THE THREE-LOOP HADRONIC VACUUM POLARIZATION IN CHIRAL PERTURBATION THEORY

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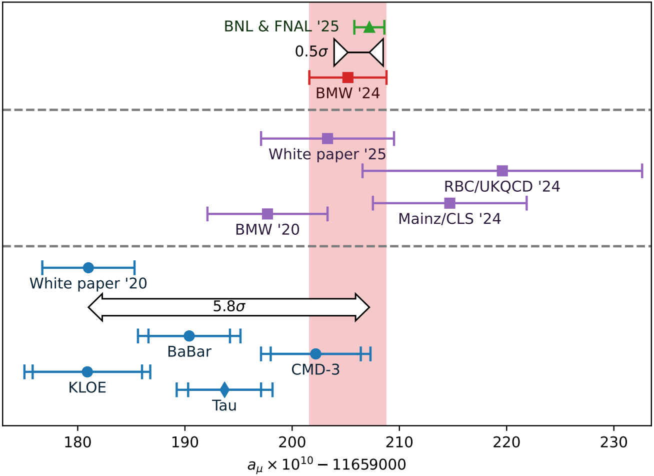
Based on arXiv:2510.12885, arXiv:2603.15252 and on going collaboration with Laurent Lellouch, Alessandro Lupo, Mattias Sjö, Kálmán Szabo

Gong show talk by Mattias Sjö at this meeting

The magnetic moment of muon

In June 2025: Fermilab announced the most precise measurement of the muon anomalous magnetic moment $a_\mu = (g_\mu - 2)/2$: 2506.03069 and 2606.17323

$$a_\mu^{\text{exp}} = 1165920715(145) \times 10^{-12}$$



BMW-DMZ collaboration Hybrid calculation of hadronic vacuum polarization in muon $g-2$ to 0.48%

Comparison with Standard Model Predictions

The final Fermilab measurement is fully compatible with the Standard Model prediction from the 2025 White Paper. However:

- The experimental result is 4.3 times more precise than the theoretical evaluation of the white paper
- 2.3 times more precise than the most precise one, performed by BMW-DMZ.

A tension remains between the leading-order hadronic contribution a_μ^{LO} evaluated via:

- dispersion relations using $e^+e^- \to$ hadrons data
- Lattice QCD

This tension must be resolved to enable a rigorous comparison between theory and experiment. To address this, we compute the low-energy contributions to the hadronic vacuum polarization using Chiral Perturbation Theory (ChPT), an effective field theory for QCD.

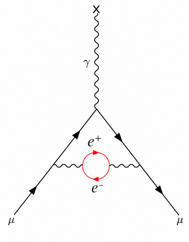
This talk and Mattias Sjö's talk at this meeting

Standard Model Contributions to a_μ

At the needed precision all the three interactions and all the standard model particles contribute to the $g-2$ of the muon

$$a_\mu^{\rm SM} = a_\mu^{\rm QED} + a_\mu^{\rm EW} + a_\mu^{\rm hadronic}$$

$=$



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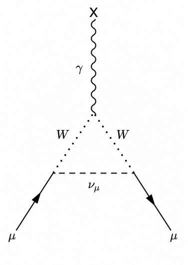

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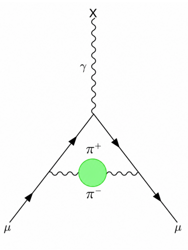


The concensus numerical QED contribution gives up to 99.994% of the result [Laporta '95; Aguilar et al. '08; Aoyama et al. '96-'19; Volkov '19-'24], which together with the Electro-Weak contribution leaves the remaining part for the hadronic contribution [the $g-2$ white papers '20]

$$a_\mu^{\rm hadronic} = a_\mu^{\rm exp} - a_\mu^{\rm QED} - a_\mu^{\rm EW} = 719.6(1.5) \times 10^{-10}$$

Chiral pertubation theory (ChPT)

We focus on low-energy hadronic contributions using ChPT, an effective field theory of QCD.

- ChPT describes the spontaneous and explicit breaking of chiral symmetry in massless QCD.
- We consider an $SU(2)$ triplet of pions (equal mass) coupled to a non-dynamical photon field.
- The Lagrangian is expanded in powers of momentum:

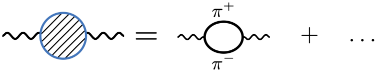
$$\mathcal{L}^{\mathrm{QCD}}(q, \bar{q}, A) \to \mathcal{L}^{\mathrm{ChPT}} = \mathcal{L}^{\mathrm{LO}}_{O(p^2)} + \mathcal{L}^{\mathrm{NLO}}_{O(p^4)} + \mathcal{L}^{\mathrm{NNLO}}_{O(p^6)} + \mathcal{L}^{\mathrm{N^3LO}}_{O(p^8)} + \cdots$$

The EFT contains low-energy constants (LECs) determined by matching physical observables [Gasser, Leutwyler '83; Bijnens, Colangelo, Ecker '99, Bijnens, Hermansson-Truedsson, Wang '18]

- NLO: 7 counterterms.
- NNLO: 57 counterterms.
- $\mathrm{N^3LO}$: 475 counterterms.

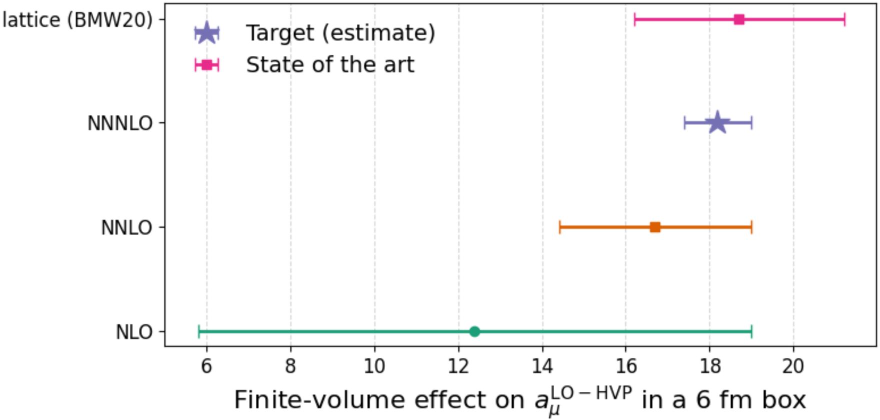
Vacuum Polarization Tensor

$$\Pi^{\mu\nu}(p) := \int d^4x\, e^{iqx}\, \mathrm{tr}\left\langle 0 \right| T\left\{ j^\mu(x) j^\nu(0) \right\} \left| 0 \right\rangle$$



Finite-volume corrections in lattice QCD

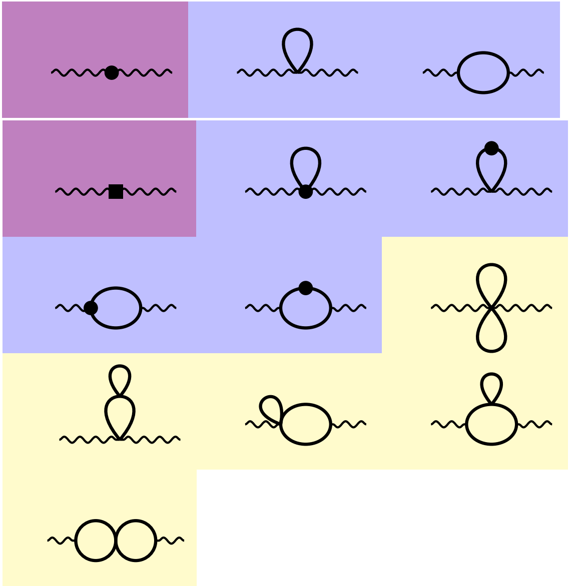
Extrapolating NLO and NNLO ChPT results for finite-volume corrections to HVP on a $(6\ \mathrm{fm})^3$ lattice shows that $\mathrm{N}^3\mathrm{LO}$ precision is required to match Fermilab's experimental accuracy.



NLO- and NNLO contributions

The NLO (one-loop) contributions

The NNLO (two-loop) contributions

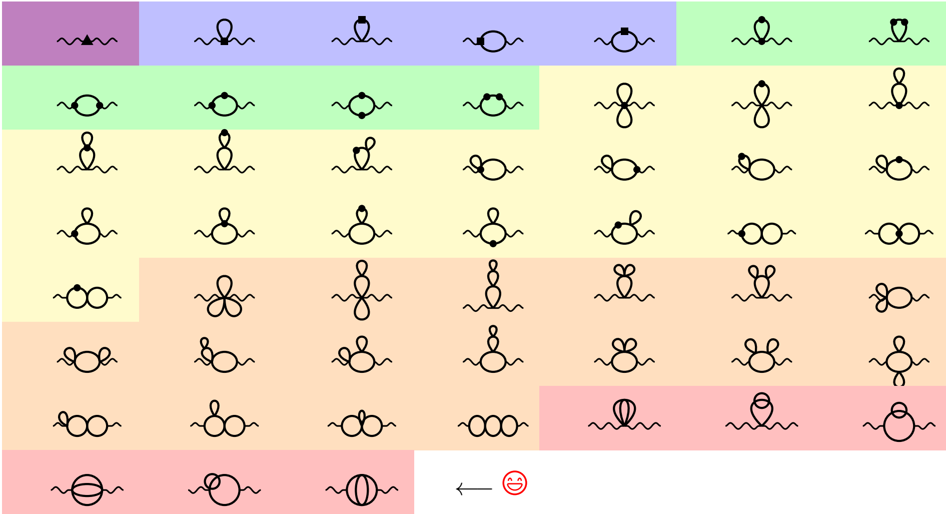


Only bubbles and tadpoles and only the 2 pions production channel

$$e^+e^- \to \gamma \to 2\pi$$

This has been evaluated previously [Gasser, Leutwyler; Bijens et al.]

N^{3}LO diagrams: arXiv:2510.12885, arXiv:2603.15252



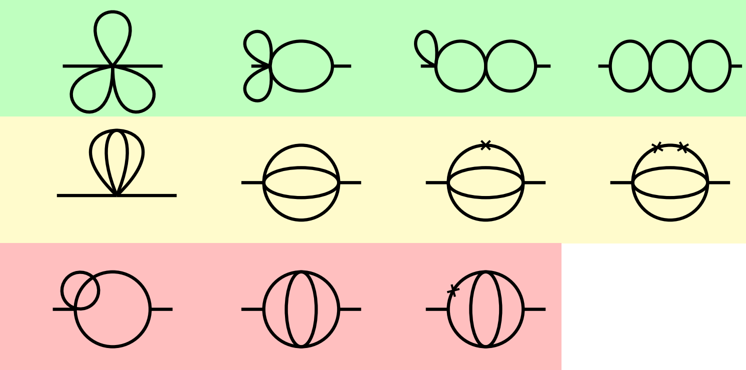
At N^{3}LO elliptic integrals appear which open the four pion production channel

$$e^+e^- \to \gamma \to 4\pi$$

This was not evaluated before

N^3LO Masters

Using **LiteRed** we obtained eleven master integrals.

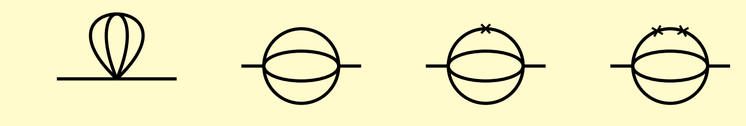


A first sector is composed of products of tadpoles and bubbles : Evaluate to multiple polylogarithms. Easy to compute analytically and numerically evaluate to high precision

A second sector composed by the three-loop all-equal mass sunset integrals

New elliptic master integrals not evaluated in the literature

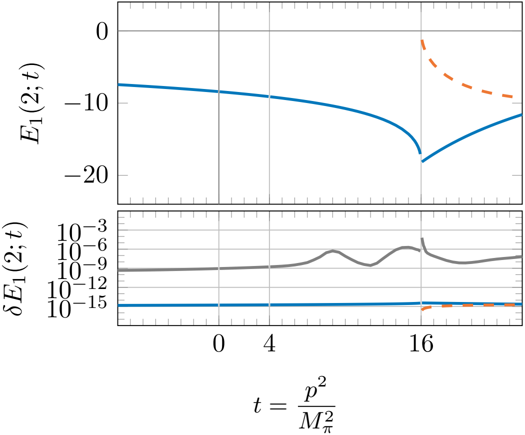
N^3LO Master : Sunset sector



The three-loop equal-mass sunset integrals known from [Block, Kerr, Vanhove; Weinzierl et al.; Duhr et al.]

The elliptic 3-log formulation of [Block, Kerr, Vanhove] allows a fast and very precise numerical evaluation for all values in the complex energy plane

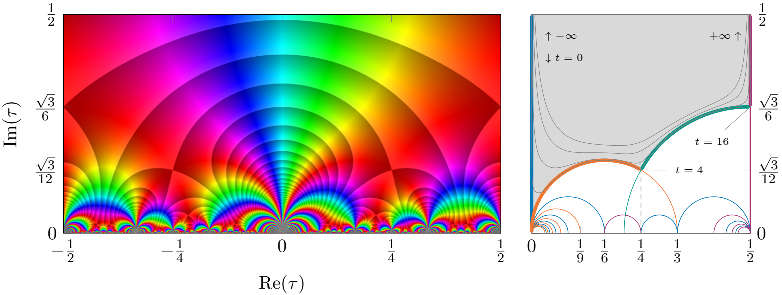
E.g. the real (——) and imaginary (– – –) parts of the sunset integral



From $t = p^2/M_\pi^2$ to elliptic nome

The elliptic master integrals are (relative) periods of a $K3$ surface with Picard number 19 expressed as elliptic polylogarithms where the natural variable is the nome of an underlying elliptic curve [Bloch, Kerr, Vanhove]

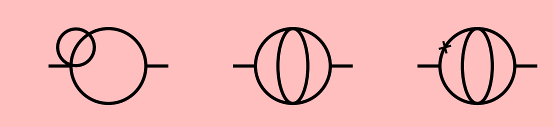
$$t = \frac{p^2}{M_\pi^2} = -\left(\frac{\eta(q)\eta(q^3)}{\eta(q^2)\eta(q^6)}\right)^6, \qquad q = \exp(2\pi i\tau)$$



a sketch of the $\Re\mathrm{e}(\tau) > 0$:

- timelike subthreshold ($0 < t < 4$ **orange**)
- two-pion ($4 < t < 16$, **green**)
- multi-pion ($16 < t$, **violet**)
- The shaded area contains all t with $\Im\mathrm{m}(t) > 0$

N^3LO Master : New elliptics integrals



New elliptic master integrals not known in the literature. Using dimension shifting relation one can expose their ϵ expansion

$$E_5(4-2\epsilon;t) = \frac{1}{3\epsilon^3} + \frac{1-3J^{(1)}_{\bigcirc}}{3\epsilon^2} + \frac{\frac{1}{2}J^{(2)}_{\bigcirc} + (J^{(1)}_{\bigcirc})^2 + \frac{\pi^2}{12} - J^{(1)}_{\bigcirc} + \frac{1}{3}}{\epsilon} + \bar{E}_5(t) + O(\epsilon)$$

- $J^{(n)}_{\bigcirc}$ are the polylogarithms from the ϵ expansion of the one-loop bubble

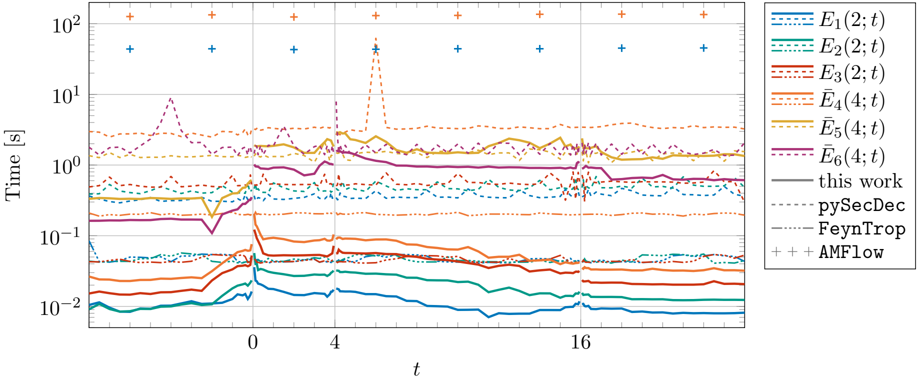
$$J^{(n)}_{\bigcirc}(t) := \int_0^1 dx\, \log^n \left[1 - x(1-x)t\right]$$

- **The ϵ poles cannot contains elliptics:** to show this one needs Schouten-like identities between the ϵ expansion of the elliptic masters near $D=2$ dimensions. These identites are not consequences of IBP relations
- Finite piece $\bar{E}_5(\beta)$ satisfies a second order differential equation with inhomogeneous term containing the elliptic sunset integral (setting $\beta^2 = 1 - 4/t$)

$$\beta^2 \left[\frac{\partial^2}{\partial\beta^2} - \frac{2\beta^2}{\beta^2-1}\right] \bar{E}_5(\beta) = S_{\text{rational}} + S_{\text{polyg}} + S_{\text{elliptic}}$$

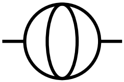
N^{3}LO Elliptic Master : numerical evaluation : timing

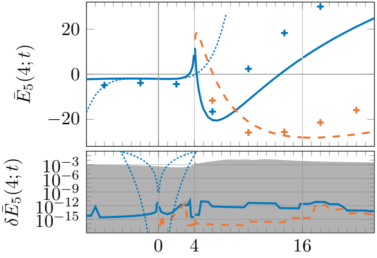
Using the elliptic 3-logarithm representation given in [Bloch, Kerr, Vanhove], we have obtained a precise and fast numerical evaluation of the elliptic masters.

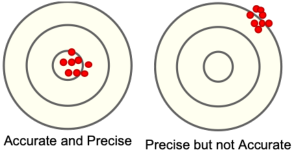


The codes for the numerical implementation are available
https://github.com/mssjo/HVP-numerics

We have as well compared to **pySecDec**, **Feyntrop** (only for convergent integrals) and **AMFlow**







Real (——) and imaginary (- - -) parts of the elliptic master integral E_5 plotted as functions of $t = p^2/M_\pi^2$.

Precision

- our implementations and `pySecDec` are indistinguishable on the scale of the plot.
- The isolated points (+ +) are obtained using `AMFlow`
- the dashed lines (·········) indicate the 8th-order series expansion around $t = 0$

Error

- error band of `pySecDec` in gray
- `AMFlow` was run with a precision goal of 10^{-12}

The HVP @ N^{3}LO: analytic result

The primary goal of this work was to have complete analytic and numerical control of the N^{3}LO contribution to the HVP.

$$16\pi^2\bar\Pi_T(q^2) = \bar\Pi_T^{\rm NLO}(t)+\frac{M_\pi^2}{16\pi^2F_\pi^2}\bar\Pi_T^{\rm NNLO}(t)+\left(\frac{M_\pi^2}{16\pi^2F_\pi^2}\right)^2\bar\Pi_T^{\rm N^3LO}(t)+\mathcal{O}\left(\left(\frac{M_\pi^2}{16\pi^2F_\pi^2}\right)^3\right),$$

$$\frac{M_\pi^2}{16\pi^2F_\pi^2}\approx 0.014,$$

$$\bar\Pi_T^{\rm NLO}(t) = 2\frac{4-t}{3t}J^{(1)}_{\ominus}(t)+\frac{4}{9},$$

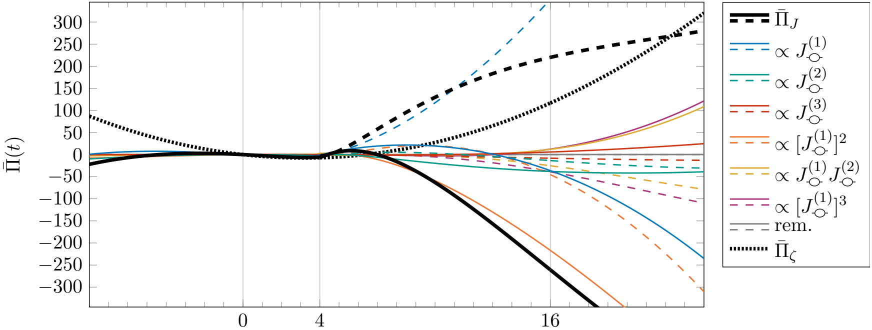
$$\bar\Pi_T^{\rm NNLO}(t) = t\left[\frac{4-t}{3t}J^{(1)}_{\ominus}(t)-\frac{1+3L_\pi}{9}\right]^2-4t\left[\frac{4-t}{3t}J^{(1)}_{\ominus}(t)-\frac{1+3L_\pi}{9}\right]l_6^q-8tc_{56}^q,$$

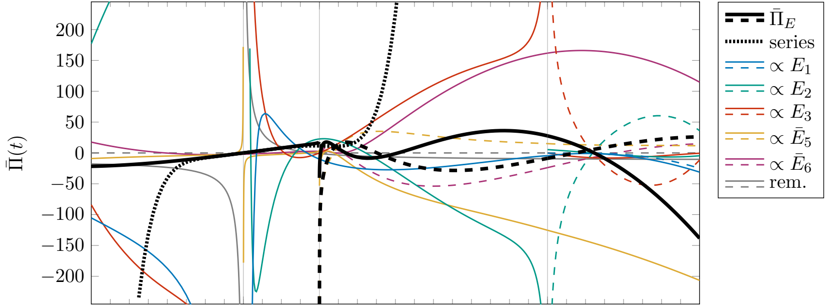
$$\bar\Pi_T^{\rm N^3LO}(t)=\bar\Pi_E(t)+\bar\Pi_J(t)+\bar\Pi_\zeta(t)+\bar\Pi_l(t)+\bar\Pi_c(t)+tr_1^q+t^2r_2^q$$

where r_1^q and r_2^q are new NNLO LECs appearing in the $\mathcal{O}(t)$ and $\mathcal{O}(t^2)$ terms undetermined in the literature.

The code for a complete derivation from scratch (derivation of Feynman graphs, reductions to master, renormalisation) upto an including the N^{3}LO order is availaible from https://github.com/mssjo/HVP-3loop

The HVP @ N^{3}LO: numerical evaluation



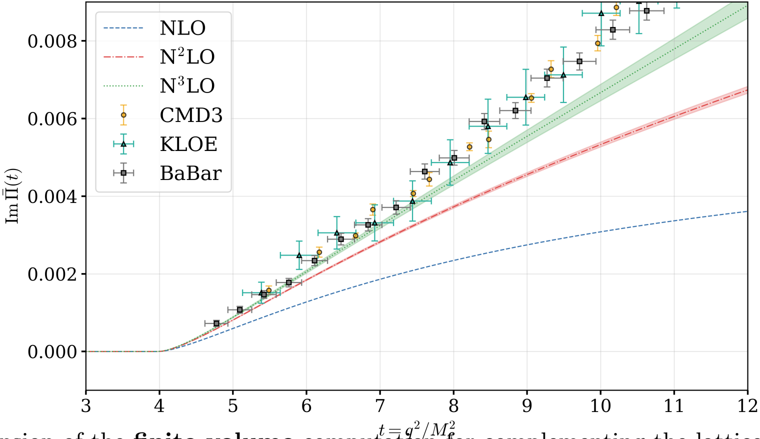


Outlook : Works in progress [Lellouch, Lupo, Sjö, Vanhove]

By achieving complete analytic and numerical control over the three-loop hadronic vacuum polarization we now have the robust mathematical tools required to properly address finite-volume effects and low $e^+e^- \to$hadrons phenomenology.

1. A study of the phenomenological impact of our N^{3}LO computation using the dispersion relation and comparison with data from Babar, CMD3 and KLOE

thanks to B. Malaescu for sharing his compilation of $e^+e^- \to \pi^+\pi^-$ cross section from Babar, CMD3 and KLOE



2. The extension of the **finite volume** computation for complementing the lattice calculation of the HVP contribution to a_μ