end{aligned}$$

This is precisely the Virasoro-Shapiro amplitude of closed string theory: the ultraviolet completion of gravitational scattering amplitudes.

The background is a dark, textured space filled with numerous small, multi-colored dots (orange, yellow, blue, white) scattered across the left and bottom portions. On the right side, there are several thick, flowing, and swirling lines in shades of orange, red, yellow, and blue. These lines form complex, looping patterns that resemble stylized ribbons or paths. In the upper right, there are faint, thin, curved lines that look like orbits or trajectories. A semi-transparent horizontal band is positioned across the middle of the image, serving as a backdrop for the title text.

Uniqueness at Five Point

Uniqueness at five point

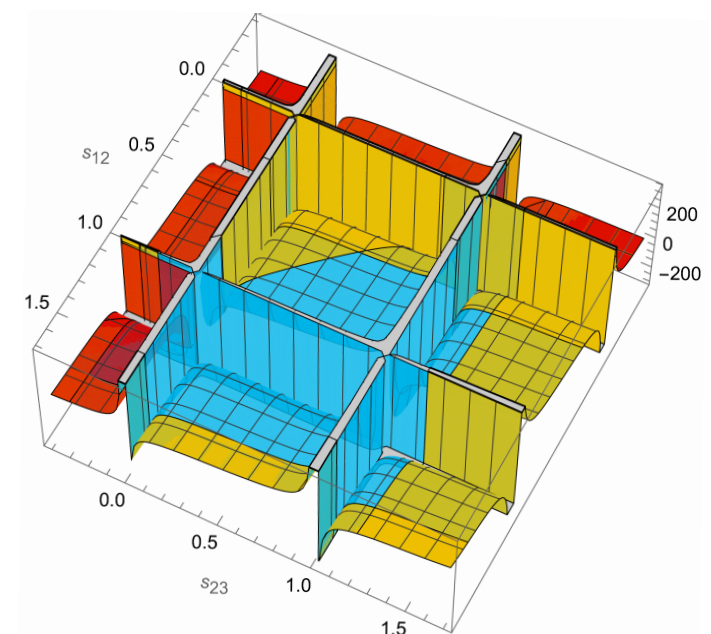
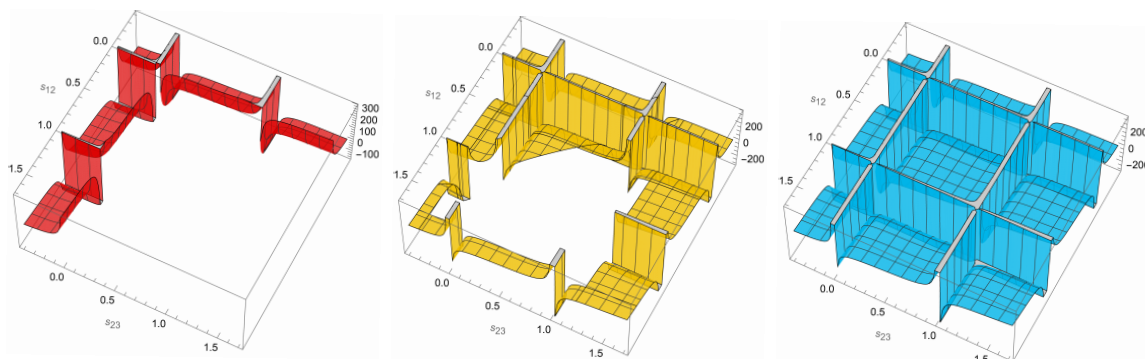
A analogous analysis uniquely establishes the *five*-point open-string amplitude.

Crucial ingredients:

- Knowledge of the five-point string amplitude itself, evaluable everywhere in kinematic space. We first found this in [Arkani-Hamed, Figueiredo, GR \[2412.20639\]](#).

$$A_{12345} = \frac{\Gamma(-s_{12})\Gamma(-s_{23})\Gamma(-s_{34})\Gamma(-s_{45})\Gamma(-s_{51})\Gamma(-s_{23} - s_{34} - s_{45})}{\Gamma(-s_{12} - s_{23})\Gamma(-s_{23} - s_{34})\Gamma(-s_{34} - s_{45})\Gamma(-s_{45} - s_{51})} \times$$

$$\times {}_6F_5 \left[\begin{matrix} -s_{23}, -s_{45}, -s_{34} - s_{45} + s_{12}, -s_{23} - s_{34} + s_{51}, -s_{23} - s_{34} - s_{45} - 1, \frac{-s_{23} - s_{34} - s_{45} + 1}{2} \\ -s_{12} - s_{23}, -s_{23} - s_{34}, -s_{34} - s_{45}, -s_{45} - s_{51}, \frac{-s_{23} - s_{34} - s_{45} - 1}{2} \end{matrix} ; -1 \right]$$



Uniqueness at five point

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Crucial ingredients:

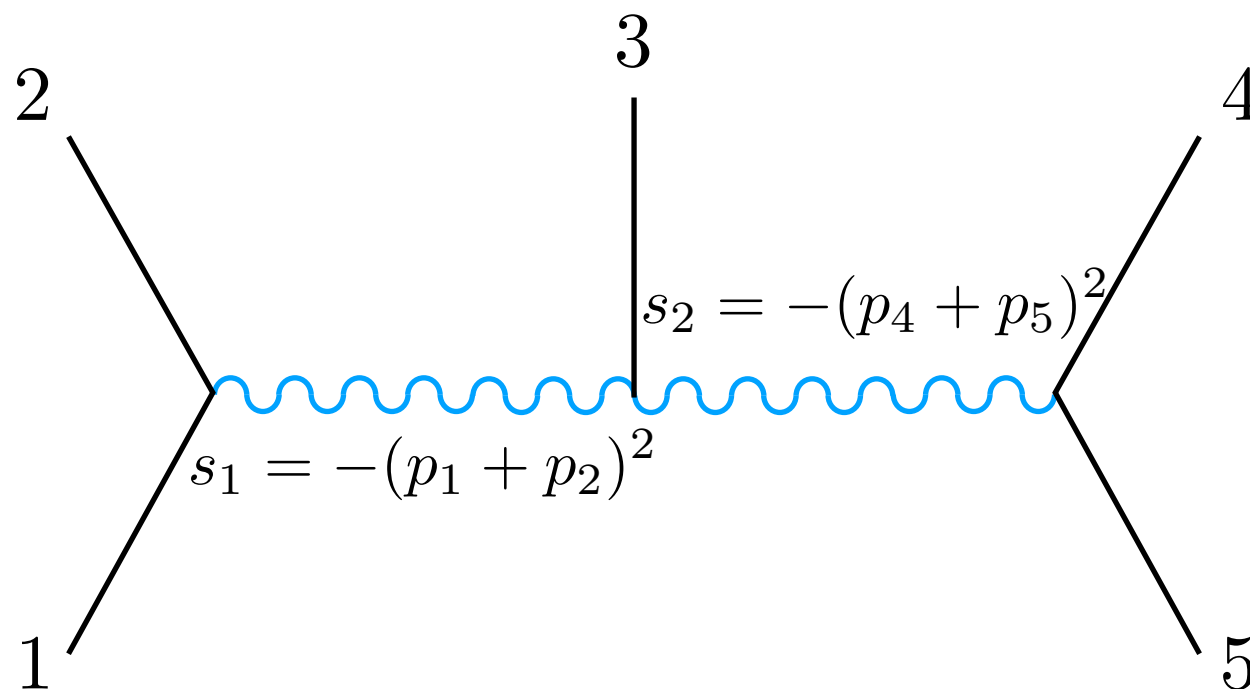
- Knowledge of the five-point string amplitude itself [Arkani-Hamed, Figueiredo, GR \[2412.20639\]](#)
- Five-point Regge limits [Arkani-Hamed, Figueiredo, GR \[2412.20639\]](#)

$$A_{\text{str},5} \stackrel{s_{1,2} \rightarrow \infty}{\sim} f(a_1, a_2, a_{12}) (-s_1)^{a_1} (-s_2)^{a_2} (-s_1 - s_2)^{a_{12}}$$

$$a_1 = t_{51} - t_{34}$$

$$a_2 = t_{51} - t_{23}$$

$$a_{12} = t_{23} + t_{34} - t_{51}$$



$$t_{23} = -(p_2 + p_3)^2$$

$$t_{34} = -(p_3 + p_4)^2$$

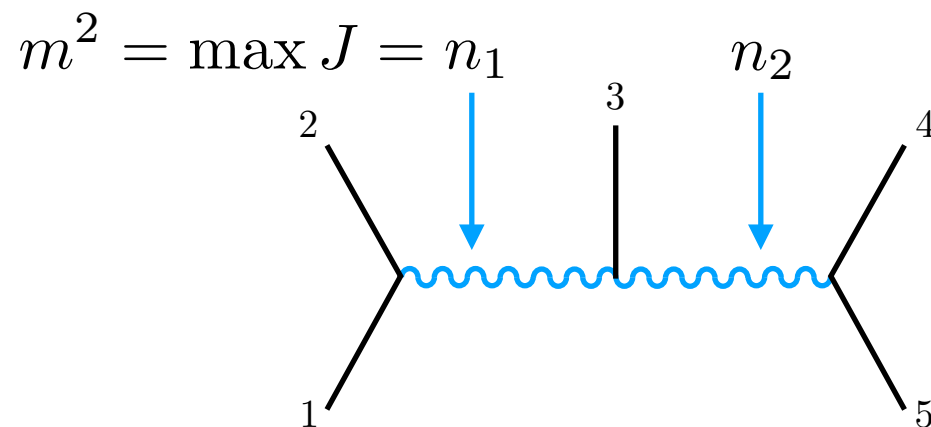
$$t_{51} = -(p_5 + p_1)^2$$

Uniqueness at five point

A analogous analysis uniquely establishes the *five*-point open-string amplitude.

Crucial ingredients:

- Knowledge of the five-point string amplitude itself [Arkani-Hamed, Figueiredo, GR \[2412.20639\]](#)
- Five-point Regge limits [Arkani-Hamed, Figueiredo, GR \[2412.20639\]](#)
- Using these Regge limits to deduce the locations of the five-point Regge zeros, we find that again these zeros uniquely fix the form of an *arbitrary* residue ansatz: [Cheung, GR, Sciotti, Tarquini \[2508.09246\]](#)



$$R(n_1, n_2, t_{23}, t_{34}, t_{51}) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{\min(n_1-i, n_2-j)} \lambda_{i,j,k}(n_1, n_2) t_{23}^i t_{34}^j t_{51}^k$$

$$\Rightarrow A_5 = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{c(n_1, n_2)}{(n_1 - s_1)(n_2 - s_2)} \frac{(1 + t_{23})_{n_1} (1 + t_{34})_{n_2}}{n_1! n_2!} {}_3F_2 \left[\begin{matrix} -n_1, -n_2, t_{51} - t_{23} - t_{34} \\ -n_1 - t_{23}, -n_2 - t_{34} \end{matrix} ; 1 \right]$$

Uniqueness at five point

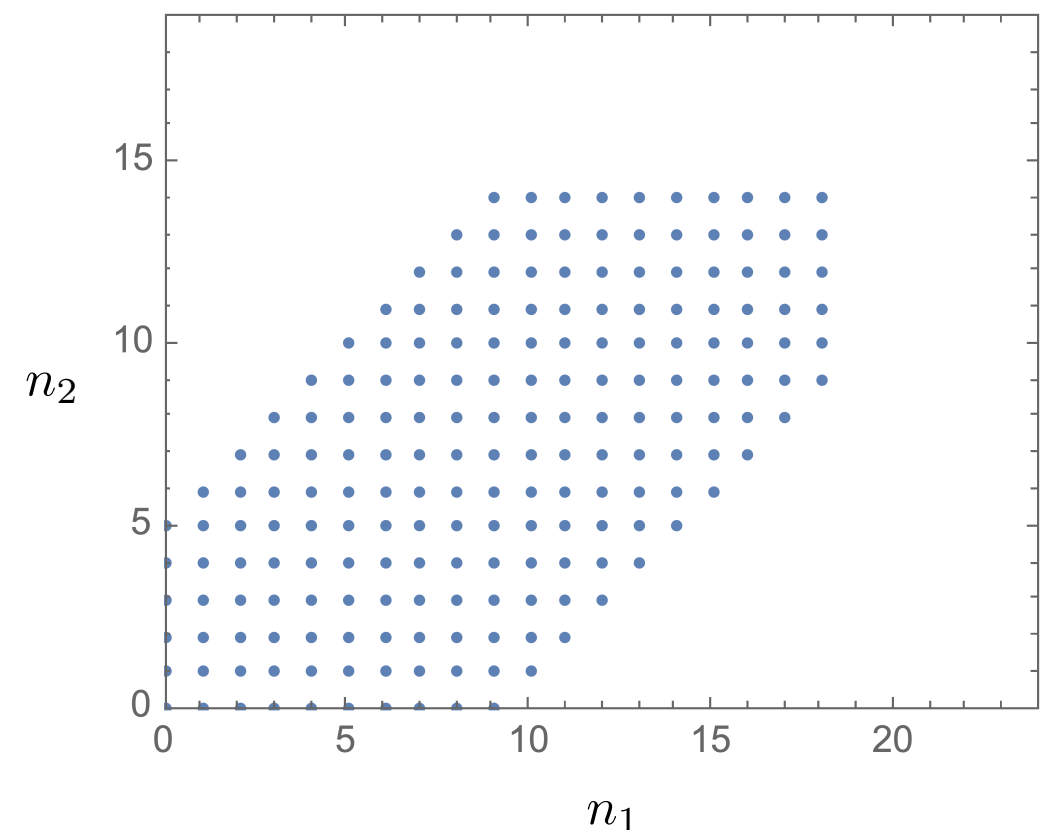
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- Five-point Regge zeros $\implies$ unique residues [Cheung, GR, Sciotti, Tarquini \[2508.09246\]](#)
- Five-point level truncation at special kinematics

$$A_5(s_1, s_2, -k_1, -k_2, -k_3) =$$

$$\sum_{n_1, n_2 \in D} \frac{c(n_1, n_2) R_{\text{str},5}(n_1, n_2, -k_1, -k_2, -k_3)}{(n_1 - s_1)(n_2 - s_2)}$$

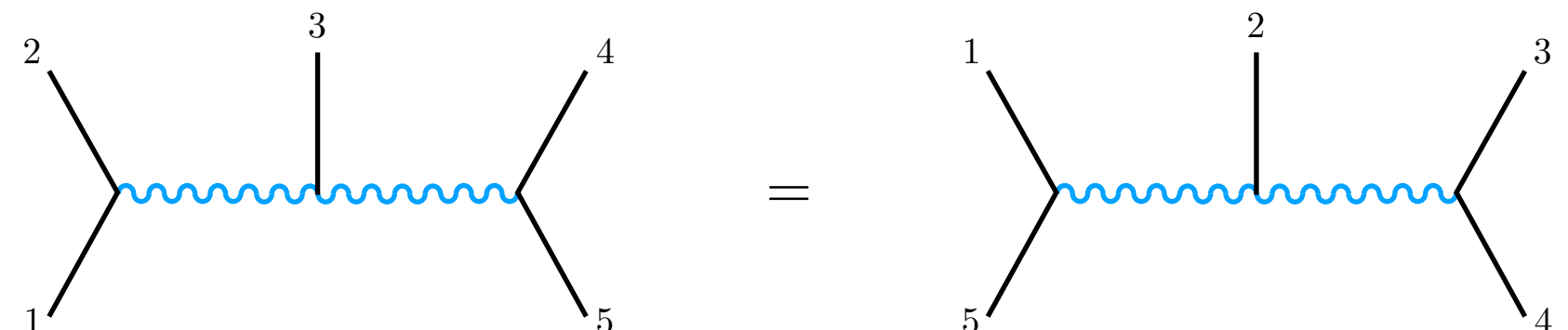


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- Five-point Regge zeros $\implies$ unique residues [Cheung, GR, Sciotti, Tarquini \[2508.09246\]](#)
- Five-point level truncation at special kinematics $\implies$ couplings fixed by cyclic invariance



$$A_5(-k_1, -k_4, -k_2, -k_3, -k_5) = A_5(-k_2, -k_5, -k_3, -k_4, -k_1)$$

$$\sum_{\substack{(n_1, n_2) \in \\ D(k_2, k_3, k_5)}} \frac{c(n_1, n_2) R_{\text{str},5}(n_1, n_2, -k_2, -k_3, -k_5)}{(n_1 + k_1)(n_2 + k_4)} = \sum_{\substack{(n_1, n_2) \in \\ D(k_3, k_4, k_1)}} \frac{c(n_1, n_2) R_{\text{str},5}(n_1, n_2, -k_3, -k_4, -k_1)}{(n_1 + k_2)(n_2 + k_5)}$$

Uniqueness at five point

Positing the same Regge scaling as the string, we find that our bootstrap ***uniquely*** gives us the amplitude for five-point planar scattering:

$$\begin{aligned} A_5(s_1, s_2, t_{23}, t_{34}, t_{51}) &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{1}{(n_1 - s_1)(n_2 - s_2)} \frac{(1 + t_{23})_{n_1} (1 + t_{34})_{n_2}}{n_1! n_2!} {}_3F_2 \left[\begin{matrix} -n_1, -n_2, t_{51} - t_{23} - t_{34} \\ -n_1 - t_{23}, -n_2 - t_{34} \end{matrix} ; 1 \right] \\ &= \frac{\Gamma(-s_1)\Gamma(-s_2)\Gamma(-t_{23})\Gamma(-t_{34})}{\Gamma(-s_1 - t_{23})\Gamma(-s_2 - t_{34})} {}_3F_2 \left[\begin{matrix} -s_1, -s_2, t_{51} - t_{23} - t_{34} \\ -s_1 - t_{23}, -s_2 - t_{34} \end{matrix} ; 1 \right] \end{aligned}$$

which is precisely the five-point open string amplitude.

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Discussion

Quantum gravity and dispersion relations

- Scattering amplitudes and bootstrap tools provide extremely powerful means of constraining and understanding the underpinnings of physics, including the possibilities for quantum gravity.
- In this work, we found how to uniquely pick out string theory by imposing conditions on the S-matrix and applying analytic dispersion relations.

Quantum gravity and dispersion relations

- Scattering amplitudes and bootstrap tools provide extremely powerful means of constraining and understanding the underpinnings of physics, including the possibilities for quantum gravity.
- In this work, we found how to uniquely pick out string theory by imposing conditions on the S-matrix and applying analytic dispersion relations.
- In another exciting application of dispersion relations, we recently demonstrated a version of the *completeness hypothesis* and *representation democracy* in quantum gravity: [Calisto, Cheung, GR, Sciotti, Tarquini \[2605.20319, 2512.11955\]](#)

- Given the existence of states charged under a symmetry G , there must exist single particles below the Planck scale occupying every single irrep ρ of G .
- These states must have interaction strengths that are democratically assigned, independent of ρ .

$$\begin{aligned}
 \sum_n \text{[t-channel exchange diagram]} &\propto -\frac{G_N \mathbb{1}}{t} \propto \sum_{\rho} \mathbb{P}_{\rho}^{(s)} \\
 \sum_n \text{[u-channel exchange diagram]} &\propto -\frac{G_N \mathbb{1}}{t} \propto \sum_{\rho} \mathbb{P}_{\rho}^{(u)}
 \end{aligned}$$

Conclusions

- In this talk, we have identified a set of two simple conditions—*ultrasoftness* and *minimality*—which allowed us to analytically bootstrap the amplitudes for string theory and prove that they are unique, without assuming a spectrum.
- We were able to accomplish this uniqueness bootstrap for both open and closed strings, and (in the open case) at both four-point and five-point.
- Open questions include all-multiplicity scattering, higher loop topologies, and whether nonperturbative features of the string can also be bootstrapped.

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- Open questions include all-multiplicity scattering, higher loop topologies, and whether nonperturbative features of the string can also be bootstrapped.

Investigations of this type allow us to understand and constrain the space of possibilities for quantum gravity. Evidence is strongly pointing to the inevitability of strings.

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Questions

Support proved by the Hertz Foundation and by
the James Arthur Postdoctoral Fellowship at NYU.