Two-Loop Amplitudes for Massive 2→3 Processes at the LHC

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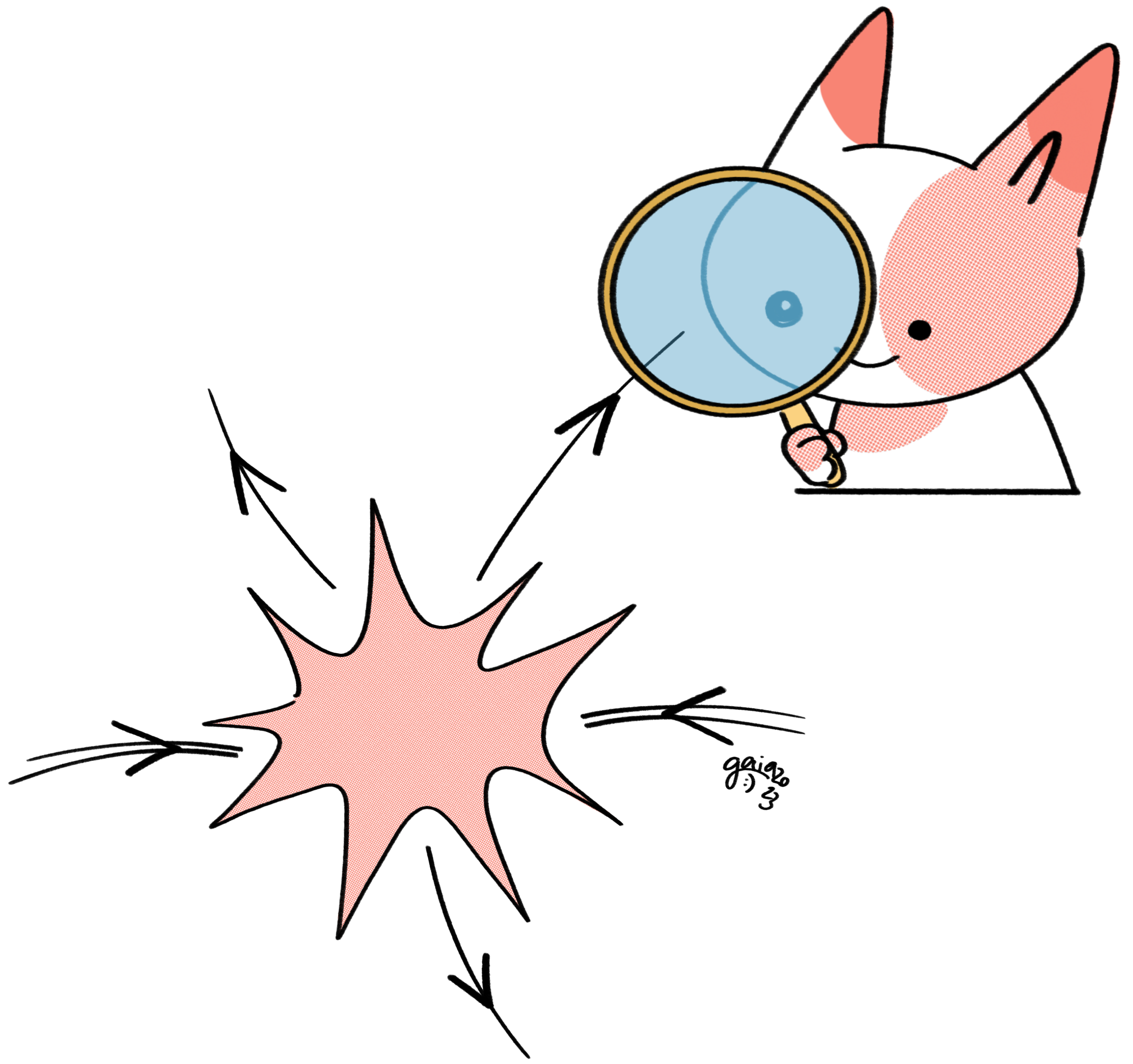
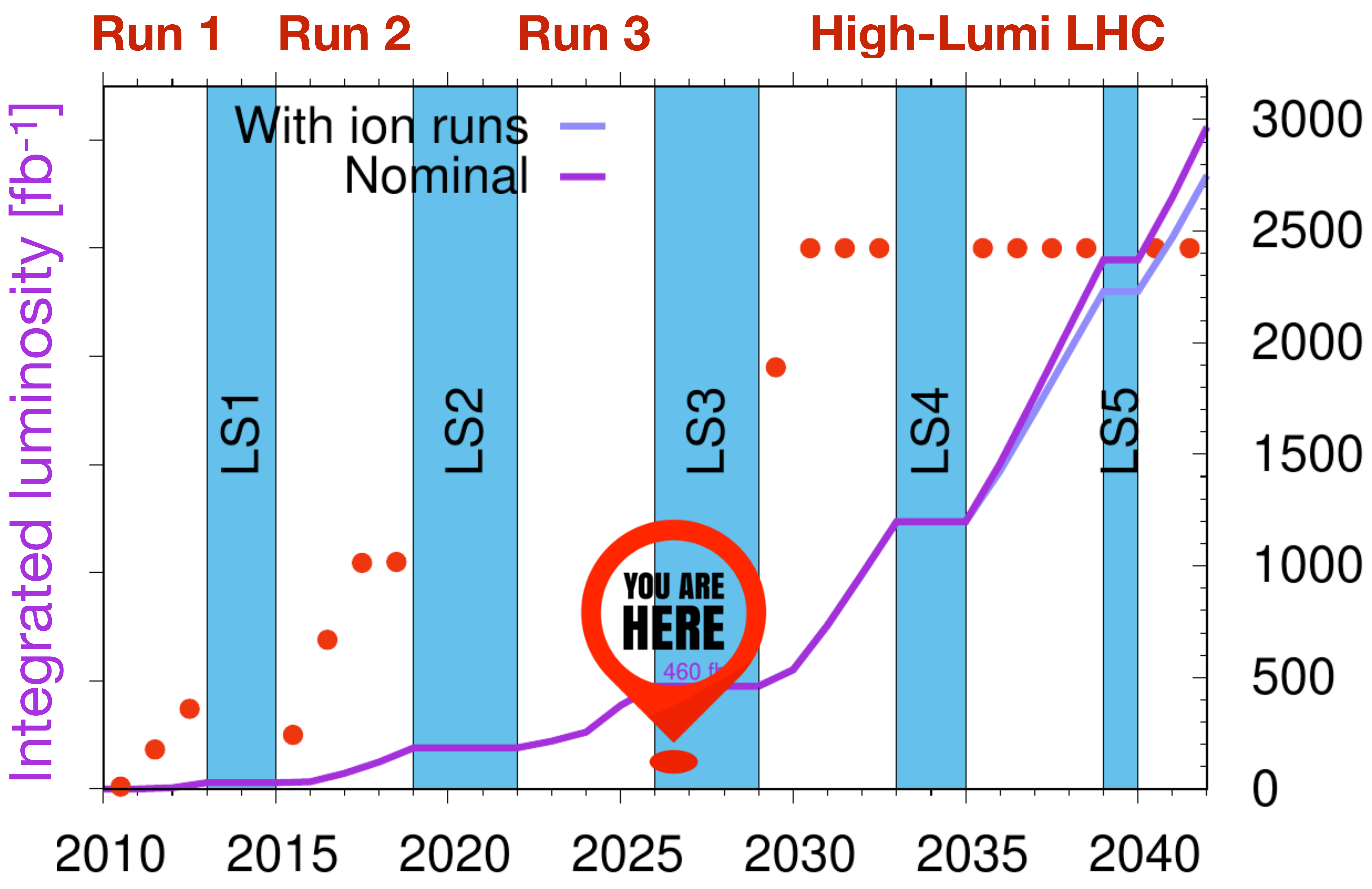
Swiss National Science Foundation

Amplitudes, 30th June 2026



Drawing credits: Gaia Fontana

LHC precision era: we're just getting started!



See Thomas Gehrmann's talk

We'll be able to test the Standard Model at (sub)-percent accuracy…

provided that the theoretical predictions can keep up!

Plot by R. T. Garcia, lhc-commissioning.web.cern.ch

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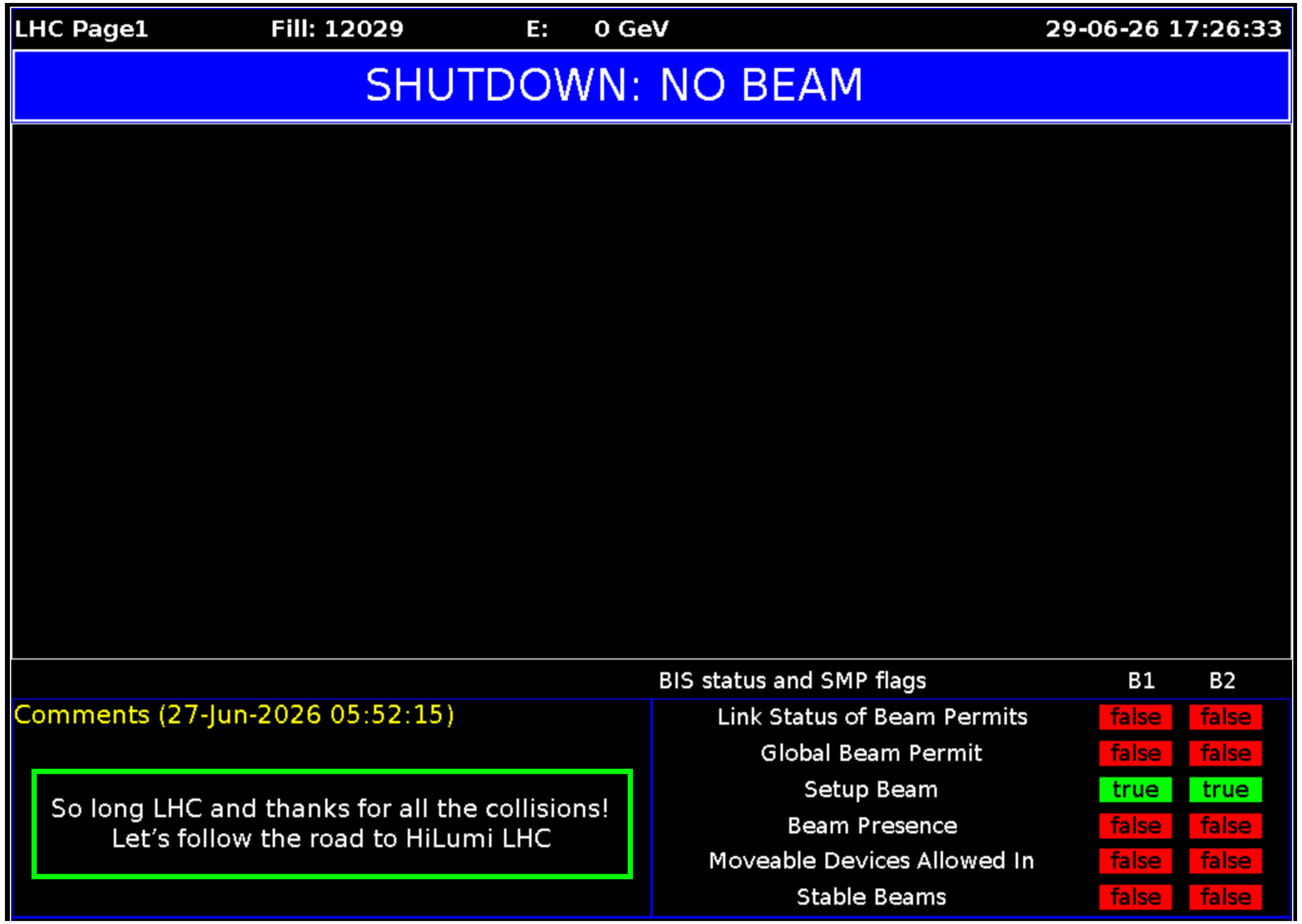

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LHC analyses require *at least* NNLO in QCD

Improvements are needed across the entire energy range

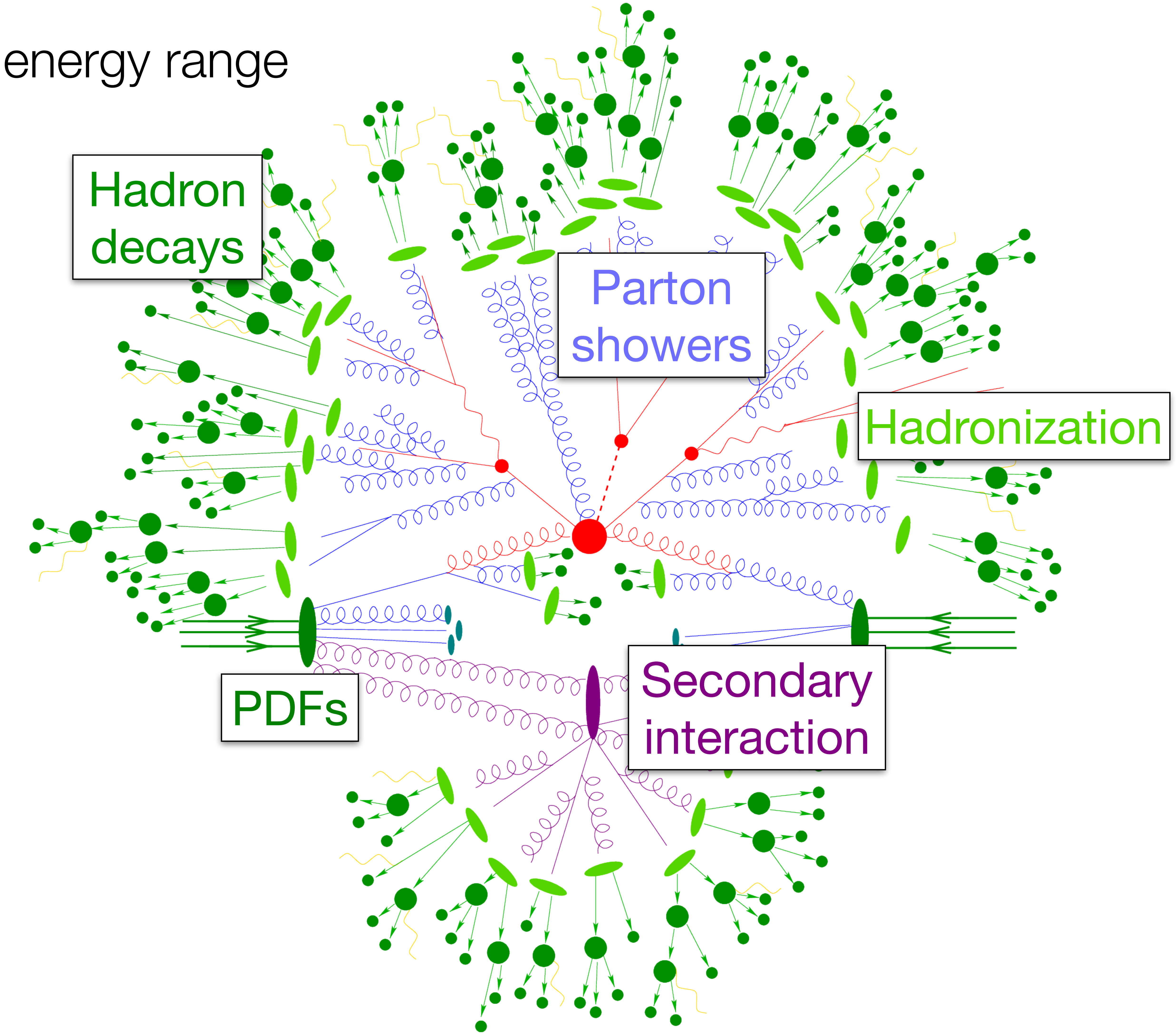

Image: Gleisberg, Höche et al. '08 (Sherpa)

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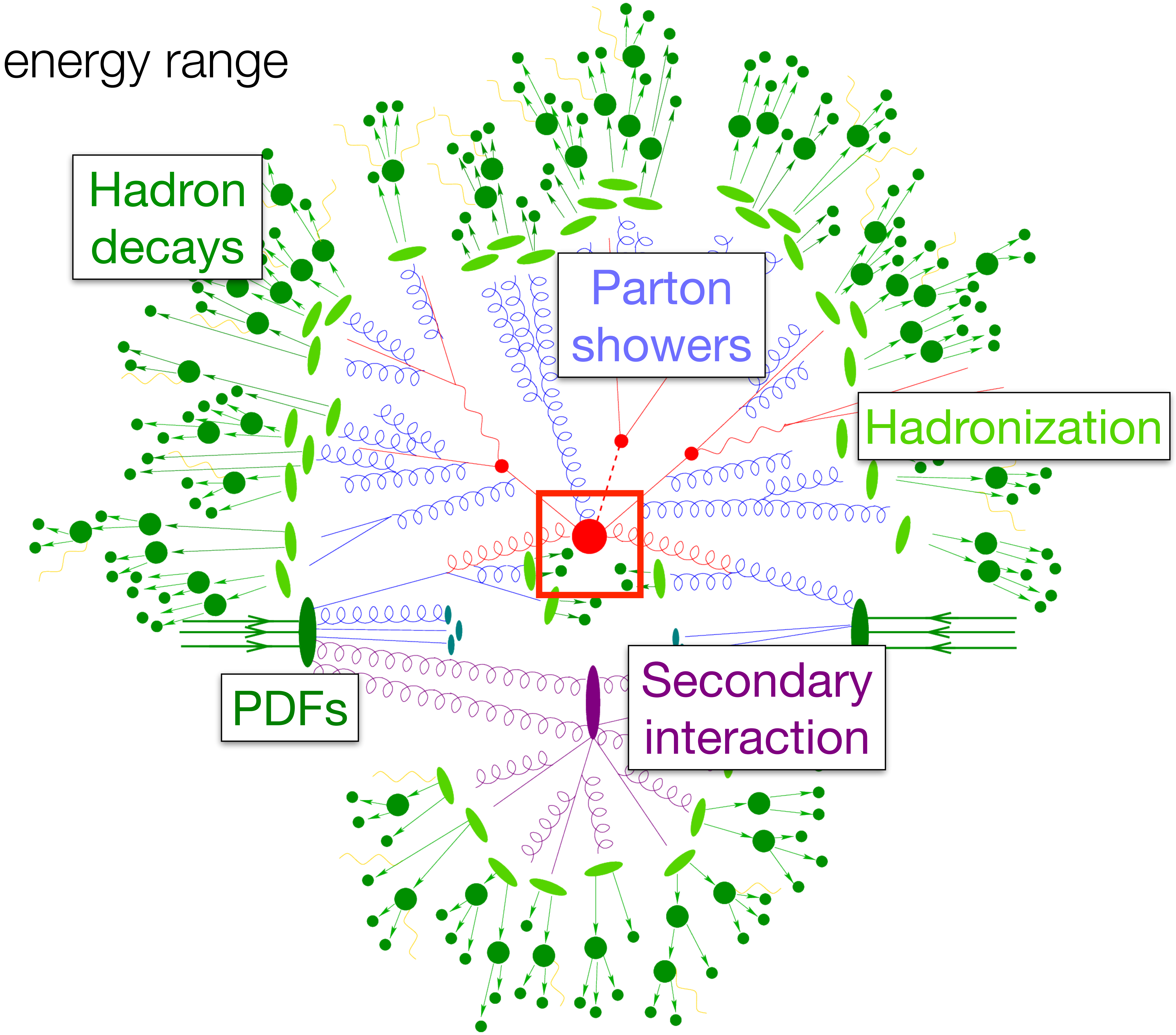
This talk: **hard interaction**

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}]\,|\mathscr{A}_{q\bar{q}\to gg}|^2$$

$$\mathrm{d}\sigma = \boxed{\boxed{\boxed{\mathrm{d}\sigma^{\mathrm{LO}}} + \alpha_S\,\mathrm{d}\sigma^{\mathrm{NLO}}} + \alpha_S^2\,\mathrm{d}\sigma^{\mathrm{NNLO}}} + \ldots$$

$\approx 10-30\,\%$ $\approx 1-10\,\%$



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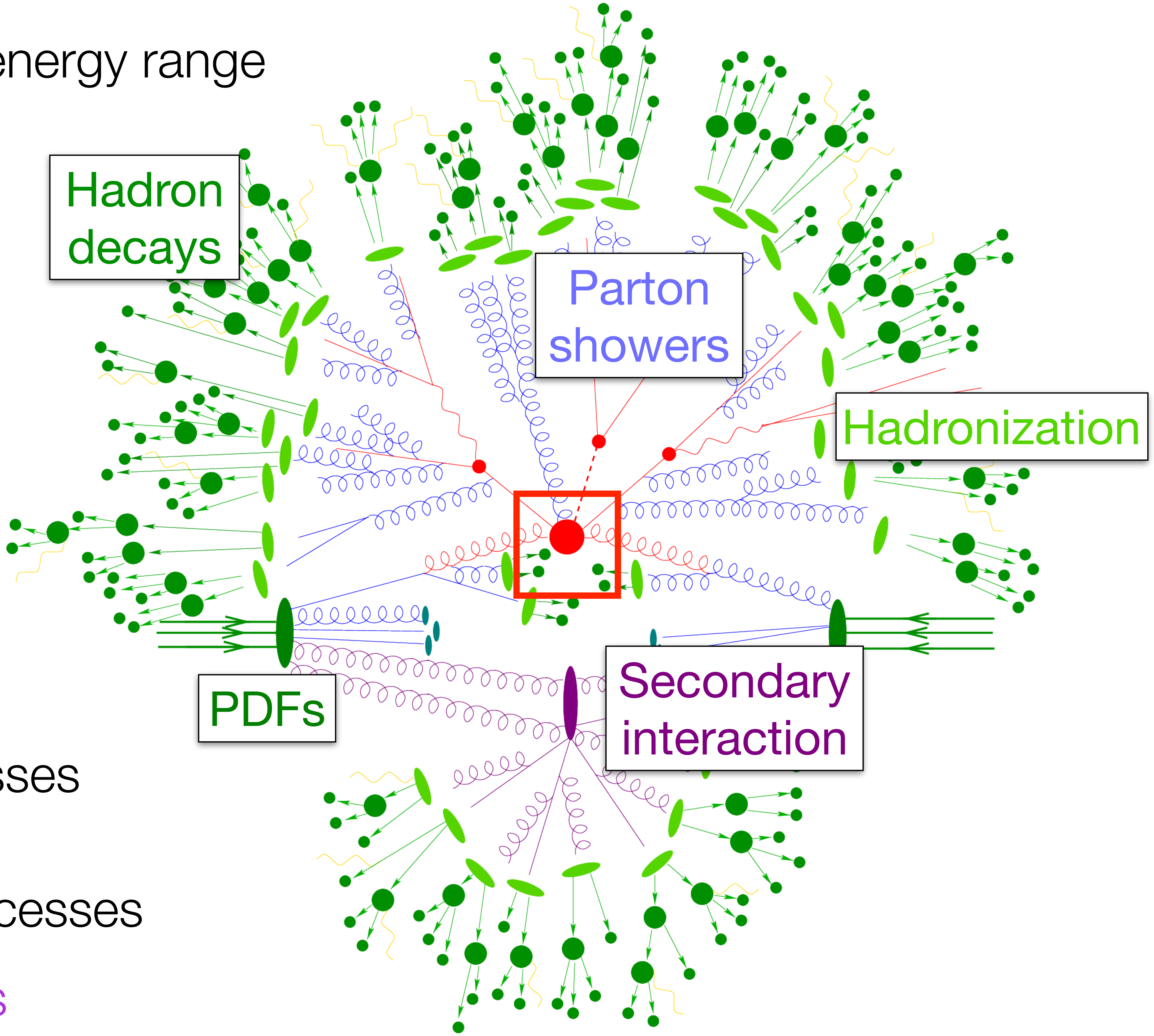
Current frontier @NNLO QCD: $2 \to 3$ processes

At amplitude level, first steps into $2 \to 4$ processes

See Y. Zhang and D. Chicherin's talks



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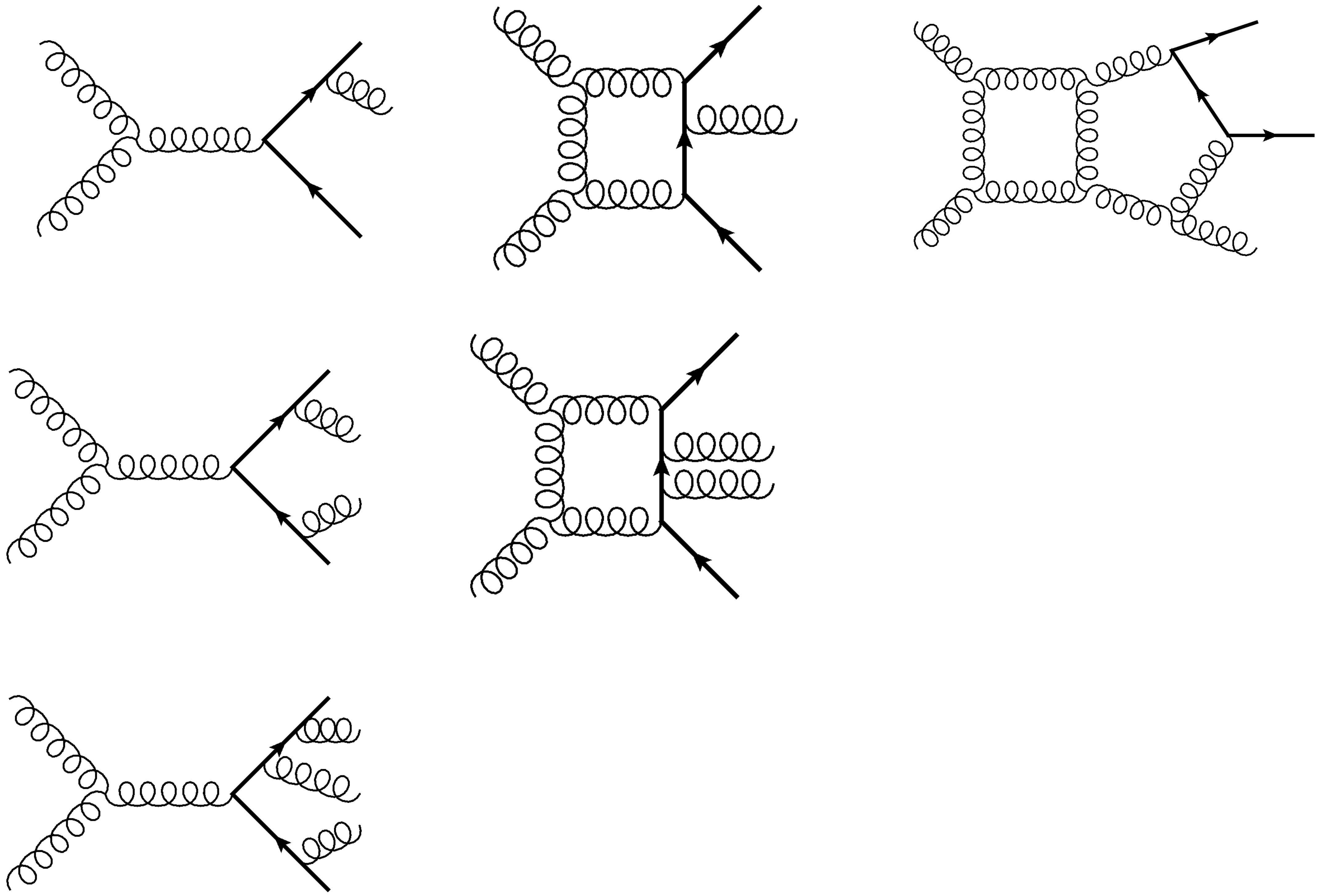
2-loop 5-point amplitudes are the main bottleneck

LO NLO NNLO

$$\sigma_{\mathrm{NNLO}} = \int_{5\mathrm{pt}} \mathrm{d}\sigma^{\mathrm{B}} + \mathrm{d}\sigma^{\mathrm{V}} + \mathrm{d}\sigma^{\mathrm{VV}} + \int_{6\mathrm{pt}} \mathrm{d}\sigma^{R} + \mathrm{d}\sigma^{RV} + \int_{7\mathrm{pt}} \mathrm{d}\sigma^{RR}$$



IR subtraction @ NNLO QCD is challenging but mature

2-loop 5-point amplitudes are the main bottleneck

LO NLO NNLO

$$
\begin{aligned}
\sigma_{\mathrm{NNLO}} &= \int_{5\mathrm{pt}} \overset{\checkmark}{\mathrm{d}\sigma^{\mathrm{B}}} + \overset{\checkmark}{\mathrm{d}\sigma^{\mathrm{V}}} + \boxed{\mathrm{d}\sigma^{\mathrm{VV}}} \\
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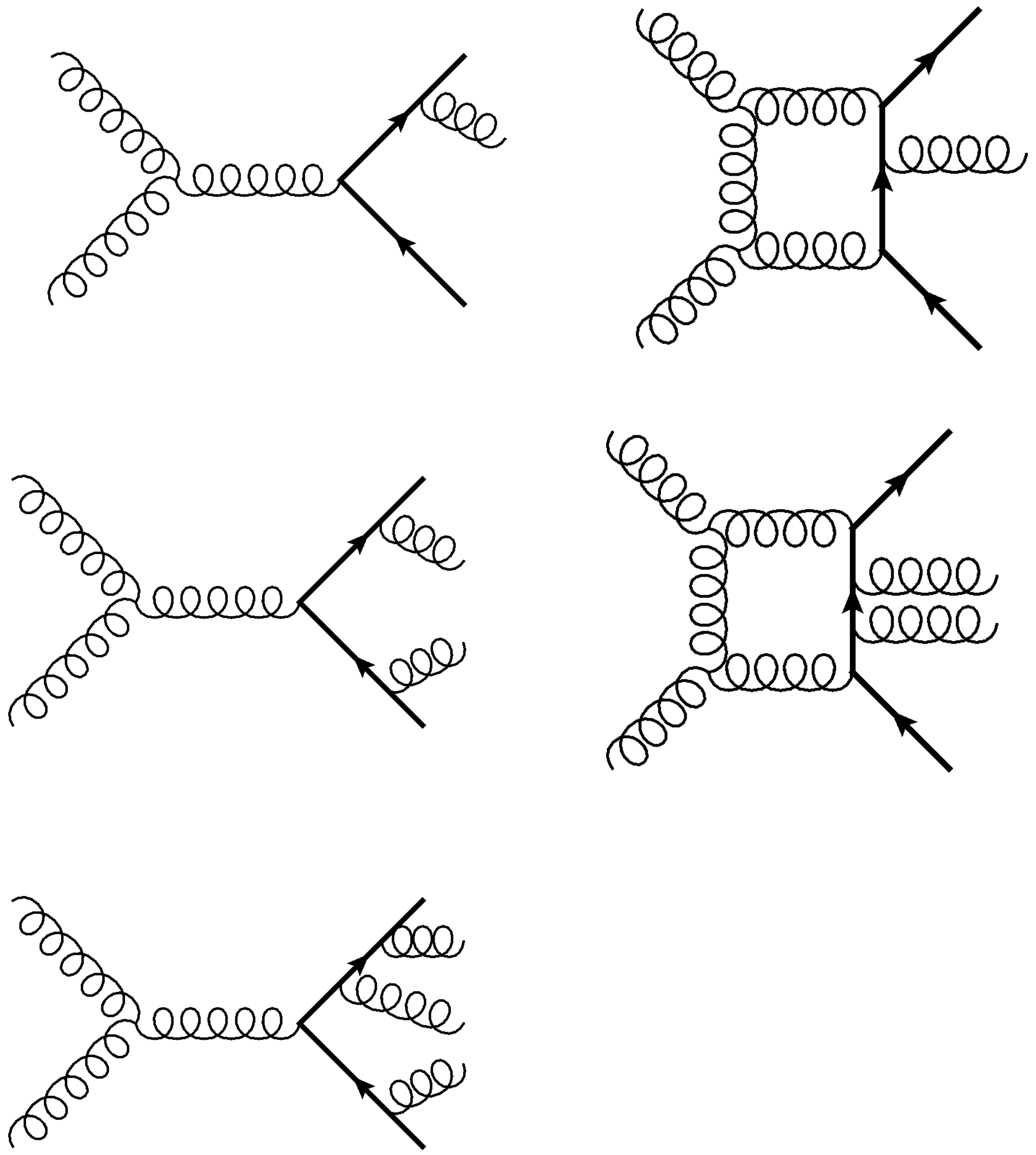
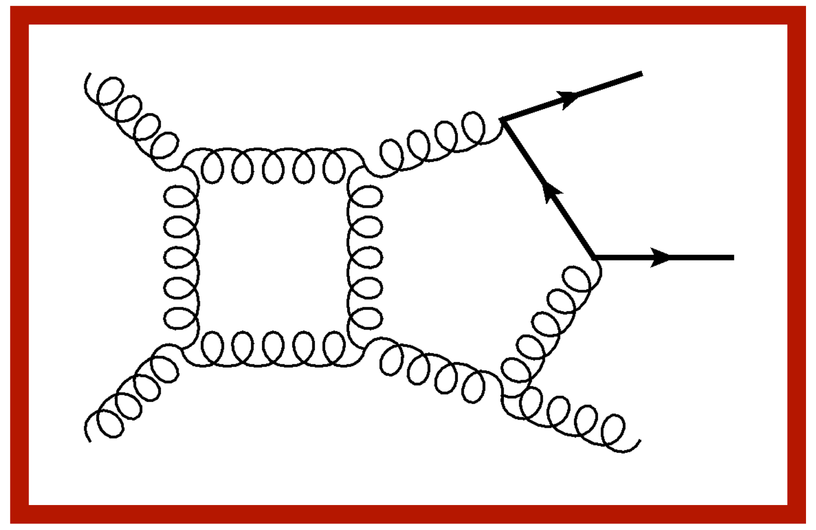
&+ \int_{7\mathrm{pt}} \overset{\checkmark}{\mathrm{d}\sigma^{RR}}
\end{aligned}
$$





Work to do for the Amplitudes community!

See Michael Ruf’s talk

IR subtraction @ NNLO QCD is challenging but mature

Why $2 \to 3$ processes @ LHC?

High centre-of-mass energy of LHC collisions $\Rightarrow$ high-multiplicity final states are common

Rich and interesting phenomenology. Two examples:

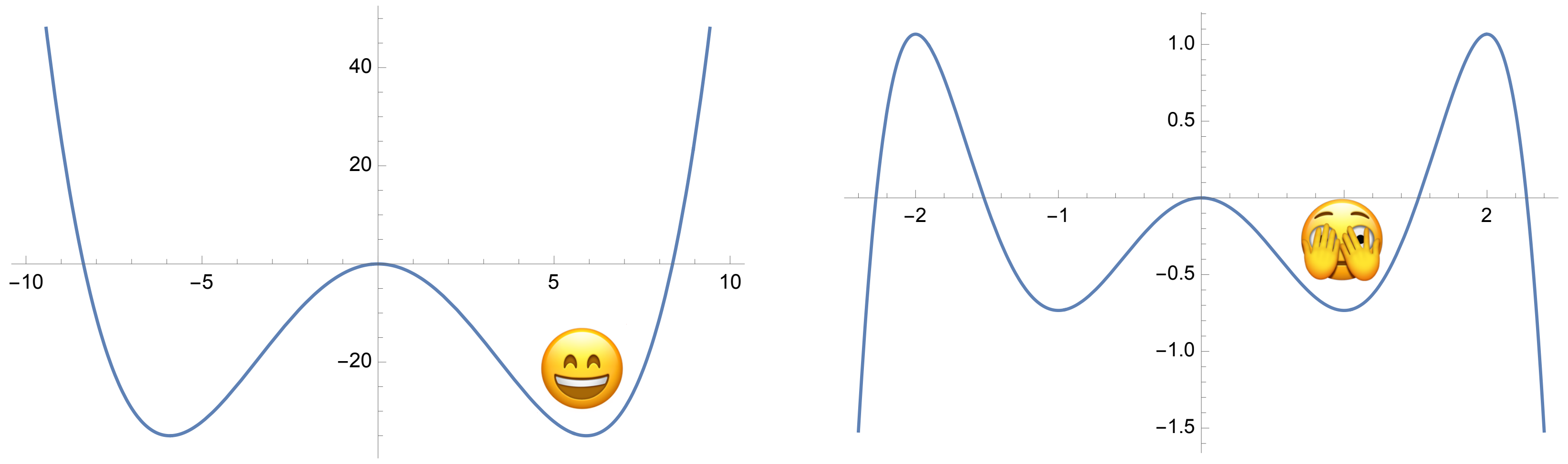
Why $2 \to 3$ processes @ LHC?

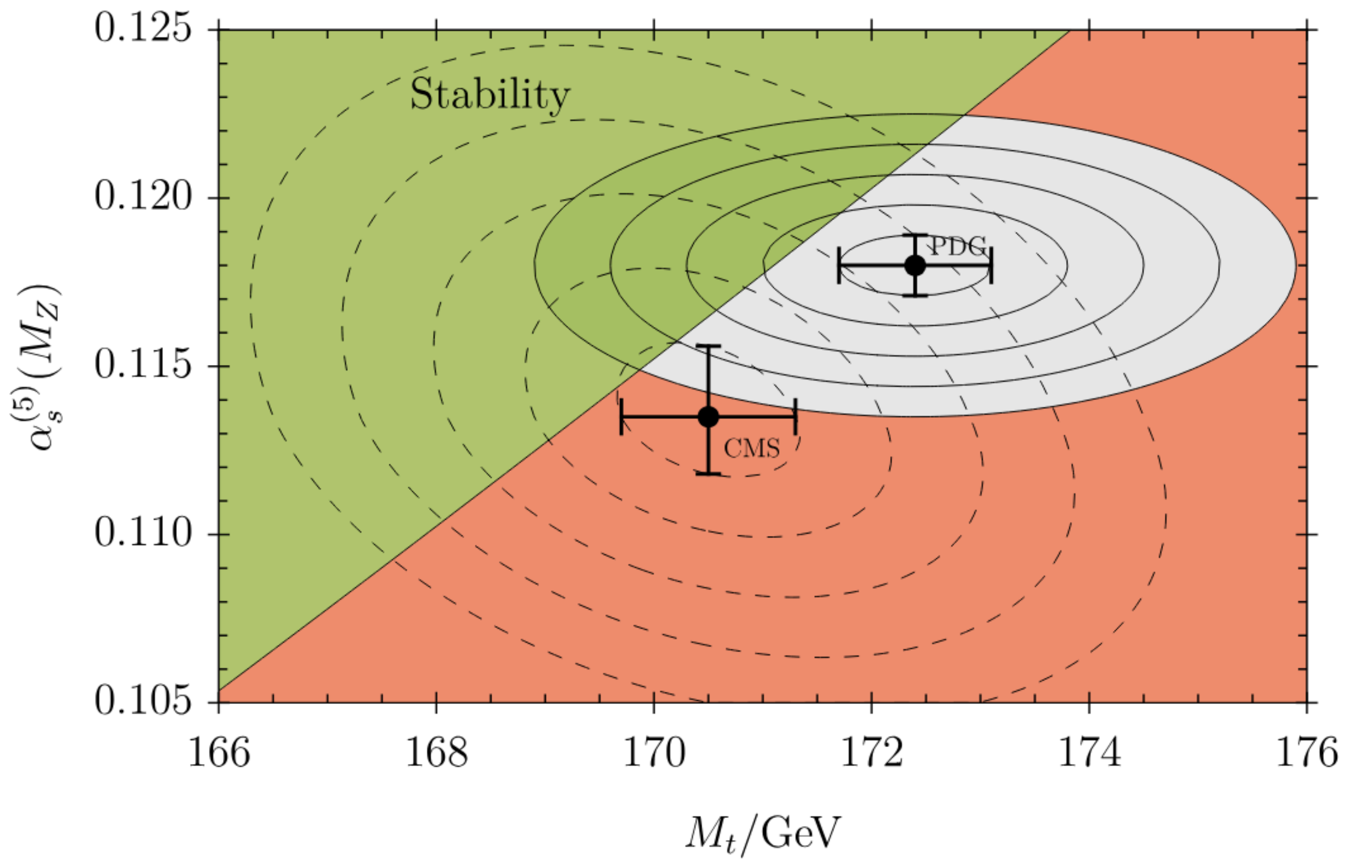
High centre-of-mass energy of LHC collisions $\Rightarrow$ high-multiplicity final states are common

Rich and interesting phenomenology. Two examples:

1) The **fate of our Universe** rests on the shape of the Higgs potential

Stability vs. metastability depends on m_{top} and α_s





$pp \to t\bar{t} + \text{jet}$ is highly sensitive to m_{top}

Alioli, Fernandez, Fuster, Irles, Moch, Uwer, Vos '13

Hiller, Höhne, Litim, Steudtner '24

m_{top} extraction limited by theoretical precision

Why $2 \to 3$ processes @ LHC?

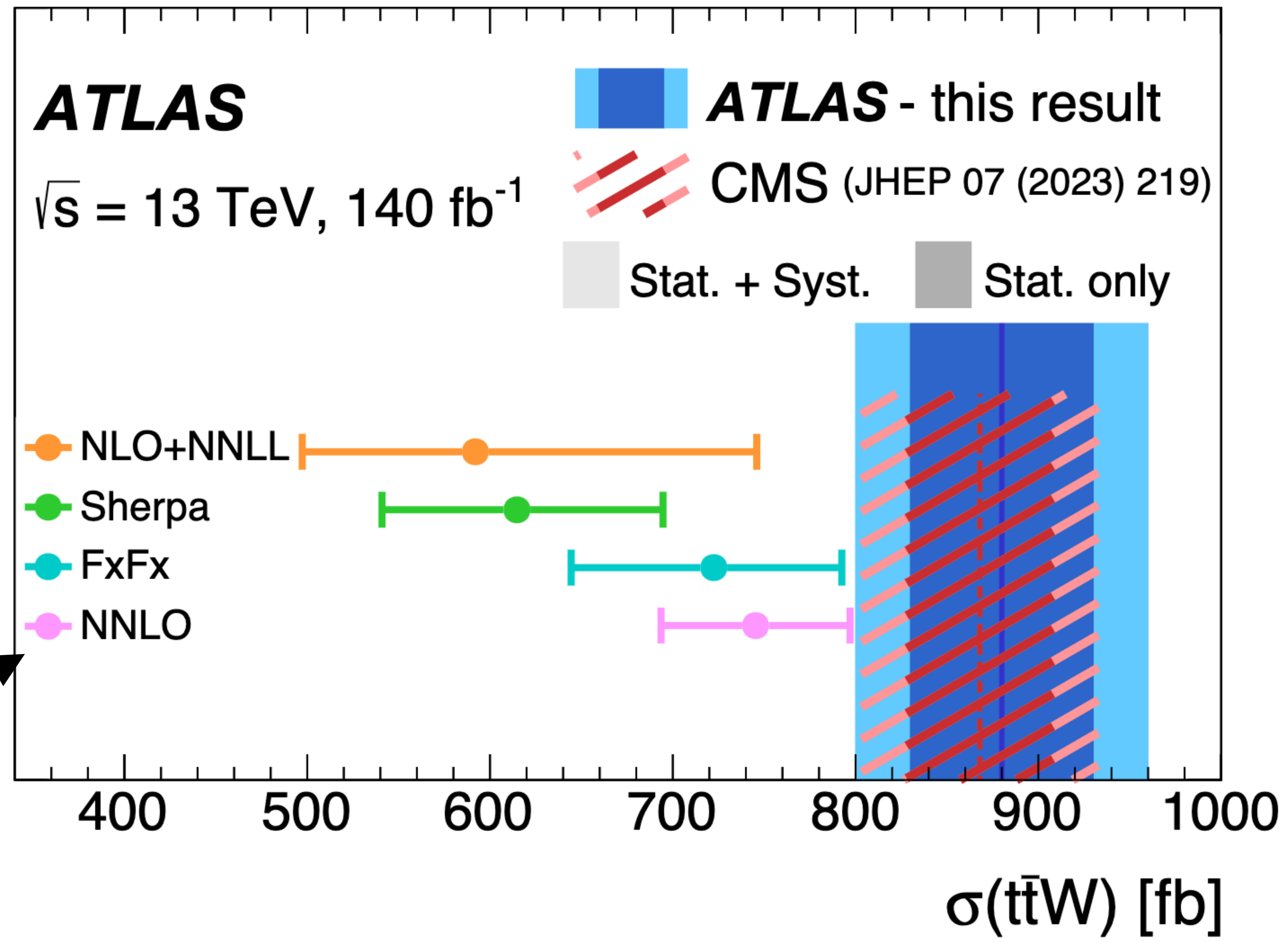
High centre-of-mass energy of LHC collisions $\Rightarrow$ high-multiplicity final states are common

Rich and interesting phenomenology. Two examples:

2) Mild but consistent **tension** EXP vs TH for $pp \to t\bar{t} + W$

Also: Important for BSM searches and background to $pp \to t\bar{t}t\bar{t}$ and $pp \to t\bar{t}H$

ATLAS '24



Dynamical approximations

Recent explosion of results for 2→3 processes @ NNLO QCD

$\mathrm{pp} \to 3\gamma$ *Kallweit, Sotnikov, Wiesemann ’20; Chawdhry, Czakon, Mitov, Poncelet ’20*

$\mathrm{pp} \to 2\gamma + \mathrm{j}$ *Chawdhry, Czakon, Mitov, Poncelet ’21; Badger, Gehrmann, Marcoli, Moodie ’21; Buccioni, Chen, Feng, Gehrmann, Huss, Marcoli ’25*

$\mathrm{pp} \to 3\mathrm{j}$ *Czakon, Mitov, Poncelet ’21; Chen, Gehrmann, Glover, Huss, Marcoli ’22; Alvarez,Cantero, Czakon, Llorente, Mitov, Poncelet ‘23*

$\mathrm{pp} \to \gamma + 2\mathrm{j}$ *Badger, Czakon, Bayu Hartanto, Moodie, Peraro, Poncelet, **SZ** ’23*

$\mathrm{pp} \to \mathrm{W} + \mathrm{b}\bar{\mathrm{b}}$ *Bayu Hartanto, Poncelet, Popescu, **SZ** ’22; Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini ’23**

$\mathrm{pp} \to \mathrm{W} + \mathrm{t}\bar{\mathrm{t}}$ *Buonocore, Devoto, Grazzini, Kallweit, Mazzitelli, Rottoli, Savoini ’23**

$\mathrm{pp} \to \mathrm{Z} + \mathrm{b}\bar{\mathrm{b}}$ *Mazzitelli, Sotnikov, Wiesemann ’24**

$\mathrm{pp} \to \mathrm{H} + \mathrm{t}\bar{\mathrm{t}}$ *Devoto, Grazzini, Kallweit, Mazzitelli, Savoini ’24*; Biello, Savoini, Signorile-Signorile, Wiesemann ’26**

$\mathrm{pp} \to \mathrm{H} + \mathrm{b}\bar{\mathrm{b}}$ *Biello, Mazzitelli, Sankar, Wesemann, Zanderighi ’25**

* dynamical approx for massive particles

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$\mathrm{pp} \to \gamma +$

$\mathrm{pp} \to \mathrm{W}$ ... *avoini '23**

$\mathrm{pp} \to \mathrm{W}$ ... approx

$\mathrm{pp} \to \mathrm{Z}$ ... particles

$\mathrm{pp} \to \mathrm{H}$ ... *emann '26**

$\mathrm{pp} \to \mathrm{H} + \mathrm{b}\bar{\mathrm{b}}$ *Biello, Mazzitelli, Sankar, Wesemann, Zanderighi '25**

Effectively, only **massless internal lines** here

Underlying formalism of

logarithmic iterated integrals *Chen '77*

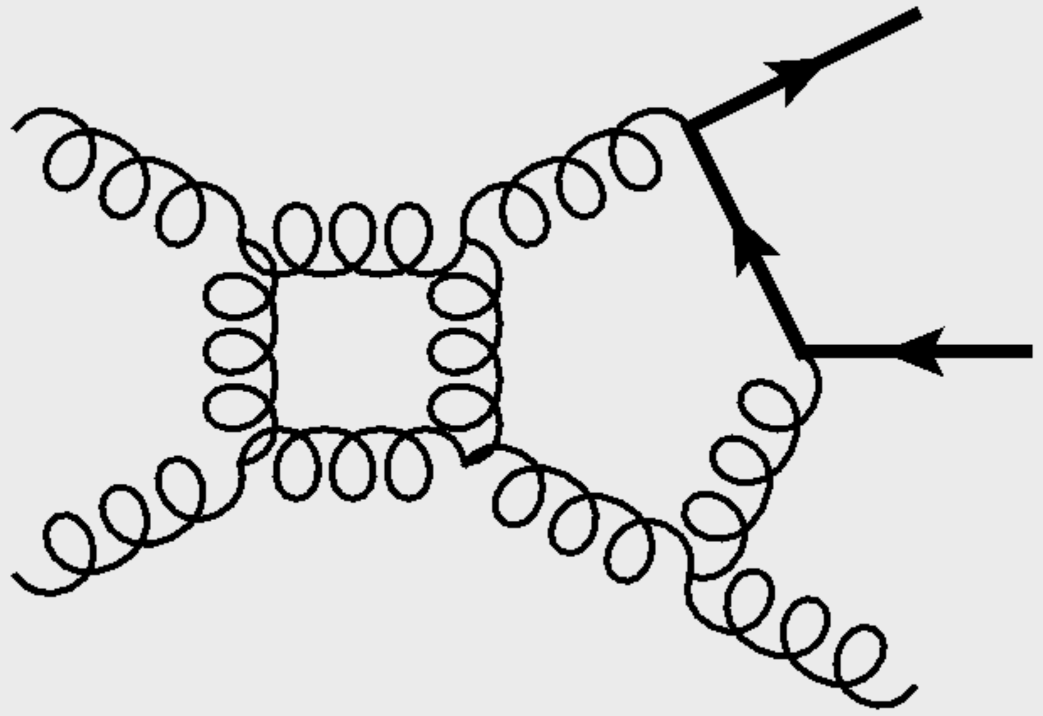
$$\mathrm{d}\left[W_{i_1}, \ldots, W_{i_{n-1}}, W_{i_n}\right] = \mathrm{d}\log W_{i_n}\left[W_{i_1}, \ldots, W_{i_{n-1}}\right]$$

Powerful mathematical techniques, such as the ***symbol***

Goncharov, Spradlin, Vergu, Volovich '10

First NNLO 2→3 results with exact virtual masses

$\mathrm{pp} \to \mathrm{t\bar{t}} + \mathrm{jet}$ @ NNLO QCD (leading colour)



with **Badger, Becchetti, Brancaccio, Hartanto**: arXiv:2412.13876

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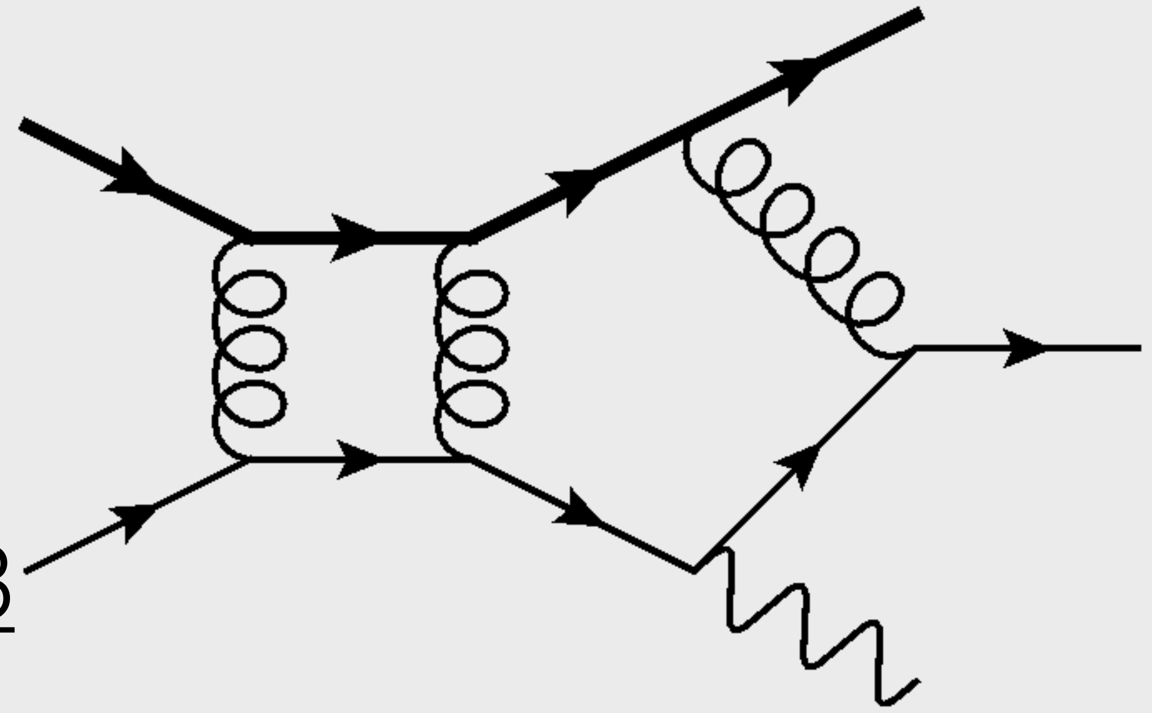

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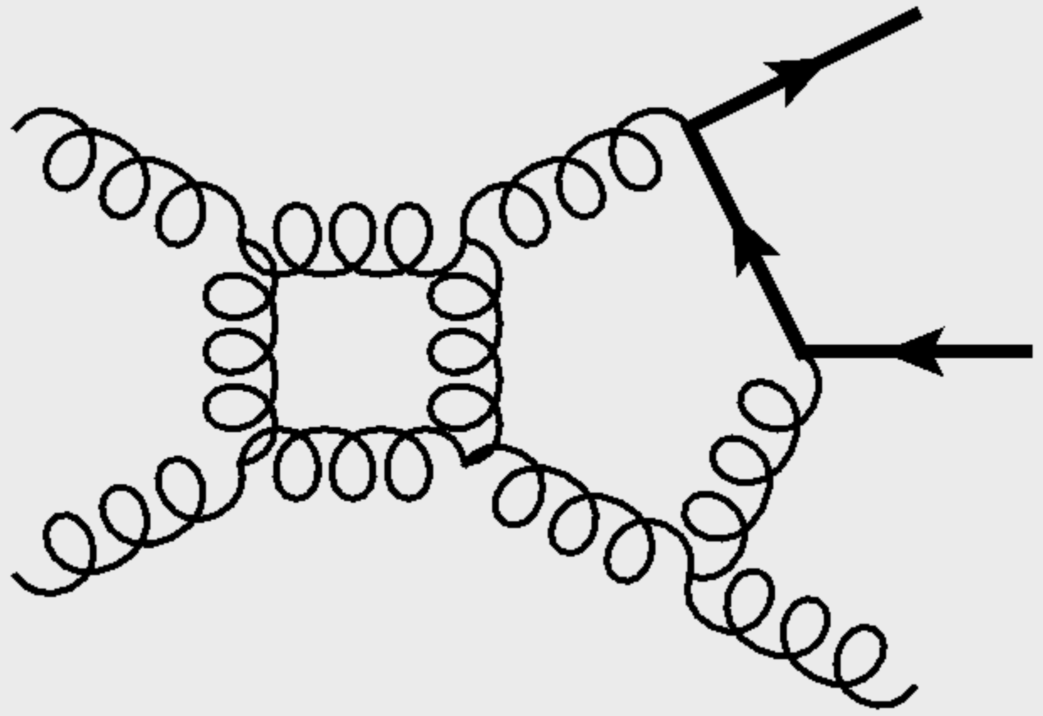

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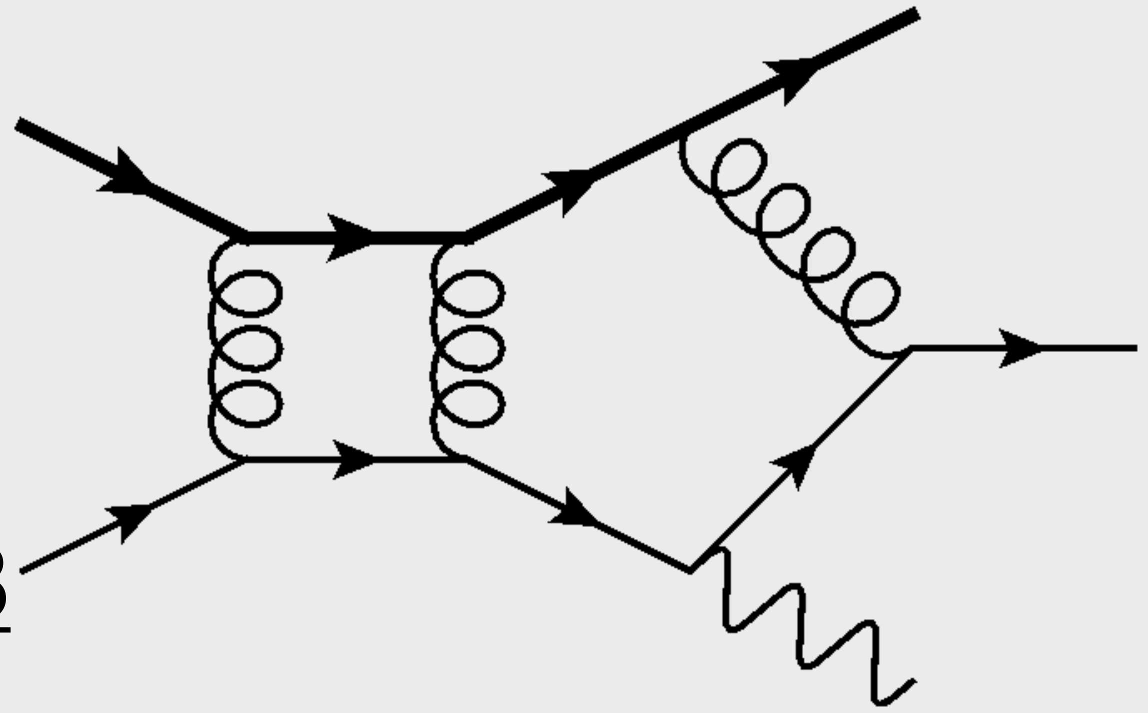
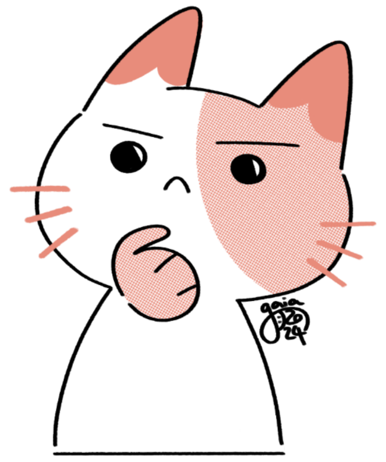


What's different with respect to the previous cases?



More masses, more variables

Escalation in algebraic complexity handled by replacing symbolic computer algebra with

numerical **finite-field** evaluation + interpolation/rational reconstruction

von Manteuffel, Schabinger 2014; Peraro 2016

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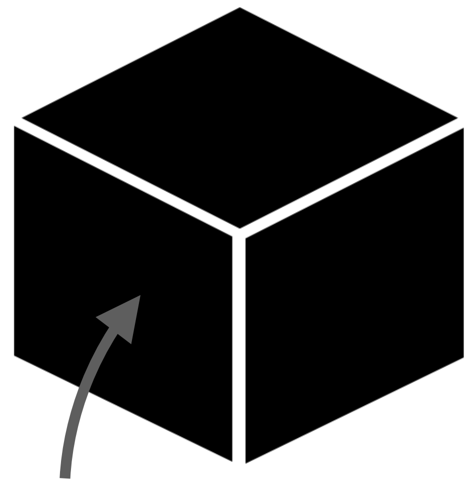
$$\mathscr{A}(x) = \sum_i \boxed{R_i(x)} \times F_i(x)$$

Rational functions Special functions

$\left(x, \mathrm{P}\right) \rightarrow$  $\rightarrow R_i(x) \bmod \mathrm{P}$

IBP reduction

$R_i(x) = ?$

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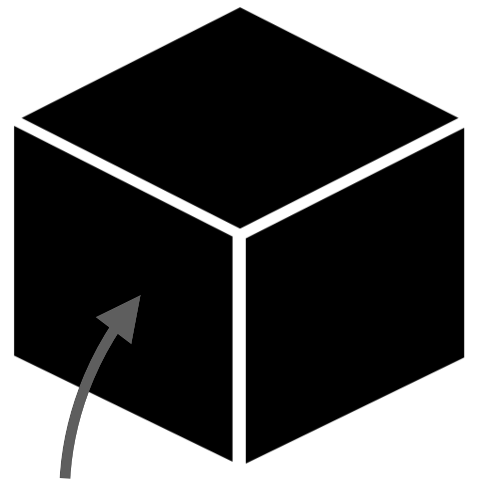


$\rightarrow R_i(x) \bmod \mathrm{P}$

$R_i(x) = \,?$

Interpolation based on partial fractioning

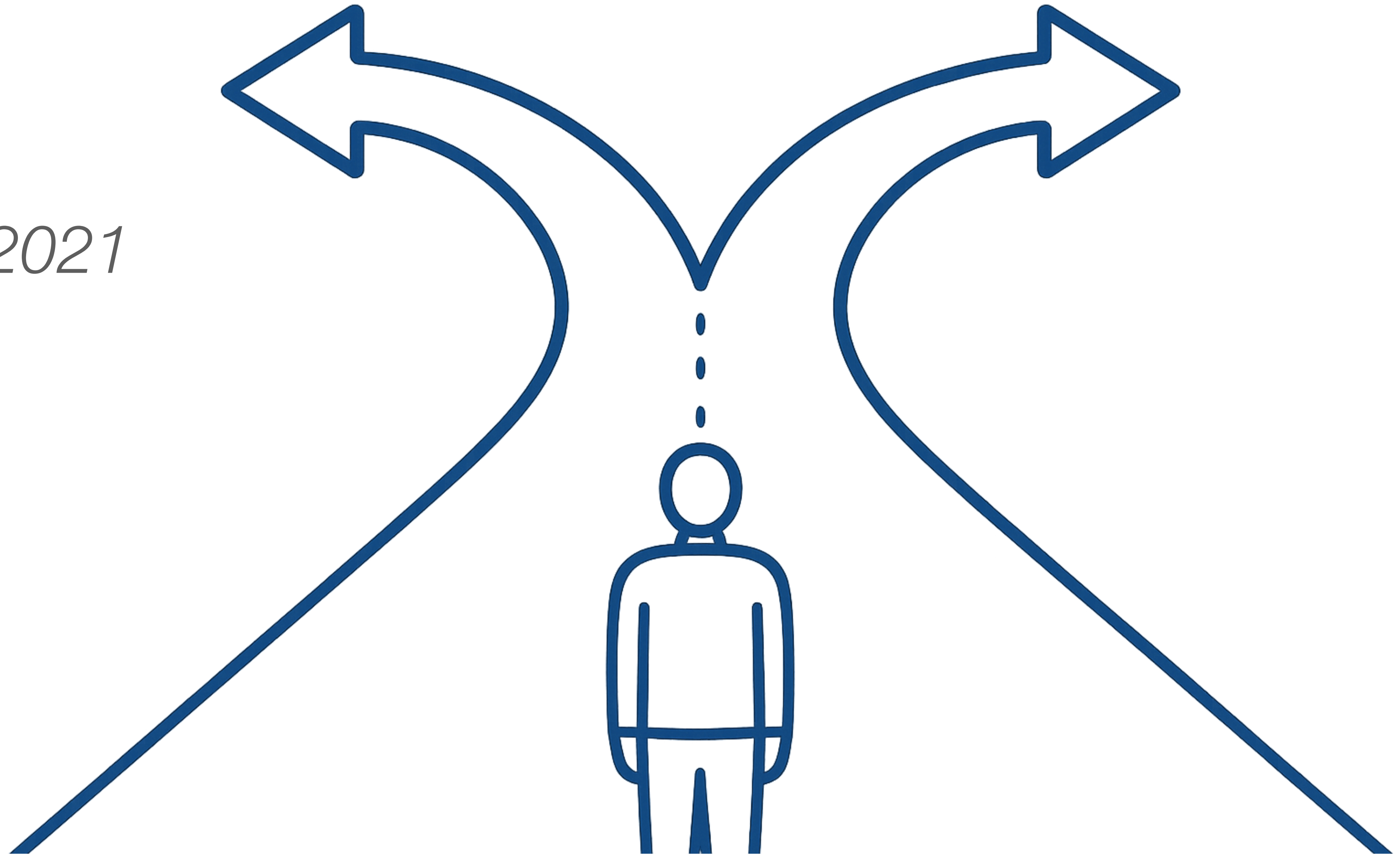
*Badger, Bayu Hartanto, **SZ** 2021*

$R_i(x) \in \mathbb{Q}(x)$

$\boxed{t\bar{t} + \mathrm{jet}}$

Analytic result

See Michael Ruf's talk



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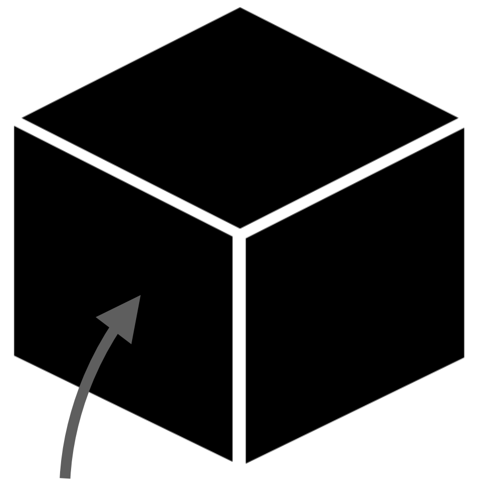

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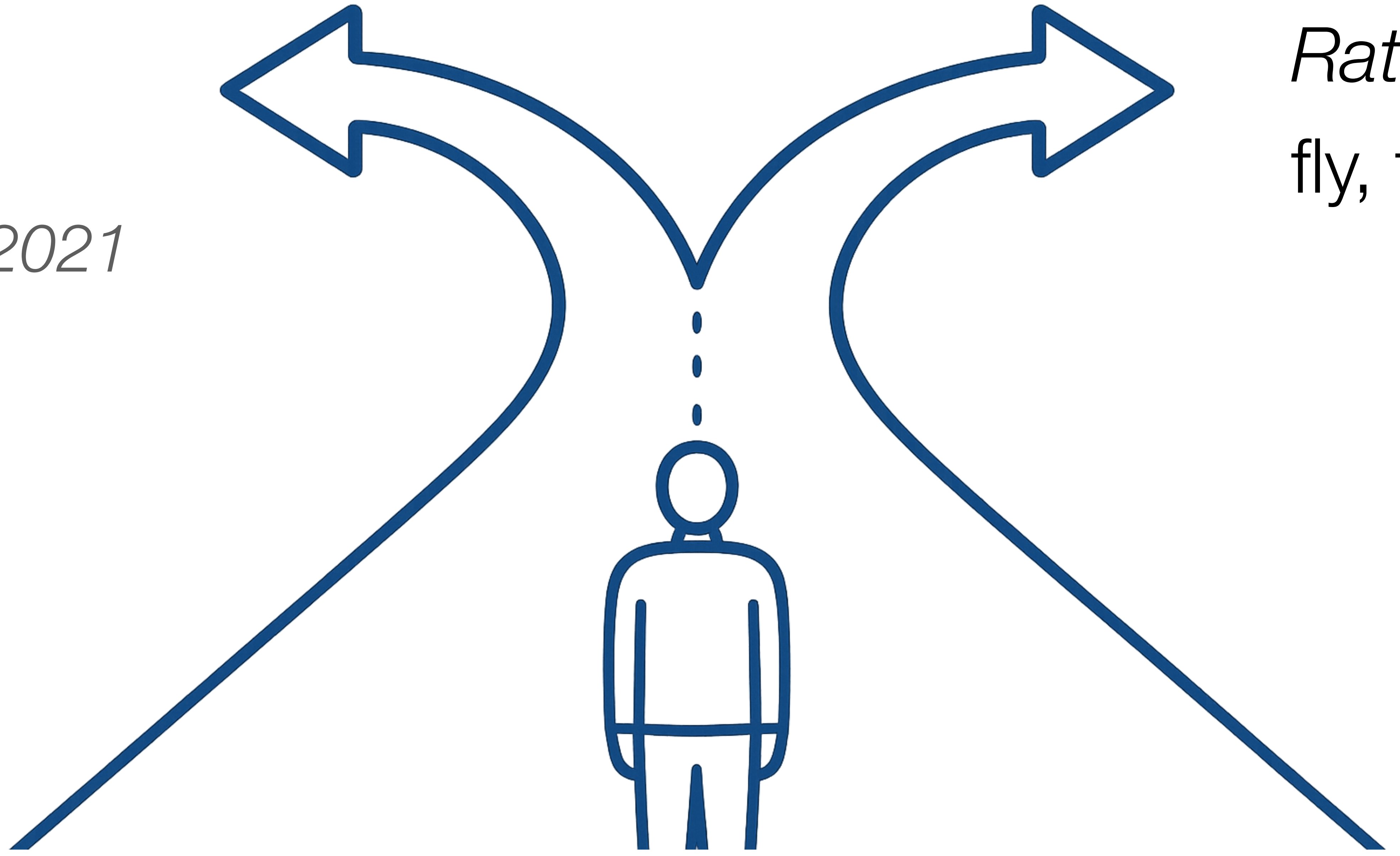

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Rational reconstruction on the fly, for each phase space point

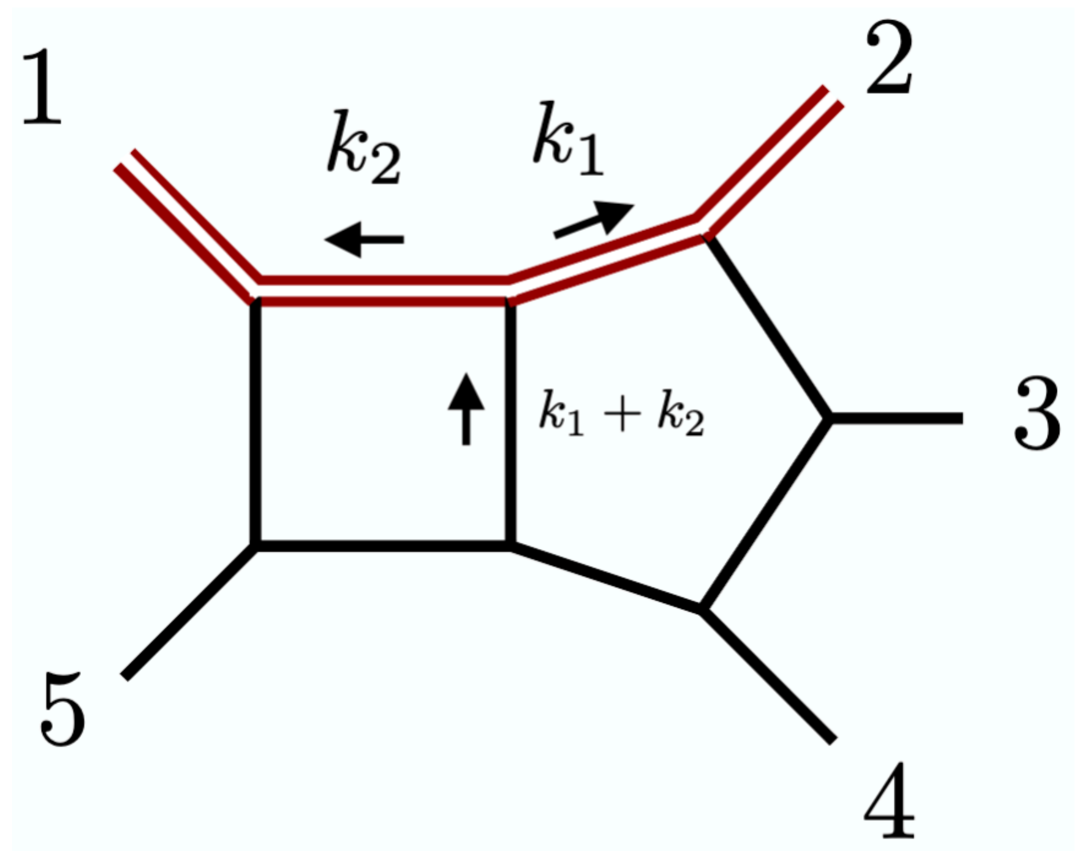
$$R_i(x_k) \in \mathbb{Q}$$

$$\boxed{t\bar{t} + W}$$

Numerical result, combined with interpolation grid

Nature laughs at the difficulties of integration

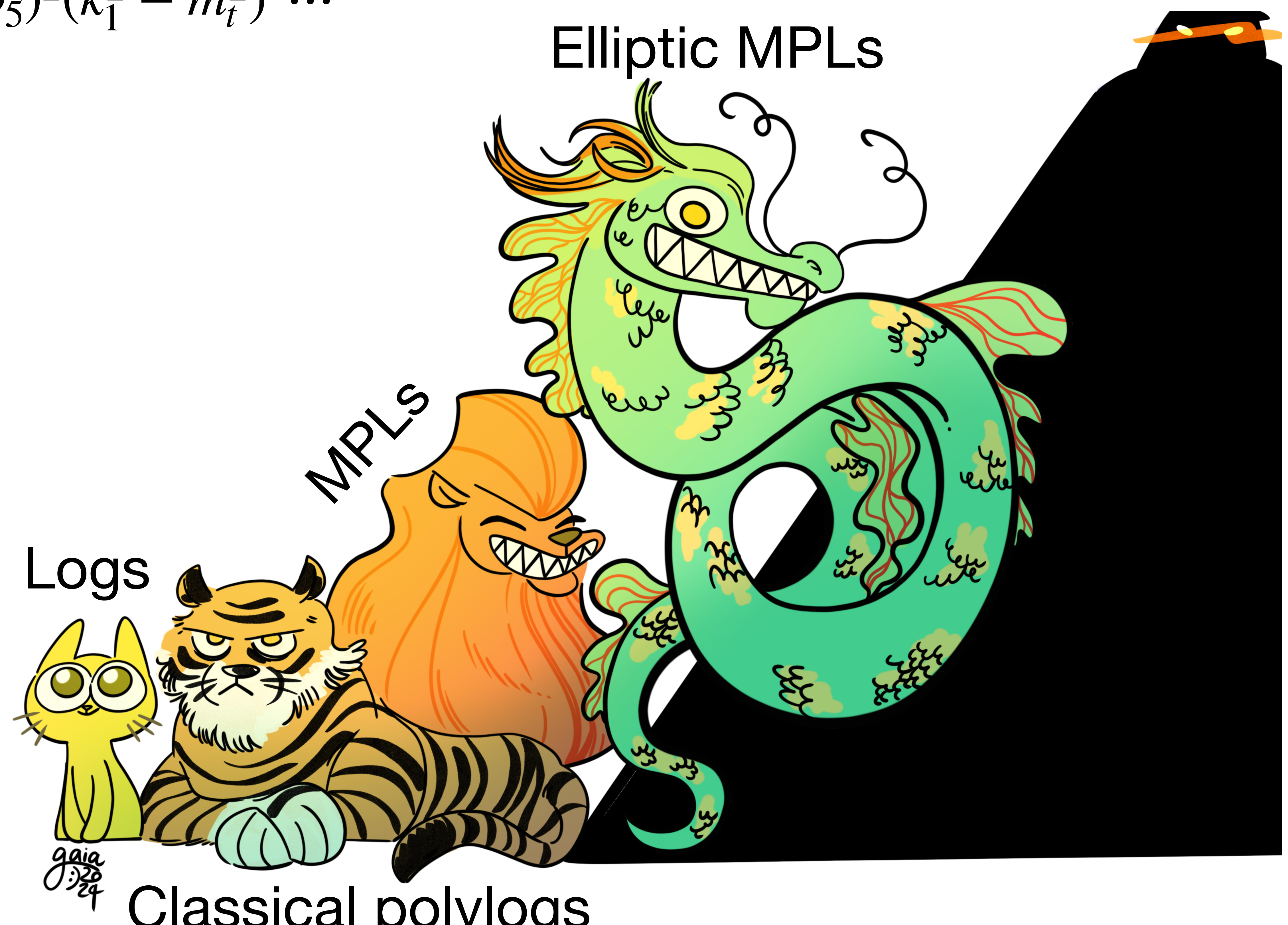
Pierre-Simon de Laplace



$$= \int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2}{(k_2^2 - m_t^2)(k_2 - p_1)^2 (k_2 - p_1 - p_5)^2 (k_1^2 - m_t^2)\ \cdots} = ???$$

Feynman integrals remain the main bottleneck

See Giulia Isabella's talk



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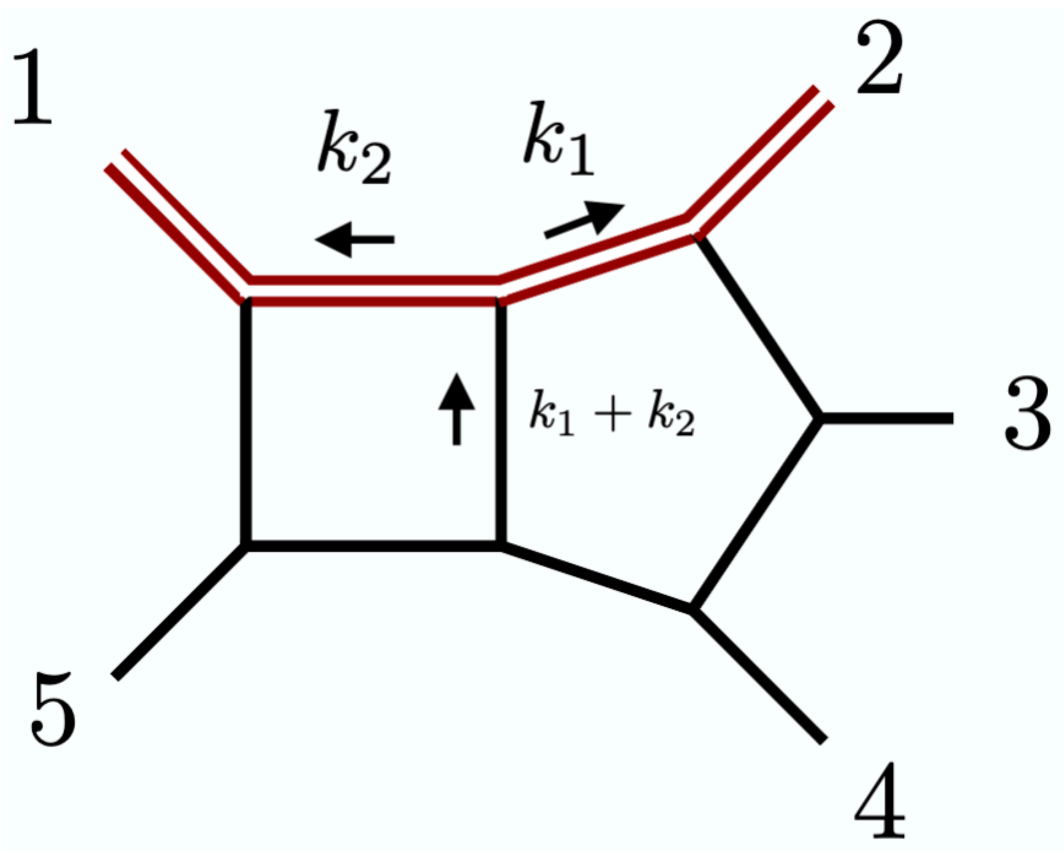

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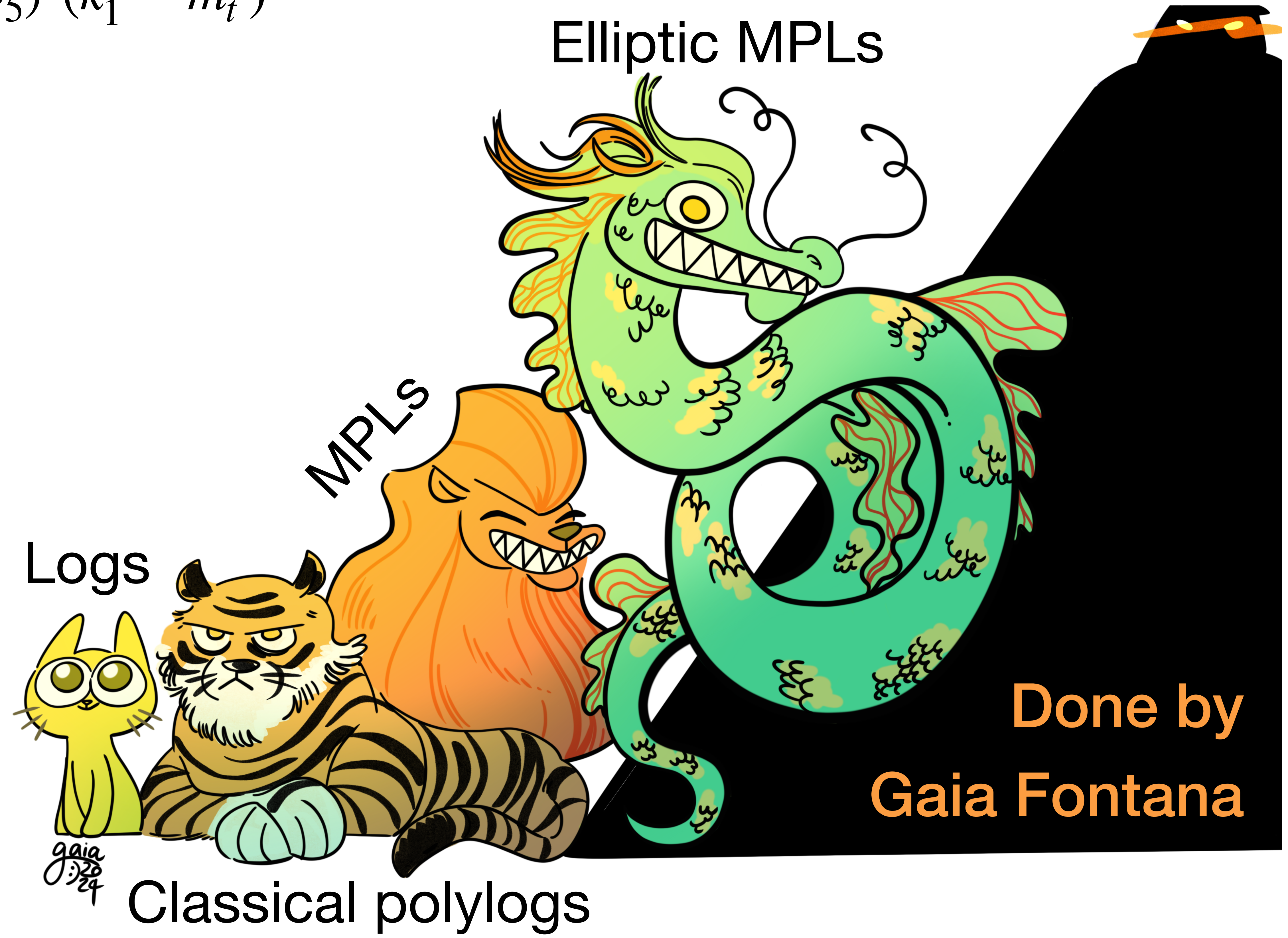
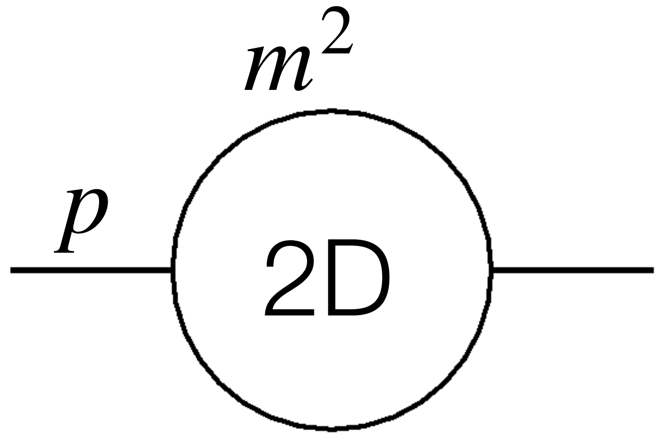


Simple poles + constant leading singularities = canonical DEs



$$\propto \int \frac{\mathrm{d}\alpha_1 \wedge \mathrm{d}\alpha_2}{p^2\,(\alpha_1\alpha_2 - x)\big[(1+\alpha_1)(1+\alpha_2) - x\big]}$$

$$\frac{\mathrm{d}x}{x-c} = \mathrm{d}\log(x-c)$$

$$= \int \frac{\mathrm{d}\alpha_1}{p^2(\alpha_1 + \alpha_1^2 + x)} \wedge \Big[\mathrm{d}\log(\alpha_1\alpha_2 - x) - \mathrm{d}\log(1 + \alpha_1 + \alpha_2 + \alpha_1\alpha_2 - x)\Big]$$

Leading singularity

$$= \frac{1}{p^2\sqrt{1-4x}} \times \int \mathrm{d}\log(\cdots) \wedge \mathrm{d}\log(\cdots)$$

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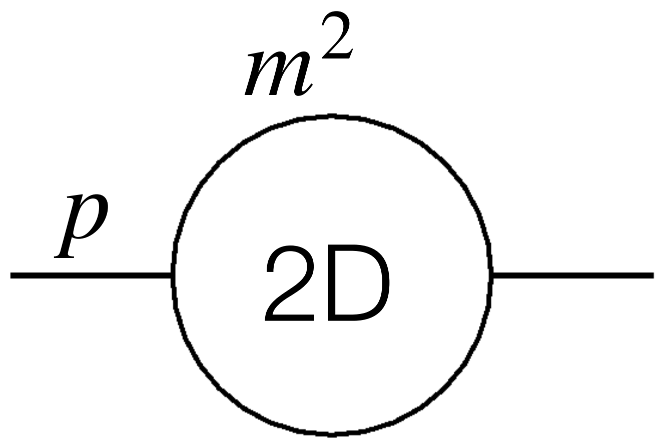

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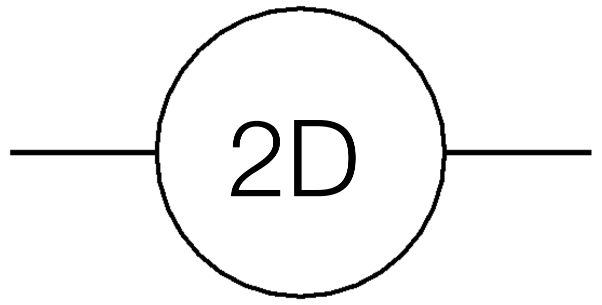
$p^2\sqrt{1-4x}\,\times$  $= \log\left(\dfrac{1-\sqrt{1-4x}}{1+\sqrt{1-4x}}\right)$ ✓

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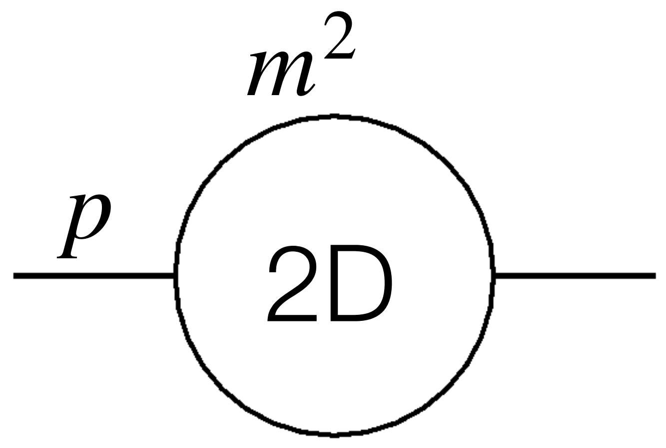

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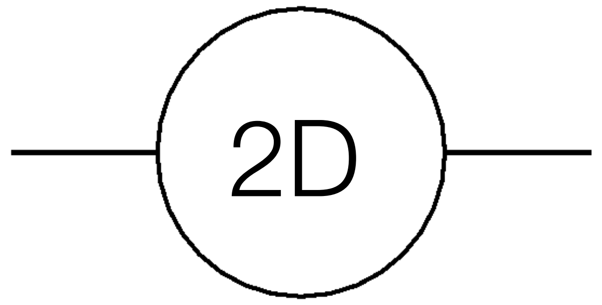
$p^2\sqrt{1-4x}\,\times$  $= \log\left(\dfrac{1-\sqrt{1-4x}}{1+\sqrt{1-4x}}\right)$ ✓

Conjecture: basis of such integrals $F(X;\epsilon)$ satisfies **canonical DEs**:

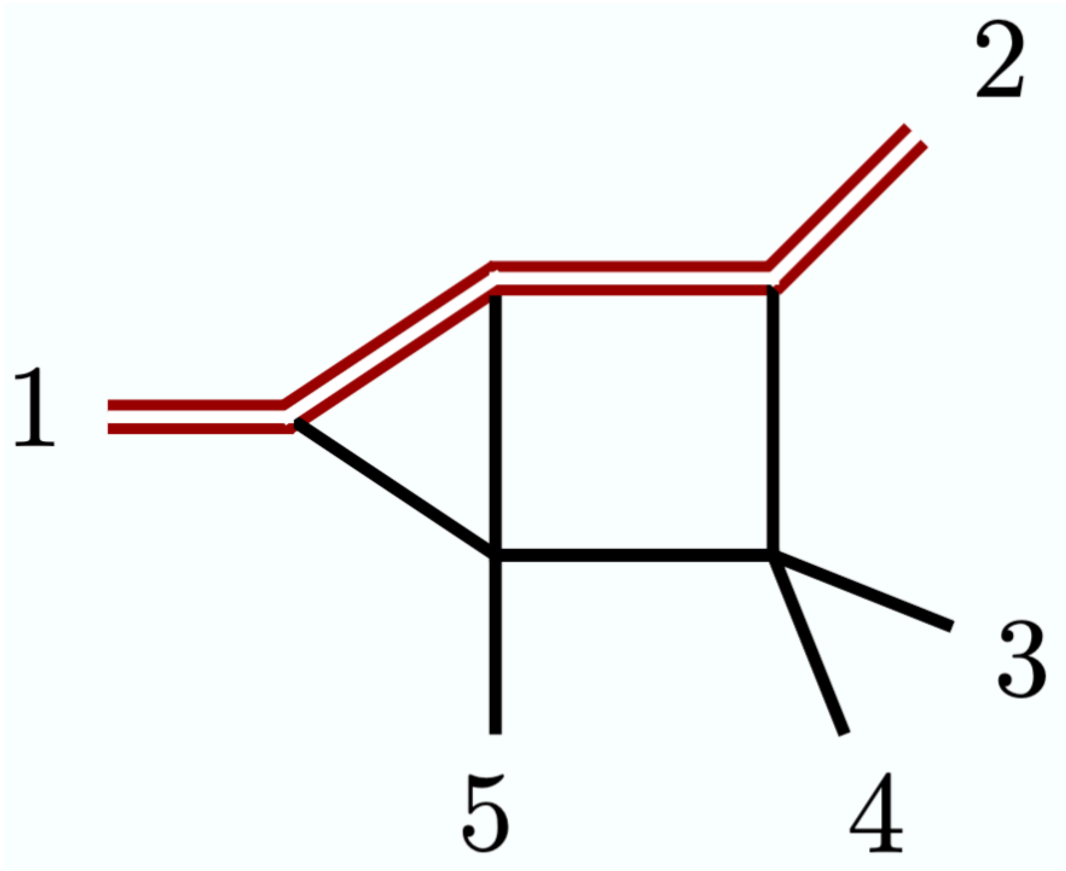
Arkani-Hamed, Bourjaily, Cachazo, Trnka 2012

$$\mathrm{d}F(X;\epsilon) = \epsilon \sum_i A_i\,\mathrm{d}\log W_i(X)\;\cdot F(X;\epsilon)$$

Henn '13

➡ Solution = logarithmic iterated integrals 🥳

The emergence of elliptic curves

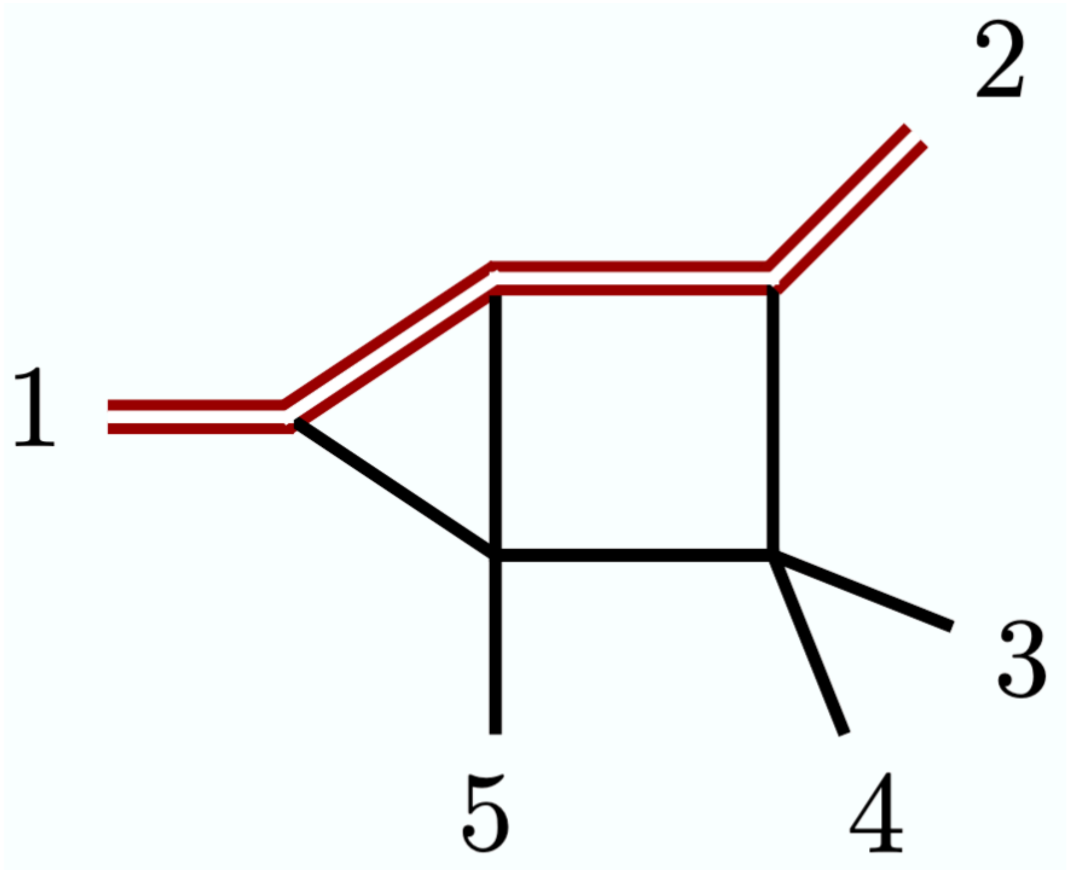


$$\propto \int \frac{\mathrm{d}z}{\sqrt{\mathscr{P}(z)}} \wedge \mathrm{dlogs}$$

$$\mathscr{P}(z) = \left(z + m_t^2\right)\left(z - 3m_t^2\right)\left(\mathscr{P}_0 + \mathscr{P}_1 z + \mathscr{P}_2 z^2\right)$$

Elliptic curve: $y^2 = \mathscr{P}(z)$

The emergence of elliptic curves

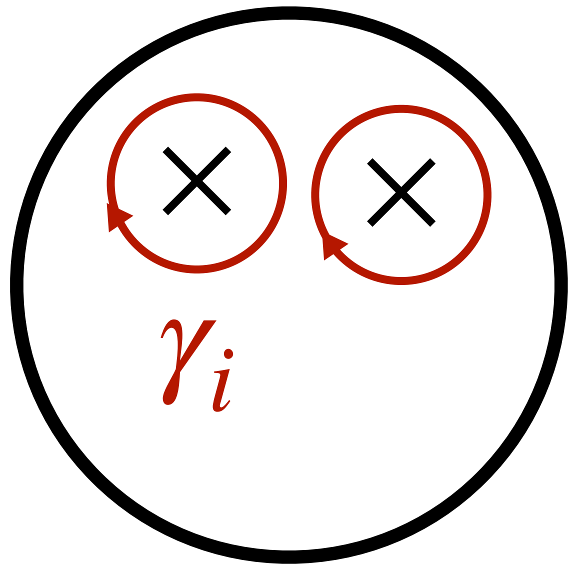


$$\propto \int \frac{\mathrm{d}z}{\sqrt{\mathcal{P}(z)}} \wedge \mathrm{dlogs}$$

$$\mathcal{P}(z) = \left(z + m_t^2\right)\left(z - 3m_t^2\right)\left(\mathcal{P}_0 + \mathcal{P}_1 z + \mathcal{P}_2 z^2\right)$$

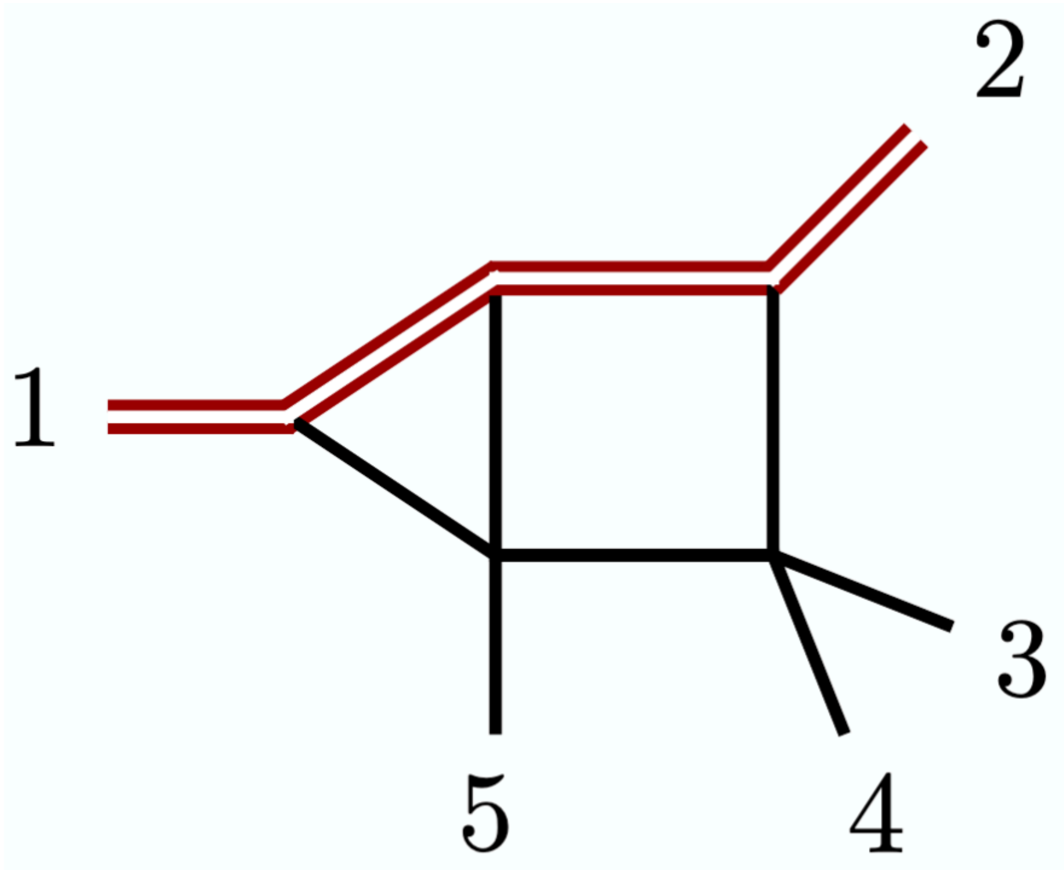
Elliptic curve: $y^2 = \mathcal{P}(z)$

$\dfrac{\mathrm{d}z}{z - c_i}$ ➡ $\mathbb{C} \cup \{\infty\} \backslash \{z = c_i\}$



➡ Algebraic leading singularities $\oint_{\gamma_i} \dfrac{\mathrm{d}z}{z - c_i}$

The emergence of elliptic curves



$$\propto \int \frac{\mathrm{d}z}{\sqrt{\mathscr{P}(z)}} \wedge \mathrm{dlogs}$$

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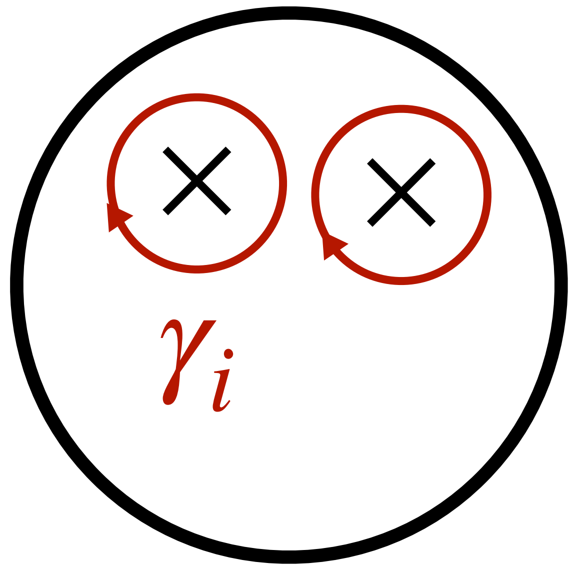
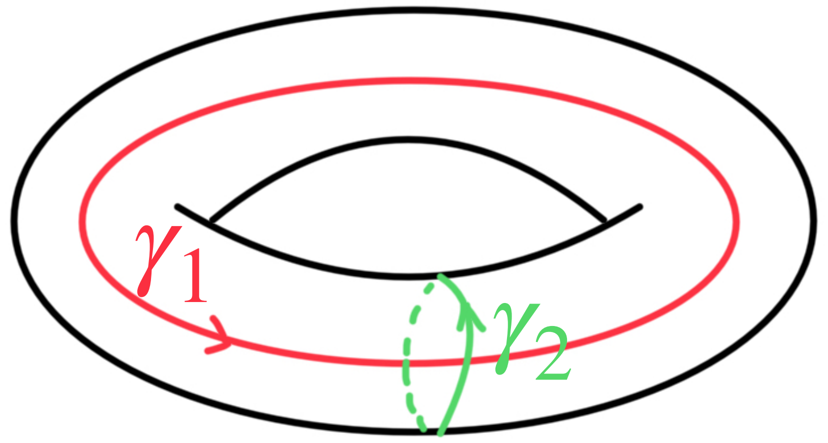


$\Rightarrow$ Algebraic leading singularities $\displaystyle\oint_{\gamma_i} \frac{\mathrm{d}z}{z - c_i}$

$\dfrac{\mathrm{d}z}{\sqrt{\mathscr{P}(z)}}, \ldots$ $\Rightarrow$



$\mathbb{C} \cup \{\infty\} \backslash \{\text{branch cuts}\}$

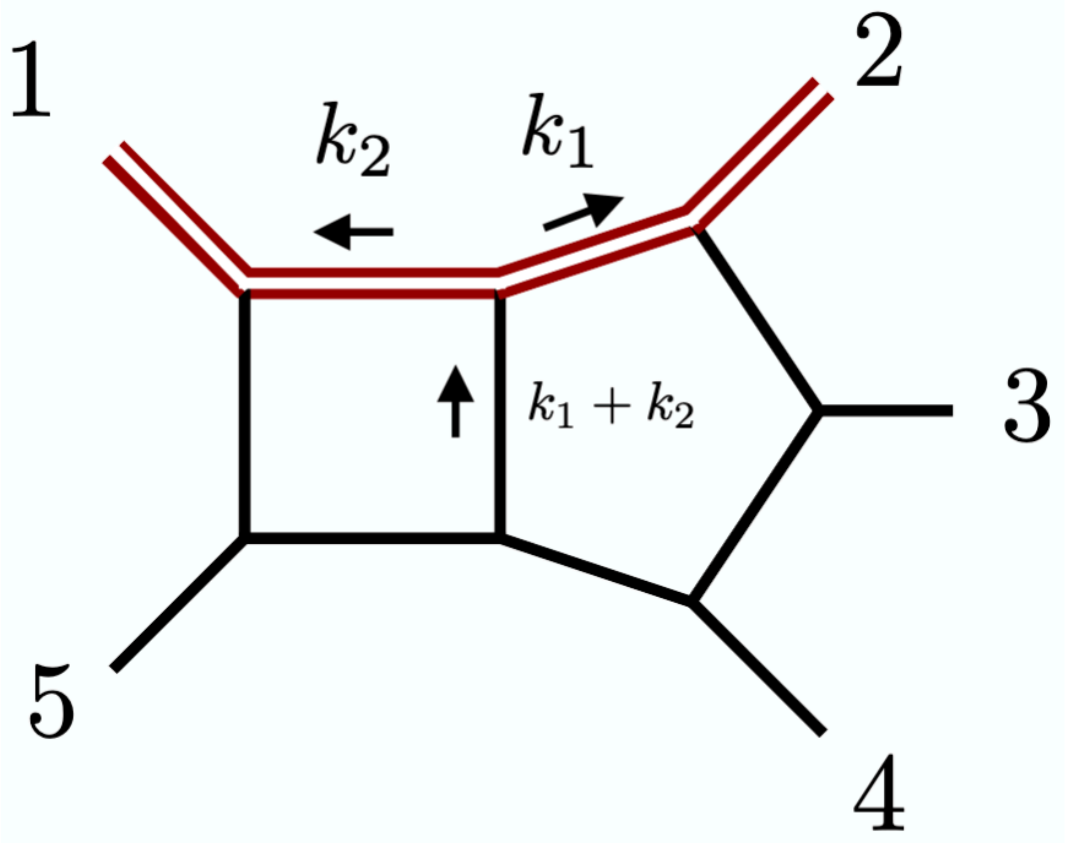
$\Rightarrow$ $\displaystyle\psi_i = \oint_{\gamma_i} \frac{\mathrm{d}z}{\sqrt{\mathscr{P}(z)}} \propto K\left(\frac{(x_3 - x_2)(x_4 - x_1)}{(x_3 - x_1)(x_4 - x_2)}\right)$

Transcendental

First canonical DEs for elliptic 5-pt. integrals ($t\bar{t}$+jet)

*Becchetti, Dlapa, **SZ** ‘25 (following method by Görges, Nega, Tancredi, Wagner ’23)*

$$\mathrm{d}\overrightarrow{\mathrm{MI}}(X;\epsilon) = \epsilon\left[\sum_{i=1}^{83} \mathrm{d}\log\left(W_i(X)\right) + \sum_{i=1}^{22}\omega_i(X)\right]\cdot\overrightarrow{\mathrm{MI}}(X;\epsilon)$$



Nested square roots: $W_\pm = \dfrac{c_1 \pm c_2\sqrt{\Delta_5} + c_3\sqrt{\beta}\sqrt{n_\pm}}{c_1 \pm c_2\sqrt{\Delta_5} - c_3\sqrt{\beta}\sqrt{n_\pm}}$ $\quad\sqrt{n_\pm} = \sqrt{d_{23}^2\,\Delta_5 - 8\,r_2\,r_4 \pm 4\,d_{23}\,r_3\sqrt{\Delta_5}}$

Transcendental functions (ψ_1, $G_{1,1}$, $G_{2,1}$, $G_{3,1}^{\pm}$), e.g. $\omega = \left[c_1\dfrac{G_{3,1}^{+}}{\psi_1^2} + \dfrac{c_2}{\psi_1}\right]\mathrm{d}x_1 + \left[\dfrac{c_3}{\psi_1}\right]\mathrm{d}x_2 + \ldots$

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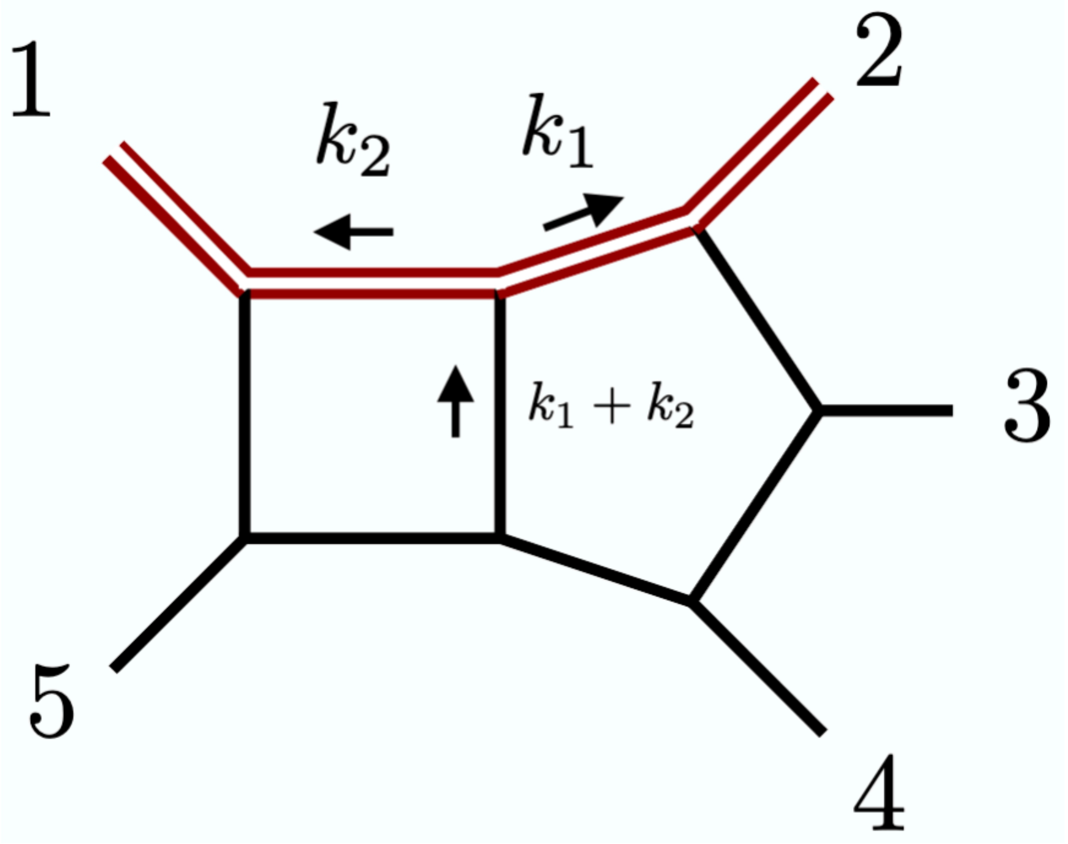

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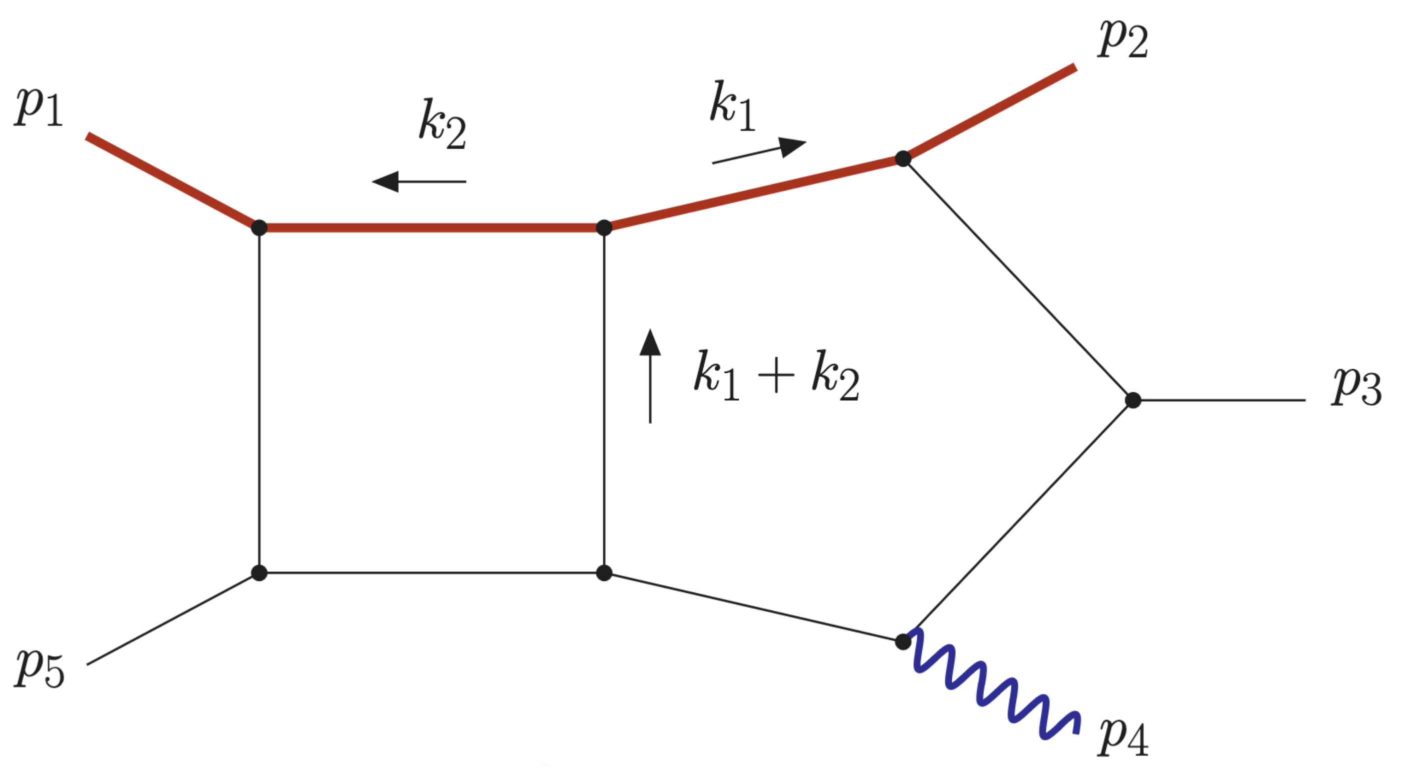
The efficient **numerical evaluation** in 3-particle phase space remains an **open problem**

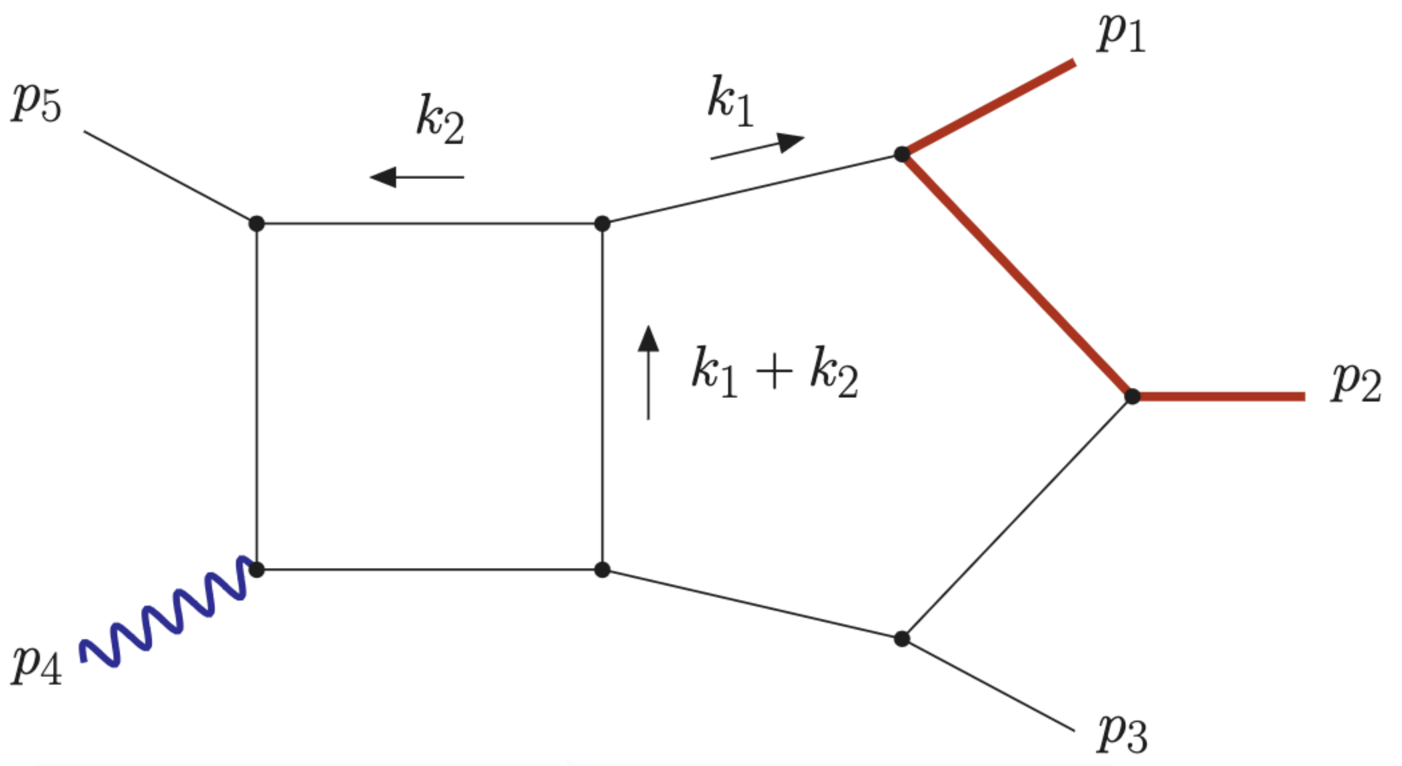
Escalation from $t\bar{t}$+jet to $t\bar{t} + W$

Becchetti, Canko, Chestnov, Peraro, Pozzoli, **SZ** ‘25



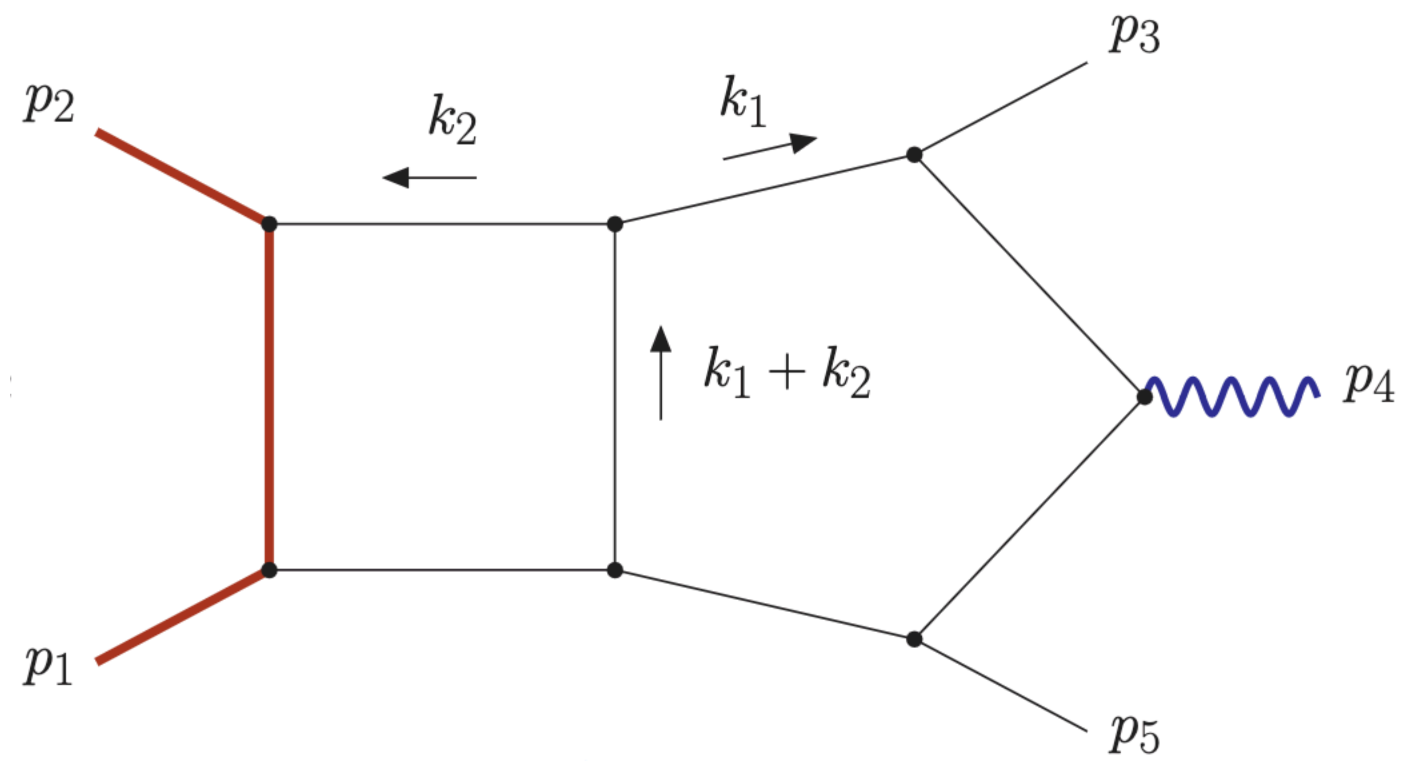
141 master integrals

2 elliptic curves



122 master integrals

1 elliptic curve



131 master integrals

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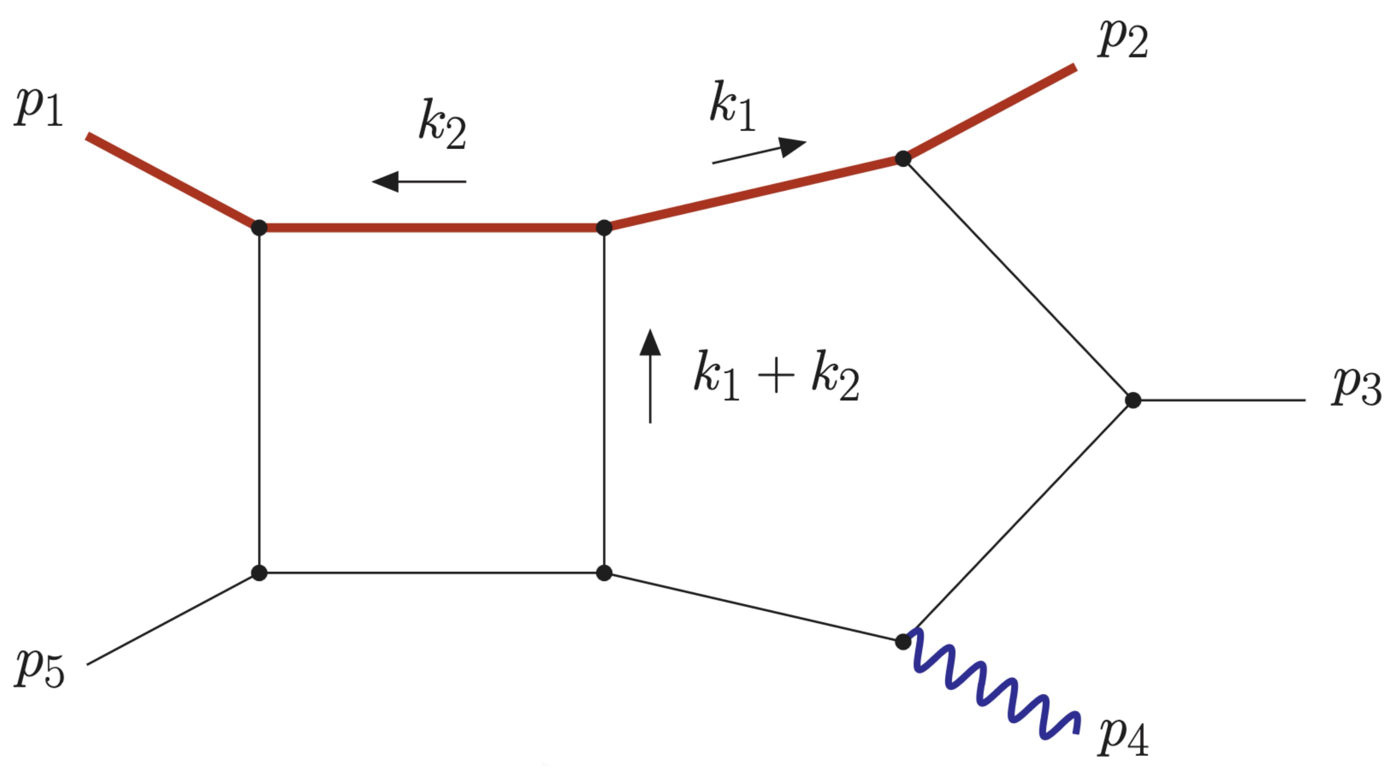

141 master integrals

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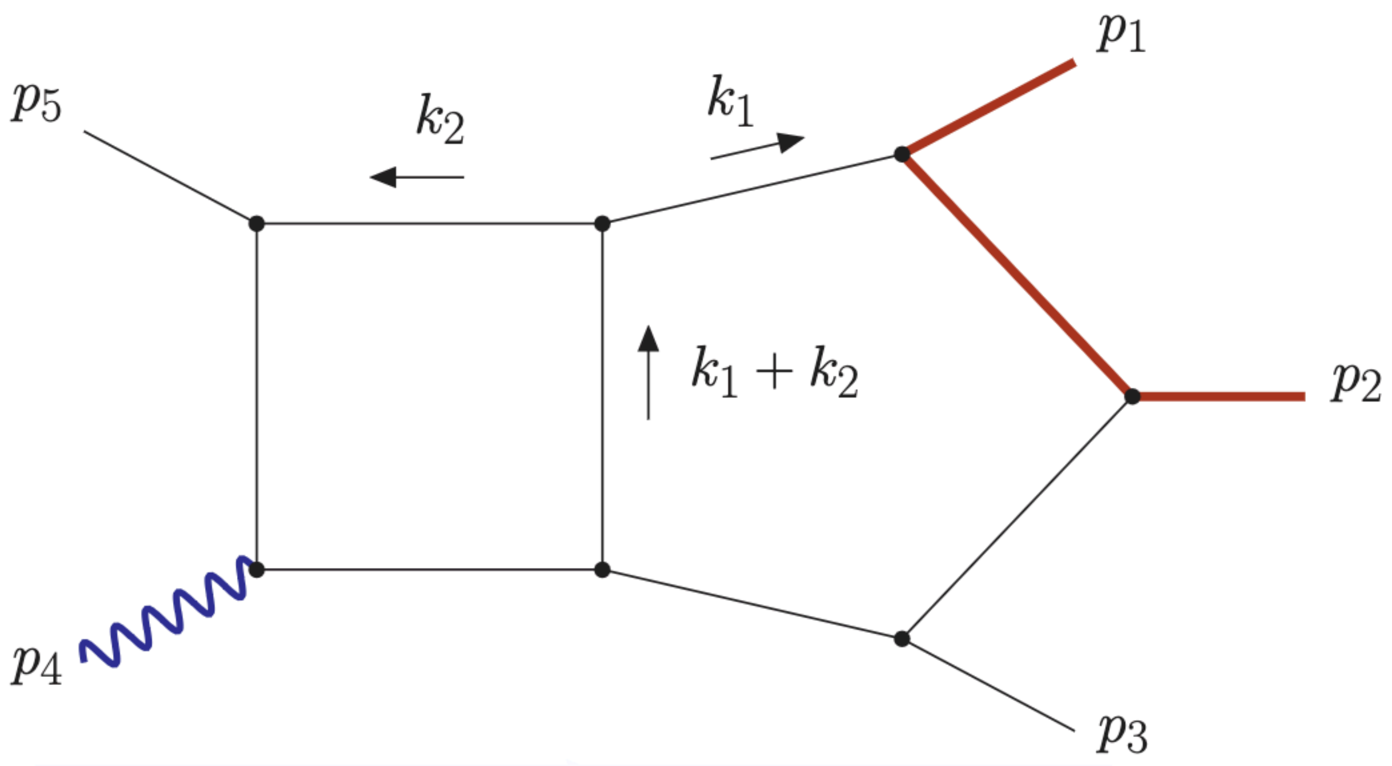

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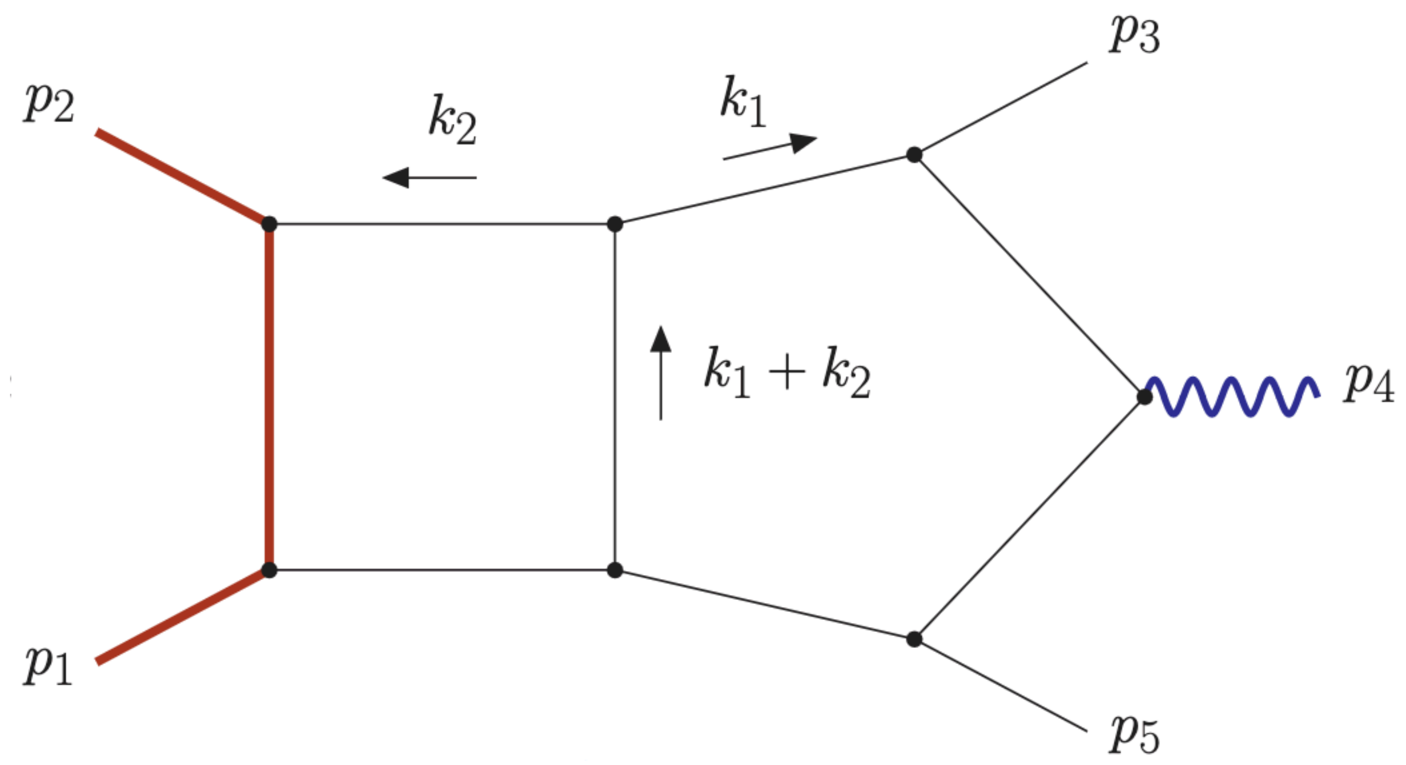

131 master integrals

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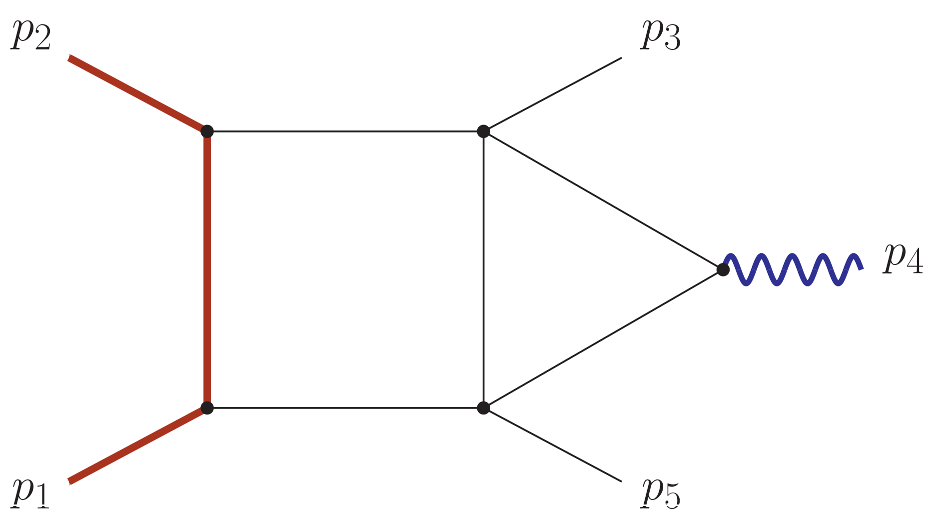
First 5-pt. elliptic curve: $y^2 = z^3 + g_1(\vec{x})\, z + g_0(\vec{x})$

degree-24 degree-36 polynomial



Discriminant contains deg.-14 irreducible polynomial in 7 variables 🤯

$P_{14}(X) =$

Correctly predicted by SOFIA *Correia, Giroux, Mizera '25*

The separation between algebraic and analytic complexity is fading away…

What is essential (for phenomenology)?

$$\overrightarrow{\mathrm{MI}}(X;\epsilon) = \overrightarrow{\mathrm{MI}}^{(0)}(X) + \epsilon\,\overrightarrow{\mathrm{MI}}^{(1)}(X) + \epsilon^2\,\overrightarrow{\mathrm{MI}}^{(2)}(X) + \ldots$$

Linearly independent

Highly **redundant** 👎

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1. **Reduce the redundancy** among the MI ϵ-expansion terms to

   - cancel the ϵ poles of the 2-loop amplitude analytically
   - simplify the expressions
   - evaluate fewer functions

$$\mathscr{A}^{(2)} = \frac{a}{\epsilon^4} + \frac{b}{\epsilon^3} + \frac{c}{\epsilon^2} + \frac{d}{\epsilon} + \mathscr{F}^{(2)}$$

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Prior belief: most MI ϵ-expansion terms are secretly of the logarithmic type!

“Almost” canonical, algebraic DEs

*Badger, Brancaccio, Becchetti, Hartanto, **SZ** ’24*

- **Give up *fully* canonical DEs**: stop right before transcendental functions appear

$$\mathrm{d}\overrightarrow{\mathrm{MI}}(X;\epsilon) = \sum_{k=0}^{2} \epsilon^k\, \Omega^{(k)}(X) \cdot \overrightarrow{\mathrm{MI}}(X;\epsilon)\,, \qquad \Omega^{(k)}(X) = \sum_i A_i^{(k)}\, \mathrm{d}\log W_i(X) \,+\, \sum_j B_j^{(k)}\, \underset{\substack{\uparrow\\ \text{Algebraic}}}{\omega_j(X)}$$

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- Choose elliptic/nested-square-root MIs so that $\mathrm{MI}_i^{(k)}(X) = 0$ for $k < 4$

  Reason: This should give a representation of the 2-loop amplitude where the ϵ poles are manifestly polylogarithmic

Identify logarithmic terms order by order in ϵ

*Badger, Brancaccio, Becchetti, Hartanto, **SZ** '24*

$$\mathrm{d}\overrightarrow{\mathrm{MI}}^{(k)}(X) = \Omega^{(0)}(X) \cdot \overrightarrow{\mathrm{MI}}^{(k)}(X) + \Omega^{(1)}(X) \cdot \overrightarrow{\mathrm{MI}}^{(k-1)}(X) + \Omega^{(2)}(X) \cdot \overrightarrow{\mathrm{MI}}^{(k-2)}(X)$$

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$\mathrm{d}\overrightarrow{\mathrm{MI}}^{(0)}(X) = 0$ ✓

Drop all vanishing $\mathrm{MI}_i^{(k)}(X)$, determined via numerical evaluations with AMFlow

Liu, Ma, Wang '18; Liu, Ma '23

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$\checkmark$ = canonical = we know how to handle it!

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$d\mathrm{MI}_i^{(4)}(X) \sim d\log \times \overrightarrow{\mathrm{MI}}^{(3)}(X)$ for most i ✓

$d\mathrm{MI}_i^{(4)}(X) \sim \omega \times \overrightarrow{\mathrm{MI}}^{(4)}(X) + \omega' \times \overrightarrow{\mathrm{MI}}^{(3)}(X) + \omega'' \times \overrightarrow{\mathrm{MI}}^{(2)}(X)$ for a few i

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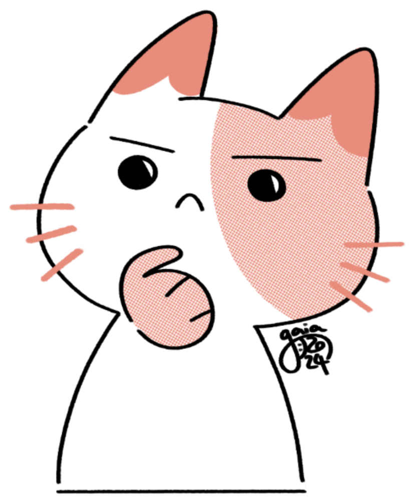

Almost everything is under full control

E.g. for $t\bar{t} + W$, only 39 $\mathrm{MI}_i^{(4)}(X)$ remain coupled

We just view them as independent functions and continue

$$\begin{aligned}\mathrm{MI}_{15}(X;\epsilon) = & \frac{1}{48} + \frac{\epsilon}{24}\left(F_1^{(1)} - 2\,F_2^{(1)} + 2\,F_4^{(1)} - 2\,F_6^{(1)}\right) \\ & - \frac{1}{48}\epsilon^2\left[7\,F_1^{(2)} + 7\,F_3^{(2)} + \frac{25}{4}\zeta_2 - \frac{15}{4}\left(F_1^{(1)}\right)^2 + 15\,F_1^{(1)}\,F_2^{(1)} - 8\left(F_2^{(1)}\right)^2 + \ldots\right] \\ & + \frac{1}{8}\epsilon^3\left[F_5^{(3)} + \frac{820}{9}\zeta_3 + \frac{7}{12}\zeta_2 F_2^{(1)} - \left(F_2^{(1)}\right)^3 + 5\left(F_3^{(1)}\right)^2 F_5^{(1)} + 6\,F_1^{(1)}F_2^{(1)}F_6^{(1)} + \ldots\right] \\ & + \epsilon^4\,F_1^{(4^*)} + \mathcal{O}\left(\epsilon^5\right)\,,\end{aligned}$$

414 independent functions

*Gehrmann, Henn, Lo Presti 2018; Chicherin, Sotnikov 2020; Chicherin, Sotnikov, **SZ** 2021; Abreu, **SZ** et al. 2023*

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414 independent functions

*Gehrmann, Henn, Lo Presti 2018; Chicherin, Sotnikov 2020; Chicherin, Sotnikov, **SZ** 2021; Abreu, **SZ** et al. 2023*

- ✓ Analytic cancellation of ϵ poles
- ✓ Fewer functions to be evaluate
- ✓ Simplification of 2-loop finite remainder

Special functions obey 1st-order algebraic DEs

$$\mathrm{d}F_i^{(w)}(X) = \text{polynomial in } \{F_k^{(w)}(X)\}$$

E.g.

$$\mathrm{d}F_1^{(4^*)} = \omega_1\, F_4^{(4^*)} + \omega_2\, F_6^{(4^*)} - \frac{1}{2}\left(\omega_{31} + \frac{1}{2}\,\mathrm{d}\log W_{69}\right) F_{39}^{(3)} + \frac{1}{4}\,\mathrm{d}\log W_{70}\left(F_7^{(3)} + F_1^{(1)} F_4^{(2)} - 2\, F_6^{(1)} F_4^{(2)}\right) + \ldots$$

Family	square roots	rat. letters	alg. letters	all letters	non-dlog one-forms	functions
Ax12	5	26	21	47	0	53
A	5	26	21	47	0	53
F1	8	42	32	74	74	239
F1x12	6	35	26	61	74	148
F2	10	36	37	73	64	178
F2x12	10	37	36	73	64	195
F2x12x34	7	31	27	58	42	124
F3	7	29	28	57	0	127
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Up to $\mathcal{O}(\varepsilon^2)$	13	52	48	100	138	

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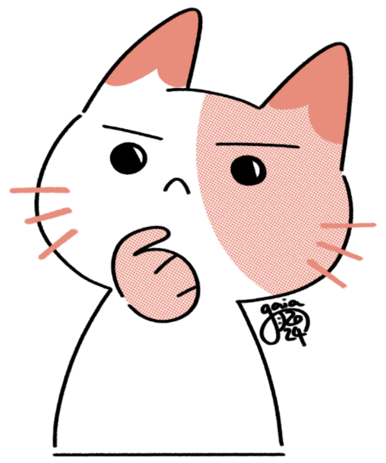
How do we solve them?



1) Numerical solution of DEs

Fully numerical: **Bulirsch-Stoer algorithm**

Czakon ’08; …; Petit Rosàs, Bobadilla ’25

- Ideal when solution is smooth & high precision is needed
- Bottleneck: evaluation of the one-forms

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$t\bar{t} + \mathrm{jet}$: evaluate all functions in ~1 s (single core, double precision)

*Badger, Becchetti, Brancaccio, Czakon, Hartanto, Poncelet, **SZ** '25*

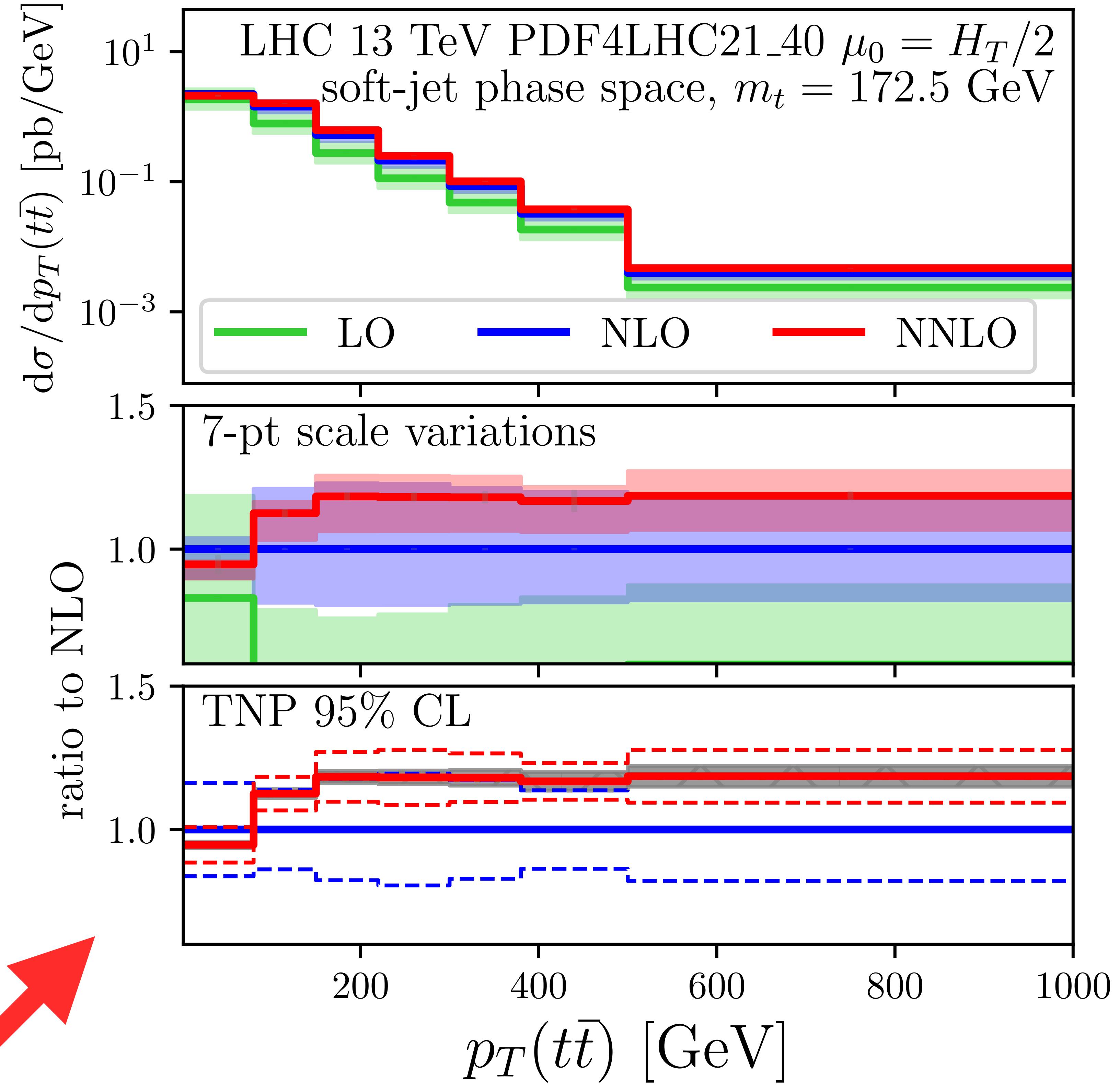
1) Numerical solution of DEs

Fully numerical: **Bulirsch-Stoer algorithm**

Czakon ’08; …; Petit Rosàs, Bobadilla ’25

- Ideal when solution is smooth & high precision is needed
- Bottleneck: evaluation of the one-forms

Substantial effort into optimisation: partial fractioning, local changes of variables, gathering common subexpressions…

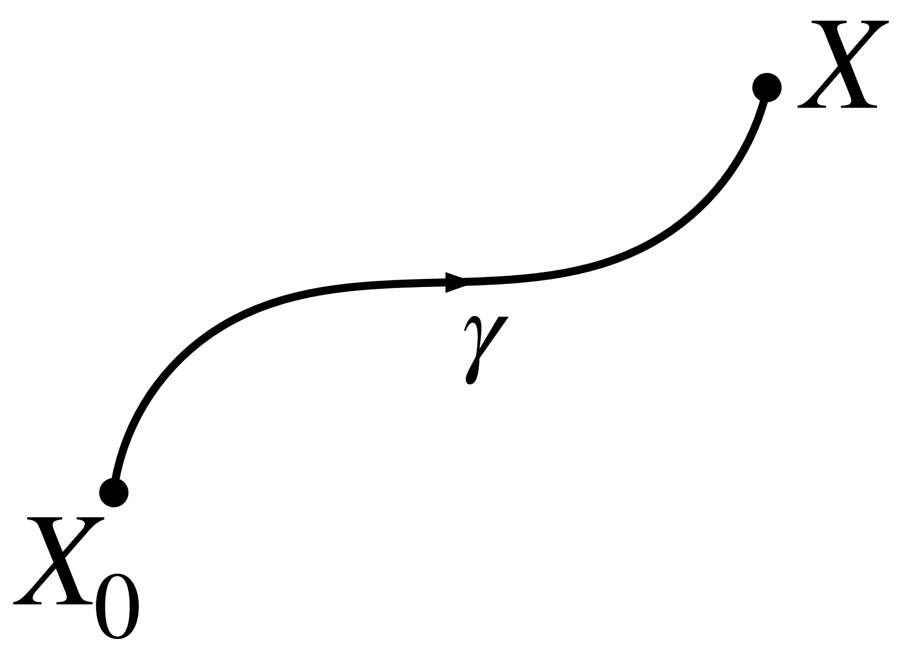


$t\bar{t} + \mathrm{jet}$: evaluate all functions in ~1 s (single core, double precision)

*Badger, Becchetti, Brancaccio, Czakon, Hartanto, Poncelet, **SZ** ’25*

2) Generalised power series expansions

Pozzorini, Remiddi, '06; …; Moriello '20



Semi-analytic: **series expansions** of the solution with coefficients fixed numerically

Can work also when **large scale variations** occur

Various public implementations. We used the solver shipped with AMFlow *Liu, Ma '22*

2) Generalised power series expansions

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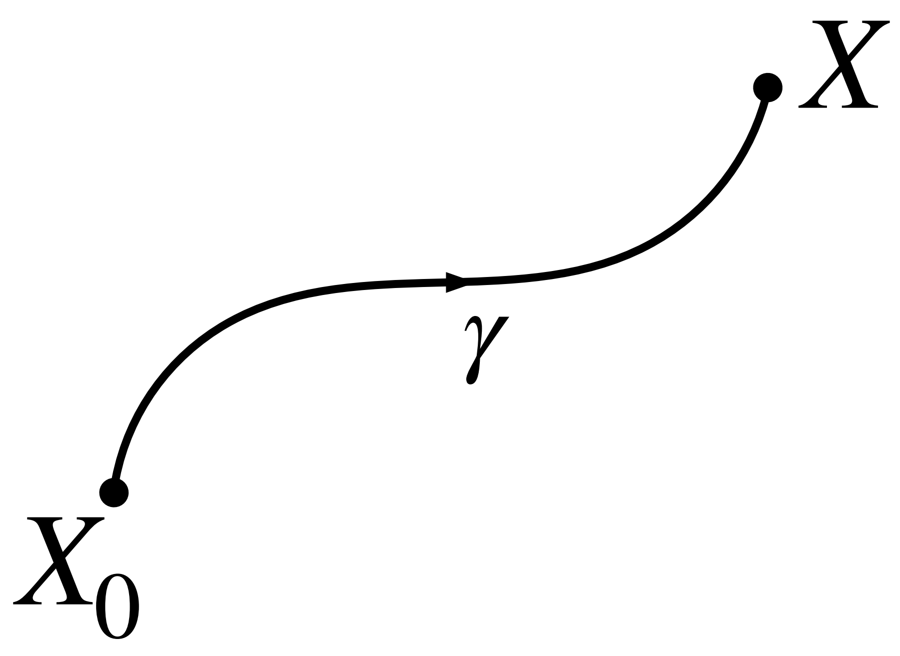

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$t\bar{t} + W$: evaluate all functions in ~20 min — 2h (single core, at least 16-digit precision)

*Becchetti, Canko, Chen, Chestnov, Delto, Ditsch, Grazzini, Kallweit, Peraro, Pozzoli, Savoini, Tancredi, **SZ** ’26*

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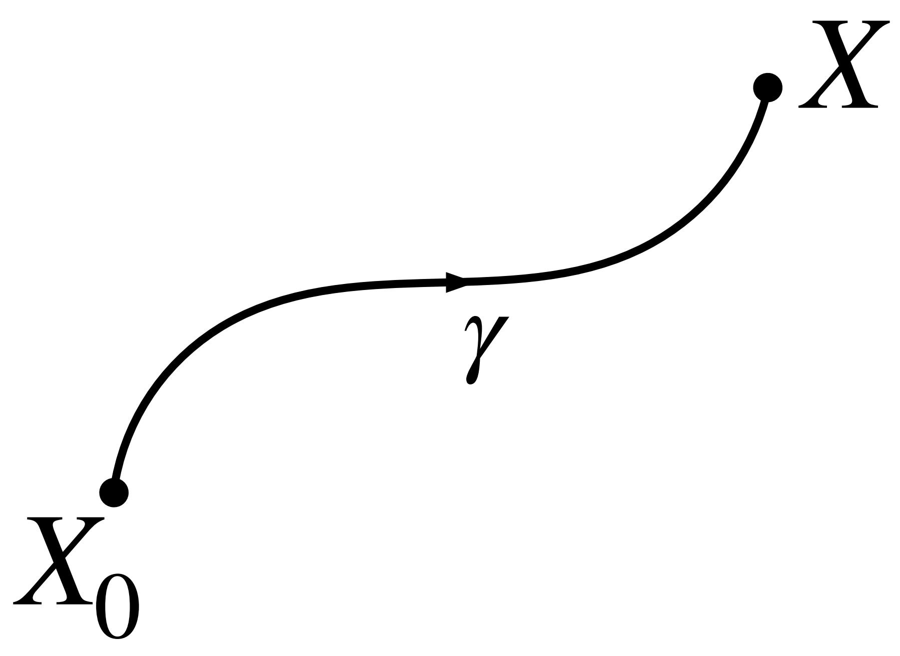

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Inclusive NNLO QCD cross section

$t\bar{t}W^-$	$t\bar{t}W^+$
$235.4(0)^{+5.1\%}_{-6.6\%} \pm 1.9\%$	$474.9(2)^{+4.8\%}_{-6.4\%} \pm 1.9\%$
$241.9(0)^{+6.4\%}_{-7.3\%} \pm 2.3\%$	$489.4(1)^{+6.3\%}_{-7.2\%} \pm 2.5\%$

$t\bar{t} + W$: evaluate all functions in ~20 min — 2h (single core, at least 16-digit precision)

*Becchetti, Canko, Chen, Chestnov, Delto, Ditsch, Grazzini, Kallweit, Peraro, Pozzoli, Savoini, Tancredi, **SZ** '26*

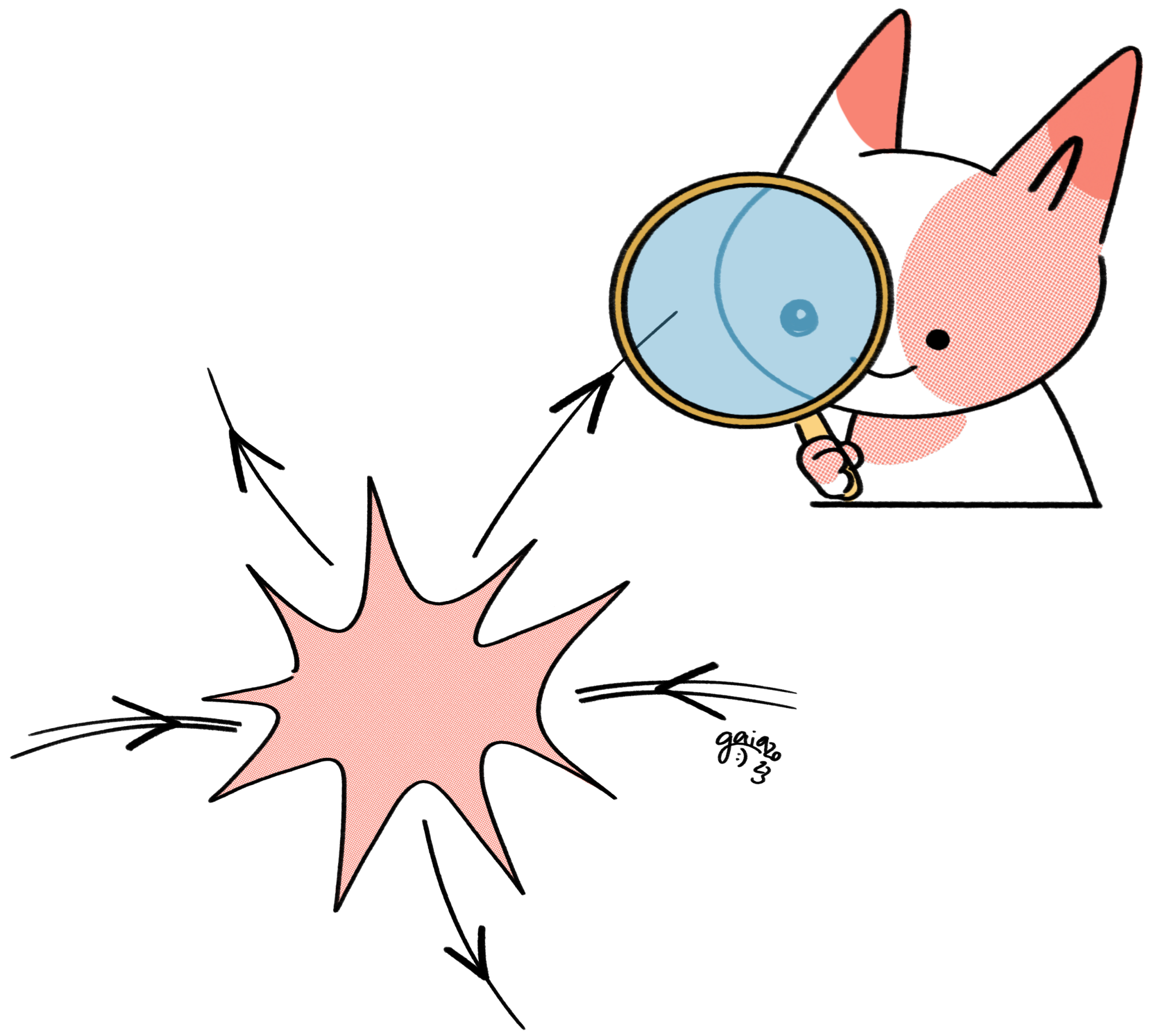
Conclusions & Outlook

✓ **NNLO QCD predictions** for $t\bar{t} + \mathrm{jet}/W$ production at LHC (leading colour)

✓ First time: **2-loop 5-particle amplitudes with internal masses** / elliptic Feynman integrals!

Advances in amplitudes from **synergy** between **numerical** methods & **analytical** insights

Still a lot of work to do for the Amplitudes community to meet the needs of the HL LHC!



Conclusions & Outlook

✓ **NNLO QCD predictions** for $t\bar{t} + \mathrm{jet}/W$ production at LHC (leading colour)

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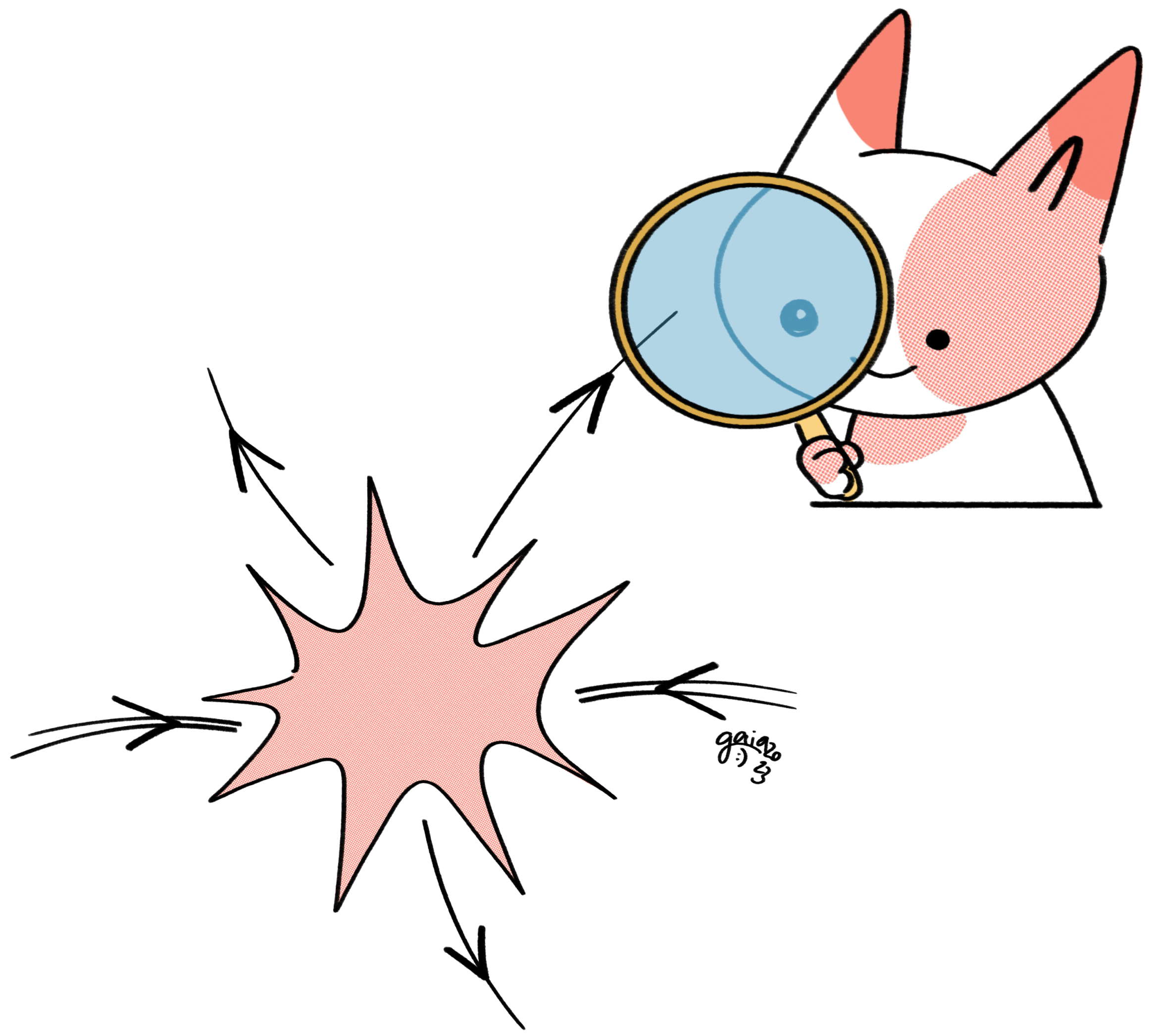
Thank you!



Why are $2 \to 3$ processes difficult?

$$\mathscr{A}(p_i, m_i, \ldots) = \sum \text{(rational function)} \times \text{(special function)}$$

Algebraic complexity
Many variables, terabytes of symbolic expressions! ⚠️

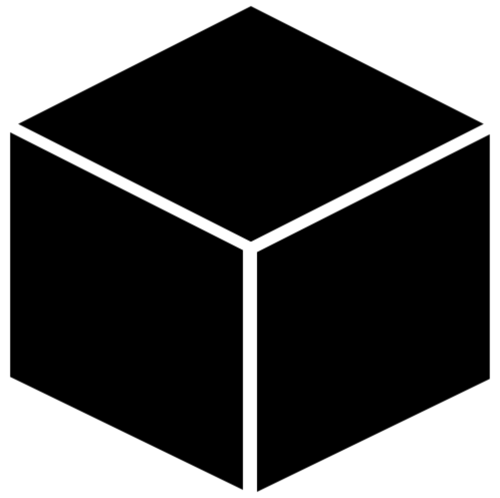
Analytic complexity
What functions? Branch cuts? Relations? Evaluation? ⚠️

5 scalar invariants $(p_i + p_j)^2$ & masses

Pseudo-scalar invariants $\epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$ exist only for $n \geq 5$ particles

"Large" phase space: **efficiency** is crucial for phenomenology

How long does it take to reconstruct a function?

$(x, \mathrm{P}) \rightarrow$  $\rightarrow f(x) \bmod \mathrm{P}$ ⏳ $f(x) = ?$

$$\frac{\text{\# points} \times \text{eval. time}}{\text{\# CPUs}}$$

Depends on the "complexity" of the functions

$f(x, y) = 1 + x + y \Rightarrow$ 5 pts.

$f(x, y) = \dfrac{(1 + x + y)^{10}}{1 - x} \Rightarrow$ 70 pts.

2-loop amplitudes for $gg \rightarrow t\bar{t}g$: degree-421 polynomials in 6 variables

$\Rightarrow$ naïvely 10^{11} million pts. 😱

⬇

Tailored reconstruction algorithms/ansätze (e.g. partial fraction decomposition)

*Peraro '19; Abreu, Dormans et al. '19; Badger, Hartanto, **SZ** '21; De Laurentis, Maître '21; Belitsky, Smirnov, Yakovlev '23; Liu '23; De Laurentis, Page '22; Abreu, De Laurentis et al. '23*

Univariate partial fraction decomposition

*Badger, Bayu Hartanto, **SZ** 2021*

$$f(x,y) = \frac{2x^4 + 4x^3y - 5x^2y^2 + xy^3 - 4y^4}{\boxed{(y-x)y^2(x^2+y^2)}}$$

Physics dictates the allowed singularities

$\Rightarrow$ Denominators can be guessed!

Abreu, Dormans et al. '19

"Bootstrap" a univariate partial fraction decomposition:

$$f(x,y) = \frac{c_1(x) + c_2(x)\,y}{x^2+y^2} + \frac{c_3(x)}{y^2} + \frac{c_4(x)}{y} + \frac{c_5(x)}{y-x}$$

Linear fit to reconstruct the coefficients $c_i(x)$ $\Rightarrow$

- One fewer variables
- Lower degrees

$$f(x,y) = \frac{3y}{x^2+y^2} - \frac{2x}{y^2} - \frac{6}{y} - \frac{1}{y-x}$$

Rational coefficients for $t\bar{t}W$

$r_{ki}(X) =$

```
21 260 549 757 479 032 455 978 413 891 272 330 463 864 721 056 811 513 695 819 036 773 481 868 217 142 386 242 810 233 902 923 925 054 456 662 581 586 169 083 561 379 832 919 428 \
  916 988 170 915 484 533 852 382 103 702 568 526 628 509 862 013 723 279 024 439 141 080 929 739 111 230 953 582 363 043 339 216 159 685 124 794 279 153 188 475 517 053 890 958 \
  576 656 986 162 393 917 744 914 362 578 049 774 887 608 978 209 945 394 595 293 739 245 415 272 329 003 417 174 542 284 462 110 825 210 757 560 258 930 439 726 139 798 937 312 \
  159 287 286 910 037 836 670 202 296 771 119 349 138 753 019 557 267 194 162 355 085 442 195 268 866 549 379 047 764 251 145 450 311 932 512 321 567 769 417 922 868 275 919 373 \
  782 432 390 220 654 078 404 799 402 482 104 759 404 500 932 314 470 872 213 751 432 690 051 290 807 598 843 160 127 574 943 933 732 471 953 466 560 629 734 785 127 409 543 327 \
  824 367 717 072 148 266 235 605 148 831 574 800 757 025 960 020 240 282 345 568 387 641 183 441 969 828 557 534 845 327 043 627 881 184 244 104 562 775 603 953 270 101 668 037 \
  187 094 806 694 420 883 380 485 145 037 665 702 947 368 628 728 225 413 828 088 938 139 781 747 713 572 348 343 550 888 644 966 488 567 208 705 634 005 174 374 380 273 548 908 \
  952 061 845 932 772 187 555 970 038 735 492 847 943 120 791 789 780 155 809 207 984 521 148 818 153 664 000 954 113 794 829 734 793 901 258 692 718 167 943 013 988 337 004 143 \
  603 758 229 639 846 495 813 306 853 810 494 576 810 811 256 326 124 987 993 341 122 535 883 833 649 526 687 488 857 335 265 576 496 663 362 086 381 686 973 774 700 509 661 853 \
  427 378 480 703 348 881 652 592 696 239 163 542 496 044 183 161 200 997 838 456 063 144 443 141 738 650 661 358 314 058 578 841 616 528 258 297 699 269 475 458 046 846 699 264 \
  759 885 525 615 227 865 974 444 586 332 159 676 480 619 761 153 427 370 936 142 766 025 986 316 451 097 425 426 059 /
 21 172 395 022 624 285 978 992 748 673 816 975 747 246 500 897 881 301 616 263 948 708 050 175 844 852 976 304 150 143 879 934 024 448 262 769 181 496 390 206 143 348 352 529 401
  850 208 116 734 482 093 028 300 973 135 856 474 642 471 040 059 838 287 511 246 604 048 396 118 553 085 597 205 328 239 275 357 120 656 107 480 094 424 193 972 588 314 506 899 \
  297 572 791 128 406 738 928 505 511 684 112 339 611 371 822 600 547 690 336 864 441 527 299 164 648 439 851 684 464 188 178 640 501 283 702 760 941 472 905 841 980 731 335 498 \
  257 925 215 940 793 329 554 991 438 813 762 310 874 726 110 854 453 956 647 977 009 967 115 364 546 313 293 587 964 548 074 332 820 504 499 675 523 622 246 938 172 698 046 075 \
  023 953 473 086 763 285 462 005 240 262 404 660 316 821 370 385 626 428 063 310 971 059 467 349 565 521 716 251 548 490 409 327 235 957 542 003 487 711 036 894 292 603 679 181 \
  523 355 608 321 836 601 959 034 315 424 196 174 912 719 641 467 896 423 079 628 182 049 447 532 259 769 274 230 697 956 643 855 441 731 773 611 890 477 862 871 411 541 941 893 \
  613 904 692 043 715 268 954 611 358 390 904 445 406 161 262 100 393 814 999 154 500 342 153 643 743 465 860 756 742 670 260 919 573 007 916 915 655 899 568 405 495 223 982 635 \
  713 668 645 792 670 312 236 455 393 880 797 894 323 703 759 132 363 225 246 433 634 729 198 705 057 744 401 611 856 681 869 489 804 974 991 408 590 033 797 743 963 975 514 693 \
  736 581 286 475 964 393 864 477 915 919 089 167 728 377 468 796 460 070 932 801 405 384 124 380 240 888 442 003 264 790 879 687 641 037 062 204 861 942 990 139 844 823 100 779 \
  257 649 446 053 188 721 025 637 021 334 903 038 296 658 288 513 893 905 260 964 963 222 889 709 004 430 867 368 277 535 127 418 268 607 653 056 709 044 478 112 909 077 993 662 \
  596 059 029 598 953 438 928 611 415 837 923 491 232 889 503 356 350 385 461 250 000 000 000
```

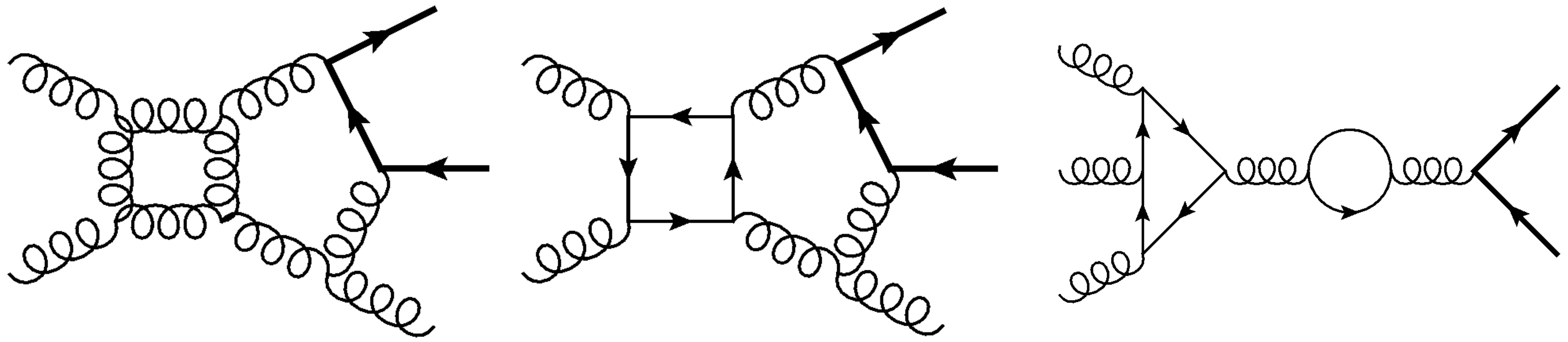
$= 1.00416 \times 10^{18}$

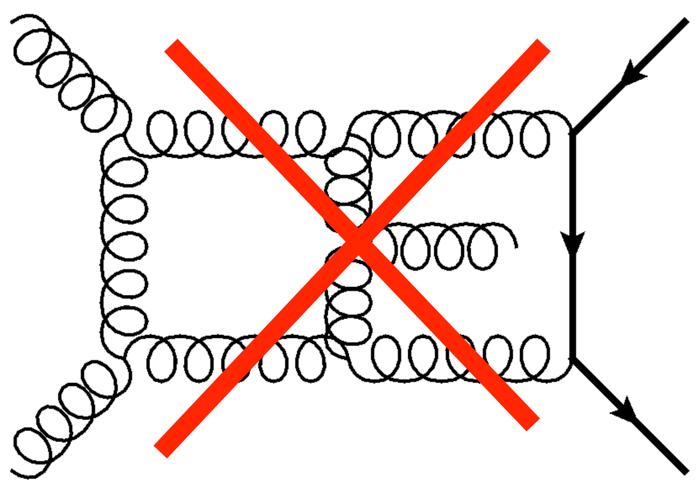
$\Rightarrow$ We need to evaluate mod up to ~250 primes to complete the rational reconstruction

Leading colour approximation

Keep only the leading terms in the limit of large $N_c = 3$ and $n_l = 5$ of the **2-loop IR-subtracted finite remainder:**

$$F^{(2)} = \left(N_c^2\, A + N_c\, n_l\, B + n_l^2\, C\right)\left[1 + \mathcal{O}\left(1/N_c\right)\right]$$

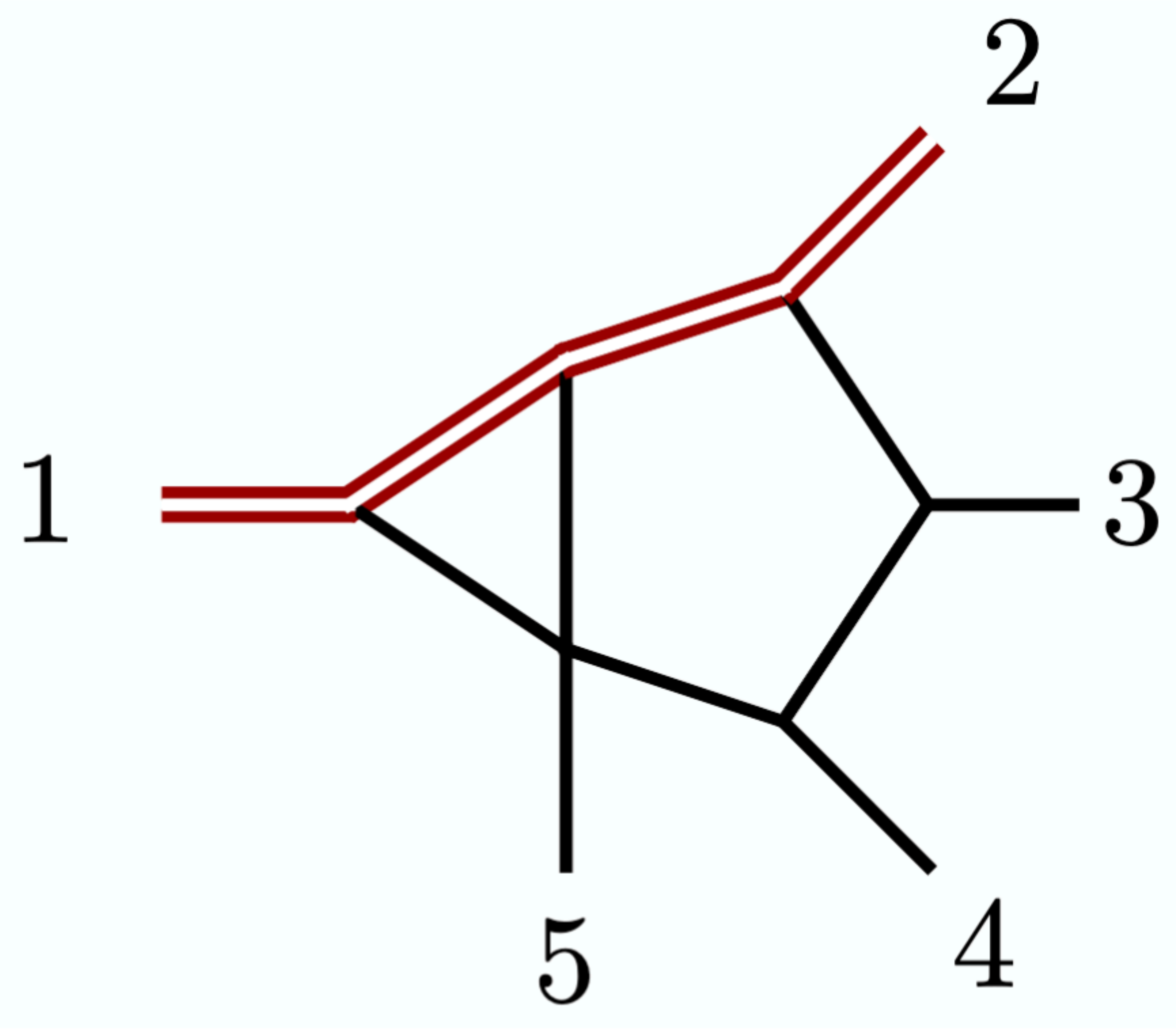




Fewer and significantly simpler diagrams

Naïvely ~30% uncertainty on $F^{(2)}$ and double virtual corrections are ~10% of σ

Nested square roots in the leading singularities



$$I_{19} = \sqrt{n_+}\,\frac{d_{34}}{a(X)\sqrt{\Delta_5}}\; I\big[a(X)\,D_9 + b(X) - c(X)\sqrt{\Delta_5}\big]$$

$$I_{20} = I_{19}\Big|_{\sqrt{\Delta_5}\to-\sqrt{\Delta_5}} \qquad\qquad \Delta_5 = \det(2\,p_i\cdot p_j)$$

$$\boxed{\sqrt{n_\pm} = \sqrt{d_{23}^2\,\Delta_5 - 8\,r_2\,r_4 \pm 4\,d_{23}\,r_3\sqrt{\Delta_5}}}$$

Yes, it’s really there! It’s not of the type $\sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$

Found also in $t\bar{t}W/t\bar{t}H$ integrals *Febres Cordero et al. ’24; Becchetti, **SZ** et al. ’25*

A new, “duplet” structure from Galois symmetry

*Becchetti, Dlapa, **SZ** ’25*

$$\sqrt{\Delta_5} \leftrightarrow -\sqrt{\Delta_5} \implies \sqrt{n_+} \leftrightarrow \sqrt{n_-}\,, \quad I_{19} \leftrightarrow I_{20}$$

Not the usual even/odd representation!

- Square roots $\to \mathbb{Z}_2$
- Nested square roots $\to \mathbb{Z}_2 \times \mathbb{Z}_2$

exterior root   interior root

Galois groups of the defining polynomials

Duplets of elements that transform into each other

Symmetry informs the construction of letters, e.g.

$$W_\pm = \frac{c_1 \pm c_2\sqrt{\Delta_5} + c_3\sqrt{\beta}\sqrt{n_\pm}}{c_1 \pm c_2\sqrt{\Delta_5} - c_3\sqrt{\beta}\sqrt{n_\pm}}$$

- $W_\pm$ both odd w.r.t. $\sqrt{\beta}$
- $W_\pm$ odd w.r.t. $\sqrt{n_\pm}$, respectively
- $W_+ \leftrightarrow W_-$ duplet w.r.t. $\sqrt{\Delta_5}$

A new, "duplet" structure from Galois symmetry

Becchetti, Dlapa, ***SZ*** *'25*

$\sqrt{\Delta_5} \leftrightarrow -\sqrt{}$ ... resentation!

- Square r... mials
- Nested s...

t

r

Symmetry

Bonus: **constraints on the DEs**! For every duplet $W_\pm$,

$$\mathrm{d}\tilde{A}(X) = \ldots + M_+ \,\mathrm{d}\log W_+ + M_- \,\mathrm{d}\log W_- + \ldots$$

$$\Rightarrow \;\; M_+ = \Sigma \cdot M_- \cdot \Sigma^{-1}$$

where Σ implements the $\sqrt{\Delta_5}$-flip on the MIs

$$W_\pm = \frac{c_1 \pm c_2\sqrt{\Delta_5} + c_3\sqrt{\beta}\sqrt{n_\pm}}{c_1 \pm c_2\sqrt{\Delta_5} - c_3\sqrt{\beta}\sqrt{n_\pm}}$$

- $W_\pm$ both odd w.r.t. $\sqrt{\beta}$
- $W_\pm$ odd w.r.t. $\sqrt{n_\pm}$, respectively
- $W_+ \leftrightarrow W_-$ duplet w.r.t. $\sqrt{\Delta_5}$

Analytic AND algebraic complexity 🤯

Elliptic curve coupled to nested square roots

$G_{3,1}^{+} =$

```
(4 sqrtDelta32 (-4 d12 d23 + 4 d23 d34 + 4 d15 d45 - 4 d34 d45 - 4 d23 mt2 - sqrtDelta5)
   EllipticK[(4 mt2 sqrtR1) / (d12^2 d15^2 + 2 d12 d34 mt2 (d15 + mt2) + mt2^2 (-d15^2 + 2 d15 d34 + 2 d34 (-d34 + mt2)) + 2 mt2 sqrtR1)] sqrtn[+]) /
 ((d15^2 - 2 d15 d34 + d34^2 - 2 d34 mt2) π (4 d12 d15 d23 - 4 d12 d23^2 + 4 d23^2 d34 + 4 d15 d23 d45 - 4 d23 d34 d45 + 4 d15 d23 mt2 - 4 d23^2 mt2 - 4 d34 d45 mt2 - d23 sqrtDelta5) sqrtR2) +
(16 d15^2 (d12 + mt2)^2 sqrtDelta32 (4 d12 d15^2 d23 - 4 d12 d15 d23^2 - 4 d12 d15 d23 d34 - 4 d12 d23^2 d34 - 4 d15 d23^2 d34 + 4 d23^2 d34^2 - 4 d15^2 d23 d45 + 8 d15 d23 d34 d45 - 4 d23 d34^2 d45 + 4 d15^2 d23 mt2 -
   4 d15 d23^2 mt2 - 4 d12 d23 d34 mt2 - 4 d15 d23 d34 mt2 - 4 d23^2 d34 mt2 + 4 d23 d34^2 mt2 + 8 d23 d34 d45 mt2 - 4 d23 d34 mt2^2 + d15 d23 sqrtDelta5 - d23 d34 sqrtDelta5 - d34 mt2 sqrtDelta5)
  EllipticPi[-((2 × (4 (d12 (d15 - d23) d23 - d23 d34 d45 + d23^2 (d34 - mt2) - d34 d45 mt2 + d15 d23 (d45 + mt2)) - d23 sqrtDelta5) sqrtR1) /
     (d23 sqrtDelta5 (d12 (-d15^2 + d15 d34 + d34 mt2) + mt2 (-d15^2 + d15 d34 + d34 (-d34 + mt2)) + sqrtR1) +
       4 (d12^2 (-d15^3 d23 + d15 d23 d34 (d23 - mt2) + d23^2 d34 mt2 + d15^2 (d23^2 + d23 d34 + d34 mt2)) + mt2 (d15^3 d23 (d45 - mt2) - d34 (d34 - mt2) (d23 d34 d45 + d34 d45 mt2 + d23^2 (-d34 + mt2)) +
             d15^2 (d23^2 (d34 + mt2) + d23 d34 (-2 d45 + mt2) + d34 mt2 (-d45 + mt2)) + d15 d34 (d34 d45 mt2 + d23^2 (-d34 + mt2) - d23 mt2 (d45 + mt2) + d23 d34 (2 d45 + mt2))) +
          d12 (d15^3 d23 (d45 - 2 mt2) + d34 mt2 (d23 d34 d45 - 2 d23^2 (d34 - mt2) + d34 d45 mt2) + d15^2 (2 d23 d34 (-d45 + mt2) + d23^2 (d34 + 2 mt2) + d34 mt2 (-d45 + 2 mt2)) +
             d15 d34 (-d23^2 (d34 - 2 mt2) + d34 d45 mt2 + d23 d34 (d45 + mt2) - d23 mt2 (d45 + 2 mt2))) + (d12 d23 (-d15 + d23) + d23 d34 d45 + d34 d45 mt2 + d23^2 (-d34 + mt2) - d15 d23 (d45 + mt2)) sqrtR1))),
   (4 mt2 sqrtR1) / (d12^2 d15^2 + 2 d12 d34 mt2 (d15 + mt2) + mt2^2 (-d15^2 + 2 d15 d34 + 2 d34 (-d34 + mt2)) + 2 mt2 sqrtR1)] sqrtn[+]) / ((d15^2 - 2 d15 d34 + d34^2 - 2 d34 mt2) π
  (4 d12 d15 d23 - 4 d12 d23^2 + 4 d23^2 d34 + 4 d15 d23 d45 - 4 d23 d34 d45 + 4 d15 d23 mt2 - 4 d23^2 mt2 - 4 d34 d45 mt2 - d23 sqrtDelta5) ×
  (4 d12^2 d15^3 d23 - 4 d12^2 d15^2 d23^2 - 4 d12^2 d15^2 d23 d34 - 4 d12^2 d15 d23^2 d34 - 4 d12 d15^2 d23^2 d34 + 4 d12 d15 d23^2 d34^2 - 4 d12 d15^3 d23 d45 + 8 d12 d15^2 d23 d34 d45 -
   4 d12 d15 d23 d34^2 d45 + 8 d12 d15^3 d23 mt2 - 8 d12 d15^2 d23^2 mt2 - 4 d12^2 d15^2 d34 mt2 + 4 d12^2 d15 d23 d34 mt2 - 8 d12 d15^2 d23 d34 mt2 - 4 d12^2 d23^2 d34 mt2 - 8 d12 d15 d23^2 d34 mt2 -
   4 d15^2 d23^2 d34 mt2 - 4 d12 d15 d23 d34^2 mt2 + 8 d12 d23^2 d34^2 mt2 + 4 d15 d23^2 d34^2 mt2 - 4 d23^2 d34^3 mt2 - 4 d15^3 d23 d45 mt2 + 4 d12 d15^2 d34 d45 mt2 + 4 d12 d15 d23 d34 d45 mt2 +
   8 d15^2 d23 d34 d45 mt2 - 4 d12 d15 d34^2 d45 mt2 - 4 d12 d23 d34^2 d45 mt2 - 8 d15 d23 d34^2 d45 mt2 + 4 d23 d34^3 d45 mt2 + 4 d15^3 d23 mt2^2 - 4 d15^2 d23^2 mt2^2 - 8 d12 d15^2 d34 mt2^2 +
   8 d12 d15 d23 d34 mt2^2 - 4 d15^2 d23 d34 mt2^2 - 8 d12 d23^2 d34 mt2^2 - 4 d15 d23^2 d34 mt2^2 - 4 d15 d23 d34^2 mt2^2 + 8 d23^2 d34^2 mt2^2 + 4 d15^2 d34 d45 mt2^2 + 4 d15 d23 d34 d45 mt2^2 -
   4 d12 d34^2 d45 mt2^2 - 4 d15 d34^2 d45 mt2^2 - 4 d23 d34^2 d45 mt2^2 + 4 d34^3 d45 mt2^2 - 4 d15^2 d34 mt2^3 + 4 d15 d23 d34 mt2^3 - 4 d23^2 d34 mt2^3 - 4 d34^2 d45 mt2^3 + d12 d15^2 d23 sqrtDelta5 -
   d12 d15 d23 d34 sqrtDelta5 + d15^2 d23 mt2 sqrtDelta5 - d12 d23 d34 mt2 sqrtDelta5 - d15 d23 d34 mt2 sqrtDelta5 + d23 d34^2 mt2 sqrtDelta5 - d23 d34 mt2^2 sqrtDelta5 + 4 d12 d15 d23 sqrtR1 -
   4 d12 d23^2 sqrtR1 + 4 d23^2 d34 sqrtR1 + 4 d15 d23 d45 sqrtR1 - 4 d23 d34 d45 sqrtR1 + 4 d15 d23 mt2 sqrtR1 - 4 d23^2 mt2 sqrtR1 - 4 d34 d45 mt2 sqrtR1 - d23 sqrtDelta5 sqrtR1) sqrtR2)
```

Transcendental functions (ψ_1, $G_{1,1}$, $G_{2,1}$, $G_{3,1}^{\pm}$) expressed in terms of complete elliptic integrals