

Frontier of Multi-loop and Multi-leg Feynman Integrals

Amplitudes 2026, Queen Mary University of London
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Outline

Introduction

Analytic Feynman integral and the methodology

Analytic Feynman integral frontier

Frontier 1: 2loop 6point Feynman integrals

Frontier 2: 3loop 5point Feynman integrals

First Applications (a brief overview)

Summary and Outlook

Based on Feynman integral Integral evaluation



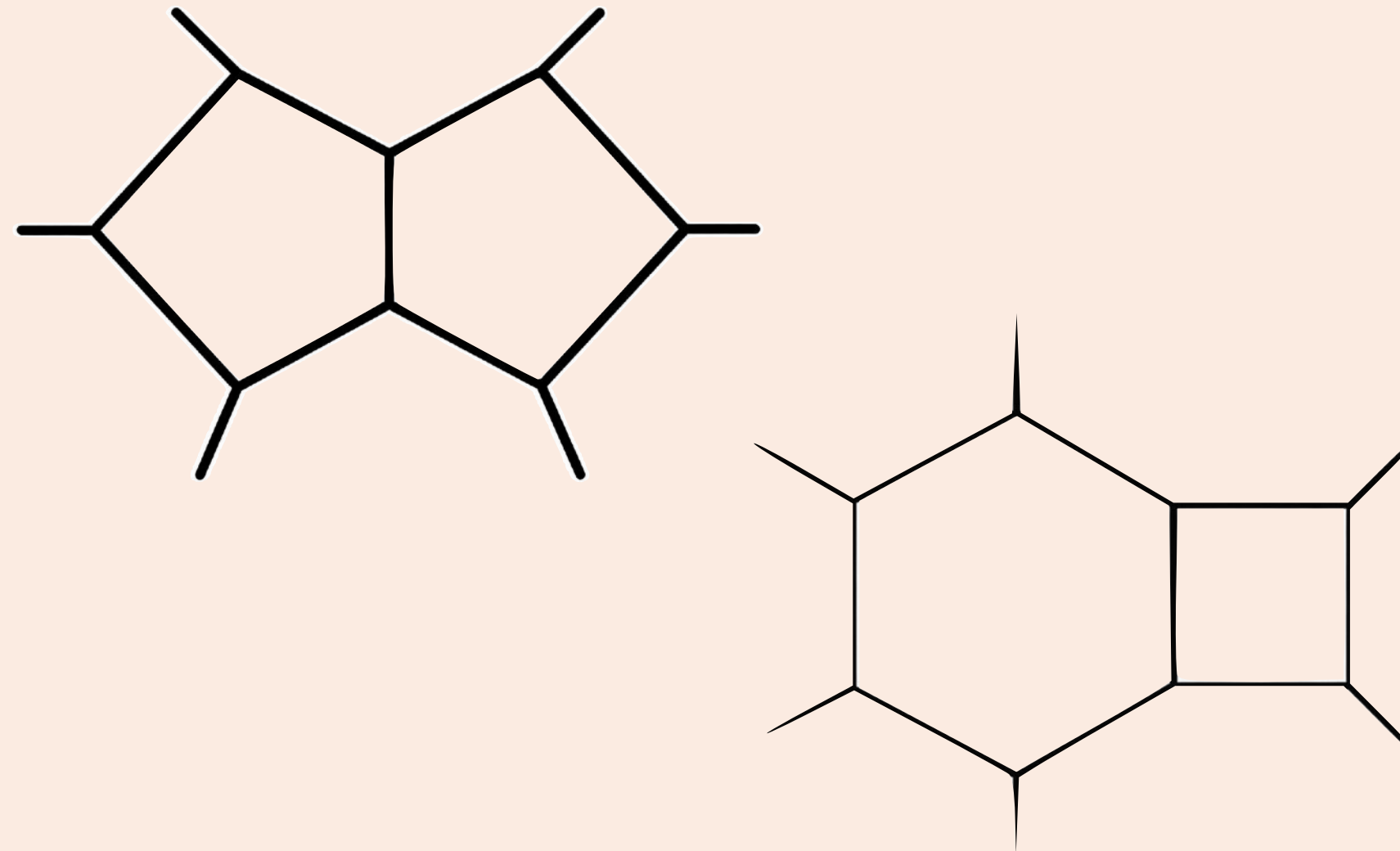
Henn



Matijašić



Miczajka



analytic computation of all **2loop 6point**
planar massless integrals



Peraro



Liu



Xu



Yang

Henn, Matijašić, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

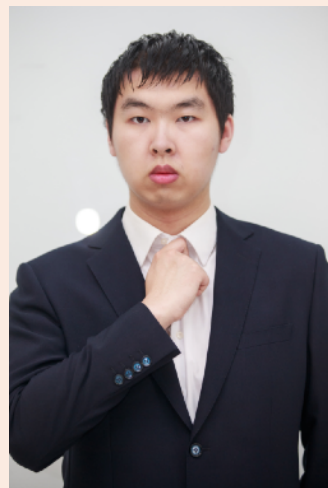
Henn, Matijašić, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027

Liu, Matijasic, Peraro, Xu, Yang, YZ, arXiv. 2603.16831, JHEP accepted

Based on Feynman integral Integral evaluation



Xu



Liu



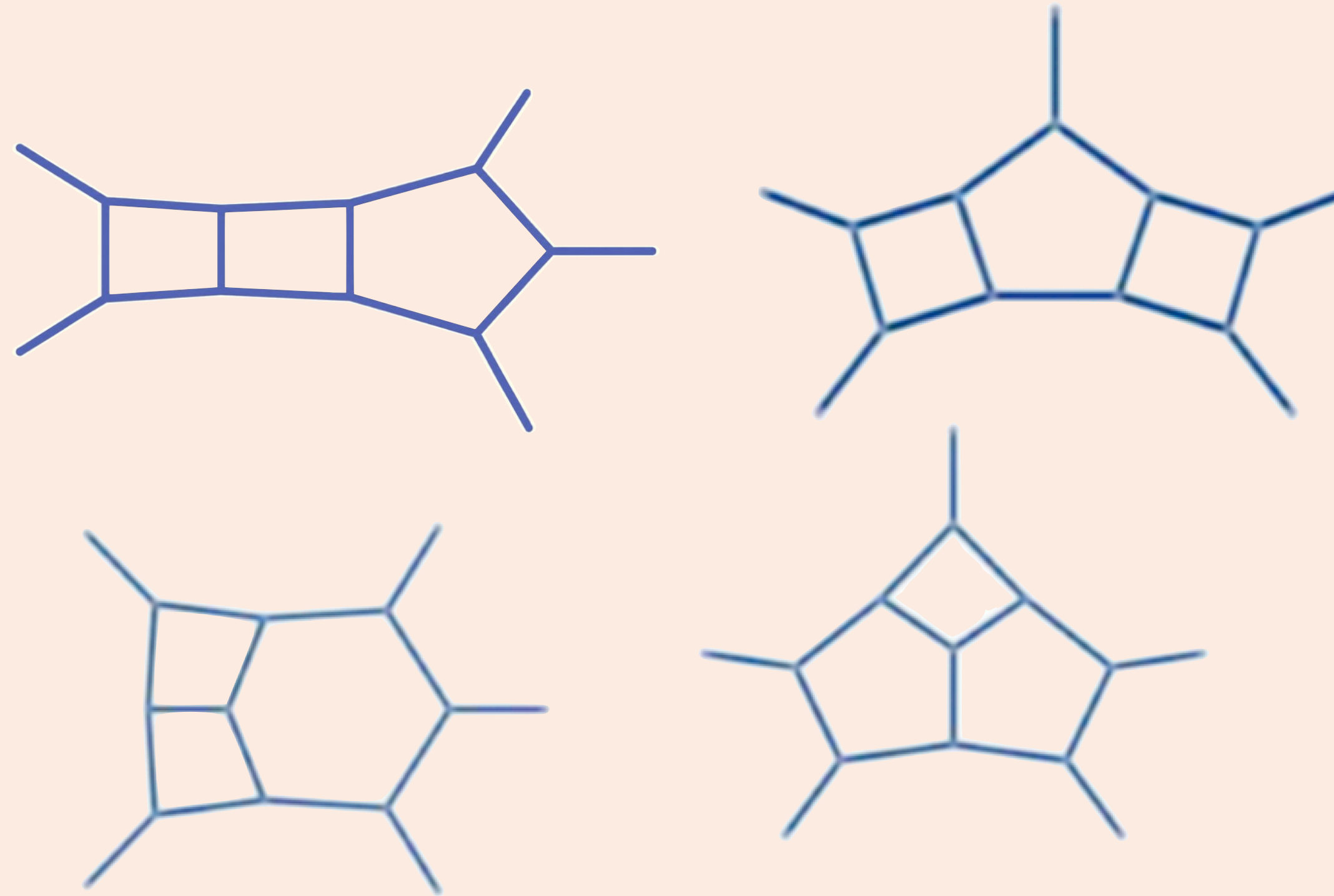
Wu



Xu



Wu



analytic computation of all **3loop 5point**
planar massless integrals



Chicherin



Zhang



Matijašić



Miczajka

Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021 (Editors' Suggestion)

Chicherin, Wu, Wu, Xu, Zhang, YZ, arXiv: 2512.17330

Based on package development

NeatIBP 1.0, a package generating small-size integration-by-parts relations for Feynman integrals
Wu, Boehm, Ma, Xu, YZ, Comput. Phys. Commun. 295 (2024), 108999

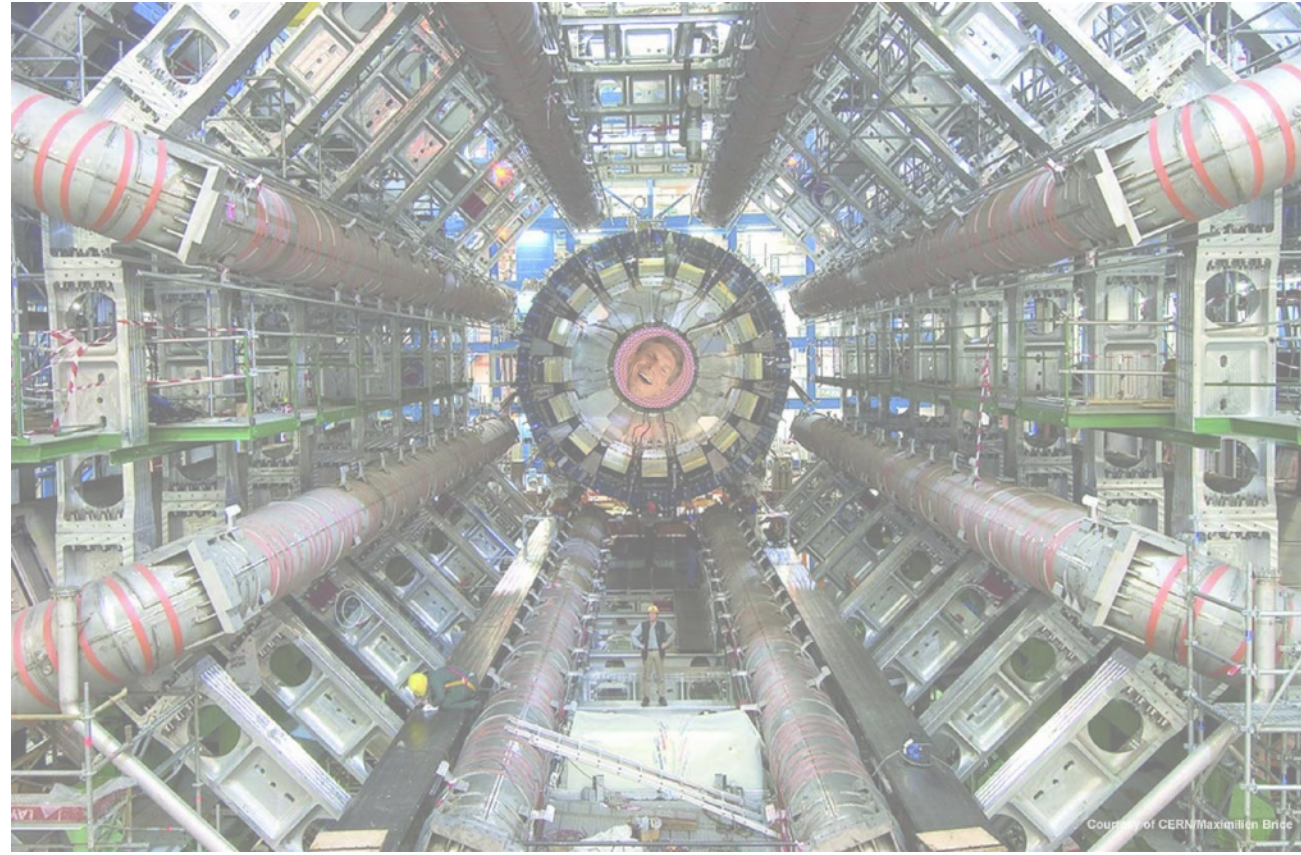
Performing integration-by-parts reductions using **NeatIBP 1.1** + Kira
Wu, Boehm, Ma, Usovitsch, Xu, YZ, Comput.Phys.Commun. 316 (2025) 109798

pfd-parallel, *a Singular/GPI-Space package for massively parallel multivariate partial fractioning*
Bendle, Boehm, Heymann Ma, Rahn, Ristau, Wittmann, Wu, Xu, YZ, Comput.Phys.Commun. 294 (2024) 108942

CHESS: CHEbyshev pSeudo-Spectral transport for Feynman integral differential equations

Liu, YZ, arXiv: 2606.26691

Introduction

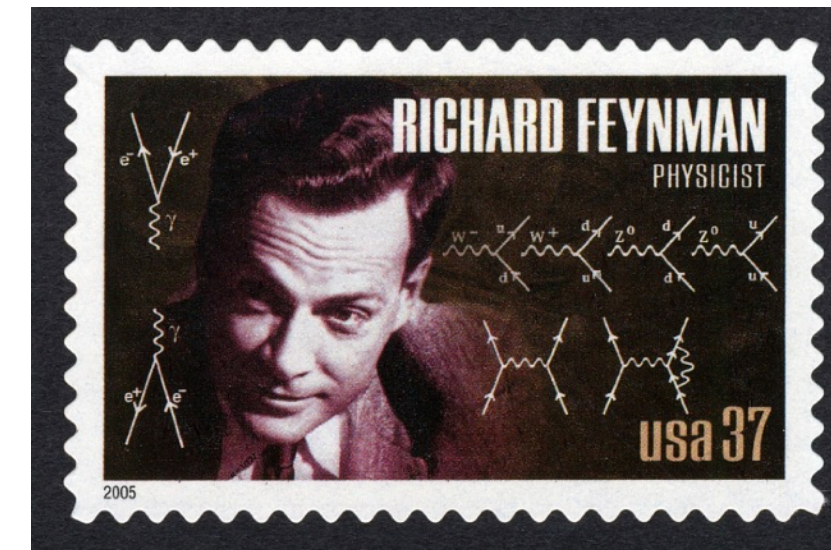


Precision physics

$$\sigma = \sigma^{LO} + \sigma^{NLO} + \sigma^{NNLO}$$

for instance,

Driesse, Jakobsen, Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch
Nature 641 (2025) 8063, 603-607



Feynman integrals



(CERN art image for supergravity)

Formal theories

N=8 supergravity UV finiteness



Gravitational wave template computations

Refer to talks by Bern, Bohnenblust, Chen
Isabella, Jakobsen, Kosower, O'Connell, Shi

Why *analytic* multi-loop Feynman integrals?

- Historically, analytic results may lead to a lot of phenomenology applications

- Theoretical aspects of quantum field theory

Refer to Gehrmann, Ruf and Zoia's talks

for example: 2loop **spacelike** splitting amplitude, factorization violation

Henn, Ma, Xu, Yan, YZ, Zhu, Phys.Rev.D 112 (2025) 7, 076003

Buccioni, Fang, Yan arXiv: 2603.27123

- Quantum field theory computation of gravitational wave

- **Function space bootstrap** for physical observable

2loop hexagonal Wilson loop + Lagrangian Carrôlo, Chicherin, Henn, Yang, YZ, JHEP 07 (2025) 214

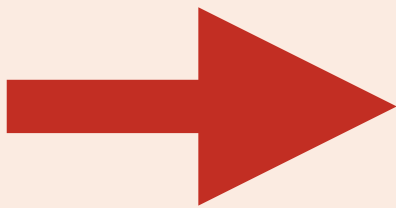
2loop six-point QCD amplitudes, weight4, Carrôlo, Chicherin, Henn, Yang, YZ, *Phys.Rev.Lett.* 136 (2026) 18, 181602

3loop pentagonal Wilson loop + Lagrangian Chicherin, Henn, Xu, Zhang, YZ arxiv: 2512.17881

Analytic Feynman integrals to phenomenology applications

Two-loop Five-point Massless integrals

Chicherin, Gehrmann, Henn, Wasser, YZ, Zoia, 2019
Abreu, Dixon, Herrmann, Page, Zeng, 2019



$pp \rightarrow \gamma\gamma\gamma$ Abreu, Page, Pascual, Sonikov 2020
Chawdhry, Czakon, Mitov, Poncelet 2021
Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov 2023

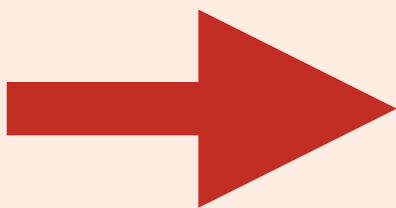
$pp \rightarrow \gamma\gamma j$ Agarwal, Buccioni, von Manteuffel, Tancredi 2021
Chawdhry, Czakon, Mitov, Poncelet 2021
Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia 2021
Buccioni, Chen, Feng, Gehrmann, Huss, Marcoli 2025

$pp \rightarrow \gamma jj$ Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet, Zoia 2023

$pp \rightarrow jjj$ Abreu, Febres Cordero, Ita, Page, Sotnikov 2021
De Laurentis, Ita, Klinkert, Sotnikov 2023
Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi 2023
De Laurentis, Ita, Sotnikov 2023

Two-loop Five-point integrals with one external mass

Abreu, Ita, Moriello, Page, Tschernow, 2020
Kardos, Papadopoulos, Smirnov, Syrrakos, Wever 2022
Abreu, Ita, Page, Tschernow 2022
Chicherin, Sotnikov, Zoia 2022
Abreu, Chicherin, Page, Sotnikov, Tschernow, Zoia 2024



... ..

$pp \rightarrow Wbb$ Badger, Hartanto, Zoia 2021 Hartanto, Poncelet, Popescu, Zoia 2022

$pp \rightarrow Wjj$ Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov 2022
De Laurentis, Ita, Page, Sotnikov 2025

$pp \rightarrow Hbb$ Badger, Hartanto, Krys, Zoia 2021
Badger, Hartanto, Poncelet, Wu, YZ, Zoia 2024

$pp \rightarrow W\gamma j$ Badger, Hartanto, Krys, Zoia 2022

$pp \rightarrow W/Zbb$ Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini 2022
Mazzitelli, Sotnikov, Wiesemann 2024

$pp \rightarrow W\gamma\gamma$ Badger, Hartanto, Wu, YZ, Zoia 2024

$pp \rightarrow ttH$ Wang, Xie, Yang, Ye 2024

$pp \rightarrow tt\gamma$ Wang, Yang 2025

...

adapt Matteo Becchetti's slide in HOCtools-II-2025

Analytic Feynman integral and the methodology

Status of analytic Feynman integrals

with dimensional regulation

Planar massless $\rightarrow$ nonplanar massless $\rightarrow$ massive

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results		?	?
Three-loop	some results		?	?	?
Four-loop	some results	?	?	?	?

Frontier

Liu, Matijasic, Miczajka, Xu, Xu, YZ,
Phys.Rev.D 112 (2025) 1, 016021, editors' suggestion
Chicherin, Wu, Wu, Xu, Zhang,YZ,
arXiv 2512.17330

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, PRL. 135, 031601
Henn, Matijasic, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027
Henn, Peraro, Xu, YZ, JHEP 03 (2022) 056
Abreu, Monni, Page, Usovitsch JHEP 06 (2025) 112

Feynman integrals: analytic functions

in terms of

Goncharov polylogarithm function

$$G(\mathbf{0}_k; z) = \frac{1}{k!} (\log z)^k, \quad G(a_1, \dots, a_k; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_k; t)$$

arguments related to **Letters**, algebraic function of kinematics

well studied function with **Hopf algebra** structure

elliptic polylogarithm

$$\widetilde{\Gamma}\left(\begin{array}{ccc} n_1, & \dots, & n_r \\ a_1, & \dots, & a_r \end{array}; \xi \Big| \tau\right) := \int_0^\xi dt \, g^{(n_1)}(t - a_1 \mid \tau) \, \widetilde{\Gamma}\left(\begin{array}{ccc} n_2, & \dots, & n_r \\ a_2, & \dots, & a_r \end{array}; t \Big| \tau\right), \quad \widetilde{\Gamma}(; \xi | \tau) = 1.$$

hyperelliptic polylogarithm

$$\widetilde{\Gamma}\left(\begin{array}{ccc} i_{n_1}, & \dots, & i_{n_k} \\ x_1, & \dots, & x_k \end{array}; z \Big| G\right) := \int_{z_0}^z \omega_{i_{n_1}}(t, x_1 \mid G) \, \widetilde{\Gamma}\left(\begin{array}{ccc} i_{n_2}, & \dots, & i_{n_k} \\ x_2, & \dots, & x_k \end{array}; t \Big| G\right), \quad \widetilde{\Gamma}(; z | G) := 1.$$

G is Schottky group uniformizing the hyperelliptic curve

Calabi-Yau periods

$$\Pi_\alpha(z) = \int_{\Gamma_\alpha(z)} \Omega(z), \quad \Gamma_\alpha(z) \in H_n(X_z, \mathbb{Z}).$$

...

Baune, Broedel, Im, Lisitsyn, Zerbini
J. Phys. A: Math. Theor. 57 445202 (2024)

Canonical Differential Equation

$$\frac{\partial}{\partial x_i} I(x, \epsilon) = \epsilon A_i(x) I(x, \epsilon) \quad \text{Henn 2013}$$

Existence (beyond the polylogarithm) and general algorithm to obtain canonical differential equation

ϵ -collaboration Phys.Rev.Lett. 136 (2026) 24, 241602

Alphabet searching

Effortless, Matijasic, Miczajka to appear

<https://github.com/antonela-matijasic/Effortless>

BaikovLetter, Jiang, Liu, Xu, Yang, Phys.Lett.B 864 (2025) 139443

SOFIA Correia, Giroux, Mizera 2503.16601

...

iterative integration reorganization (to one-fold integral)

Liu, Matijasic, Miczajka, Xu, Xu, YZ, *Phys.Rev.D* 112 (2025) 1, 016021

numerical solution

DiffExp, Hidding *Comput.Phys.Commun.* 269 (2021) 108125

Line, Prisco, Ronca, Tramontano JHEP07(2025)219

CHESS, Liu YZ, arXiv: 2606.26691

...

$$I^{(0)} = B^{(0)}$$

$$I^{(1)} = B^{(1)} + \int_{\gamma} (d\tilde{A}) I^{(0)}$$

$$I^{(2)} = B^{(2)} + \int_{\gamma} (d\tilde{A}) I^{(1)}$$

$$\vdots$$

$$I^{(n)} = B^{(n)} + \int_{\gamma} (d\tilde{A}) I^{(n-1)}$$

*for Next-to-Next-to-leading order
production of 4 jets, 4 photons ...*

2loop 6point Feynman integrals

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *Phys. Rev. Lett.* 135 (2025) 3, 031601

Abreu, Monni, Page, Usovitsch, *JHEP* 06 (2025) 112

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *JHEP* 08(2024) 027

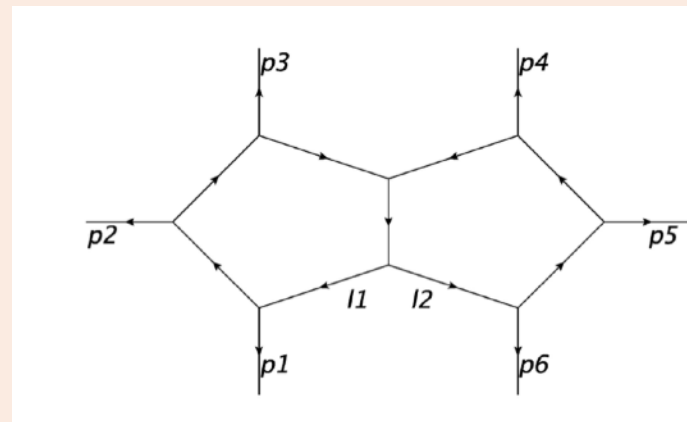
Liu, Matijasic, Peraro, Xu, Yang, YZ, *arXiv* 2603.16831, *JHEP* accepted

All planar massless 2loop 6point integrals

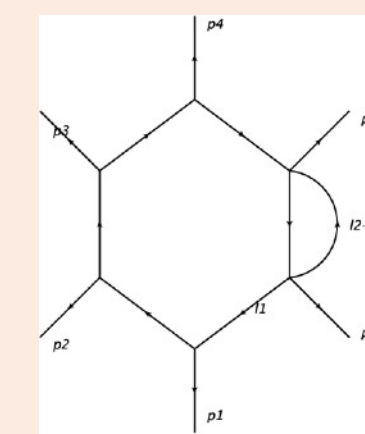
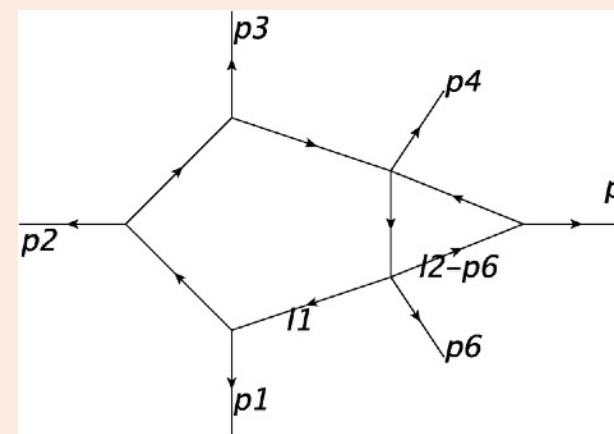
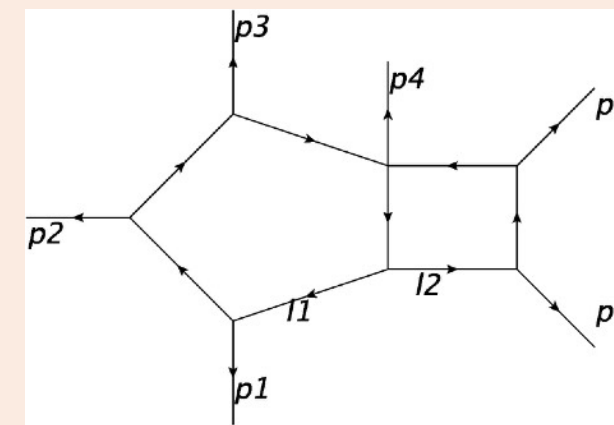
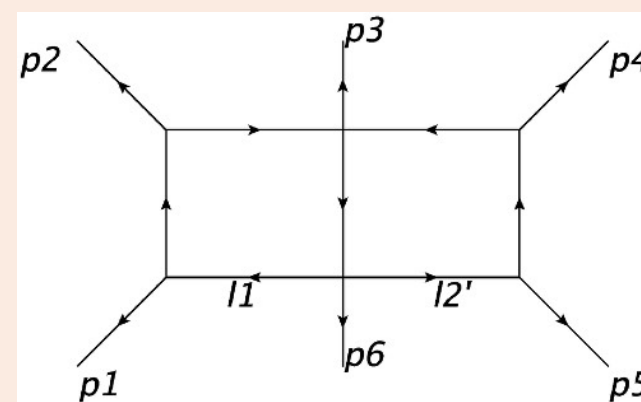
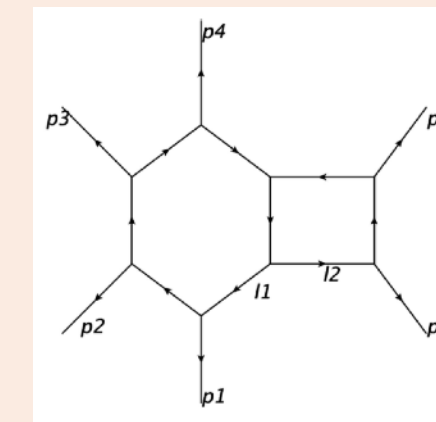
Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *Phys. Rev. Lett.* 135 (2025) 3, 031601

Abreu, Monni, Page, Usovitsch, *JHEP* 06 (2025) 112

267
pure integrals



202
pure integrals



Momentum Twistor

external momenta

$d=4$

$$Z_i = \begin{pmatrix} \lambda_{i,\alpha} \\ \mu_{i,\dot{\beta}} \end{pmatrix}, \quad i = 1, \dots, 6$$

$$\tilde{\lambda}_i = \frac{\langle i, i+1 \rangle \mu_{i-1} + \langle i+1, i-1 \rangle \mu_i + \langle i-1, i \rangle \mu_{i+1}}{\langle i, i+1 \rangle \langle i-1, i \rangle},$$

A particular parameterization

Badger, Frellesvig, YZ 2013

$$Z = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_2 x_1} + \frac{1}{x_1} & \frac{1}{x_2 x_1} + \frac{1}{x_2 x_3 x_1} + \frac{1}{x_1} & \frac{1}{x_2 x_1} + \frac{1}{x_2 x_3 x_1} + \frac{1}{x_2 x_3 x_4 x_1} + \frac{1}{x_1} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{x_5}{x_2} & x_6 & 1 \\ 0 & 0 & 1 & 1 & x_7 & 1 - \frac{x_8}{x_5} \end{pmatrix}$$

$x_1 = s_{12}$

$x_2 = -\frac{\text{Tr}_+(1234)}{2s_{12}s_{34}}$

$x_3 = -\frac{\text{Tr}_+(1345)}{2s_{45}s_{13}}$

$x_4 = -\frac{\text{Tr}_+(1456)}{2s_{56}s_{14}}$

$x_5 = \frac{s_{23}}{s_{12}}$

$x_6 = -\frac{\text{Tr}_+(1532) + \text{Tr}_+(1542)}{2s_{15}s_{12}}$

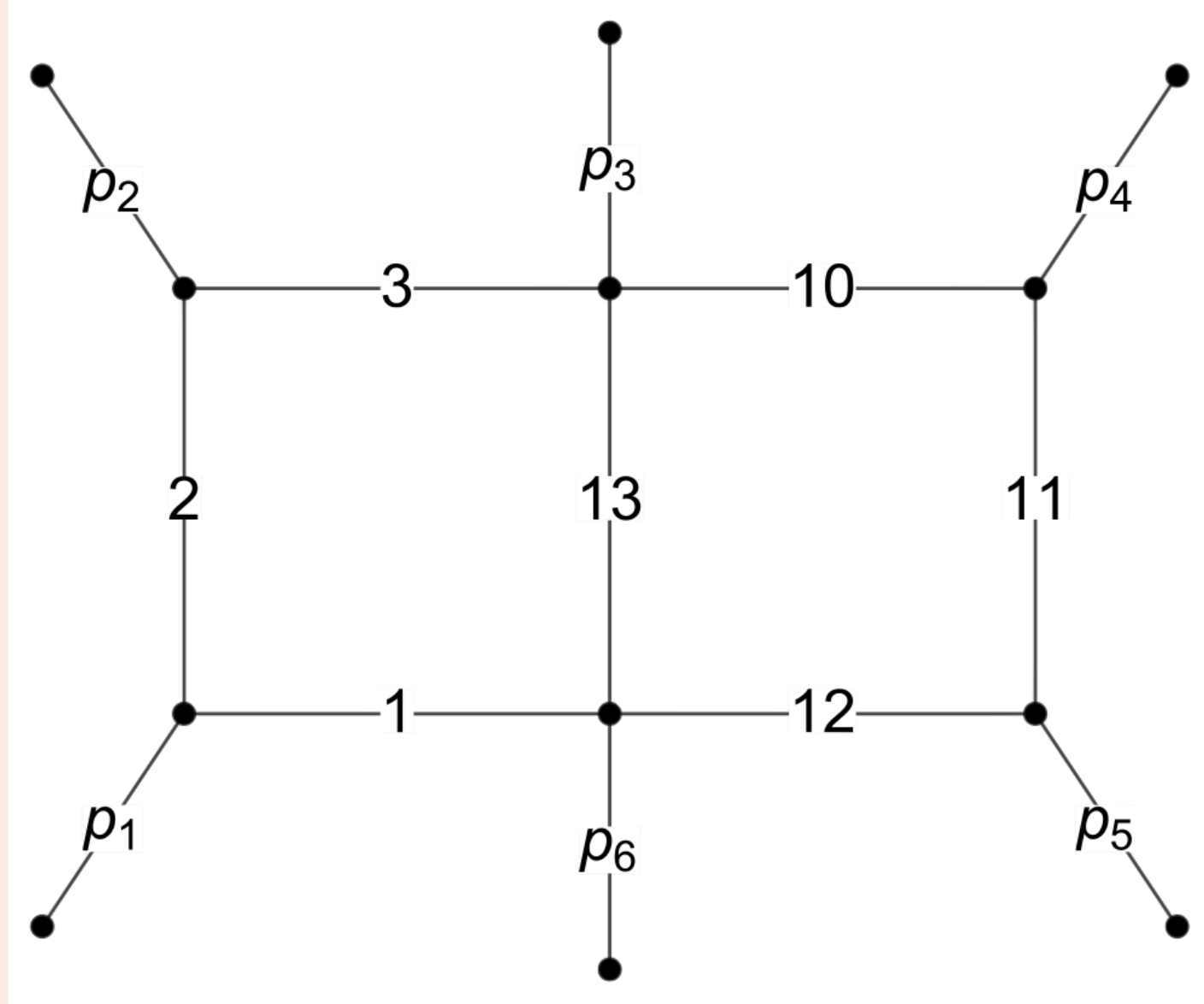
$x_7 = 1 + \frac{\text{Tr}_+(1542) + \text{Tr}_+(1543)}{2s_{15}s_{23}}$

$x_8 = \frac{s_{123}}{s_{12}}$

Momentum parametrization rationalizes all pseudo scalars

$\epsilon_{ijkl} \equiv 4i\epsilon_{\mu\nu\rho\sigma} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma, \quad \epsilon_{ijkl}^2 = G_{ijkl}$

Uniformly transcendental (UT or pure) basis determination



Chiral numerator

(Arkani-Hamed, Bourjaily, Cachazo, Trnka 2011)

/ Gram determinant

correspondence $\Delta_6 = \langle 12 \rangle [23] \langle 34 \rangle [45] \langle 56 \rangle [61] - \langle 23 \rangle [34] \langle 45 \rangle [56] \langle 61 \rangle [12] .$

$$I_{\text{db},i} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_i}{D_1 D_2 D_3 D_{10} D_{11} D_{12} D_{13}} , \quad i = 1, \dots, 7$$

$$N_1 = -s_{12} s_{45} s_{156} ,$$

$$N_2 = -s_{12} s_{45} (l_1 + p_5 + p_6)^2 ,$$

$$N_3 = \frac{s_{45}}{\epsilon_{5126}} G \left(\begin{array}{cccc} l_1 & p_1 & p_2 & p_5 + p_6 \\ p_1 & p_2 & p_5 & p_6 \end{array} \right) ,$$

$$N_4 = \frac{s_{12}}{\epsilon_{1543}} G \left(\begin{array}{cccc} l_2 - p_6 & p_5 & p_4 & p_1 + p_6 \\ p_1 & p_5 & p_4 & p_3 \end{array} \right) ,$$

$$N_5 = -\frac{1}{4} \frac{\epsilon_{1245}}{G(1, 2, 5, 6)} G \left(\begin{array}{ccccc} l_1 & p_1 & p_2 & p_5 & p_6 \\ l_2 & p_1 & p_2 & p_5 & p_6 \end{array} \right) ,$$

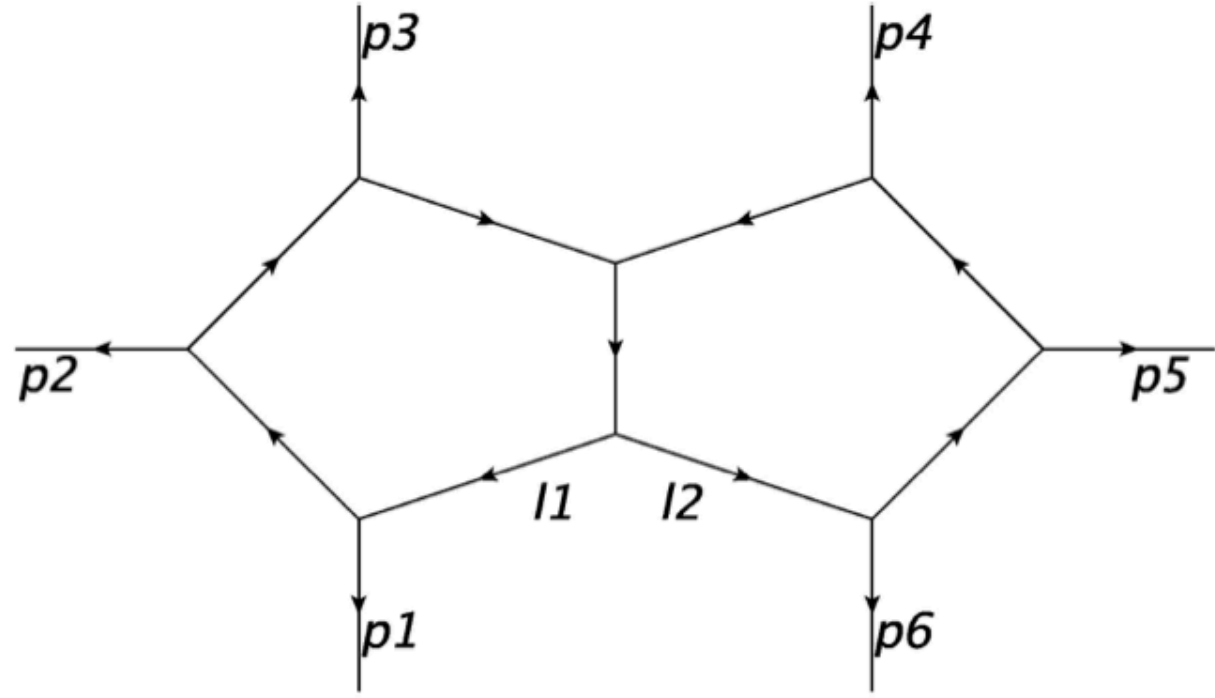
$$N_6 = \frac{1}{8} G \left(\begin{array}{ccc} l_1 & p_1 & p_2 \\ l_2 - p_6 & p_4 & p_5 \end{array} \right) + \frac{D_2 D_{11} (s_{123} + s_{126})}{8} ,$$

$$N_7 = -\frac{1}{2\epsilon} \frac{\Delta_6}{G(1, 2, 4, 5) D_{13}} G \left(\begin{array}{ccccc} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{array} \right) .$$

see also Abreu, Monni, Page, Usovitsch *JHEP* 06 (2025) 112 , *a systematic way of finding pure basis*

2loop 6point top sector, pure integrals

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601



UT integrals list

$$I_1^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} - N_4^{\text{DP-a}}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1 \dots D_9}$$

$$I_3^{\text{DP-a}} = F_3 \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{\mu_{12}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

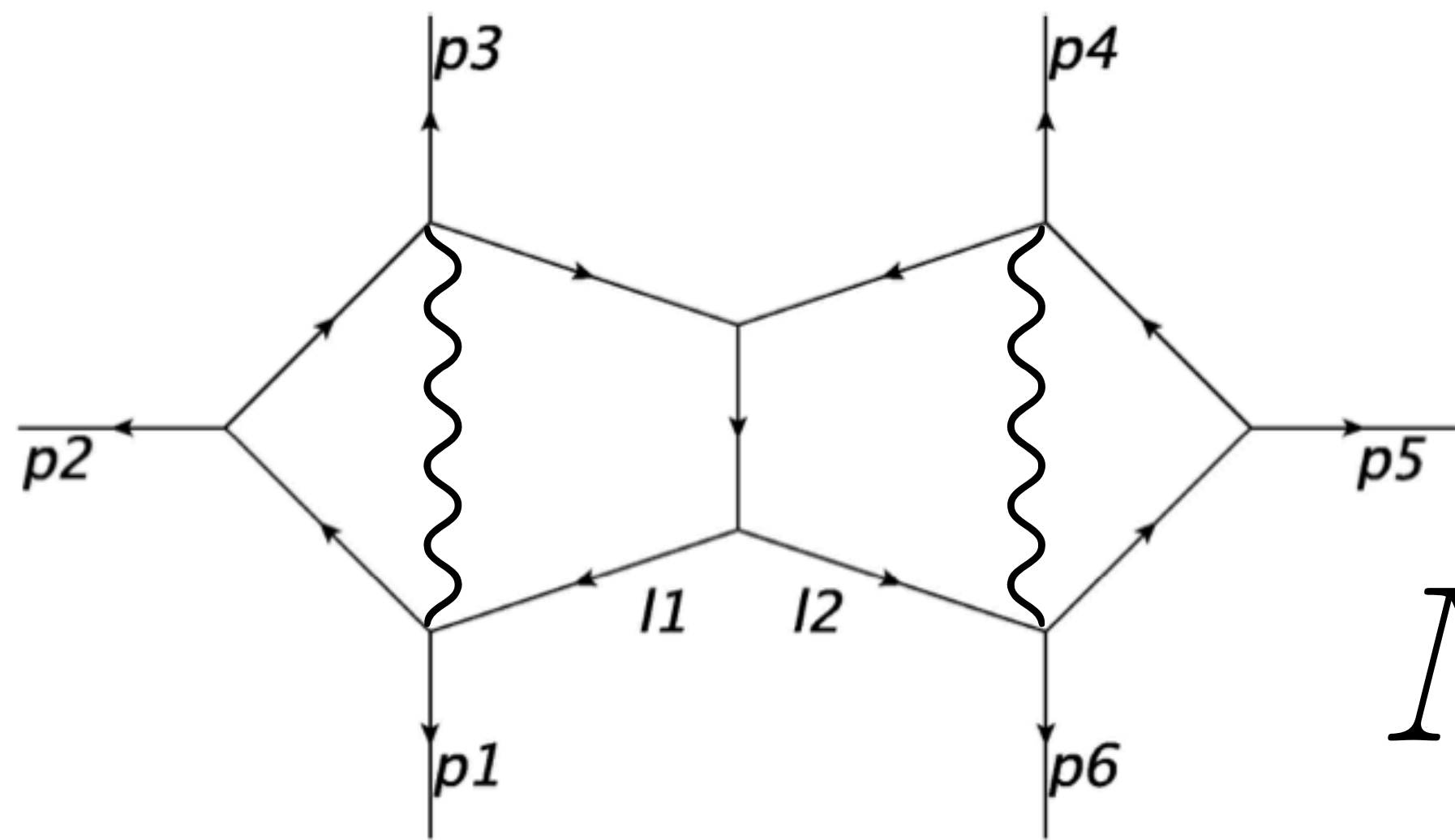
$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \longrightarrow \text{evanescent, 6D weight-6 integral}$$

$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5 \mu_{12}}{D_1 \dots D_9}$$

5 MIs (this sector)
267 MIs (whole family)

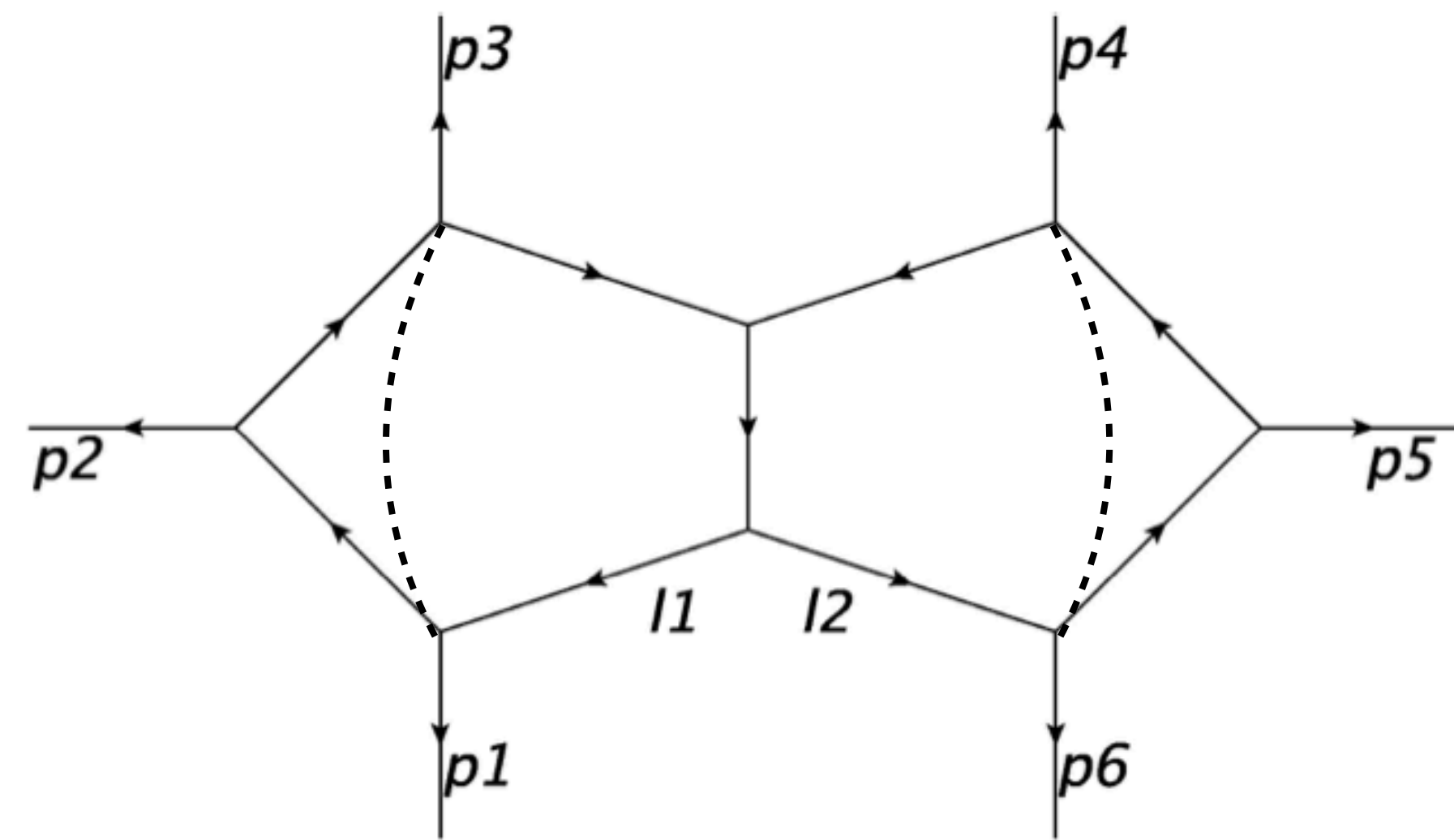
N_1, N_2, N_3 and N_4 are chiral numerators

The weight-4 parts of those integrals known from DCI studies



N_1

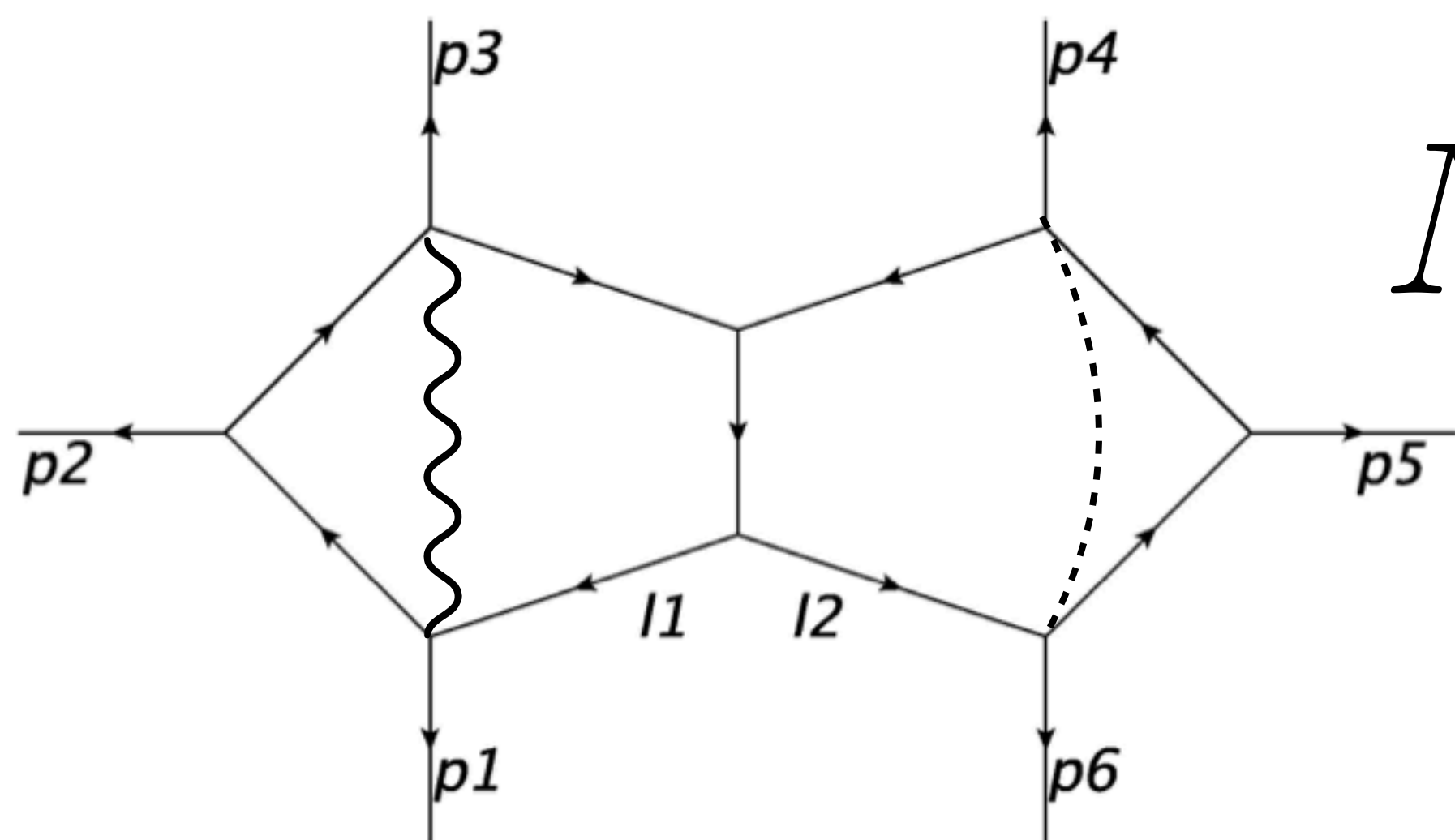
N_4



$$ut_2 = I[N_2 - N_3] = -2\tilde{\Omega}_{\text{odd}} + O(\epsilon),$$

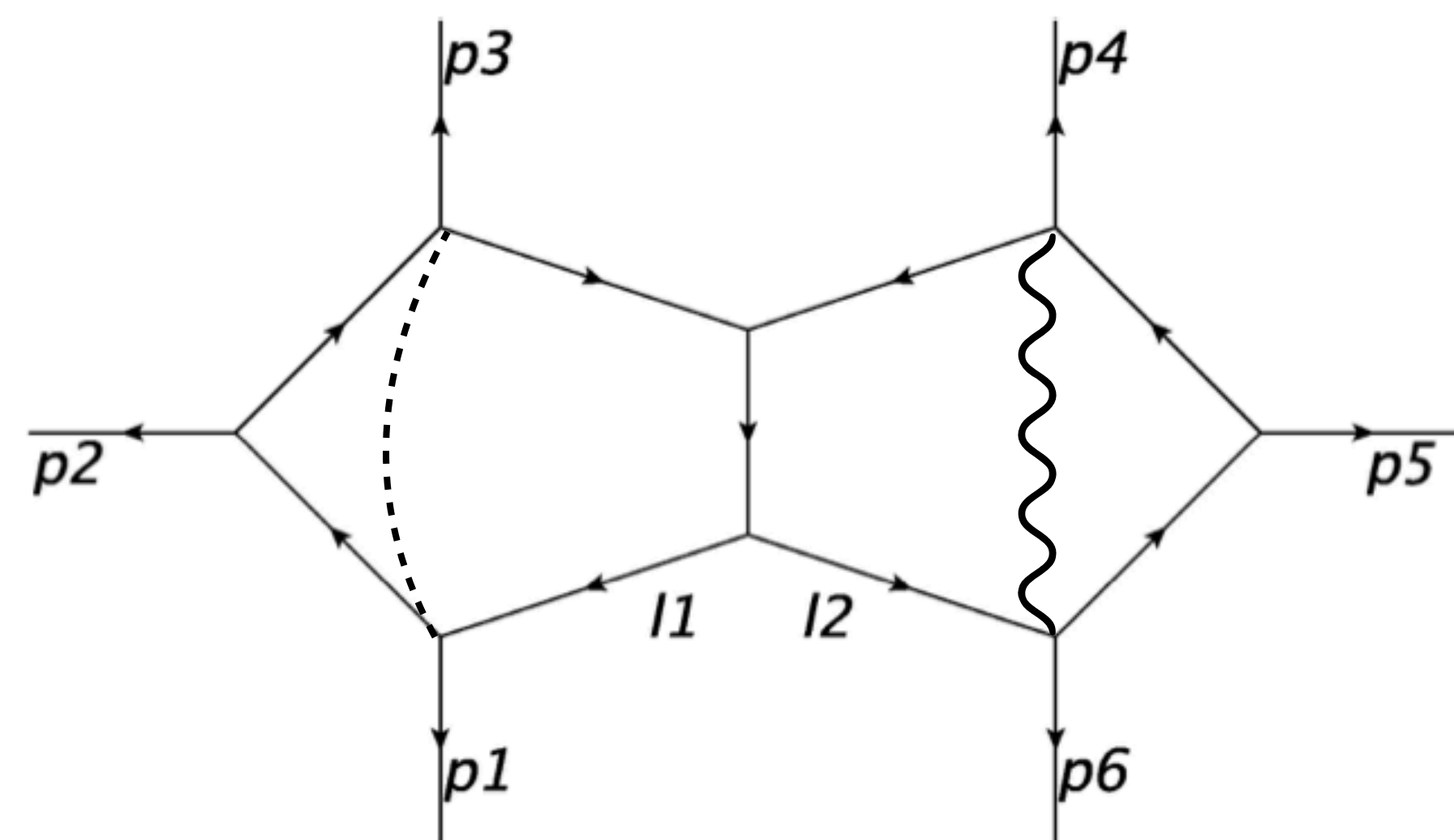
$$ut_5 = I[N_1 + N_4] = 2\Omega_{\text{even}} + O(\epsilon)$$

chiral-numerator integrals are finite and
analytically calculated to weight-4 long time ago
Dixon, Drummond, Henn 2011



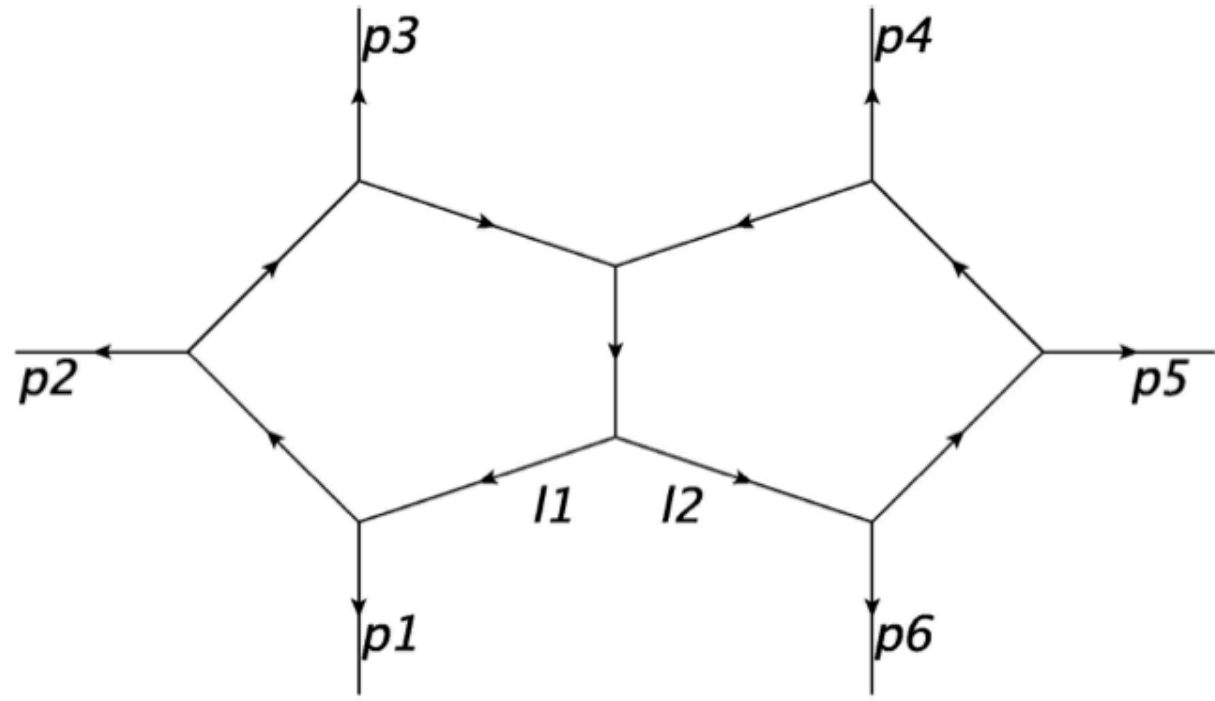
N_2

N_3



2loop 6point top sector, UT integrals

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601



UT integrals list

$$I_1^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} - N_4^{\text{DP-a}}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1 \dots D_9}$$

$$I_3^{\text{DP-a}} = F_3 \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{\mu_{12}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \longrightarrow \text{evanescent, 6D weight-6 integral}$$

$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5 \mu_{12}}{D_1 \dots D_9}$$

Abreu, Monni, Page, Usovitsch *JHEP* 06 (2025) 112

Those integrals provide **no independent** weight-4 functions ...
not needed for two-loop amplitudes to the $O(\epsilon^0)$ order

This sector has no contribution
to divergent and finite part of two-loop
gauge theory amplitudes

5 MIs (this sector)
267 MIs (whole family)

Alphabet, 2loop 6point massless planar integrals, up to weight 4

Even letter $F(s)$ a polynomial in Mandelstam variables
or homogeneously linear in square roots

a Feynman integral's even letters are connected to Landau singularity

Refer to Vergu's talk

Odd letter $\frac{P(s) - \sqrt{Q(s)}}{P(s) + \sqrt{Q(s)}}$ $\log(W) \mapsto -\log(W)$ under the sign change of the square root

“square roots”: $\epsilon_{ijkl}, \Delta_6, \sqrt{\lambda(s_{12}, s_{34}, s_{56})}$

pseudo
scalar

leading
singularity
hexagon

Källin function
from massive triangle
diagrams

More
complicated
letter

$$\frac{P(s) - \sqrt{Q_1(s)}\sqrt{Q_2(s)}}{P(s) + \sqrt{Q_1(s)}\sqrt{Q_2(s)}}$$

Matijasic, Miczajka, to appear

Effortless

<https://github.com/antonela-matijasic/Effortless>

Even letter, Odd letter and the more complicated ...

Cluster algebra structure

156 Even letters

$$\begin{aligned} & s_{12}, \quad s_{123} \\ & s_{12} - s_{123} \\ & \dots \\ & -s_{12}s_{45} + s_{123}s_{345} \\ & \dots \\ & \sqrt{\lambda(s_{12}, s_{34}, s_{56})}, \quad \epsilon_{ijkl} \end{aligned}$$

Bossinger, Drummond, Glew, Gürdoğan, arXiv: 2507.01015
 Pokraka, Spradlin, Volovich, Weng, JHEP 03 (2026) 120

79 Odd letters

$$\begin{aligned} & \frac{s_{12} + s_{34} - s_{56} - \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{s_{12} + s_{34} - s_{56} + \sqrt{\lambda(s_{12}, s_{34}, s_{56})}} \\ & \dots \\ & \frac{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) - \epsilon_{1234}}{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) + \epsilon_{1234}}, \\ & \dots \quad \dots \\ & \frac{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) - \Delta_6}{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) + \Delta_6} \end{aligned}$$

10 More complicated letters

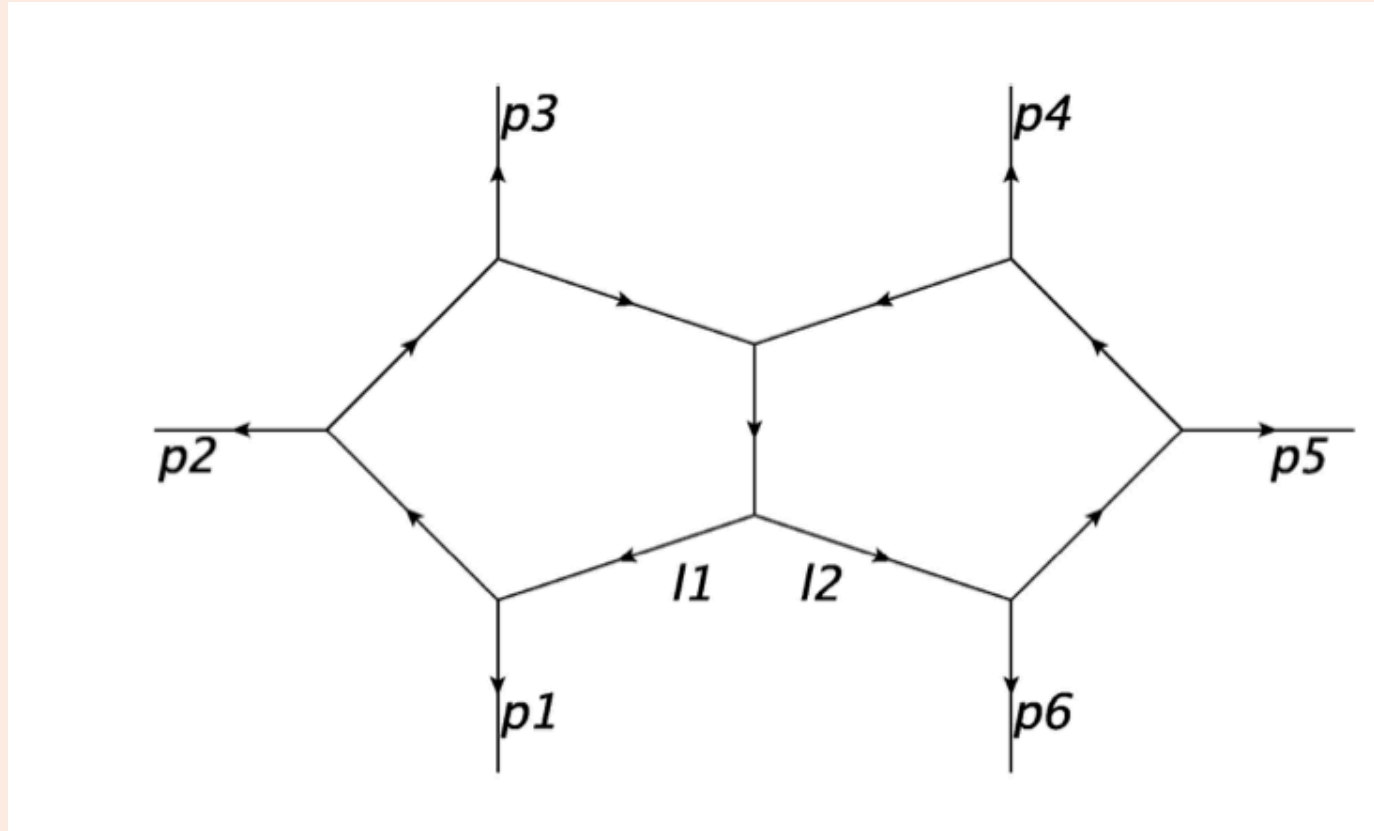
$$\frac{P - \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{P + \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}} \quad \dots$$

245 letters
 up to the weight four

Then the canonical DE is obtained analytically
 after ~50 times of numeric IBP running

Two-loop Six-point integrals to higher orders of ϵ

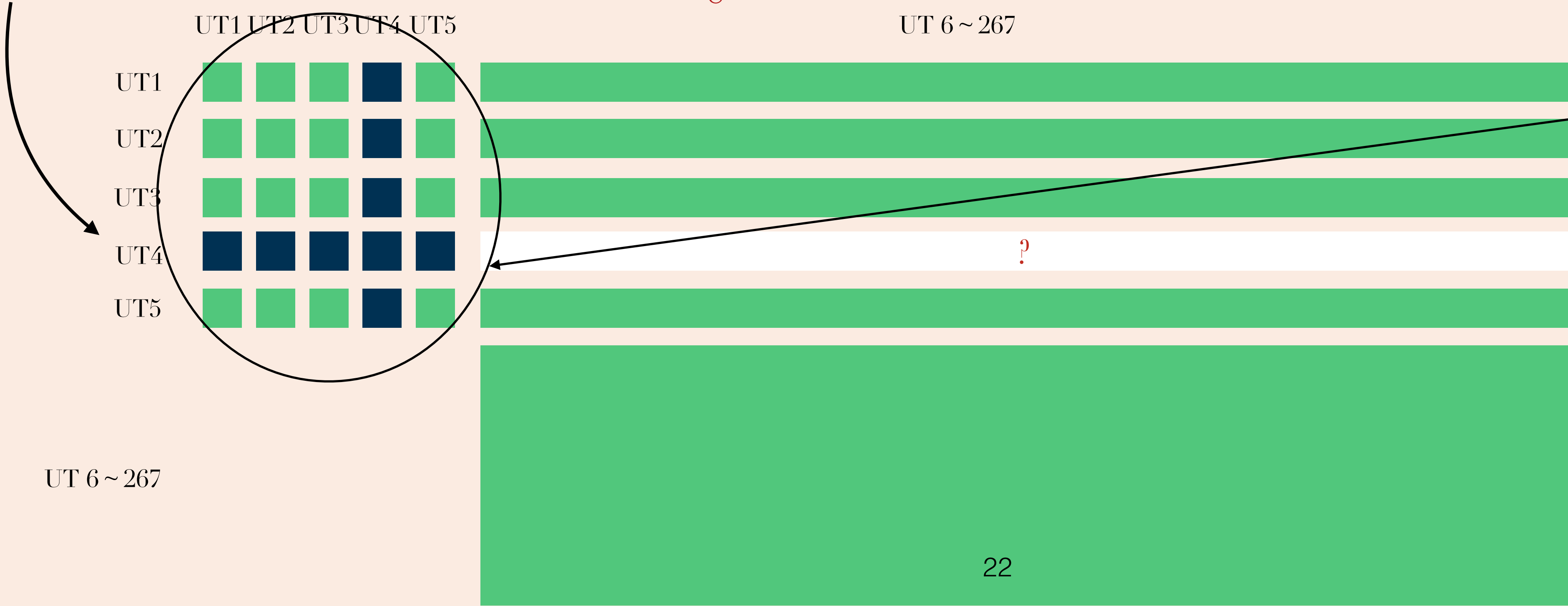
Liu, Matijasic, Peraro, Yang, Xu, YZ, arXiv. 2603.16831



6D double pentagon, starts with weight six

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \sim O(\epsilon^2)$$
$$F_4 = \frac{1}{8} \left(G(1, 2, 3, 6) s_{45}^2 + G(1, 2, 3, 5) s_{46}^2 + G(1, 2, 3, 4) s_{56}^2 \right. \\ \left. - 2\epsilon_{1235} \epsilon_{1236} s_{45} s_{46} - 2\epsilon_{1234} \epsilon_{1236} s_{45} s_{56} - 2\epsilon_{1234} \epsilon_{1235} s_{46} s_{56} \right)^{1/2},$$
$$= \frac{1}{8} \sqrt{\lambda(s_{45} \epsilon_{1236}, s_{46} \epsilon_{1235}, s_{56} \epsilon_{1234})}$$

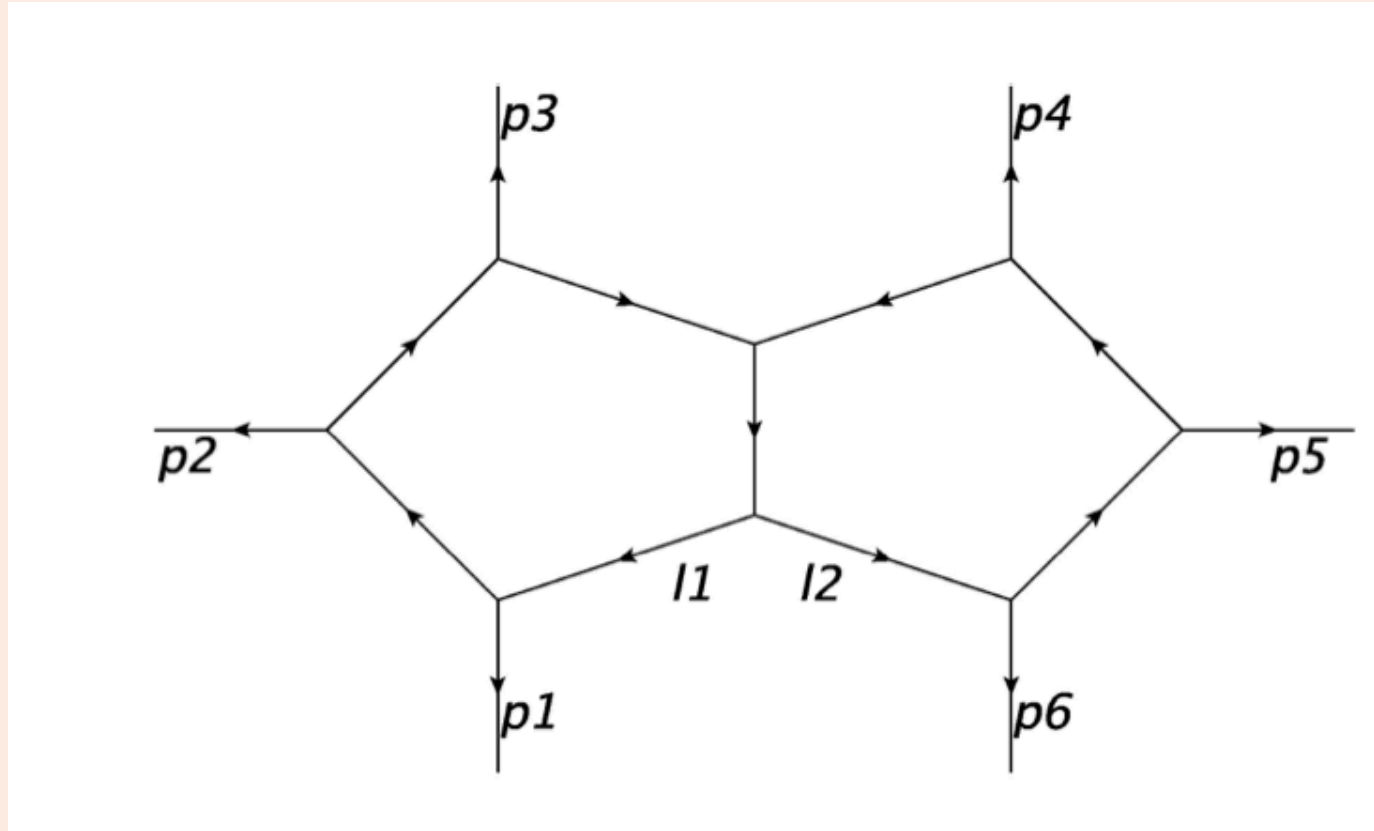
a new square root
not rationalised
by momentum twistor



$$\frac{F_4 + R_i}{F_4 - R_i}, \quad i = 1, \dots, 5$$

18 more letters
from the diagonal block

x-point integrals to higher orders of ϵ



6D double pentagon, starts with weight six

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \sim O(\epsilon^2)$$

Calico: Bertolini, Fontana, Peraro

FiniteFlow: Peraro

UT1 UT2 UT3 UT4 UT5

UT 6 ~ 267

UT1

UT2

UT3

UT4

UT5

UT 6 ~ 267

6 more letters
from the off-diagonal block

Two-loop Six-point integrals to higher orders of ϵ

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

Liu, Matijasic, Peraro, Yang, Xu, YZ, arXiv: 2603.16831

closed under the
dihedral group action

$$245 + 18 + 6 = 269 \text{ letters in total}$$

up to weight-4

2loop 6point massless planar integrals
calculated to **weight-six**

Henn, Peraro, Xu, YZ, *JHEP* 03 (2022) 056

Chebyshev pseudospectral transport,
to get weight-6 numerics in one minute

Boundary Values

Euclidean region $X_0 : \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\} \rightarrow \{-1, -1, -1, -1, -1, -1, -1, -1, -1\}$

Boundary values at the symmetric Euclidean point, are obtained up to weight 4: Zeta values and polylogarithm of roots of unity

$$\epsilon^4 I_{\text{db},1}(X_0) = 1 + \frac{\pi^2}{6} \epsilon^2 + \frac{38}{3} \zeta_3 \epsilon^3 + \left(\frac{49\pi^4}{216} + \frac{32}{3} \text{Im} [\text{Li}_2(\rho)]^2 \right) \epsilon^4,$$

$$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\epsilon^4 I_{\text{db},2}(X_0) = 1 + \frac{\pi^2}{6} \epsilon^2 + \frac{34}{3} \zeta_3 \epsilon^3 + \left(\frac{71\pi^4}{360} + 20 \text{Im} [\text{Li}_2(\rho)]^2 \right) \epsilon^4,$$

$$I_{\text{db},3}(X_0) = I_{\text{db},4}(X_0) = I_{\text{db},5}(X_0) = 0,$$

$$\epsilon^4 I_{\text{db},6}(X_0) = - \left(\frac{\pi^4}{540} + \frac{4}{3} \text{Im} [\text{Li}_2(\rho)]^2 \right) \epsilon^4,$$

$$\epsilon^4 I_{\text{db},7}(X_0) = 0.$$

from the ordinary differential equation
spurious pole asymptotic analysis

Physical region, $12 \rightarrow 3456$

obtained with AMFlow (about 3 days), ~ 200 digits, to **weight 6**

$$\{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\} \rightarrow \left\{ \frac{325}{16}, -\frac{181}{16}, \frac{65}{8}, \frac{9}{4}, -\frac{19}{8}, 7, -\frac{109}{16}, \frac{115}{8} \right\}$$

Solution of canonical DE, two-loop six-point integrals

weight-1, 2

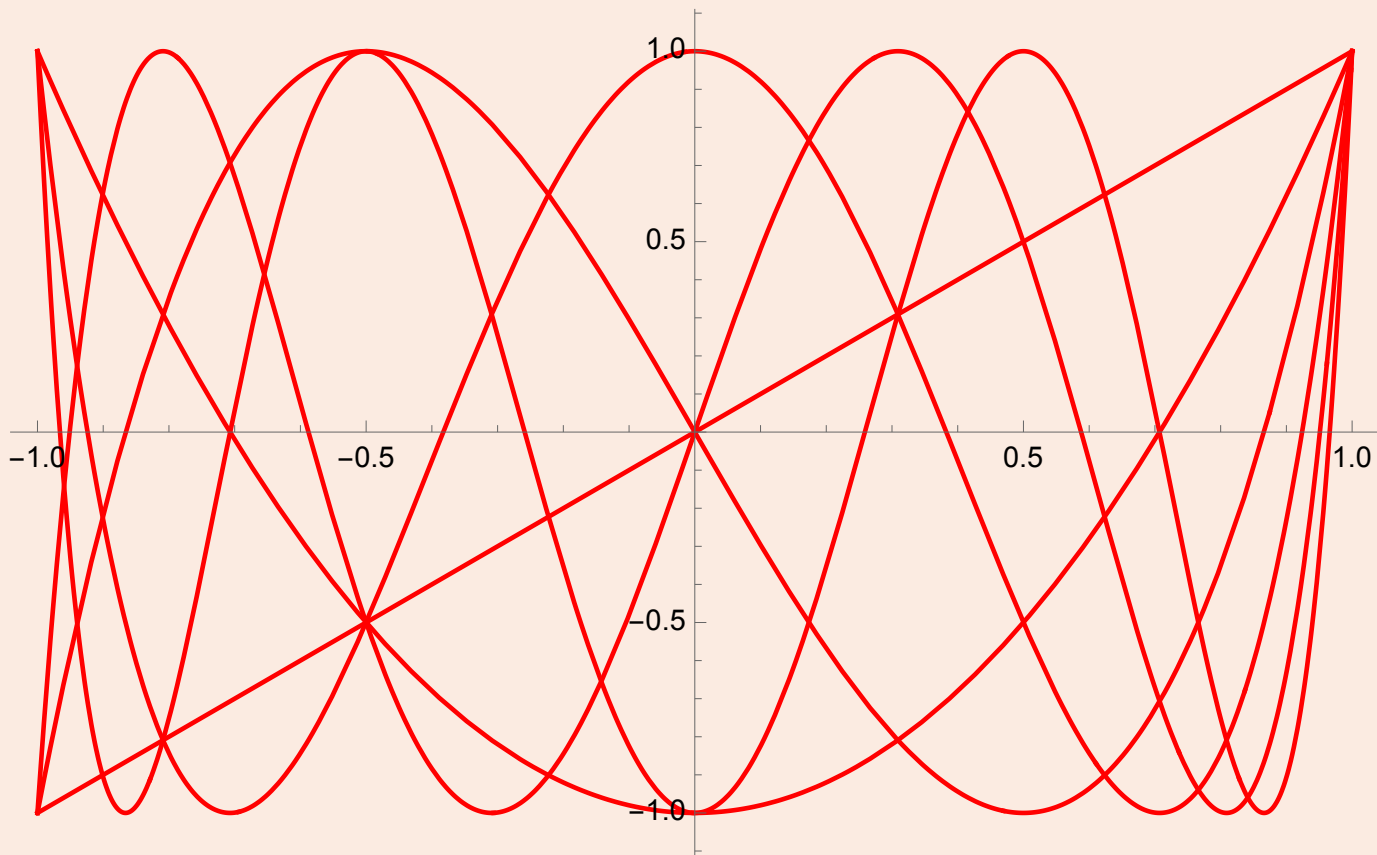
logarithm and classical poly-logarithm

weight-3, 4

one-fold integration

$$\begin{aligned}\vec{I}^{(4)} &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 dt \frac{d\tilde{A}}{dt} \vec{I}^{(3)}(\vec{x}_0) + \int_0^1 dt_1 \int_0^{t_1} dt_2 \frac{d\tilde{A}}{dt_1} \frac{d\tilde{A}}{dt_2} \vec{f}^{(2)}(t_2) \\ &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 dt \left(\frac{d\tilde{A}}{dt} \vec{I}^{(3)}(\vec{x}_0) + \left(\tilde{A}(1) - \tilde{A}(t) \right) \frac{d\tilde{A}}{dt} \vec{f}^{(2)}(t) \right).\end{aligned}$$

CHEbyshev pSeudo-Spectral transportation (CHESS) Liu and YZ, arXiv:2606.26691
 Refer to Gongshow talk by Yuanche Liu



approximate the Feynman integrals by
 Chebyshev polynomial expansion

An efficient way to solve (canonical) differential equation numerically
 Physical region, up to weight 6, all 267 integrals in double pentagon family

Chebyshev Nodes	absolute error	time	RAM
12	$\sim 7.58 \times 10^{-16}$	18.09 s	2.9 GB
24	$\sim 1.36 \times 10^{-31}$	144.76 s	26.1 GB

for Next-to-Next-to-Next-to-leading order

production of 3 jets, 3 photons ... with 2loop 6point Feynman integrals

3loop 5point Feynman integrals

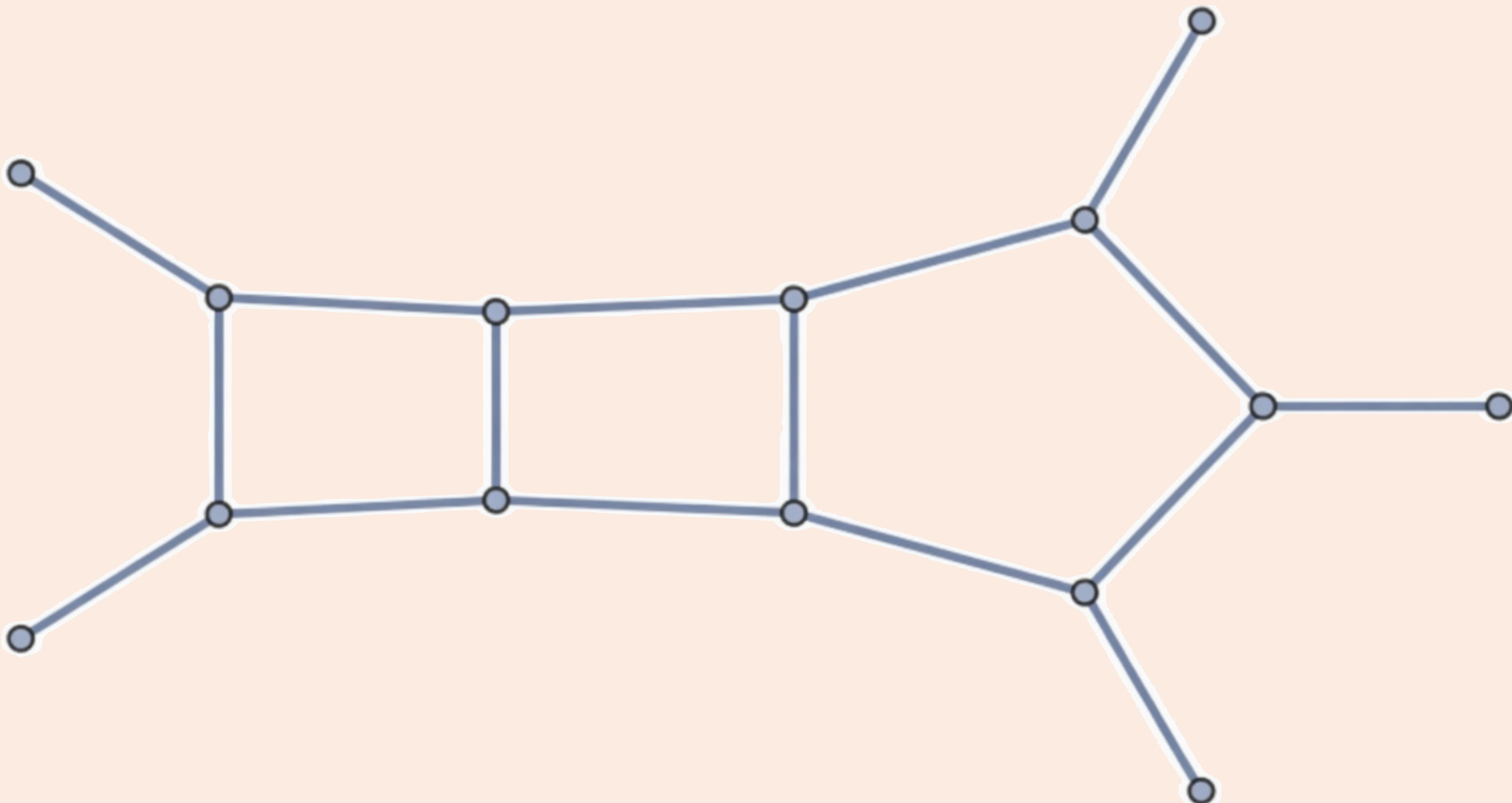
Liu, Matijasic, Miczajka, Xu, Xu, YZ

Phys.Rev.D 112 (2025) 1, 016021, editors' suggestion

Chicherin, Wu, Wu, Xu, Zhang, YZ

arXiv 2512.17330

3loop 5point planar family



5 scales

$s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021
Editors' suggestion

UT basis found!

Baikov analysis
Gram determinant

Canonical
differential
equation

We use **NeatIBP 1.0** to
derive the differential equation
 ~ 100 million IBPs $\rightarrow$ 85000 IBPs

weight-1,2,3 classical polylogarithm, weight-4,5,6 one-fold integration
It takes **2 minutes on a laptop** to get 10 digits from our analytic solution

Using NeatIBP

“NeatIBP 1.0, a package generating small-size integration-by-parts relations for Feynman integrals”

Wu, Boehm, Ma, Xu, YZ, *Comput. Phys. Commun.* 295 (2024), 108999

“Performing integration-by-parts reductions using NeatIBP 1.1 + Kira”

Wu, Boehm, Ma, Usovitsch, Xu, YZ, *Comput.Phys.Commun.* 316 (2025) 109798

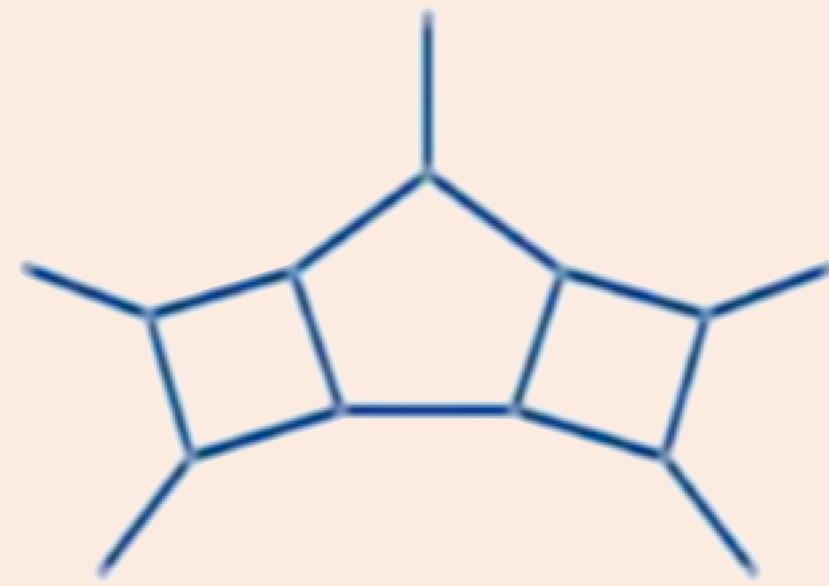
Using **algebraic geometry**
to find short IBP system,
2 or 3 orders of magnitude **shorter**
than that from Laporta algorithm.

(selected)	Studies that used NeatIBP	Reference
	Differential Equations for Energy Correlators in Any Angle	arXiv:2506.02061
	One-loop amplitudes for $t\bar{t}j$ and $t\bar{t}\gamma$	
	productions at the LHC through $O(\epsilon^2)$	arXiv:2505.10406
	Two-loop Feynman integrals for leading color $Wt\bar{t}$ production	JHEP 07 (2025) 001
	Two-loop QCD helicity amplitudes for $gg \rightarrow g t\bar{t}$ at leading color	JHEP 03 (2025) 070
	Full-color double-virtual amplitudes for $q\bar{q} \rightarrow b\bar{b}H$	JHEP 03 (2025) 066
	Three-loop five-point pentagon-box-box Feynman diagram	arXiv:2411.18697
	Two-loop QCD corrections for $pp \rightarrow t\bar{t}j$	arXiv:2411.10856
	Two-loop amplitudes for $W\gamma\gamma$ production at LHC	JHEP 12 (2025) 221
	NLO corrections to $J/\Psi c\bar{c}$ photoproduction	Phys.Rev.D 110 (2024) 9, 094047
	Two-loop five-point two-mass planar integrals	JHEP 10 (2024) 167
	Two-loop integrals for $t\bar{t}j$ production at hadron colliders	
	in the leading color approximation	JHEP 07 (2024) 073

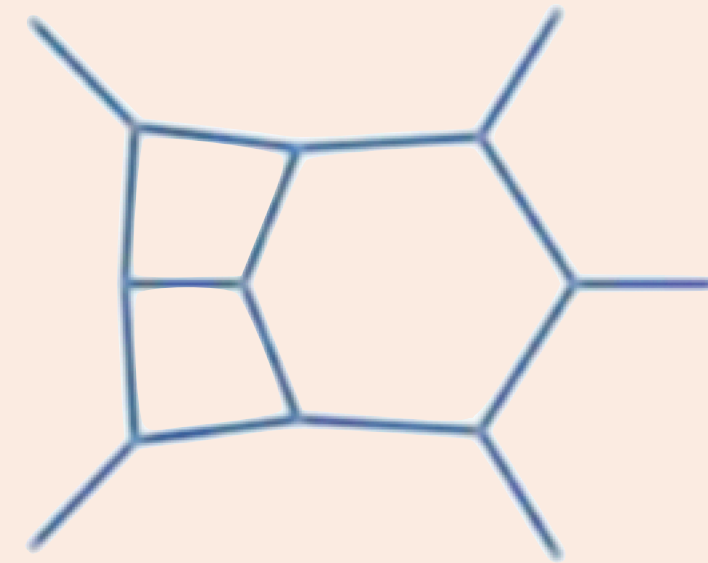
phenomenology application of NeatIBP

Complete result of all 3loop 5point planar massless integrals

Chicherin, Wu, Wu, Xu, Zhang, YZ,
arXiv 2512.17330



367 MIs



431 MIs



734 MIs

56 letters ...

IBP reduction is extremely complicated, we have to use the latest **NeatIBP 1.1** version

NeatIBP 1.1: Syzygy + Spanning Cuts

Most complicated IBP system with cuts, consists of $\sim 10^5$ equations, ~ 100 cuts

Complete result of all 3loop 5point planar massless integrals

Chicherin, Wu, Wu, Xu, Zhang, YZ,
arXiv 2512.17330

pure basis determination

top sector has 19 MIs

loop-by-loop dlog integrand generation, dimensional shift
module lift method



$$\mathcal{N} = \sum_I c_I ISP^{\alpha_I}$$

$\sum_I c_I \text{residue}_j(ISP^{\alpha_I}) = r_j,$
 c_I **polynomial** in Mandelstam variables, r_j **rational numbers**

$$\mathcal{N}_4 = \frac{\varepsilon(s_{12} - s_{45})}{(1 + 2\varepsilon)\epsilon_5} G(\ell_2, \ell_3, k_1, k_2, k_3, k_4),$$

$$\mathcal{N}_{13} = s_{12}s_{34}s_{45}(\ell_2 - k_1)^2(\ell_3 - k_{123})^2,$$

$$\mathcal{N}_{17} = s_{12}s_{45}(\ell_3 + k_5)^2 \left((\ell_2 - k_{123})^2(\ell_3 - k_{12})^2 - (\ell_3 - k_{123})^2(\ell_2 - k_{12})^2 \right),$$

$$\mathcal{N}_{19} = s_{12}s_{15}s_{45} \left((\ell_2 - k_{123})^2(\ell_3 - k_{12})^2 - (\ell_3 - k_{123})^2(\ell_2 - k_{12})^2 \right).$$

3loop 5point integrals, symbology

Chicherin, Wu, Wu, Xu, Zhang,YZ,
arXiv 2512.17330

From the canonical differential equation

We confirmed the 3loop 5point letters from Chicherin, Henn, Trnka, Zhang *JHEP* 04 (2025) 022

56 Letters

$$\mathbb{A}_{\text{P}}^{3\text{-loop}} = \mathbb{A}_{\text{P}}^{2\text{-loop}} \cup \{\widehat{W}_i\}_{i=1}^5 \cup \{\widetilde{W}_i\}_{i=16}^{20} \cup \{\widetilde{W}_i\}_{i=41}^{55} \cup \{\widetilde{W}_i\}_{i=76}^{80}$$

26 Letters

Symbols

$$\begin{aligned} \Delta_4^{(1)} = & s_{15}^2 s_{12}^2 + s_{23}^2 s_{12}^2 - 2 s_{15} s_{23} s_{12}^2 - 2 s_{23}^2 s_{34} s_{12} \\ & + 2 s_{15} s_{23} s_{34} s_{12} + 2 s_{15} s_{34} s_{45} s_{12} + 2 s_{23} s_{34} s_{45} s_{12} \\ & + s_{23}^2 s_{34}^2 + s_{34}^2 s_{45}^2 - 2 s_{23} s_{34}^2 s_{45}, . \end{aligned}$$

$$\begin{aligned} \widehat{W}_1 &= s_{23} s_{34} - s_{34} s_{45} + s_{45} s_{15} , \\ \widetilde{W}_{16} &= \Delta_4^{(1)} , \\ \widetilde{W}_i &= \frac{q_i - \sqrt{\Delta_4^{(1)}}}{q_i + \sqrt{\Delta_4^{(1)}}}, \; i \in \{41, 46, 51\} , \\ \widetilde{W}_{76} &= \frac{q_{76} - \epsilon_5 \sqrt{\Delta_4^{(1)}}}{q_{76} + \epsilon_5 \sqrt{\Delta_4^{(1)}}} , \end{aligned}$$

weight	1	2	3	4	5	6
3loop 5point Symbols	5	20	76	285	1000	2220

small number of symbols
good news for bootstrap

Solution of canonical DE, three-loop five-point integrals

weight-1, 2, 3

logarithm and classical poly-logarithm

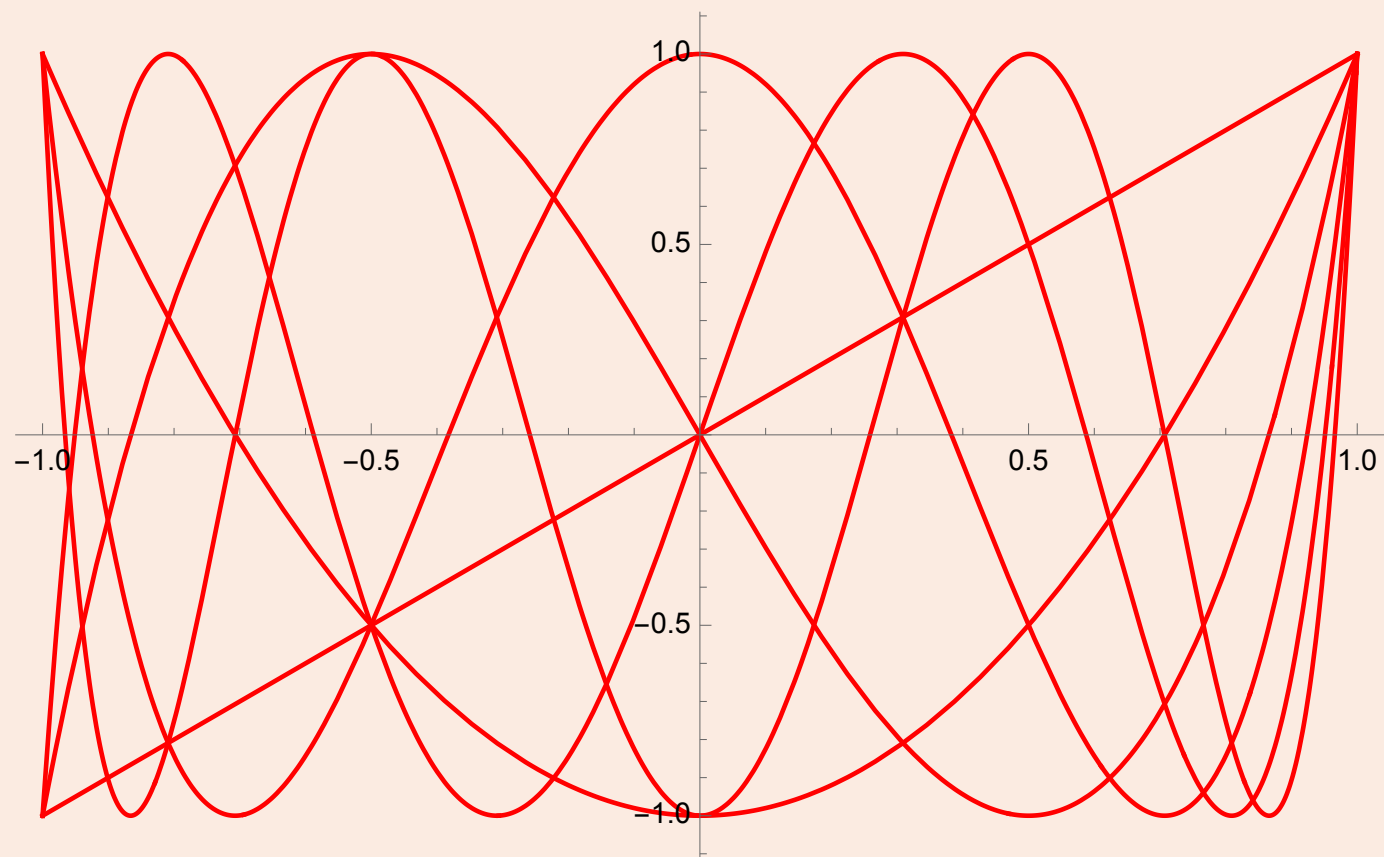
weight-3, 4, 6

one-fold integration

$$\begin{aligned} \mathbf{I}^{(n+3)}(x) = & \mathbf{I}^{(n+3)}(x_0) + \int_0^1 \frac{d\tilde{A}}{dt} \mathbf{I}^{(n+2)}(x_0) dt \\ & + \int_0^1 \left(\tilde{A}(1) - \tilde{A}(t) \right) \frac{d\tilde{A}}{dt} \mathbf{I}^{(n+1)}(x_0) dt \\ & + \int_0^1 \left(\tilde{A}(t) - \tilde{A}(1) \right) \tilde{A}(t) \frac{d\tilde{A}}{dt} \mathbf{I}^{(n)}(t) dt \\ & + \int_0^1 \left(\tilde{B}(1) - \tilde{B}(t) \right) \frac{d\tilde{A}}{dt} \mathbf{I}^{(n)}(t) dt. \end{aligned}$$

$$d\tilde{B} = (d\tilde{A})\tilde{A},$$

CHEbyshev pSeudo-Spectral transportation (CHESS)

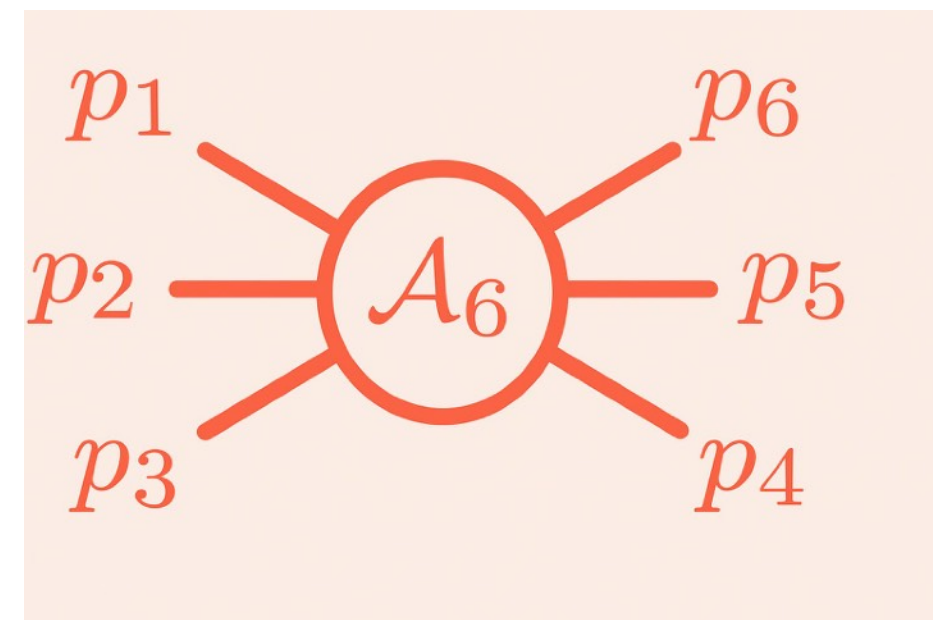


On a MacBook Pro with a 14-core M4 Pro,
we obtain the values with 10 significant digits of the 734 pure integrals,
in about 39 seconds at the following generic point in the Euclidean region.

Applications (A Brief Preview)

Refer to Chicherin's talk

2loop 6point amplitude, **bootstrap**



“Bootstrapping Six-Gluon QCD Amplitudes”

— — + + + + helicity, with fermion loops
weight-4, symbol level

Carrôlo, Chicherin, Henn, Yang, YZ
Phys.Rev.Lett. 136 (2026) 18, 181602

“QCD Scattering Amplitudes and Prescriptive Unitarity”

All MHV-helicity gluon amplitudes, with fermion loops
weight-4, symbol level

Carrôlo, Chicherin, Henn, Yang, YZ
arXiv:2602.02783 JHEP accepted

2loop hexagonal Wilson loop with Lagrangian insertion

“Hexagonal Wilson loop with Lagrangian insertion at two loops in N=4 super Yang-Mills theory”

Carrôlo, Checherin, Henn, Yang, YZ JHEP 07 (2025) 214

3loop pentagonal Wilson loop with Lagrangian insertion

“Three-loop pentagonal Wilson loop with Lagrangian insertion”

Chicherin, Henn, Xu, Zhang, YZ arXiv:2512.17881

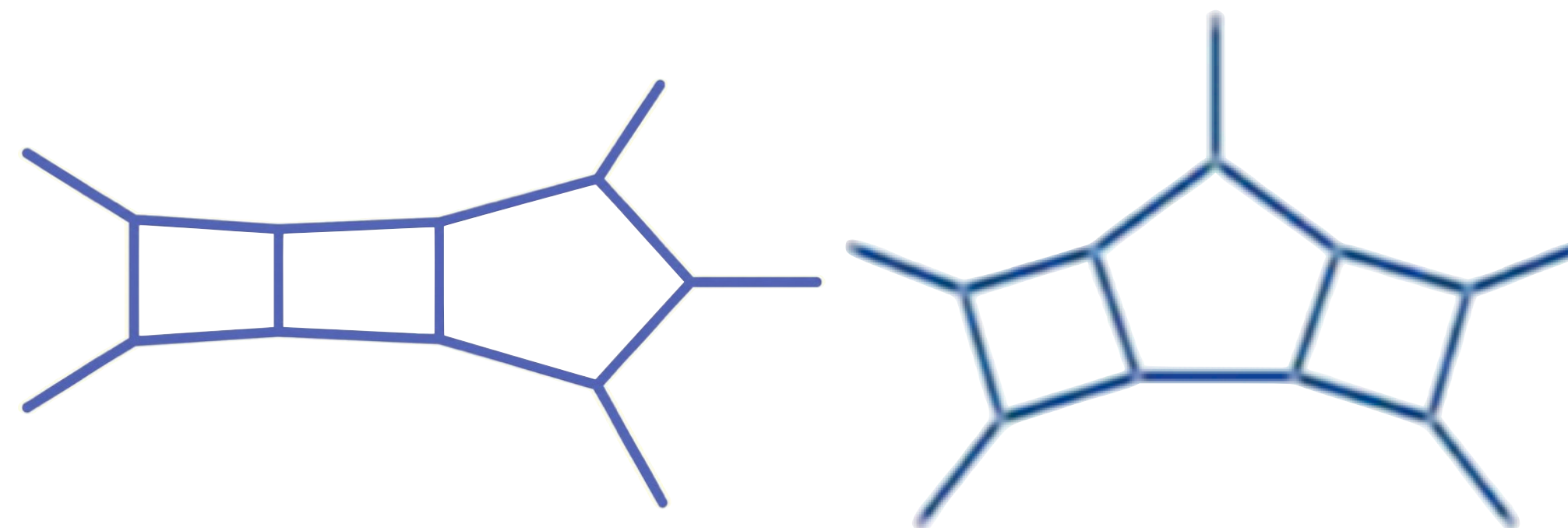
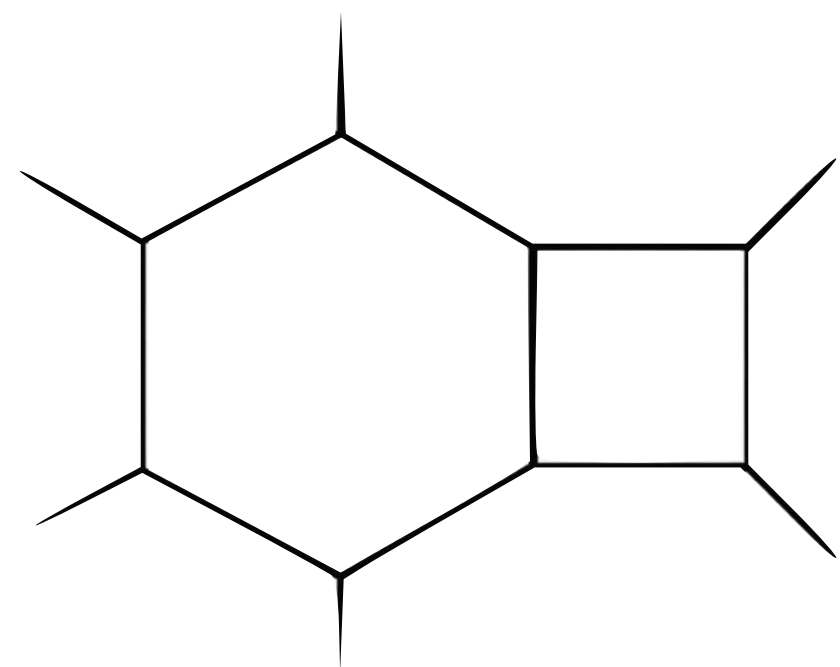
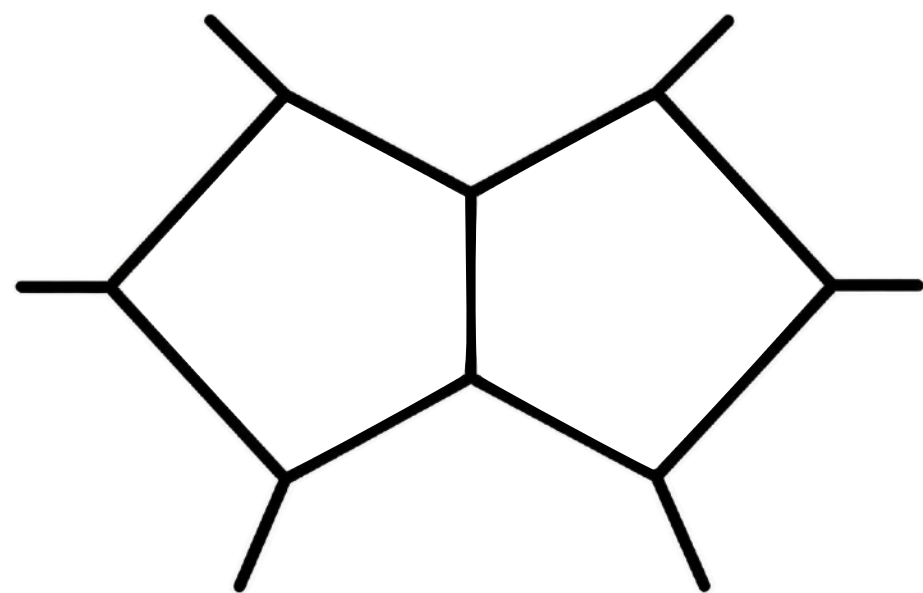
Summary and Outlook

Analytic computation of

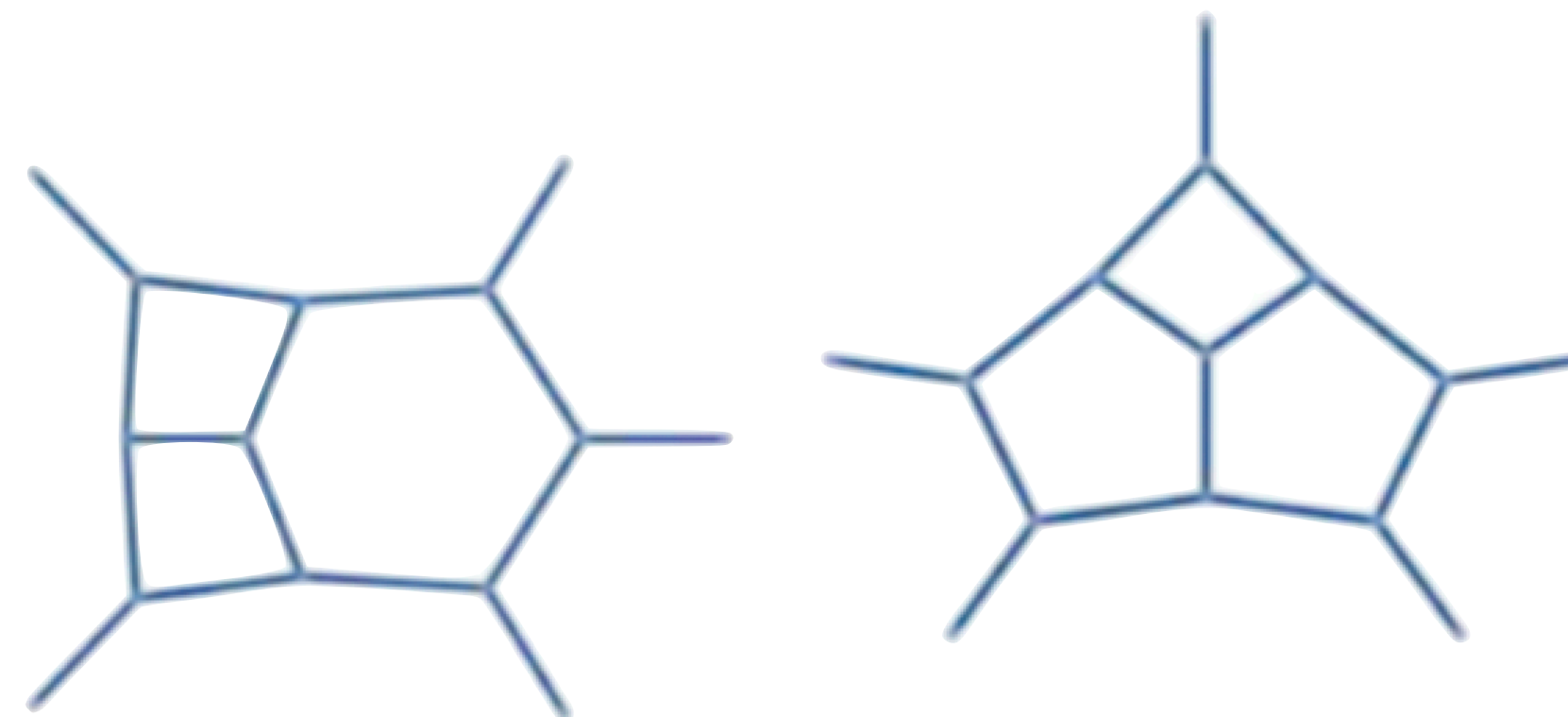
- ✓ all 2loop 6point planar massless integrals
- ✓ all 3loop 5point planar massless integrals

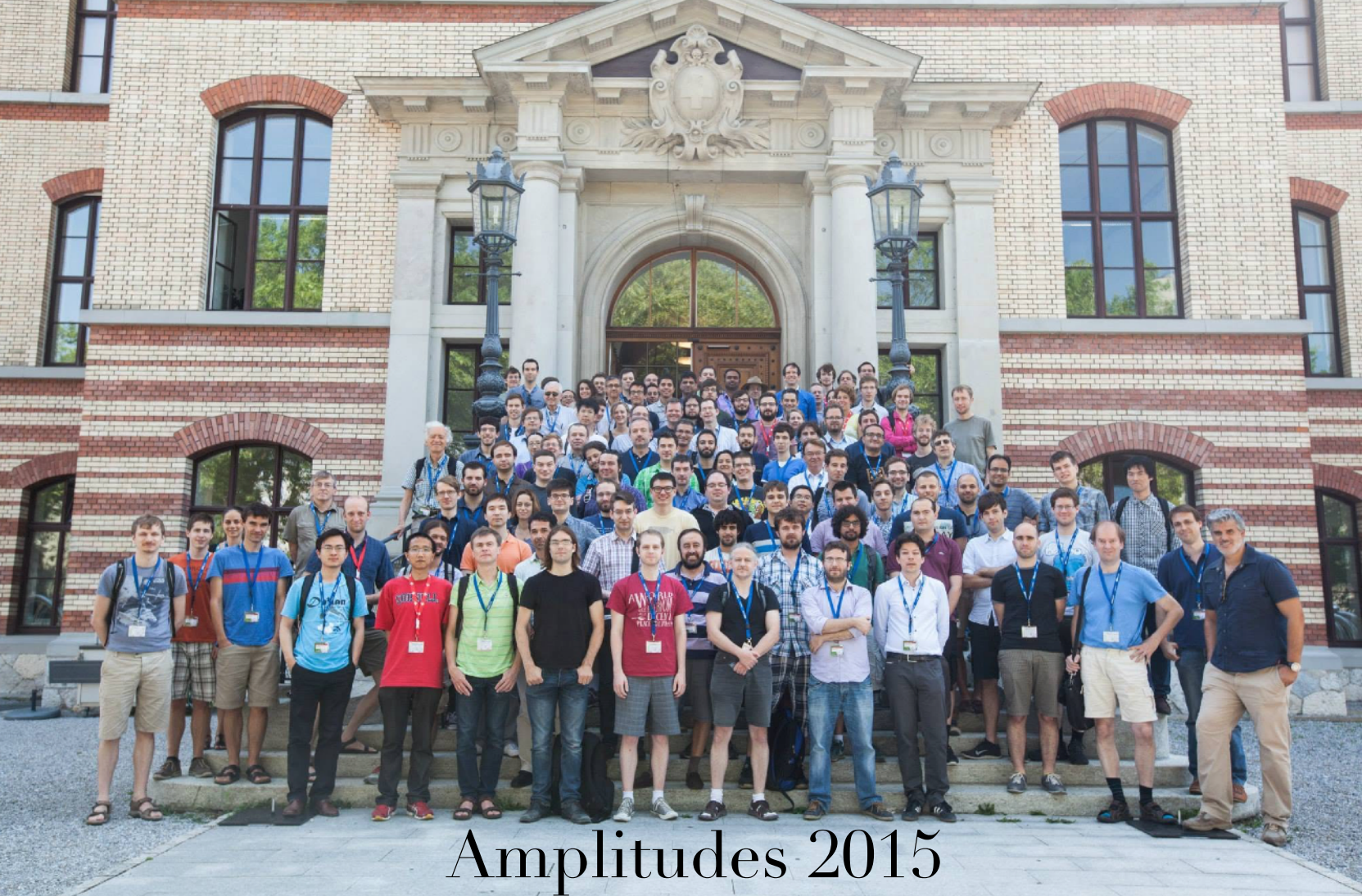
NeatIBP, a powerful package for cutting-edge IBP reduction for precision physics

We wish that a lot more applications based on
2loop 6point, 3loop 5point integrals would emerge.



Multi-loop multi-leg Feynman integrals
are no longer that difficult!





Amplitudes 2015



Amplitudes 2016



Amplitudes 2017



Amplitudes 2013



Amplitudes 2023

It has been my great pleasure to take the Amplitudes conference group photo five times before.
I am delighted to do it again this year. :)