Tropical quantum field theory

Amplitudes 2026
Queen Mary University London

arXiv:2508.14263

June 30, 2026

Michael Borinsky — Perimeter Institute

QFT and Amplitudes

Standard perturbative QFT workflow

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_G \frac{1}{|\mathrm{Aut}(G)|} I_G$$

QFT observable/
amplitude

Sum over graphs with
L loops and n legs

Symmetry
factor

Feynman integral

Example

PHYSICAL REVIEW LETTERS **130,** 071801 (2023)

Editors' Suggestion **Featured in Physics**

Measurement of the Electron Magnetic Moment

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(Received 27 September 2022; revised 7 December 2022; accepted 7 December 2022; published 13 February 2023)

The electron magnetic moment, $-\mu/\mu_B = g/2 = 1.001\,159\,652\,180\,59\,(13)$ [0.13 ppt], is determined 2.2 times more accurately than the value that stood for fourteen years. The most precisely determined property of an elementary particle tests the most precise prediction of the standard model (SM) to 1 part in 10^{12}. The test would improve an order of magnitude if the uncertainty from discrepant measurements of the fine structure constant α is eliminated since the SM prediction is a function of α. The new measurement and SM theory together predict $\alpha^{-1} = 137.035\,999\,166\,(15)$ [0.11 ppb] with an uncertainty 10 times smaller than the current disagreement between measured α values.

DOI: 10.1103/PhysRevLett.130.071801

… theoretical computations up to
12 digits come from
perturbative quantum field theory

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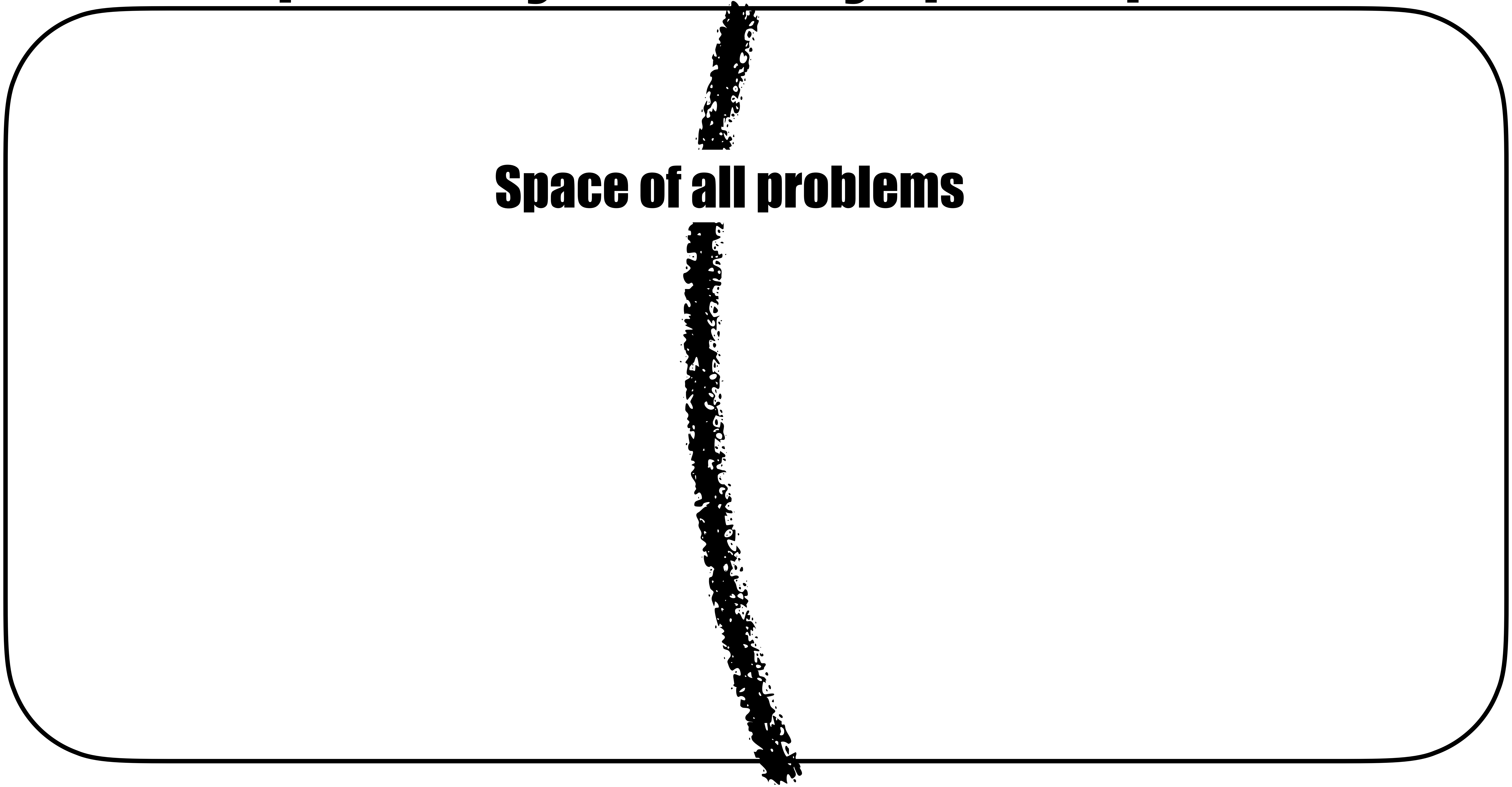
⇒ Using the standard approach predictions take **factorial** computer-time

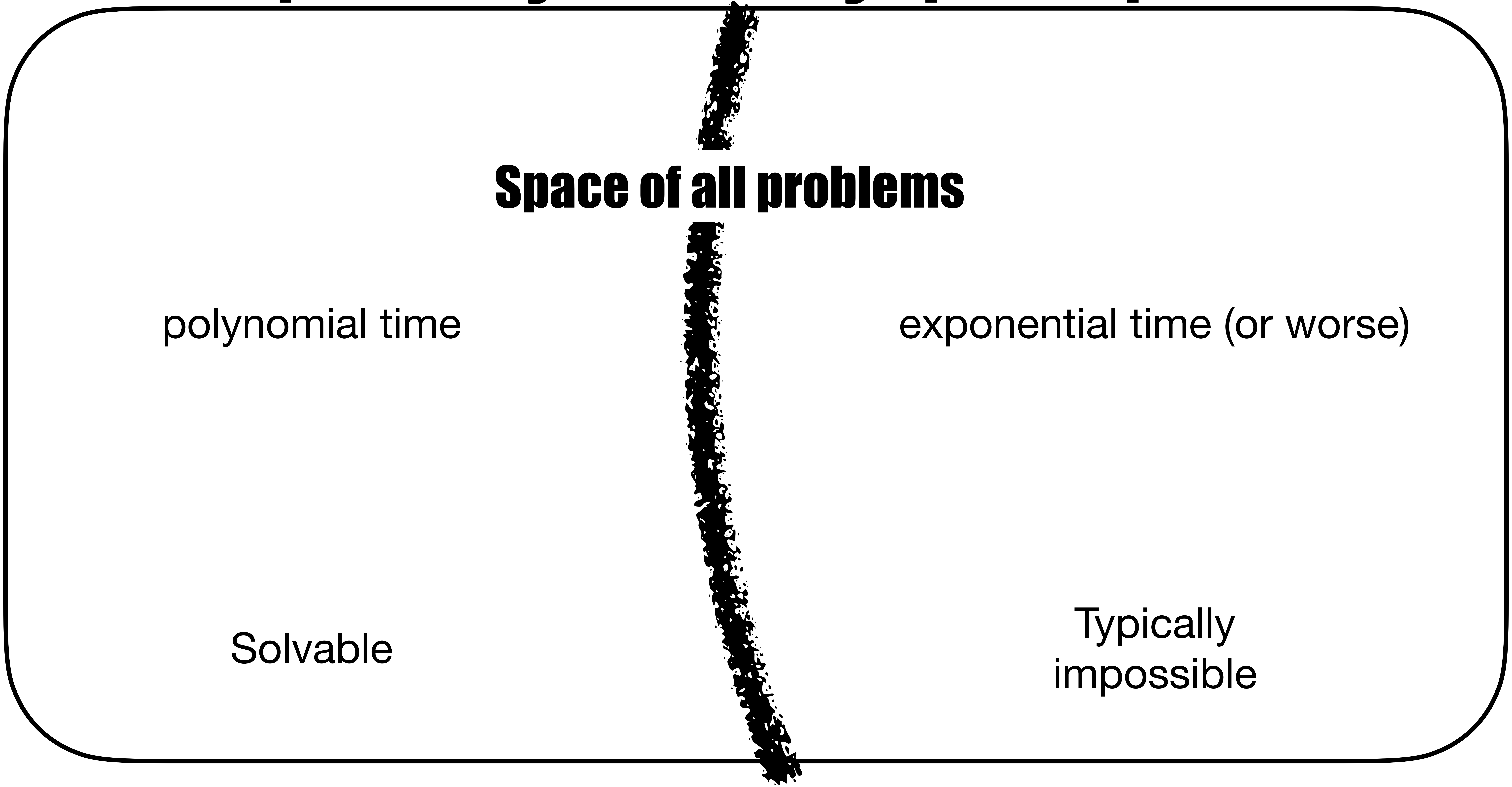
Complexity theory perspective

Space of all problems

Complexity theory perspective



Complexity theory perspective



Space of all problems

polynomial time

Solvable

exponential time (or worse)

Typically impossible

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_G \frac{1}{|\mathrm{Aut}(G)|} I_G$$

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Major problem: The number of graphs grows as $(L!)$

⇒ Using the standard approach predictions take **factorial** computer-time

⇒ Predictions eventually become more expensive than measurement

Message of this talk: It is possible to do much better.

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Theorem: (MB 2025)

In all Euclidean, scalar and finite QFTs,

$\mathscr{A}_{L,n}(p_1, \ldots, p_n)$ can be be computed to finite accuracy in

polynomial-time in L and n

independently of the spacetime dimension.

$$\mathscr{A}_{L,n}(p_1,\ldots,p_n) = \sum_G \frac{1}{|\mathrm{Aut}(G)|} I_G$$

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amplitude

Sum over graphs with
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Symmetry
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Feynman integral

Feynman integrals

$$I_G = \int_{\mathbb{M}^{DL}} \frac{d^D k_1 \cdots d^D k_L}{\prod_e (q_e^2 - m_e^2 + i\varepsilon)}$$

- Hard to evaluate: There is no entirely general, effective evaluation algorithm.

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- Fastest algorithm works for finite, scalar, Euclidean Feynman integrals+:

Runtime (exponential): $\mathcal{O}(|E_G| 2^{|E_G|} + \delta^{-2} |V_G|^3)$

MB 2020

⇒ Evaluation (of 5-6 digits) is relatively fast up to ~17 loops

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MB 2020

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⇒ Even evaluating an individual Feynman integral takes exponential time in L, n.

Recap: ‘Fast’ tropical Feynman integration

… using tropical geometry

MB 2020; drawing from: Schultka 2018, Panzer 2019, Aguiar-Ardila 2017

$$I_G$$

Feynman integral
(complicated)

Recap: ‘Fast’ tropical Feynman integration

… using tropical geometry

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$$\begin{tikzcd} I_G \arrow[r, dotted, "\text{Tropicalization}"'] & I_G^{\mathrm{tr}} \end{tikzcd}$$

Feynman integral
(complicated)

Tropical Feynman integral
(computable in exponential runtime)

Recap: ‘Fast’ tropical Feynman integration

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MB 2020; drawing from: Schultka 2018, Panzer 2019, Aguiar-Ardila 2017

$$\begin{tikzcd}
I_G \arrow[r, dotted, "\text{Tropicalization}"'] & I_G^{\mathrm{tr}} \arrow[l, dotted, bend right=60, "\substack{\text{Fast (numerical) integration algorithm} \\ \text{(tropical sampling)}}"']
\end{tikzcd}$$

Feynman integral
(complicated)

Tropical Feynman integral
(computable in exponential runtime)

New result

New result

We can lift this cycle to the global level of the whole QFT

New result

We can lift this cycle to the global level of the whole QFT

… and it is much more effective

Tropical QFT and global tropical sampling

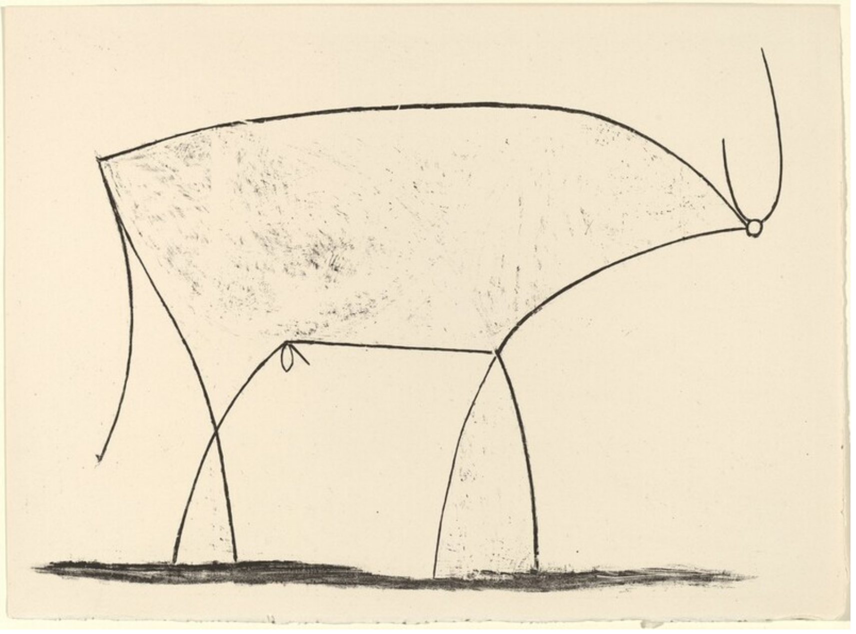
MB 2025

QFT

(complicated)

Tropical QFT and global tropical sampling

MB 2025



QFT $\cdots\cdots\cdots\cdots\blacktriangleright$ **Tropical QFT**

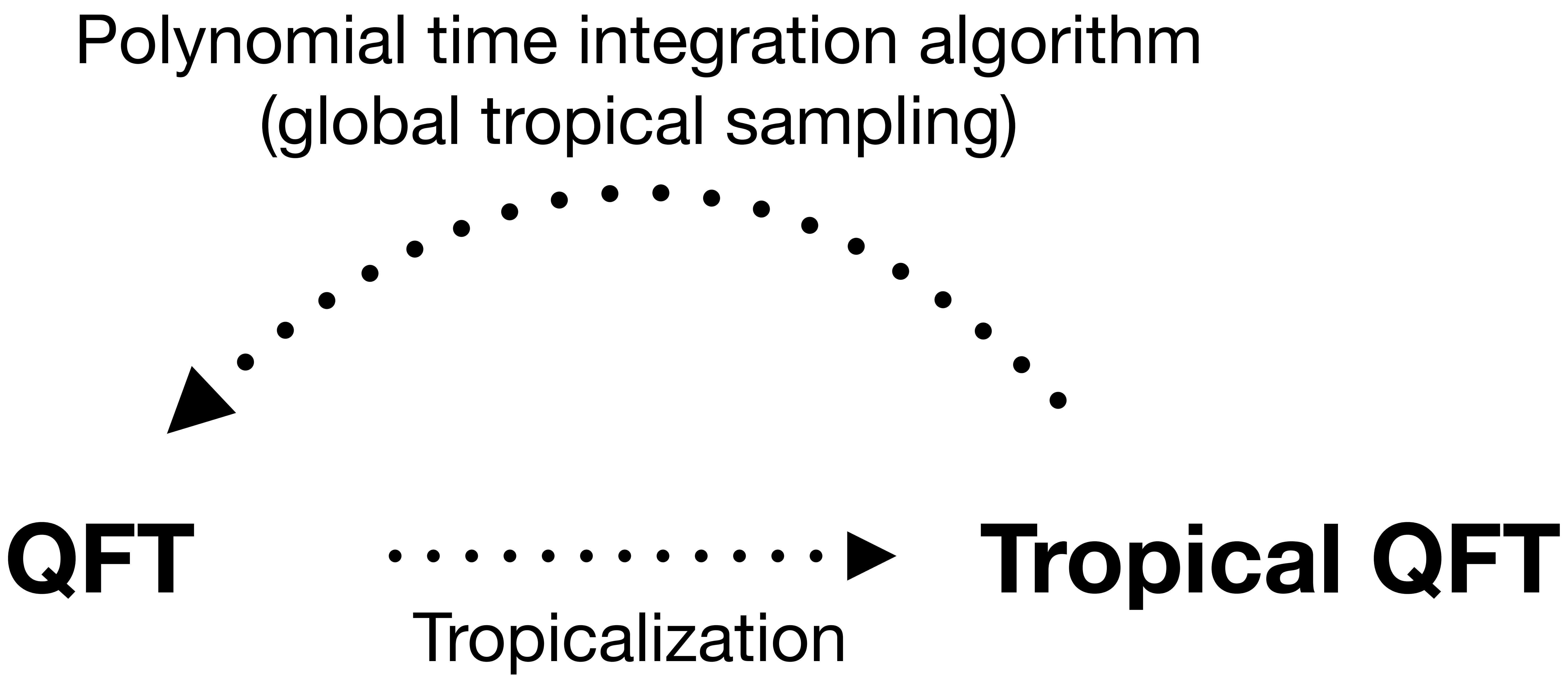
Tropicalization

(complicated)

Exactly solvable (in polynomial time)

Tropical QFT and global tropical sampling

MB 2025



Polynomial time integration algorithm
(global tropical sampling)

QFT

Tropicalization

Tropical QFT

(complicated)

Exactly solvable (in polynomial time)

Tropical QFT

MB arXiv:2508.14263

Inspiration from and relations to **Surfaceology**

Arkani-Hamed, Figueiredo, Frost, Salvatori, Plamondon, Thomas, … 2023—

Particularly

Salvatori 2025

Important empirical data:

Balduf 2023, Balduf—Shaban 2024, MB-Favorito 2025, Balduf—Thürigen 2025

Tropical QFT

Two complementary viewpoints— Lagrangian and Geometric

Tropical QFT

Lagrangian viewpoint

Massive scalar quantum field theory

$$V[\Phi] = \sum_{k\geq 3} \lambda_k \Phi^k / k!$$

Action $$\mathcal{S}[\Phi, J] = \int_{\mathbb{R}^D} \mathrm{d}^D x \left(\frac{1}{2}\Phi(x)\left(\Box + m^2\right)\Phi(x) - V[\Phi](x) - J(x)\Phi(x) \right);$$

Massive scalar quantum field theory

$$V[\Phi] = \sum_{k\geq 3} \lambda_k \Phi^k / k!$$

Action

$$\mathcal{S}^{\xi}[\Phi, J] = \int_{\mathbb{R}^{D\cdot\xi}} \mathrm{d}^{D\cdot\xi}x \left(\frac{1}{2}\Phi(x)\left(\Box + m^2\right)^{\xi}\Phi(x) - V[\Phi](x) - J(x)\Phi(x) \right),$$

Deformation: $D \to D \cdot \xi$ and $(\Box + m^2) \to (\Box + m^2)^{\xi}$

(Relative mass scalings of all operators remain constant.)

Massive scalar quantum field theory

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Action $$\mathcal{S}^{\xi}[\Phi, J] = \int_{\mathbb{R}^{D\cdot\xi}} \mathrm{d}^{D\cdot\xi} x \left(\frac{1}{2} \Phi(x) \left(\Box + m^2\right)^{\xi} \Phi(x) - V[\Phi](x) - J(x)\Phi(x) \right);$$

Partition function $$Z^{\xi}[J] = \int \exp\left(-\mathcal{S}^{\xi}[\Phi, J]\right) \mathcal{D}[\Phi]$$

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Initial QFT

$$\xi = 1$$

Tropical QFT

$$\lim_{\xi \to 0^+} Z^\xi = Z^{\mathrm{tr}}$$

Tropical QFT is a **deformation limit** of traditional QFT.

The tropical QFT is exactly solvable

Effective action (a generating function of tropicalized amplitudes):

$$\Gamma^{\mathrm{tr}}(\hbar, \varphi) = \sum_{L,n} \mathscr{A}^{\mathrm{tr}}_{L,n} \hbar^L \varphi^n$$

Recursive solution ↔ exact solvability

Theorem MB 2025

The tropical QFT's effective action is completely fixed by the nonlinear PDE

$$\mathscr{P}_D \Gamma^{\mathrm{tr}} = \left(1 - \frac{\partial^2 \Gamma^{\mathrm{tr}}}{\partial \varphi^2} \right)^{-1} - 1, \qquad \text{where}$$

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where

$$\mathcal{P}_D = \left(-D - \left(1 - \frac{D}{2}\right) \varphi \frac{\partial}{\partial \varphi} + \sum_{k \geq 3} \left(k - D\left(\frac{k}{2} - 1\right)\right) \lambda_k \frac{\partial}{\partial \lambda_k} \right)$$

(+ trivial boundary conditions.)

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(+ trivial boundary conditions.)

Relation to functional renormalization group methods

Wetterich 1993

Tropical QFT

Geometric viewpoint

Amplitudes as an integration problem on the **moduli space of graphs**

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_G \frac{1}{|\mathrm{Aut}(G)|} I_G$$

Amplitude

Sum over graphs with L loops and n legs

Symmetry factor

Feynman integral

$$\mathscr{A}_{L,n}(p_1,\ldots,p_n) = \sum_G \frac{1}{|\mathrm{Aut}G|}\int_{\mathbb{R}^{E_G}_{>0}} \frac{d\alpha_1\ldots d\alpha_{E_G}}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})^{D/2}}$$

Amplitude

Sum over graphs with
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Symmetry
factor

Rational integrand,
over positive orthand.

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_G \frac{1}{|\mathrm{Aut}G|} \int_{\mathbb{R}_{>0}^{E_G}} \frac{d\alpha_1 \ldots d\alpha_{E_G}}{\mathscr{P}_G(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}}$$

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Integral over a **moduli space of graphs.**

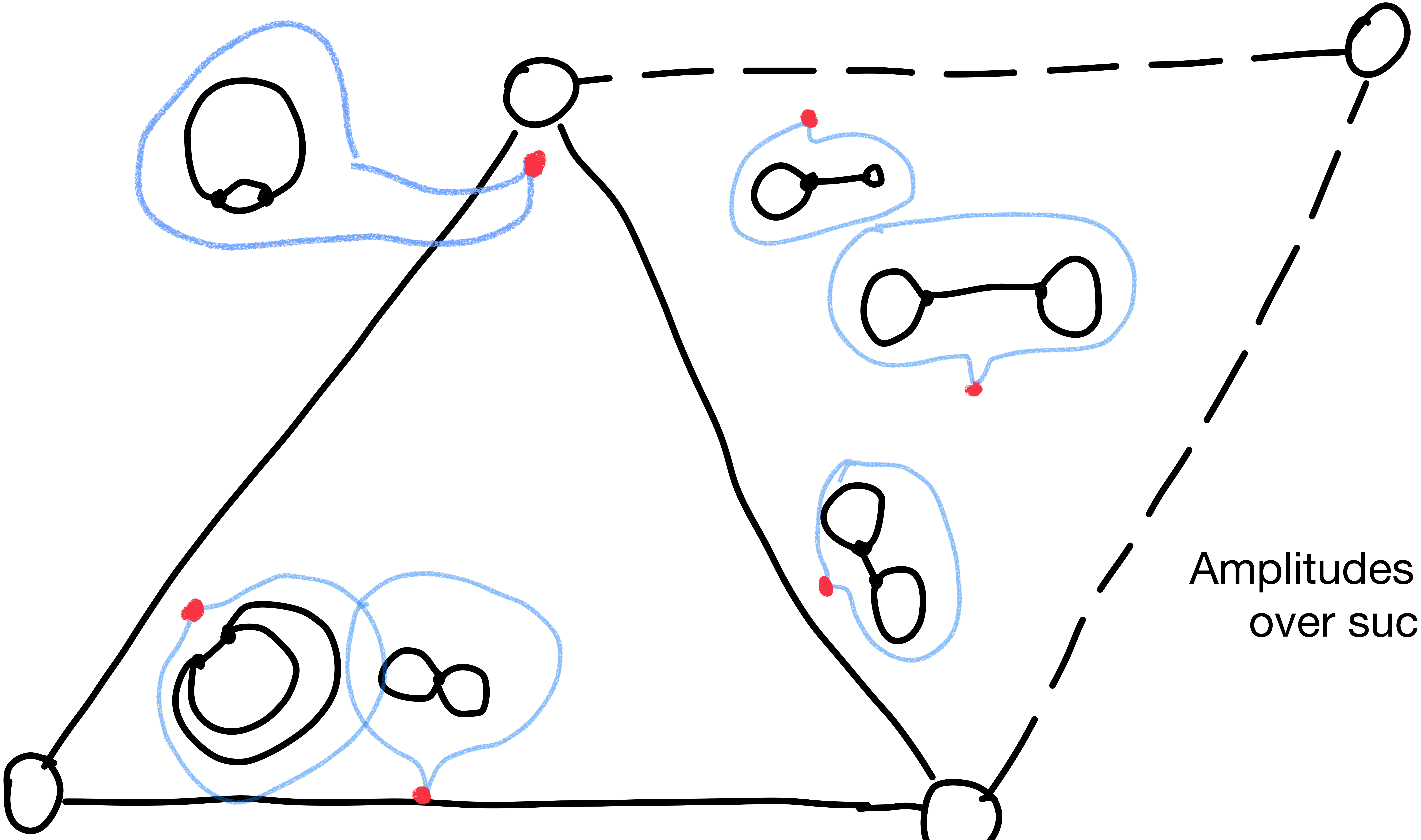
$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_G \frac{1}{|\mathrm{Aut}G|} \int_{\mathbb{R}_{>0}^{E_G}} \frac{d\alpha_1 \ldots d\alpha_{E_G}}{\mathscr{P}_G(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}}$$

$$= \int_{\mathscr{M}_{L,n}^{\mathrm{tr}}} \mu_D(\boldsymbol{p})$$

where $\mathscr{M}_{L,n}^{\mathrm{tr}} = \bigsqcup_G \left(\mathbb{R}_{>0}^{E_G} \,/\, \mathrm{Aut}(G) \right)$

is (almost) the moduli space of graphs (and also the moduli space of tropical curves).

$\mathscr{M}^{\text{tr}}_{2,0}$



Amplitudes are integrals over such spaces.

Feynman volume forms on the moduli space of graphs have a tropical analogue

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$$\mu_D(\boldsymbol{p})\,|_{\Delta_G} = \frac{d\alpha_1\ldots d\alpha_{E_G}}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})^{D/2}} \qquad \mu^{\mathrm{tr}}_D(\boldsymbol{p})\,|_{\Delta_G} = \frac{d\alpha_1\ldots d\alpha_{E_G}}{\mathscr{P}^{\mathrm{tr}}_G(\boldsymbol{\alpha},\boldsymbol{p})^{D/2}}$$

Tropicalization of $\mathscr{P}_G$

$$\mathscr{A}_{L,n}(p_1,\ldots,p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu_D(\boldsymbol{p})$$

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$$
\begin{aligned}
\mathscr{A}_{L,n}(p_1,\ldots,p_n) &= \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu_D(\boldsymbol{p}) \\
&= \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \left(\frac{\mathscr{P}^{\mathrm{tr}}_G(\boldsymbol{\alpha},\boldsymbol{p})}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})} \right)^{D/2} \mu^{\mathrm{tr}}_D(\boldsymbol{p}) \\
&= \mathscr{A}^{\mathrm{tr}}_{L,n} \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \left(\frac{\mathscr{P}^{\mathrm{tr}}_G(\boldsymbol{\alpha},\boldsymbol{p})}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})} \right)^{D/2} \widehat{\mu}^{\mathrm{tr}}_D(\boldsymbol{p})
\end{aligned}
$$

Tropical amplitude

Key: this term is bounded.

Probability density

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \mathscr{A}^{\mathrm{tr}}_{L,n} \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \left(\frac{\mathscr{P}^{\mathrm{tr}}_G(\boldsymbol{\alpha}, \boldsymbol{p})}{\mathscr{P}_G(\boldsymbol{\alpha}, \boldsymbol{p})} \right)^{D/2} \widehat{\mu}^{\mathrm{tr}}_D(\boldsymbol{p})$$

Key: this term is bounded.

Sampling algorithm for

$$\int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \widehat{\mu}^{\mathrm{tr}}_D(\boldsymbol{p}) = 1.$$

Monte Carlo

Numerical evaluation of $\mathscr{A}_{L,n}(p_1, \ldots, p_n)$

Theorem

MB 2025

There is a **global sampling** algorithm for the measure $\widehat{\mu}_D^{\mathrm{tr}}(\boldsymbol{p})$ on $\mathscr{M}_{L,n}^{\mathrm{tr}}$.

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Proof: uses exact solution of tropical QFT.

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There is a **global sampling** algorithm for the measure $\widehat{\mu}^{\mathrm{tr}}_{D}(\boldsymbol{p})$ on $\mathscr{M}^{\mathrm{tr}}_{L,n}$.

It’s runtime and memory requirements scale **polynomially** with L, n .

Proof: uses exact solution of tropical QFT.

$\Rightarrow \mathscr{A}_{L,n}(p_1, \ldots, p_n)$ can be be estimated to finite accuracy in **polynomial-time** in L and n.

Fastest algorithms for Feynman integration run in exponential time.

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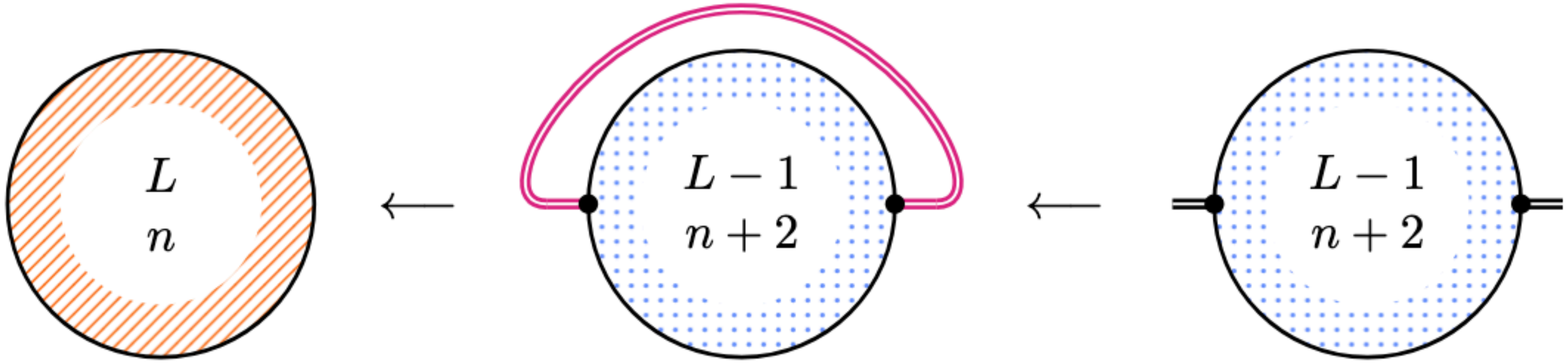
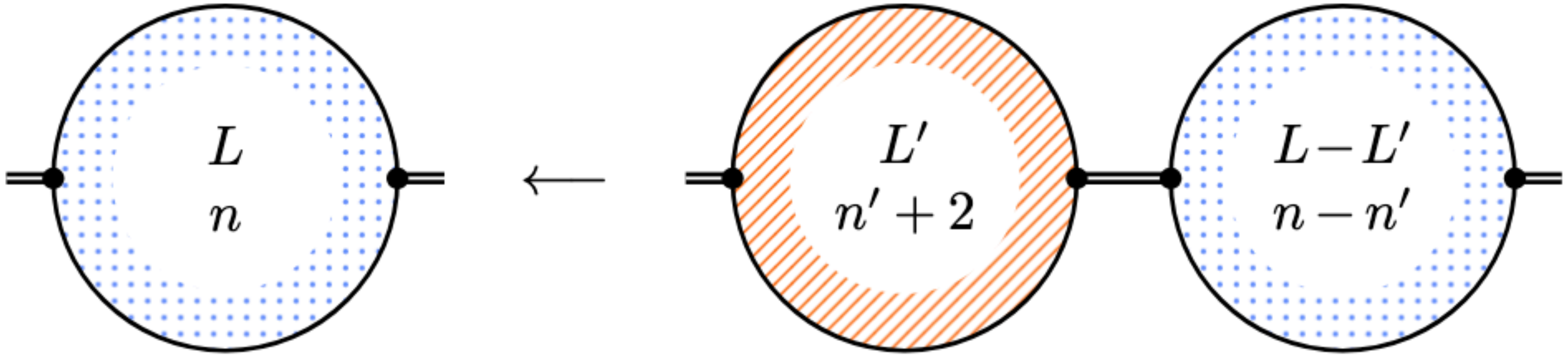
$\Rightarrow$ Evaluation of **individual Feynman integrals** quickly becomes slower than evaluating the **whole amplitude** $\mathscr{A}_{L,n}(p_1, \ldots, p_n)$.



(a) Illustration of the $L > 0$ case of Algorithm 18.



(b) Illustration of the (L', n') case in Algorithm 19.

The algorithm recursively produces metric graphs with the correct probability distribution. Most graphs will never be generated.

Example computations

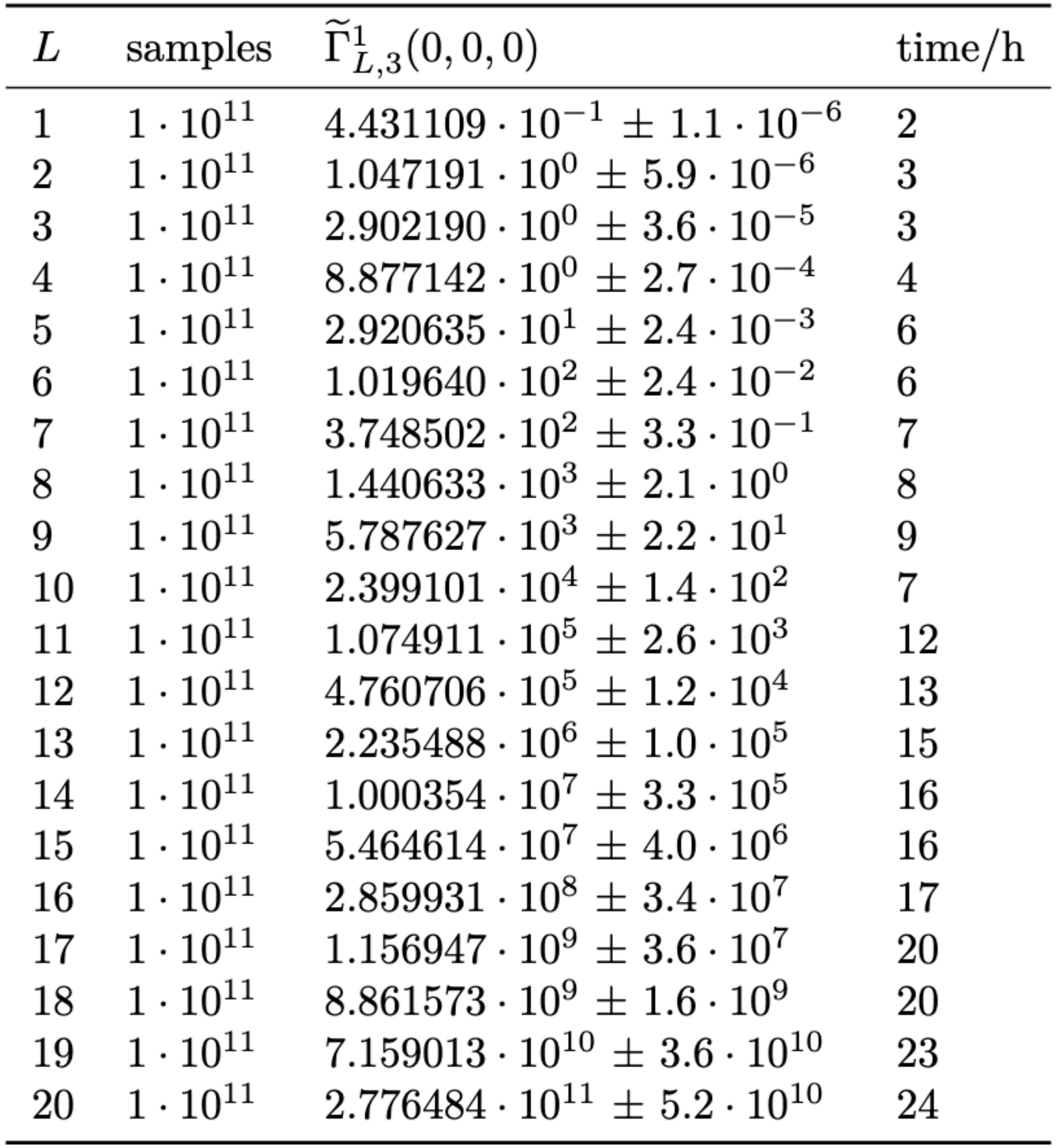
Massive ϕ^3 theory in $D = 3$, with $m^2 = 1$.

Example computation for $L \leq 20$ using global tropical sampling

Relations to **Lee—Yang 1952** phenomenon

Known (numerically) up to $L \leq 7$

(**Sberveglieria, Spada 2024**)



L	samples	$\widetilde{\Gamma}^1_{L,3}(0,0,0)$	time/h
1	$1 \cdot 10^{11}$	$4.431109 \cdot 10^{-1} \pm 1.1 \cdot 10^{-6}$	2
2	$1 \cdot 10^{11}$	$1.047191 \cdot 10^{0} \pm 5.9 \cdot 10^{-6}$	3
3	$1 \cdot 10^{11}$	$2.902190 \cdot 10^{0} \pm 3.6 \cdot 10^{-5}$	3
4	$1 \cdot 10^{11}$	$8.877142 \cdot 10^{0} \pm 2.7 \cdot 10^{-4}$	4
5	$1 \cdot 10^{11}$	$2.920635 \cdot 10^{1} \pm 2.4 \cdot 10^{-3}$	6
6	$1 \cdot 10^{11}$	$1.019640 \cdot 10^{2} \pm 2.4 \cdot 10^{-2}$	6
7	$1 \cdot 10^{11}$	$3.748502 \cdot 10^{2} \pm 3.3 \cdot 10^{-1}$	7
8	$1 \cdot 10^{11}$	$1.440633 \cdot 10^{3} \pm 2.1 \cdot 10^{0}$	8
9	$1 \cdot 10^{11}$	$5.787627 \cdot 10^{3} \pm 2.2 \cdot 10^{1}$	9
10	$1 \cdot 10^{11}$	$2.399101 \cdot 10^{4} \pm 1.4 \cdot 10^{2}$	7
11	$1 \cdot 10^{11}$	$1.074911 \cdot 10^{5} \pm 2.6 \cdot 10^{3}$	12
12	$1 \cdot 10^{11}$	$4.760706 \cdot 10^{5} \pm 1.2 \cdot 10^{4}$	13
13	$1 \cdot 10^{11}$	$2.235488 \cdot 10^{6} \pm 1.0 \cdot 10^{5}$	15
14	$1 \cdot 10^{11}$	$1.000354 \cdot 10^{7} \pm 3.3 \cdot 10^{5}$	16
15	$1 \cdot 10^{11}$	$5.464614 \cdot 10^{7} \pm 4.0 \cdot 10^{6}$	16
16	$1 \cdot 10^{11}$	$2.859931 \cdot 10^{8} \pm 3.4 \cdot 10^{7}$	17
17	$1 \cdot 10^{11}$	$1.156947 \cdot 10^{9} \pm 3.6 \cdot 10^{7}$	20
18	$1 \cdot 10^{11}$	$8.861573 \cdot 10^{9} \pm 1.6 \cdot 10^{9}$	20
19	$1 \cdot 10^{11}$	$7.159013 \cdot 10^{10} \pm 3.6 \cdot 10^{10}$	23
20	$1 \cdot 10^{11}$	$2.776484 \cdot 10^{11} \pm 5.2 \cdot 10^{10}$	24

Table 1: 3-point function computation in massive ϕ^3 theory in $D = 3$.

Primitive β function in 4-dim ϕ^4 theory

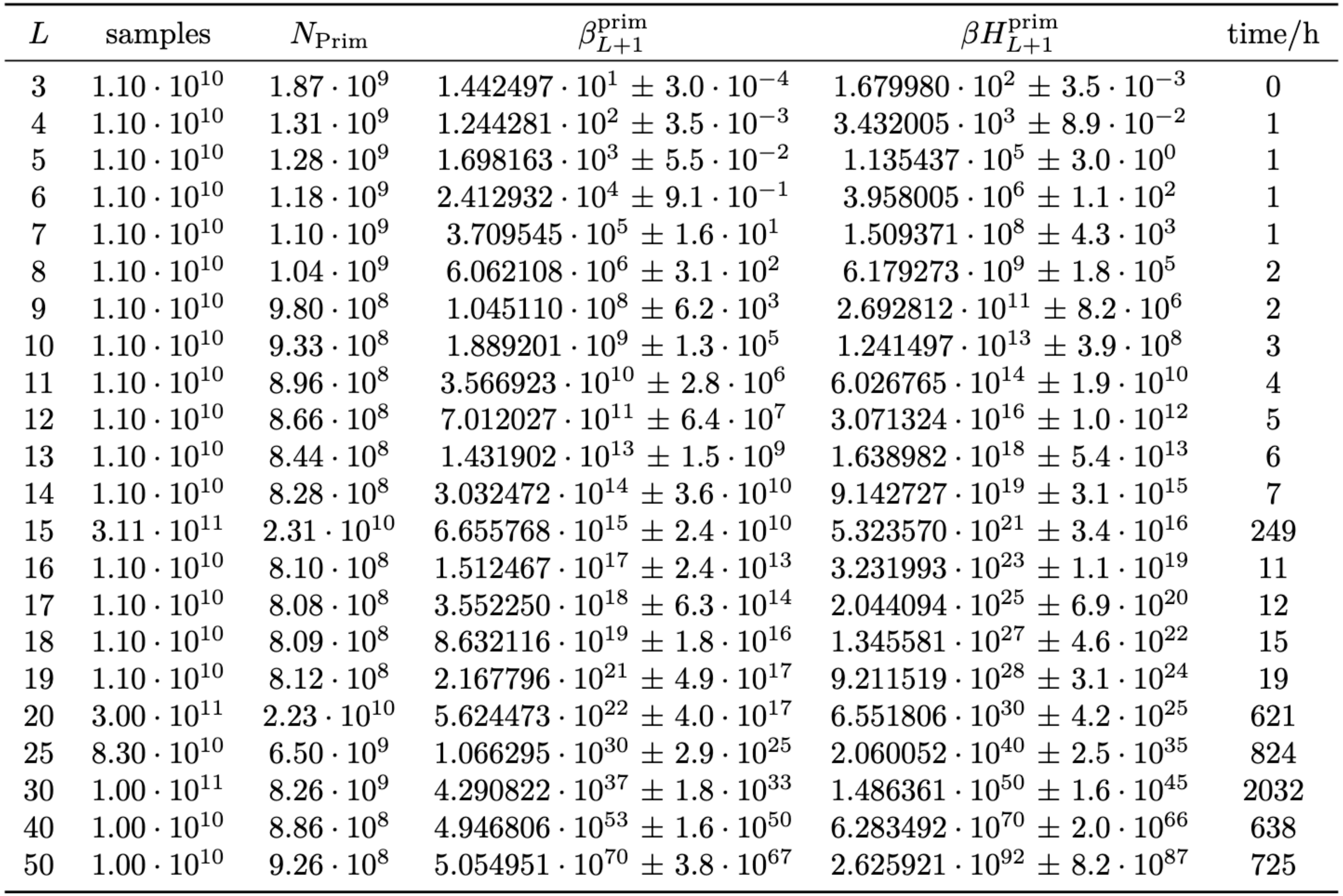
(Conjectured to be equal to full MS β function for large L.)

Relation to the **Ising model**

Known before (analytically) for $L \leq 7$ **Schnetz 2022.**

With tropical sampling (numerically) up to $L \leq 18$ **Balduf 2023**.

Example computation for $L \leq 50$ using global tropical sampling



L	samples	N_{Prim}	$\beta^{\text{prim}}_{L+1}$	$\beta H^{\text{prim}}_{L+1}$	time/h
3	$1.10 \cdot 10^{10}$	$1.87 \cdot 10^{9}$	$1.442497 \cdot 10^{1} \pm 3.0 \cdot 10^{-4}$	$1.679980 \cdot 10^{2} \pm 3.5 \cdot 10^{-3}$	0
4	$1.10 \cdot 10^{10}$	$1.31 \cdot 10^{9}$	$1.244281 \cdot 10^{2} \pm 3.5 \cdot 10^{-3}$	$3.432005 \cdot 10^{3} \pm 8.9 \cdot 10^{-2}$	1
5	$1.10 \cdot 10^{10}$	$1.28 \cdot 10^{9}$	$1.698163 \cdot 10^{3} \pm 5.5 \cdot 10^{-2}$	$1.135437 \cdot 10^{5} \pm 3.0 \cdot 10^{0}$	1
6	$1.10 \cdot 10^{10}$	$1.18 \cdot 10^{9}$	$2.412932 \cdot 10^{4} \pm 9.1 \cdot 10^{-1}$	$3.958005 \cdot 10^{6} \pm 1.1 \cdot 10^{2}$	1
7	$1.10 \cdot 10^{10}$	$1.10 \cdot 10^{9}$	$3.709545 \cdot 10^{5} \pm 1.6 \cdot 10^{1}$	$1.509371 \cdot 10^{8} \pm 4.3 \cdot 10^{3}$	1
8	$1.10 \cdot 10^{10}$	$1.04 \cdot 10^{9}$	$6.062108 \cdot 10^{6} \pm 3.1 \cdot 10^{2}$	$6.179273 \cdot 10^{9} \pm 1.8 \cdot 10^{5}$	2
9	$1.10 \cdot 10^{10}$	$9.80 \cdot 10^{8}$	$1.045110 \cdot 10^{8} \pm 6.2 \cdot 10^{3}$	$2.692812 \cdot 10^{11} \pm 8.2 \cdot 10^{6}$	2
10	$1.10 \cdot 10^{10}$	$9.33 \cdot 10^{8}$	$1.889201 \cdot 10^{9} \pm 1.3 \cdot 10^{5}$	$1.241497 \cdot 10^{13} \pm 3.9 \cdot 10^{8}$	3
11	$1.10 \cdot 10^{10}$	$8.96 \cdot 10^{8}$	$3.566923 \cdot 10^{10} \pm 2.8 \cdot 10^{6}$	$6.026765 \cdot 10^{14} \pm 1.9 \cdot 10^{10}$	4
12	$1.10 \cdot 10^{10}$	$8.66 \cdot 10^{8}$	$7.012027 \cdot 10^{11} \pm 6.4 \cdot 10^{7}$	$3.071324 \cdot 10^{16} \pm 1.0 \cdot 10^{12}$	5
13	$1.10 \cdot 10^{10}$	$8.44 \cdot 10^{8}$	$1.431902 \cdot 10^{13} \pm 1.5 \cdot 10^{9}$	$1.638982 \cdot 10^{18} \pm 5.4 \cdot 10^{13}$	6
14	$1.10 \cdot 10^{10}$	$8.28 \cdot 10^{8}$	$3.032472 \cdot 10^{14} \pm 3.6 \cdot 10^{10}$	$9.142727 \cdot 10^{19} \pm 3.1 \cdot 10^{15}$	7
15	$3.11 \cdot 10^{11}$	$2.31 \cdot 10^{10}$	$6.655768 \cdot 10^{15} \pm 2.4 \cdot 10^{10}$	$5.323570 \cdot 10^{21} \pm 3.4 \cdot 10^{16}$	249
16	$1.10 \cdot 10^{10}$	$8.10 \cdot 10^{8}$	$1.512467 \cdot 10^{17} \pm 2.4 \cdot 10^{13}$	$3.231993 \cdot 10^{23} \pm 1.1 \cdot 10^{19}$	11
17	$1.10 \cdot 10^{10}$	$8.08 \cdot 10^{8}$	$3.552250 \cdot 10^{18} \pm 6.3 \cdot 10^{14}$	$2.044094 \cdot 10^{25} \pm 6.9 \cdot 10^{20}$	12
18	$1.10 \cdot 10^{10}$	$8.09 \cdot 10^{8}$	$8.632116 \cdot 10^{19} \pm 1.8 \cdot 10^{16}$	$1.345581 \cdot 10^{27} \pm 4.6 \cdot 10^{22}$	15
19	$1.10 \cdot 10^{10}$	$8.12 \cdot 10^{8}$	$2.167796 \cdot 10^{21} \pm 4.9 \cdot 10^{17}$	$9.211519 \cdot 10^{28} \pm 3.1 \cdot 10^{24}$	19
20	$3.00 \cdot 10^{11}$	$2.23 \cdot 10^{10}$	$5.624473 \cdot 10^{22} \pm 4.0 \cdot 10^{17}$	$6.551806 \cdot 10^{30} \pm 4.2 \cdot 10^{25}$	621
25	$8.30 \cdot 10^{10}$	$6.50 \cdot 10^{9}$	$1.066295 \cdot 10^{30} \pm 2.9 \cdot 10^{25}$	$2.060052 \cdot 10^{40} \pm 2.5 \cdot 10^{35}$	824
30	$1.00 \cdot 10^{11}$	$8.26 \cdot 10^{9}$	$4.290822 \cdot 10^{37} \pm 1.8 \cdot 10^{33}$	$1.486361 \cdot 10^{50} \pm 1.6 \cdot 10^{45}$	2032
40	$1.00 \cdot 10^{10}$	$8.86 \cdot 10^{8}$	$4.946806 \cdot 10^{53} \pm 1.6 \cdot 10^{50}$	$6.283492 \cdot 10^{70} \pm 2.0 \cdot 10^{66}$	638
50	$1.00 \cdot 10^{10}$	$9.26 \cdot 10^{8}$	$5.054951 \cdot 10^{70} \pm 3.8 \cdot 10^{67}$	$2.625921 \cdot 10^{92} \pm 8.2 \cdot 10^{87}$	725

Table 2: Primitive β function computation in ϕ^4 theory.

Summary

- Tropical QFT is a, **exactly solvable,** deformation of (scalar) QFT.
- Gives an interesting new toy model for interacting QFTs.
- Gives a **polynomial-time** algorithm for estimation of scattering amplitudes and other perturbative QFT quantities.
- Tropical QFT solves the **factorial-time** problem of QFT (in the scalar+finite case).

Many open questions: E.g.

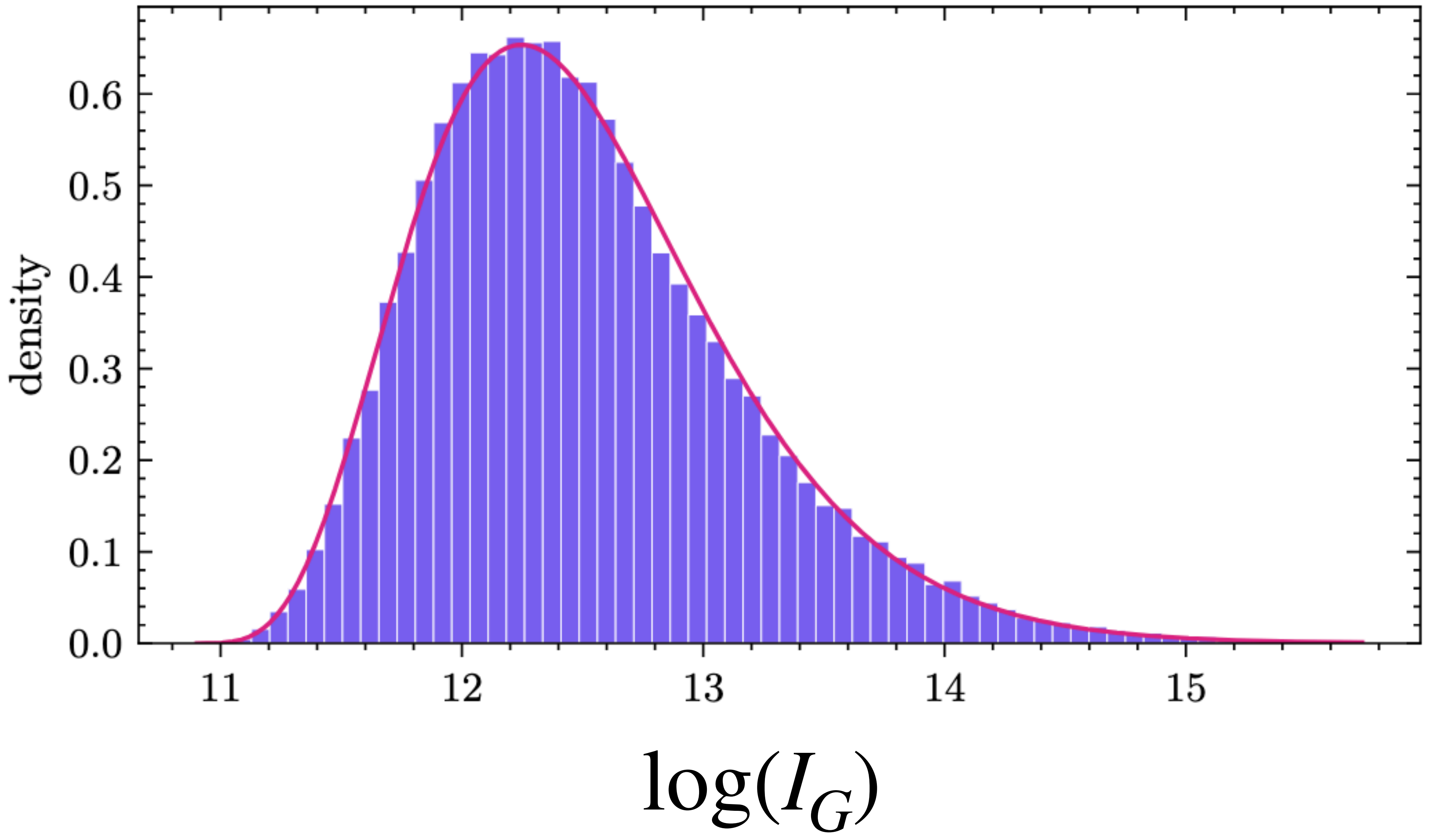
- What is the tropicalization of non-scalar theories?
- What QFT properties are visible from its 'tropical' shadow?

Studying tropical QFT already gave gives illuminating results and conjectures for ordinary QFT — **Balduf—Panzer 2025** (see Paul's short talk on Thursday)

Why are these difficult? — **Problem 2**

Obvious idea — Avoid the super-exponential number of terms by sampling

$$A_L = \sum_{G,\, L(G) = L} I_G$$



MB, Favorito 2025

I_G follows a heavy-tailed distribution

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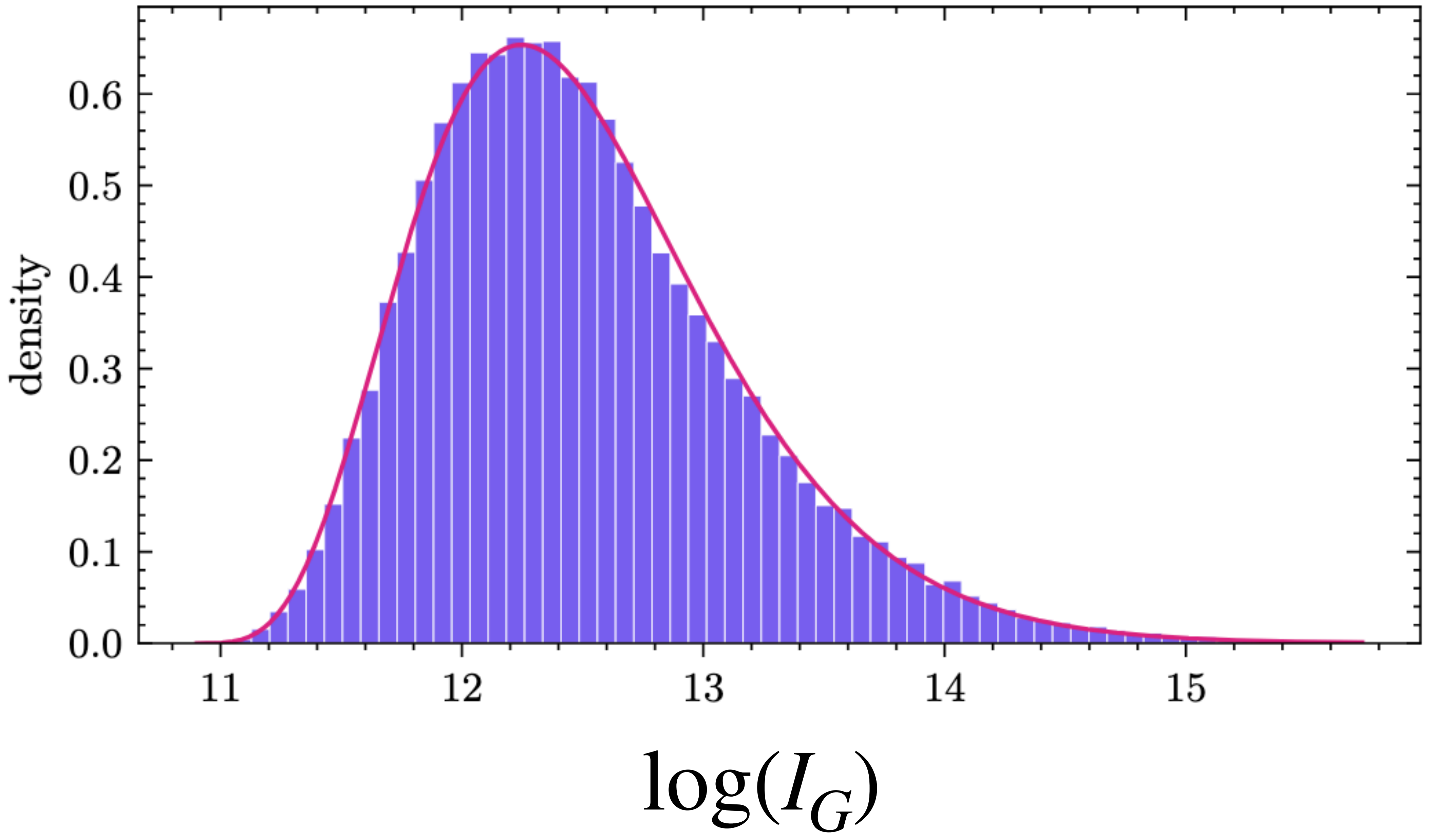

MB, Favorito 2025

I_G follows a heavy-tailed distribution ⇒ Naive sampling also **fails**

Tropical approximation

Polynomial $$P(x_1, \ldots, x_n) = \sum_{\boldsymbol{k} \in M} a_{\boldsymbol{k}} \prod_{i=1}^{n} x_i^{k_i} \quad \in \mathbb{R}_+[x_1, \ldots, x_n]$$

Definition
Tropical approximation $$P^{\mathrm{tr}}(x_1, \ldots, x_n) := \max_{\boldsymbol{k} \in M} \prod_{i=1}^{n} x_i^{k_i}$$

Rule: $\sum \to \max$ and $a_{\boldsymbol{k}} \to 1$.

'Typical' tropicalization is recovered in $\log$ coordinates.

Many shapes of Feynman integrals

$$I_G = (\,\star\,) \int_{\mathbb{M}^{DL}} \frac{d^D k_1 \cdots d^D k_L}{\prod_e (q_e^2 - m_e^2 + i\varepsilon)}$$

Many shapes of Feynman integrals

$$I_G = (\star)\int_{\mathbb{M}^{DL}} \frac{d^D k_1 \cdots d^D k_L}{\prod_e (q_e^2 - m_e^2 + i\varepsilon)} = (\star)\int_{\mathbb{R}_{>0}^{E_G}} \exp(-F/U)\frac{d\alpha_1 \ldots d\alpha_{E_G}}{U^{D/2}}$$

Many shapes of Feynman integrals

$$I_G = (\star)\int_{\mathbb{M}^{DL}} \frac{d^Dk_1 \cdots d^Dk_L}{\prod_e (q_e^2 - m_e^2 + i\varepsilon)} = (\star)\int_{\mathbb{R}_{>0}^{E_G}} \exp(-F/U)\frac{d\alpha_1 \ldots d\alpha_{E_G}}{U^{D/2}} = \int_{\mathbb{R}_{>0}^{E_G}} \frac{d\alpha_1 \ldots d\alpha_{E_G}}{\mathscr{P}_G(\alpha, p)^{D/2}}$$

Here: Lee—Pomeransky polynomial/ representation

(i.e. Feynman/Schwinger/parametric representation)