What can Landau analysis do for you

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- Can be used to do Landau bootstrap
- There is a large web of relationships between monodromies around various singularities (hierarchy, Steinmann and Pham-Steinmann).
- Can be used to study a given singularity (no need to study the full problem). In particular, can study some singularities of divergent integrals without worrying about regularization.
- Is needed to perform analytic continuation.
- Non-effective Landau loci (work with Adolfo Hilario Garcia and Johannes Henn).
- Can determine the location of a singularity in the symbol via the Landau exponent (asymptotic behavior) [Hannesdóttir, McLeod, Schwarz, Vergu].

- Can determine the singularities in various dimensions separately (higher order in ϵ)
- In principle, a factorization property for every Landau diagram. Potentially very powerful.
- IR singularities and permanent pinches (work with Dimitri Corradini and Shun-Qing Zhang).
- Interesting interplay with the Galois group of various polynomial equations already for the zero-dimensional on-shell space case. Extension to higher dimensions?

Analytic Continuation

Finding the letters of the symbol is not enough. One needs to find the version with the correct physical singularities.
For example, should we use $\log x$ or $\frac{1}{n}\log(x^n)$?
Or, for a more extreme example

$$\log\frac{a+\sqrt{b}}{a-\sqrt{b}}, \qquad 2\log\frac{\sqrt{a+\sqrt{a^2-b}}+\sqrt{a-\sqrt{a^2-b}}}{\sqrt{a+\sqrt{a^2-b}}-\sqrt{a-\sqrt{a^2-b}}}$$

Starting point of iterated integral

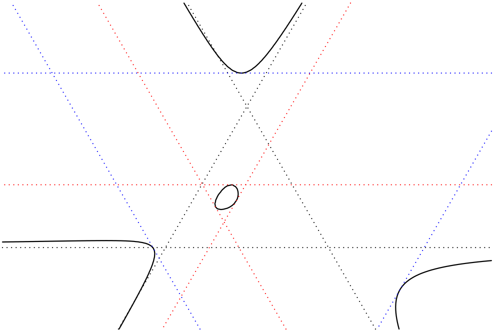
When expressed as an iterated integral the path ends at the point whose coordinates are the kinematics where we want to evaluate the process. But what should be the starting point?

There may not be a preferred starting point ([CV, 2512.12763])!

For example, for a bubble integral we may start the integral at (the real points of) pseudo-threshold and the answer does not depend on the position on the pseudo-threshold (as long as we don't cross other singularities).

Analytic continuation around Landau loci

Real points of Landau loci are of two types: *effective* (which correspond to non-negative α and real physical processes [Coleman-Norton]) and *non-effective*. There should be no singularity along the non-effective points (on the real points of the physical sheet).
But for the effective points we need to make a choice of complex detour. The exact prescription has been described by Pham and it involves explicitly the solution of the Landau singularity problem.



Physical region for a four-point scattering. Thresholds in blue and pseudo-thresholds in red. Thresholds intersect the physical region at *points*. Could argue by continuity that Pham's prescription applies in this case as well.

Second type singularities

[Hannesdóttir, McLeod, Schwarz, CV], [CV, 2512.12763], one can study second type singularities by inversion (or other changes of coordinate which map regions at infinity to zero). These changes of coordinates have Jacobians such as $\frac{1}{x_0^{D-4}}$ which produce a permanent pinch when $D > 0$. In the case of UV-finite integrals this can be used to compute contributions to terms of order $\mathcal{O}(\epsilon)$.

Divergences

Most of the integrals we are dealing with for physics applications are divergent. They are regularized (in most cases) using dimensional regularization.
Various ways to improve the situation: prescriptive bases [Bourjaily, Hermann, Langer, Patatoukos, Trnka, Zheng] forming finite linear combinations [De Angelis, Gambuti, Kosower, Novichkov, Ma, Tancredi, Wu, Zhang], [Henn, Mistlberger, Wasser, Smirnov], subtractions [Anastasiou, Sterman].

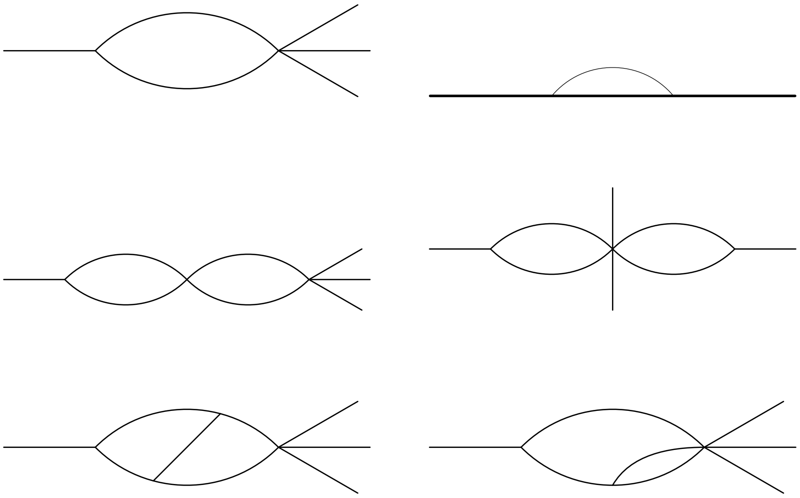
IR divergences

The IR singularities can be understood as some sort of "permanent pinches" (Kinoshita, 1962) but he called them "mass singularities". The terminology of permanent pinches seems to originate from (Boyling, 1968), but he originally studied it in Feynman parameterization.

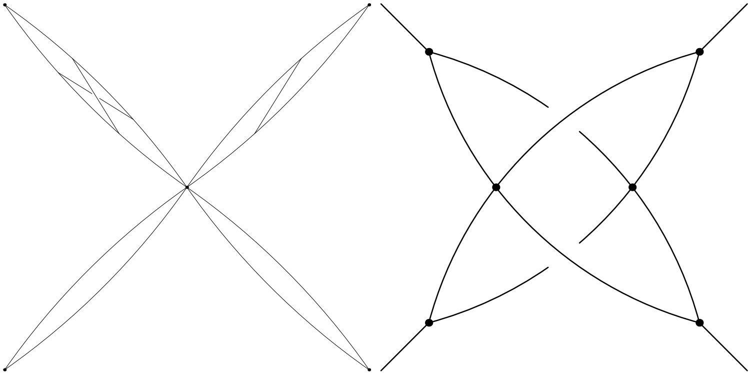
Permanent pinches are configurations of loop momenta in an associated permanent pinch Landau diagram so that the Landau equations are satisfied for *all* values of external kinematics.

In contrast, Landau singularities only appear for special values of external kinematics.

Examples of permanent pinches



General structure of permanent pinches



The decomposition into jet, soft and hard parts becomes unsurprising when considered from the perspective of permanent pinches.

A bubble residue

[Dimitri Corradini, CV, Shun-Qing Zhang], to appear.
Consider the integral

$$I(1,0,1,1) = \int \frac{d^4 x_0}{x_{01}^2 (x_{03}^2 - m^2) x_{04}^2}.$$

It has a permanent pinch where $x_0 = (1+z)x_1 - z x_4$ and its bubble residue is a one-form

$$\mathrm{res}_{\mathrm{bubble}}\, \omega = -\frac{dz}{x_{13}^2 z + m^2}.$$

Integrating this over $z \in [0, -1]$ we find

$$\int_0^{-1} \frac{-dz}{x_{13}^2 z + m^2} = -\frac{1}{x_{13}^2} \log\left(\frac{m^2 - x_{13}^2}{m^2}\right).$$

For comparison, the divergent part in dimensional regularization is

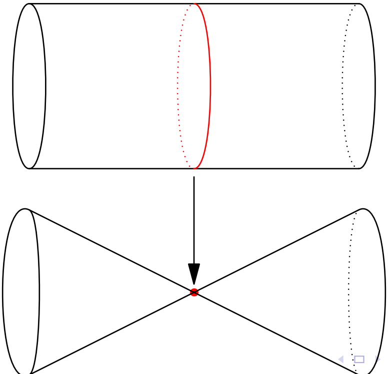
$$I(1,0,1,1)_D = -\frac{i\pi^2 \log(1 - \frac{x_{13}^2}{m^2})}{\epsilon x_{13}^2} + \cdots .$$

Squared tadpole residue

Consider the integral

$$I(2,1,1,1) = \int \frac{d^4 x_0}{(x_{01}^2)^2 (x_{02}^2 - m^2) x_{03}^2 (x_{04}^2 - m^2)}.$$

Taking the residue in the tadpole x_{01} implies $x_0 \to x_1$ and we parameterize $x_0 = x_1 + z(1, \vec{y})$ so $x_{01}^2 = z^2(1 - \vec{y}^2)$, *etc.* (see [CV, 2512.12763]).



Squared tadpole residue computation

$$\omega = \frac{d^4 x_0}{(x_{01}^2)^2 (x_{02}^2 - m^2) x_{03}^2 (x_{04}^2 - m^2)} =$$

$$\frac{dz d^3 \vec{y}}{z(1-\vec{y}^2)^2(-m^2 + \mathcal{O}(z))^2 (x_{13}^2 + \mathcal{O}(z))},$$

so

$$\operatorname{res}_{z=0} \omega = \frac{d^3 \vec{y}}{(1-\vec{y}^2)^2 m^4 x_{13}^2}.$$

We take *another* residue. Change variables

$$\vec{y} = \sigma \Big(\frac{2t}{1+t^2+u^2}, \frac{2u}{1+t^2+u^2}, \frac{-1+t^2+u^2}{1+t^2+u^2} \Big)$$

so

$$\operatorname{res}_{z=0} \omega = \frac{1}{m^4 x_{13}^2} \frac{-4\sigma^2 d\sigma dt du}{(1-\sigma^2)^2 (1+t^2+u^2)^2}.$$

Since

$$\operatorname{res}_{\sigma=1} \frac{\sigma^2 d\sigma}{(1-\sigma^2)^2} = \frac{1}{4},$$

we have

$$\operatorname{res}_{\sigma=1} \operatorname{res}_{z=0} \omega = -\frac{1}{m^4 x_{13}^2} \frac{dtdu}{(1+t^2+u^2)^2}$$

and

$$\int_{\mathbb{R}^2} \operatorname{res}_{\sigma=1} \operatorname{res}_{z=0} \omega = -\frac{\pi}{m^4 x_{13}^2}.$$

Compare to the divergent part

$$I(2,1,1,1)_D = \frac{i\pi^2}{m^4 x_{13}^2} \frac{1}{\epsilon} + \cdots$$

This can be used to show that the combination $I(2,1,1,1) - \frac{1}{m^4} I(2,0,1,0)$ is finite. The permanent pinches cancel in this combination.

Surprising behavior

One-mass triangles (with massless propagators):

$$I(1,1,0,1)_D = \mathcal{O}(\epsilon^{-2}),$$

$$I(2,1,0,1)_D = \mathcal{O}(\epsilon^{-1}),$$

$$I(2,1,-1,1)_D = \mathcal{O}(\epsilon^{-2}).$$

Parallels the results:

$$\operatorname{res} \frac{dz}{z} = 1, \qquad \operatorname{res} \frac{dz}{z^2} = 0, \qquad \operatorname{res} \frac{f(z)dz}{z^2} = f'(0).$$

This can not be captured by simple power counting arguments.

An example

Consider the box integral $I(3,1,1,1)$ with one cubic propagator (and the third propagator massive). In D dimensions it behaves as

$$I(3,1,1,1)_D = \epsilon^{-2} \frac{i\pi^2}{(x_{24}^2)^3(m^2 - x_{13}^2)^5} \Big(m^4 (x_{24}^2)^2 + 6m^4 x_{24}^2 x_{13}^2 + 4m^2 (x_{24}^2)^2 x_{13}^2 + 6m^4 (x_{13}^2)^2 + 6m^2 x_{24}^2 (x_{13}^2)^2 + (x_{24}^2)^2 (x_{13}^2)^2 \Big) + \mathcal{O}(\epsilon^{-1}). \quad (1)$$

The same coefficient can be computed taking the triangle residue, which is the leading permanent pinch.

Conjecture: it is possible to extract the leading divergence of a Feynman integral from the "pinch kinematics" by taking a certain kind of (Leray) residues to obtain a differential form and by parameterizing the (real points) of the pinch kinematics to obtain an integration cycle.
There is an operation which takes you from a pair $[\omega], [\gamma]$ of cohomology and homology class to a pair $[\omega_p], [\gamma_p]$ for any permanent pinch p. Then, if p is maximal (not contained in another one), then there is a contribution to the divergence in ϵ given by $\int_{\gamma_p} \omega_p$.
If true, it becomes possible to manipulate divergent integrals consistently (without using dimensional regularization) as long as the final result is finite. In principle, this method allows us to distinguish between an integral which is *convergent* and one for which the singularities cancel between different regions.

Thank You

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UNIVERSE+ is funded by the European Union (ERC, UNIVERSE PLUS, 101118787). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.