Amplitudes 2026, QMUL, June 29 2026

Rational Functions beyond Scattering

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University of Edinburgh

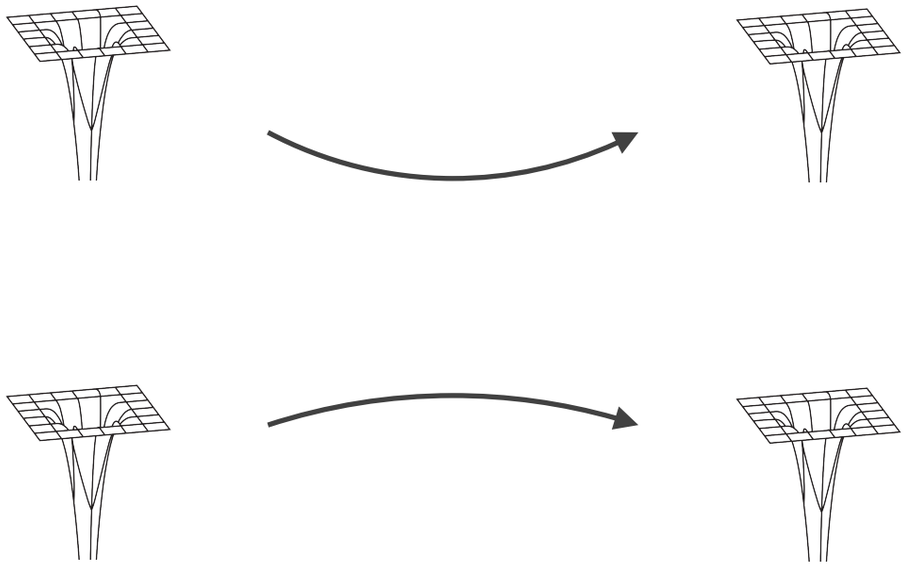
Funded by the European Union

European Research Council
Established by the European Commission

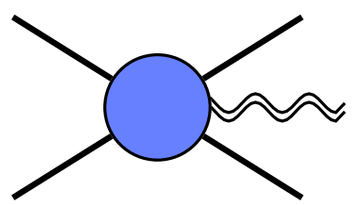
In progress with:
Majed Khalaf, David Kosower,
Ofri Telem, Shen Yan

Motivation

Amplitudes for gravitational waves:



$$\text{waveform} = \frac{1}{4\pi r}\int \mathrm{d}\omega\, e^{-i\omega u}$$



$$k \Big|_{k=\omega(1,\vec{n})} + \cdots$$

Cristofoli, Gonzo, Kosower, DOC

Motivation

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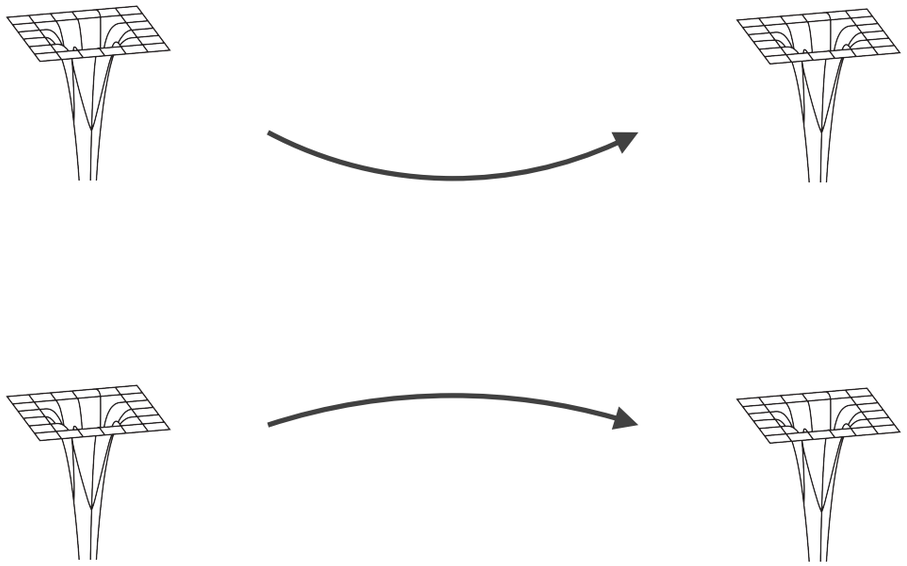

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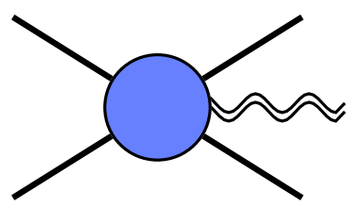


$$k\,\Big|_{k=\omega(1,\vec{n})} + \cdots$$

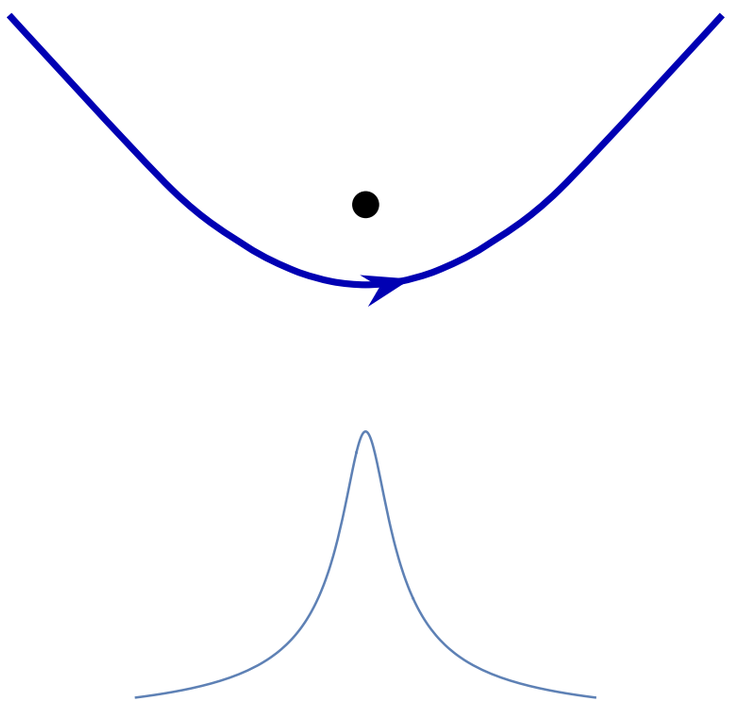
Cristofoli, Gonzo, Kosower, DOC

NLO known — NNLO?

Herderschee, Roiban, Teng; Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini; Georgoudis, Heissenberg, Vazquez-Holm; Elkhidir, Sergola, Vazquez-Holm, DOC

Motivation

Scattering



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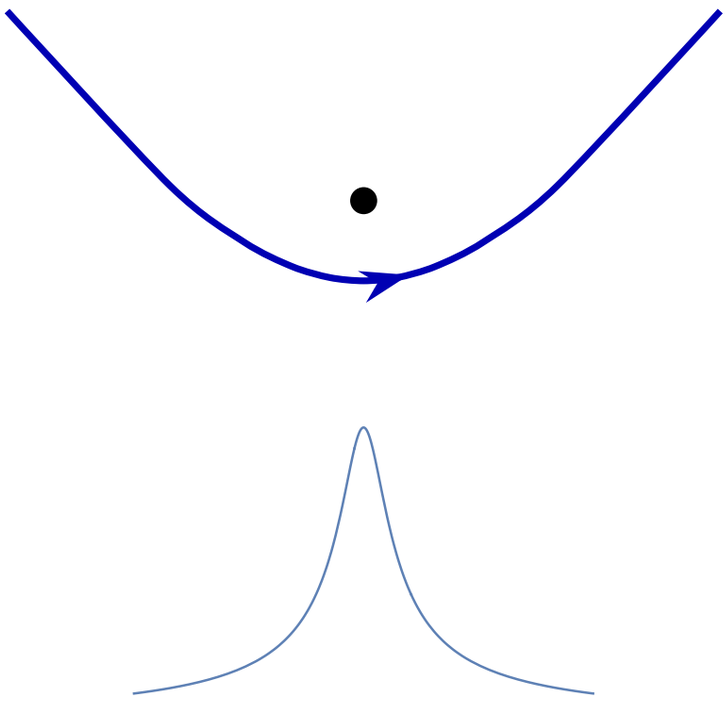
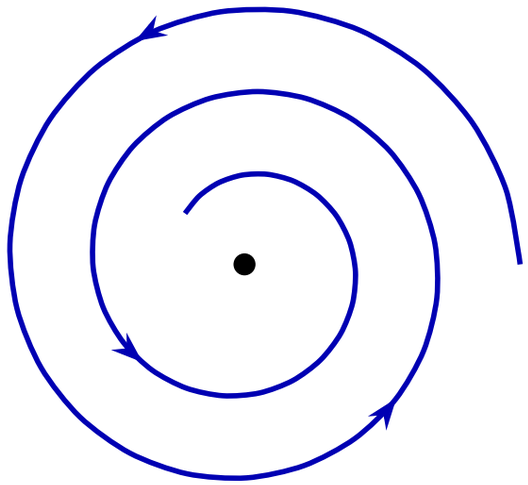
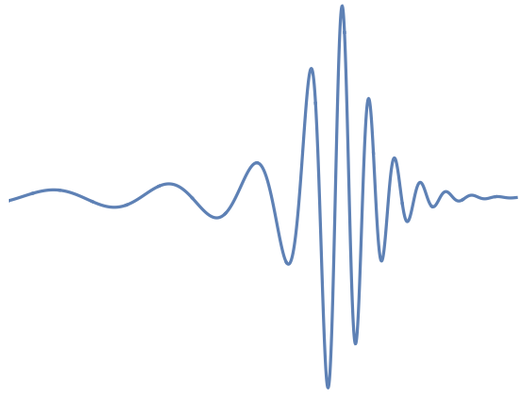


Bound





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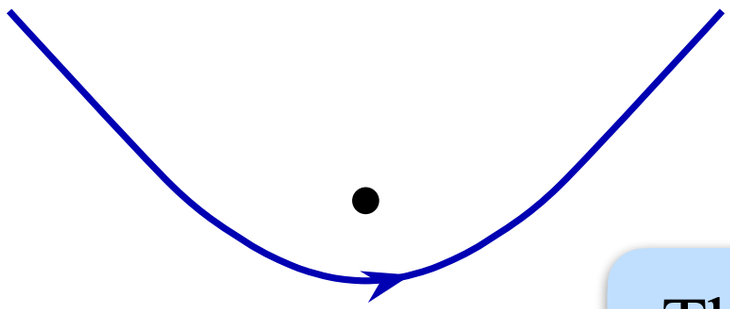

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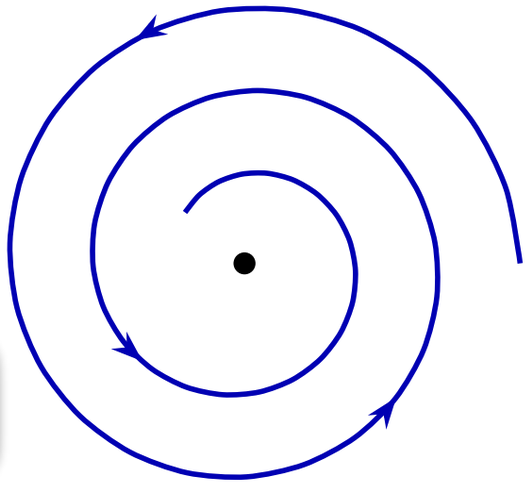
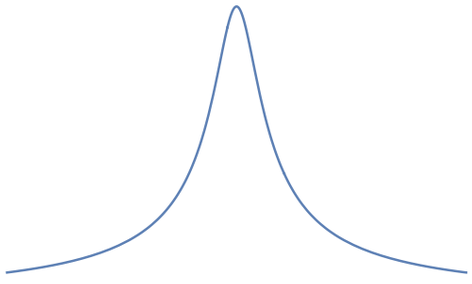
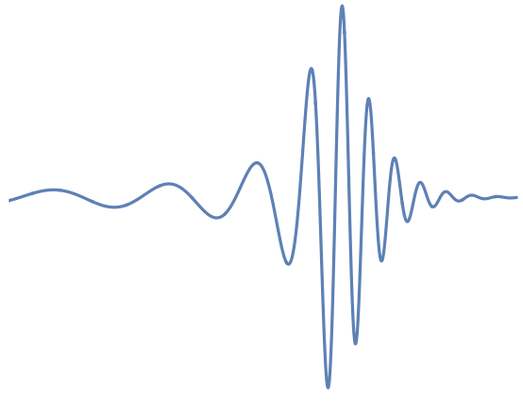


These are different!





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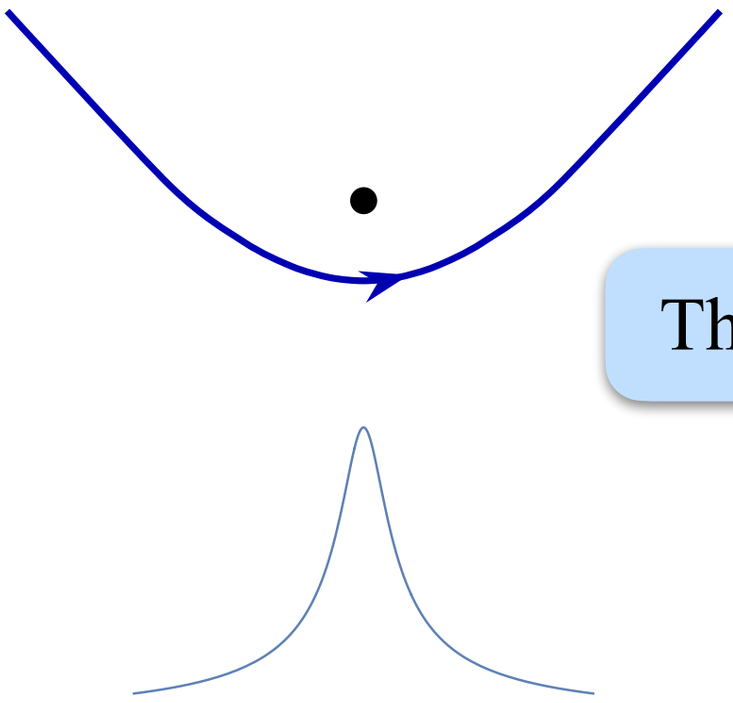

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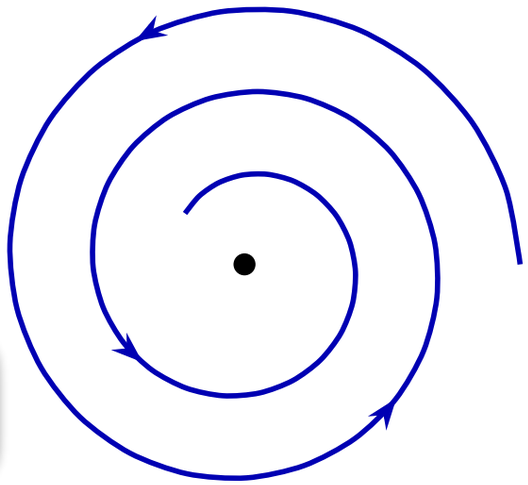

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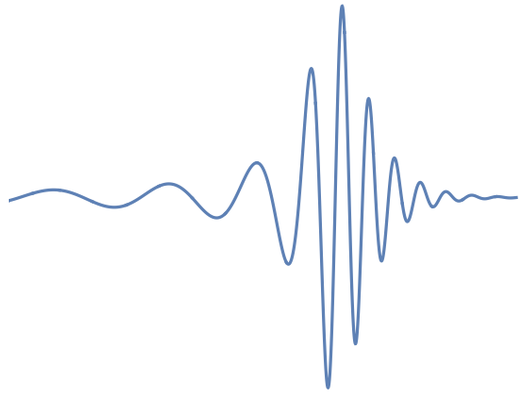
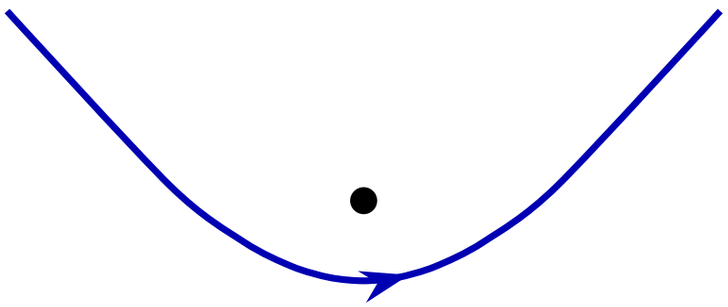
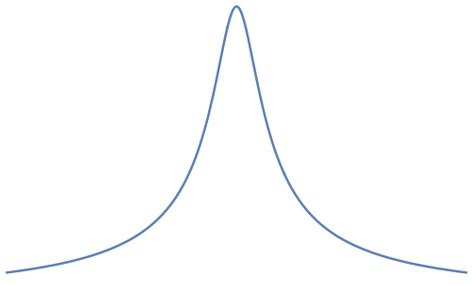


Technical obstruction: “tails”, integrals over history of system

This talk

Scattering





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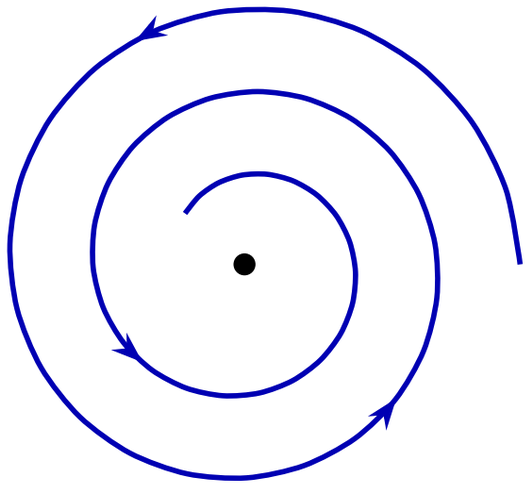

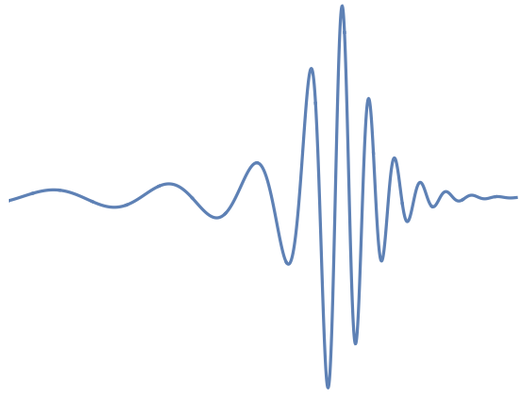

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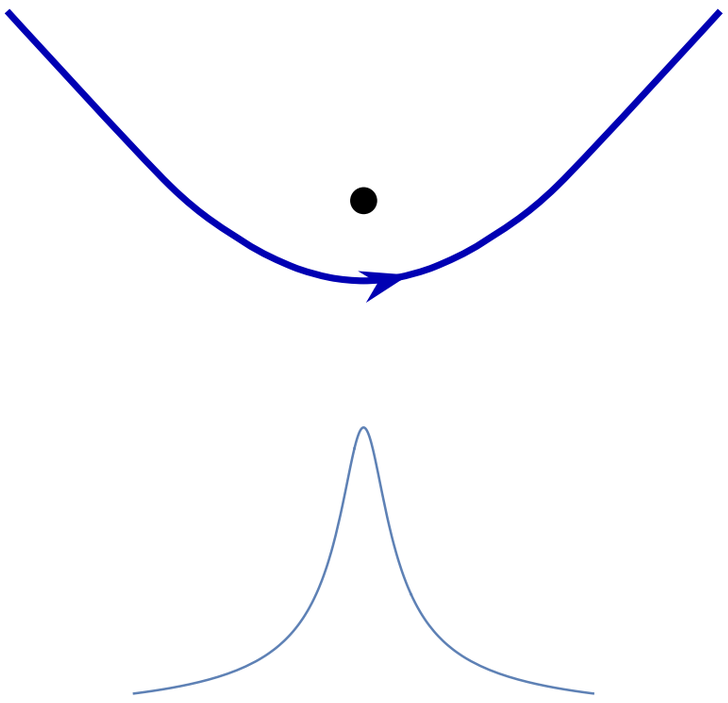

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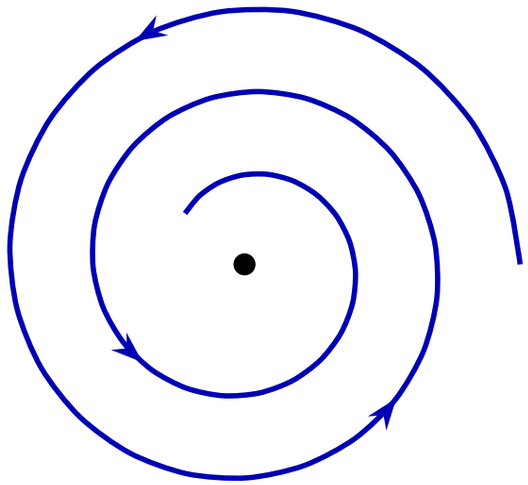

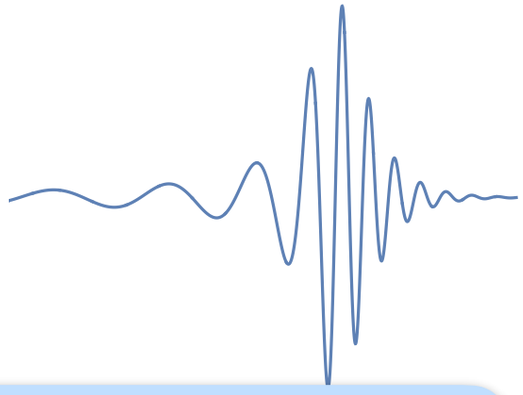

Bound observables still close cousins of amplitudes

Outline

1. Backgrounds in worldline EFT
2. Bound diagrams
3. Bound integrands as decorated rational functions

Worldline EFT

Backgrounds in worldline EFT

$$I = -\frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu} - \sum_i \int \mathrm{d}\tau \left(\tfrac{1}{2}m_i\dot{x}_i^2 + eQ_i\, A(x_i)\cdot\dot{x}_i\right)$$

Goldberger and Rothstein
Kälin, Porto; Jakobsen, Mogull, Plefka, Steinhoff

Worldline EFT

Backgrounds in worldline EFT

$$\dot{x}_i = \frac{\mathrm{d}x_i}{\mathrm{d}\tau}$$

EM proof of concept

Unit of charge e

$$I = \boxed{-\frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu}} - \sum_i \int \mathrm{d}\tau \left(\tfrac{1}{2}m_i\dot{x}_i^2 + eQ_i\, A(x_i)\cdot\dot{x}_i\right)$$

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Scattering expansion:

$$x_i^\mu = b_i^\mu + u_i^\mu\tau + z_i^\mu(\tau)$$

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Scattering expansion:

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Fluctuation

Goldberger and Rothstein

Kälin, Porto; Jakobsen, Mogull, Plefka, Steinhoff

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Vertices: order $e, e^2, \ldots$

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Worldline EFT

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Vertices: order $e, e^2, \ldots$

Repeat for more general background

Goldberger and Rothstein

Kälin, Porto; Jakobsen, Mogull, Plefka, Steinhoff

Worldline EFT

Backgrounds in worldline EFT

$$x_i^\mu = \boxed{b_i^\mu + u_i^\mu \tau} + z_i^\mu(\tau)$$

Fixed background

Worldline EFT

Backgrounds in worldline EFT

$$x_i^\mu = \boxed{r_i^\mu(\tau)} + z_i^\mu(\tau)$$

Fixed background

Worldline EFT

Backgrounds in worldline EFT

$$x_i^\mu = \boxed{r_i^\mu(\tau)} + z_i^\mu(\tau)$$

Fixed background

$$I_{\rm pp} = -\sum_i \int \mathrm{d}\tau \Big(\tfrac{1}{2} m_i \dot{x}_i^2 + e Q_i \, A(x_i) \cdot \dot{x}_i \Big)$$

$$= -\sum_i \int \mathrm{d}\tau \Big(\tfrac{1}{2} m_i \dot{z}_i^2 - z_{i\,\mu} \big(m_i \ddot{r}_i^\mu - e Q_i \, F^{\mu\nu}(r_i) \dot{r}_{i\,\nu} \big) + \cdots \Big)$$

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In general not order e!

Worldline EFT

Backgrounds in worldline EFT

$$x_i^\mu = \boxed{r_i^\mu(\tau)} + z_i^\mu(\tau)$$

Fixed background

$$I_{\rm pp} = -\sum_i \int ($$

Require background "close" to solution

$$= -\sum_i \int \mathrm{d}\tau \Big(\tfrac{1}{2} m_i \dot{z}_i^2 - z_{i\,\mu} \big(m_i \ddot{r}_i^\mu - e Q_i \, F^{\mu\nu}(r_i) \dot{r}_{i\,\nu} \big) + \cdots \Big)$$

In general not order e!

Worldline EFT

We choose covariant "Schild solution"

$$\bar{A}^{\mu}(x) = \int \mathrm{d}^4 y \; \tfrac{1}{2}\big(G_{\rm ret}(x-y) + G_{\rm adv}(x-y)\big) \; J^{\mu}_{\rm pp}(y)$$

$$m_i \ddot{r}_i^{\mu} = e Q_i \bar{F}^{\mu\nu}(r_i)\dot{r}_{i\,\nu}$$

Worldline EFT

We choose covariant "Schild solution"

Schild field

Time-symmetric Green's function

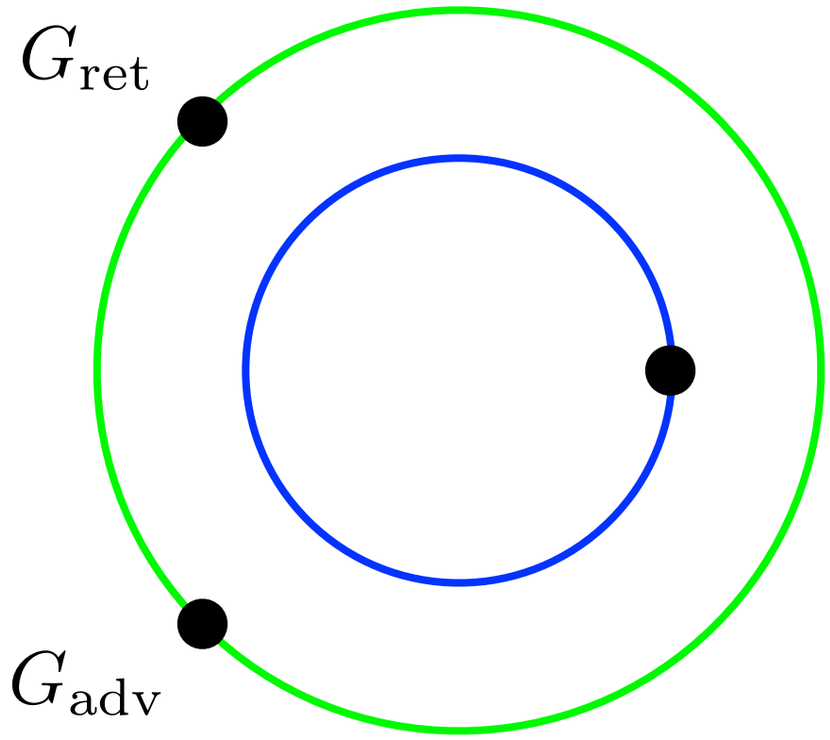
$$\bar{A}^\mu(x) = \int \mathrm{d}^4 y \, \frac{1}{2}\left(G_{\rm ret}(x-y) + G_{\rm adv}(x-y)\right) J^\mu_{\rm pp}(y)$$

$$m_i \ddot{r}_i^\mu = e Q_i \bar{F}^{\mu\nu}(r_i)\dot{r}_{i\,\nu}$$

$$\bar{F}^{\mu\nu} = \partial^\mu \bar{A}^\nu - \partial^\nu \bar{A}^\mu$$

Background Choices

We choose covariant "Schild solution"



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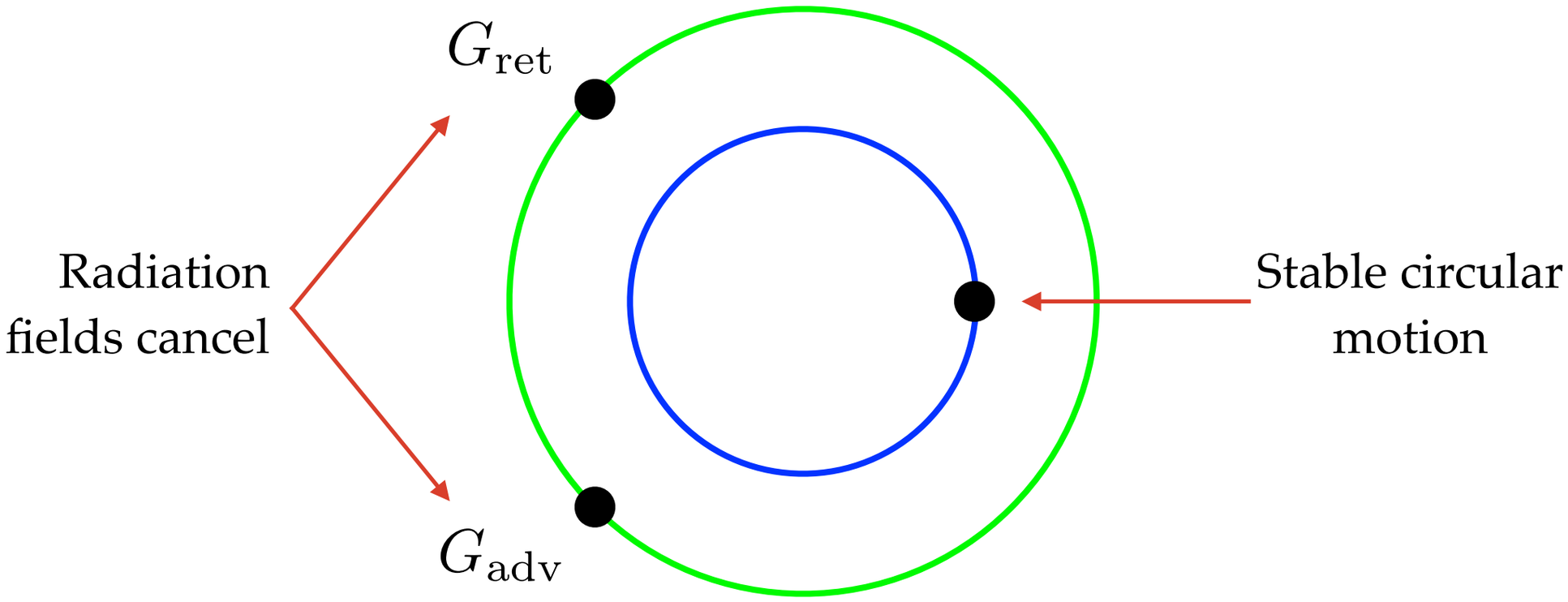

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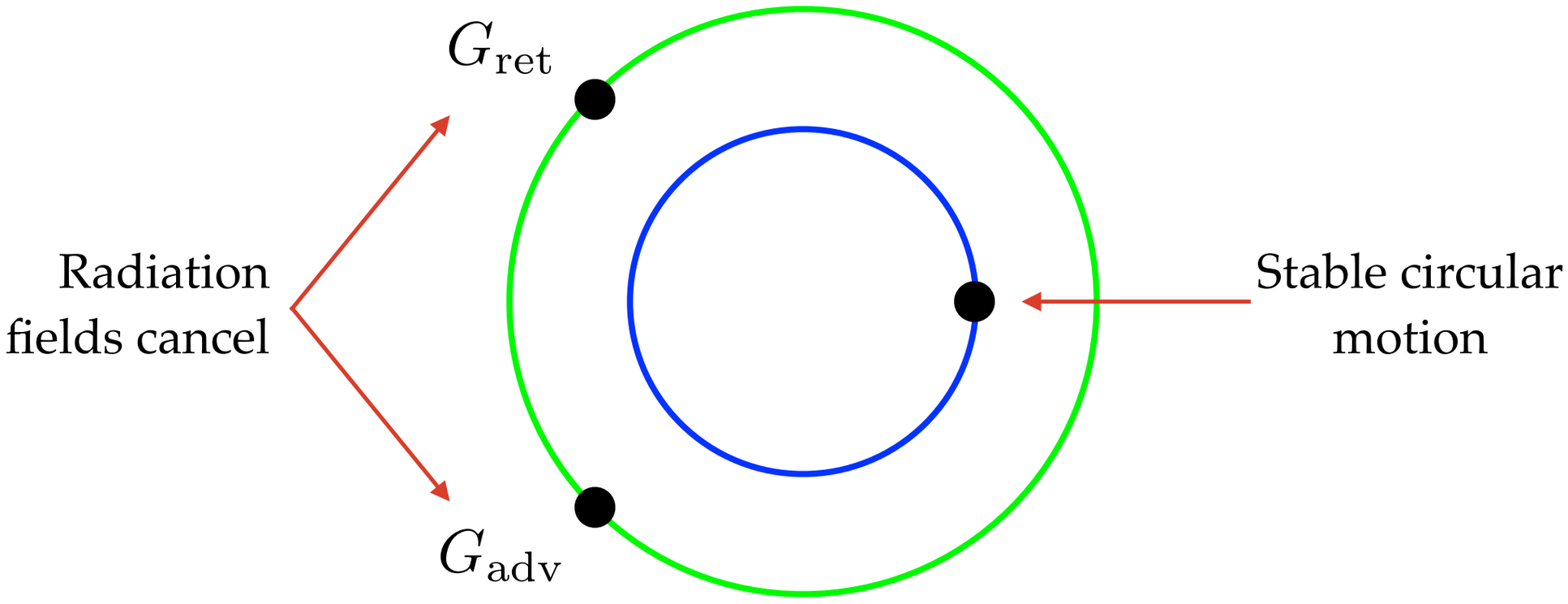

Zeroth order post-Minkowski binary in circular motion

Background Choices

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$$I_{\rm pp} = -\sum_i \int \mathrm{d}\tau \Big(\tfrac{1}{2} m_i \dot{z}_i^2 - z_{i\,\mu} \big(m_i \ddot{r}_i^\mu - e Q_i \, F^{\mu\nu}(r_i) \dot{r}_{i\,\nu} \big) + \cdots \Big)$$

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Radiative combination

Background Choices

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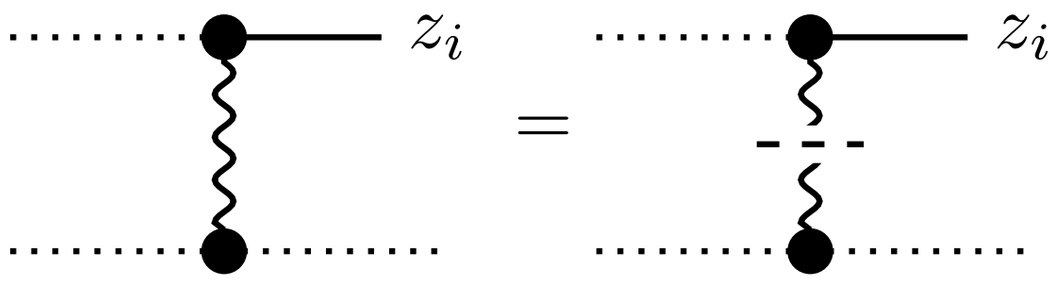
Radiative combination



Deflection sourced by radiation

Bound diagrams

Diagrams

Feynman rules: expand action about background

$$I_{\rm pp} = -\sum_i \int \mathrm{d}\tau \Big(\tfrac{1}{2} m_i \dot{z}_i^2 + eQ_i A(r_i) \cdot \dot{r}_i + eQ_i z_{i\,\mu}\, F^{\mu\nu}_{\rm cut}\, \dot{r}_{i\,\nu} + \mathcal{O}(z^2) \Big)$$

Vertices depend on r_i, $\dot{r}_i$

Diagrams

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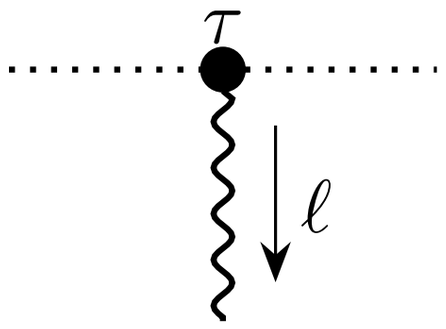

$$= \int \mathrm{d}\tau \, \left(-i\, eQ_i\, e^{i\ell\cdot r_i}\, \dot{r}_i^{\mu} \right)$$

Diagrams

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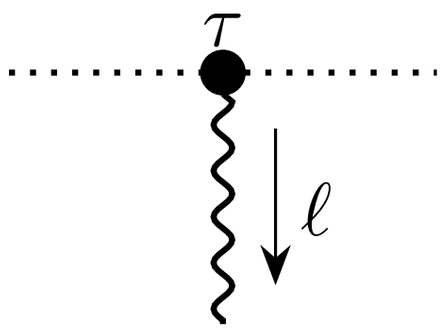


$$= \int \mathrm{d}\tau \left(-i\, eQ_i\, e^{i\ell\cdot r_i}\, \dot{r}_i^{\mu}\right) \quad \to \hat{\delta}(\ell\cdot u)\left(-i\, eQ_i\; e^{i\ell\cdot b_i}\, u_i^{\mu}\right)$$

$$r_i = b_i + u_i\tau$$

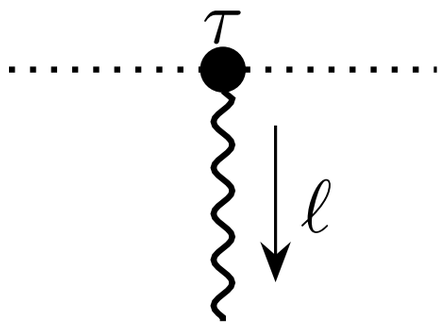
Diagrams

Feynman rules: expand action about background

$$I_{\rm pp} = -\sum_i \int {\rm d}\tau \Big(\tfrac{1}{2} m_i \dot{z}_i^2 + eQ_i A(r_i)\cdot \dot{r}_i + eQ_i z_{i\,\mu}\, F^{\mu\nu}_{\rm cut}\, \dot{r}_{i\,\nu} + \mathcal{O}(z^2) \Big)$$

Vertices depend on r_i, $\dot{r}_i$

Form of rules same for bound and unbound



$$= \int {\rm d}\tau \left(-i\, eQ_i\, e^{i\ell\cdot r_i}\, \dot{r}_i^{\mu} \right) \quad \to \hat{\delta}(\ell\cdot u) \left(-i\, eQ_i\, e^{i\ell\cdot b_i}\, u_i^{\mu} \right)$$

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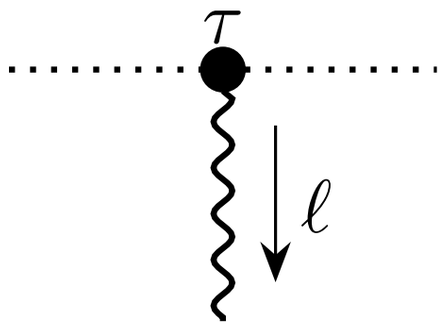

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"On-shell condition"

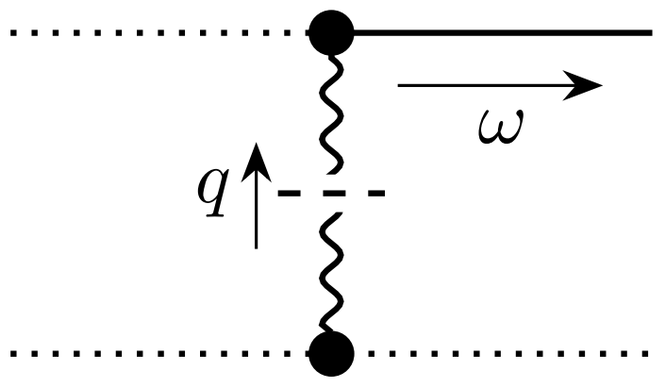
Total time derivatives: zero

Diagrams

Same rules, same diagrams

Gonzo and Mogull

$z^{\mu}_{(1)} =$



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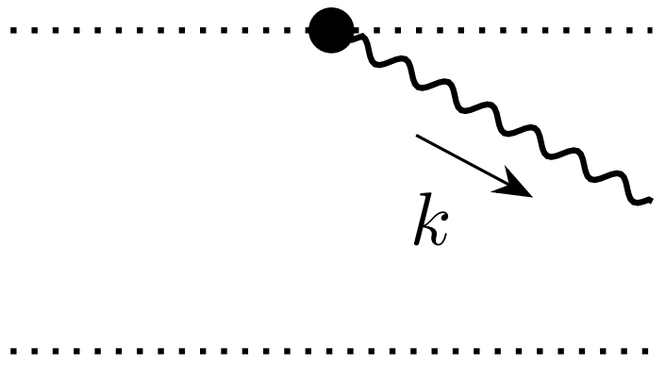

Diagrams

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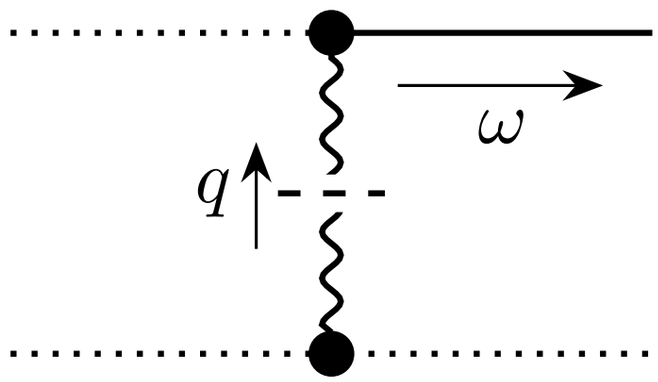

Scattering: zero after integration

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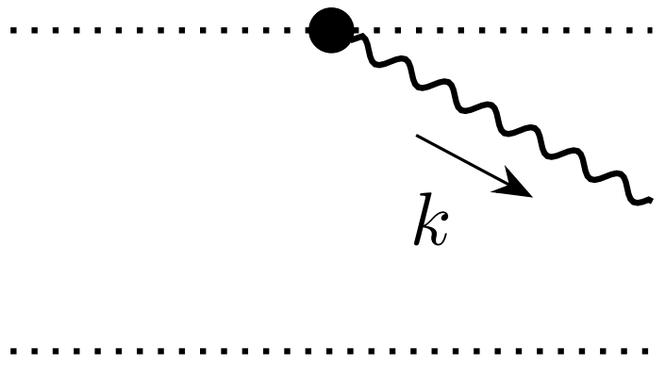

Diagrams

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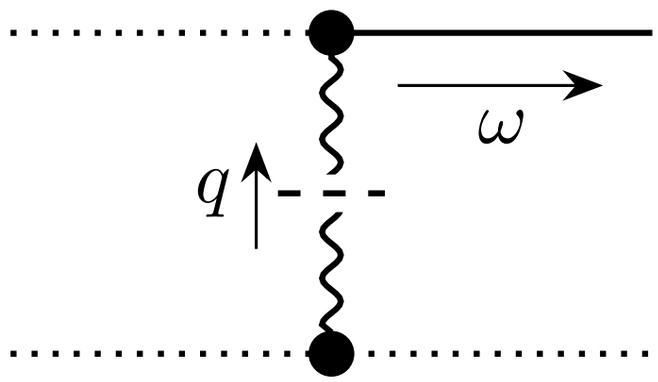

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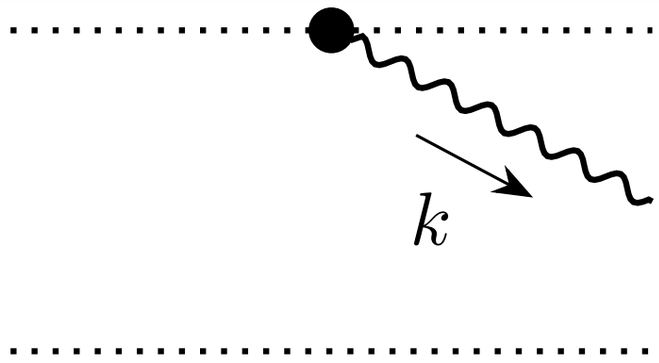

Bound: non-zero

Diagrams

Same rules, same diagrams

Gonzo and Mogull

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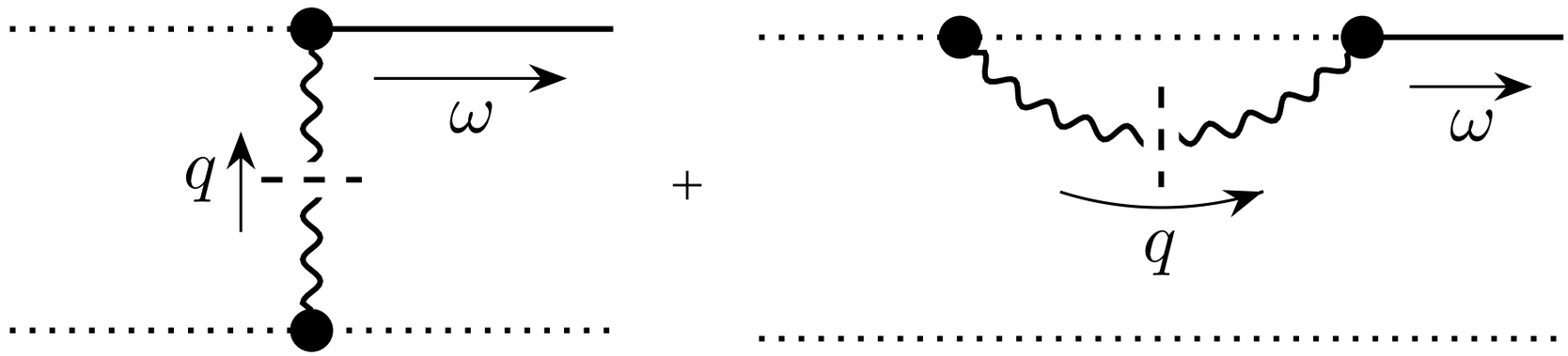

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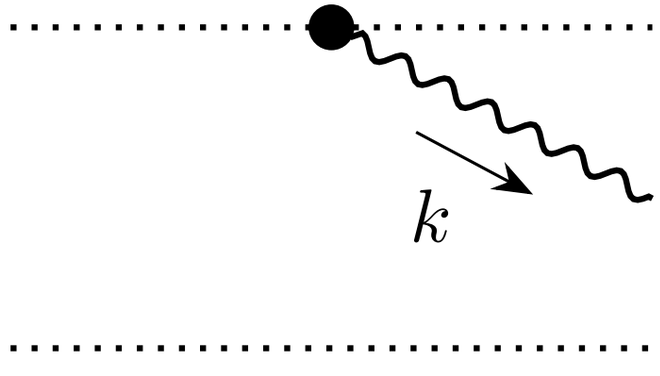
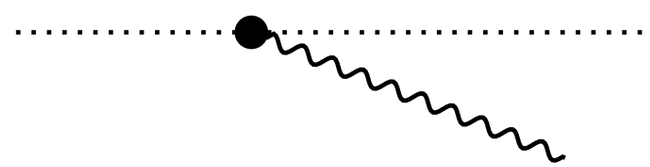


Diagrams

Small parameter: dimensional analysis

$$A^{\mu}_{(1)} =$$



Diagrams

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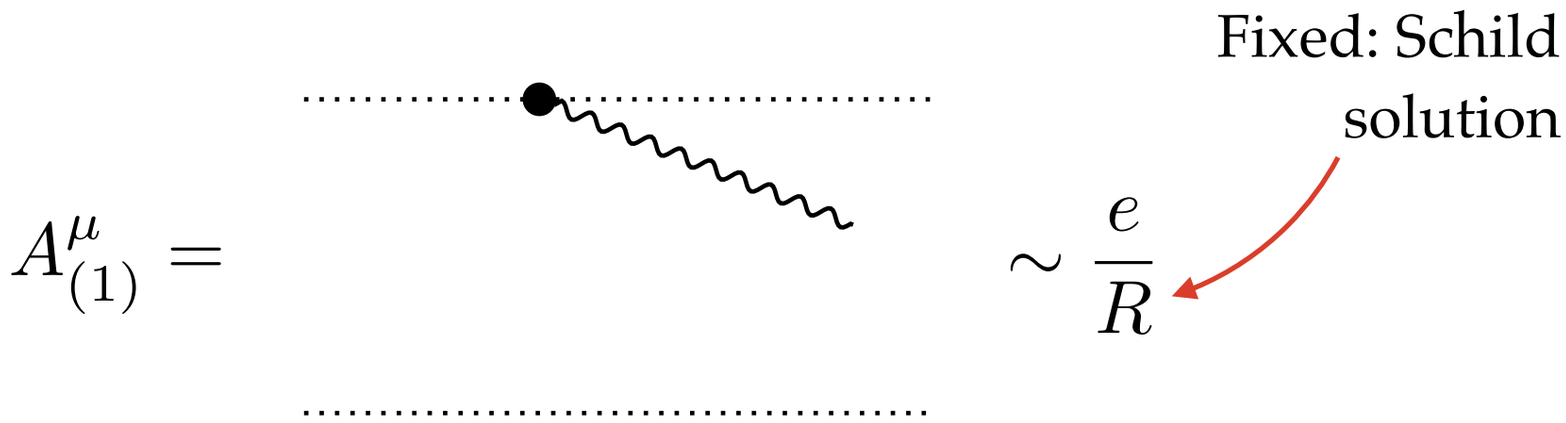
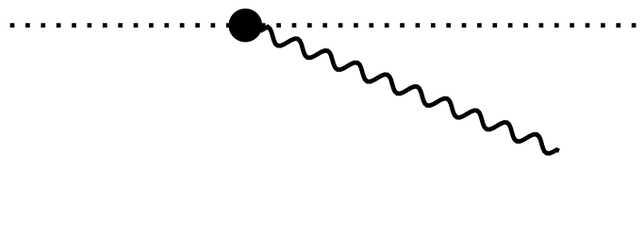


Diagrams

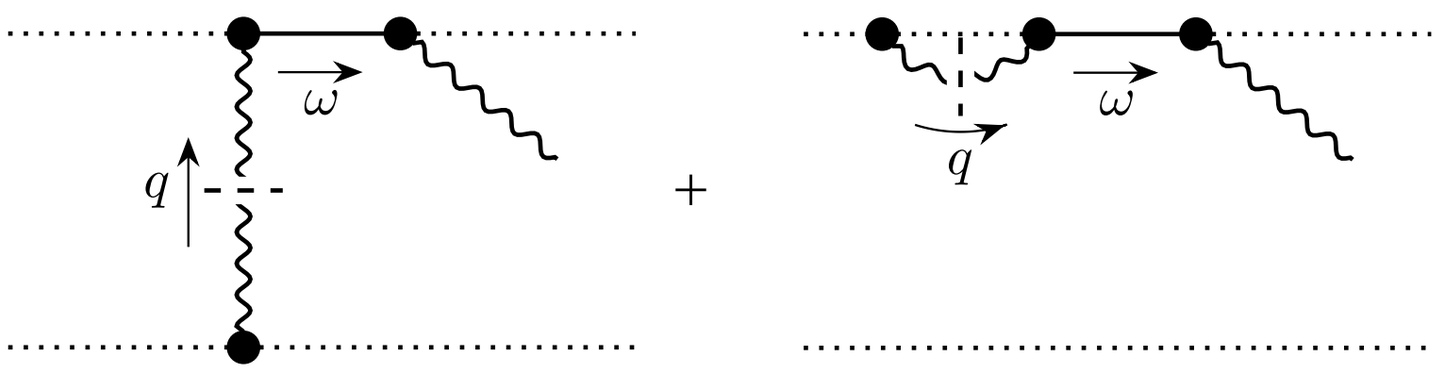
Small parameter: dimensional analysis

$$A^{\mu}_{(1)} = \quad \sim \frac{e}{R}$$



Fixed: Schild solution

$$A^{\mu}_{(3)} = \quad + \quad \sim \frac{e^3}{mR^2}$$



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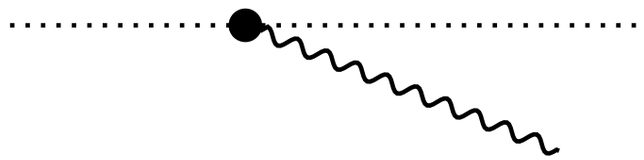

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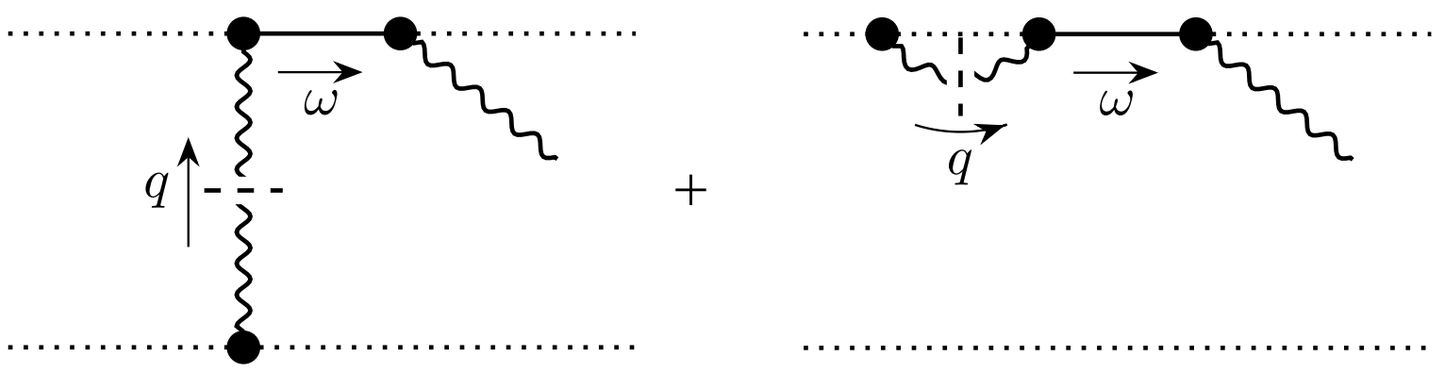

Ratio $\dfrac{e^2}{mR} \ll 1$ cf unbound $\dfrac{e^2}{mb} \ll 1$

Diagrams

Small parameter: dimensional analysis

$$\frac{e^2}{mR} \ll 1$$

Diagrams

Small parameter: dimensional analysis

$$\frac{e^2}{mR} \ll 1$$

Bound case: virial theorem

$$\frac{e^2}{mR} \sim v^2 \ll 1$$

Diagrams

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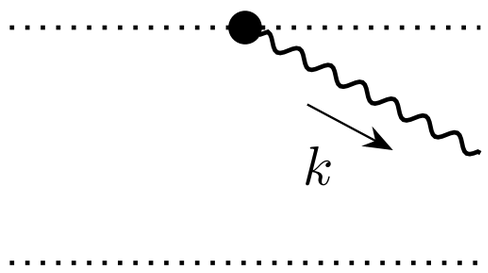
Post-Newtonian — relativistic formulation to access PN corrections

Integrands as Rational Functions

Integrands

Leading order radiation

$$k^2\,\epsilon\cdot\tilde{A}_{(1)}(k) =$$



$$= \sum_i \frac{1}{m_i}\int \mathrm{d}\tau\, e^{-ik\cdot r_i}\, eQ_i\, p_i\cdot\epsilon$$

$$p_i = m_i\dot{r}_i$$

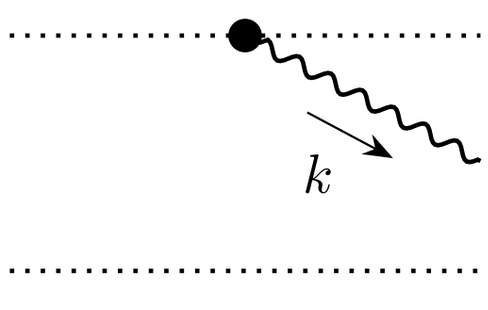
Integrands

Leading order radiation

Gauge invariant

$$k \cdot p_i \to \frac{\mathrm{d}}{\mathrm{d}\tau} k \cdot r_i$$

$$k^2\, \epsilon \cdot \tilde{A}_{(1)}(k) =$$



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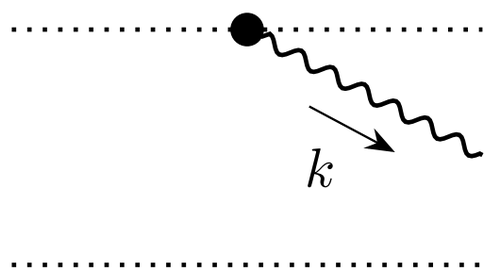

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Integrands

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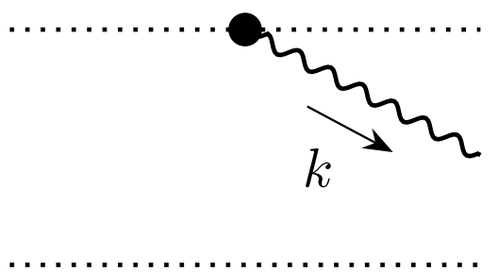

Perform integrals:

$$A^\mu_{(1)}(x) = \sum_i \frac{eQ_i}{4\pi\,R}\;\frac{\dot{r}_i^\mu}{n\cdot\dot{r}_i}\bigg|_{\mathrm{ret}}$$

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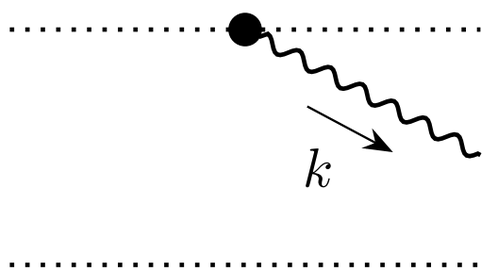

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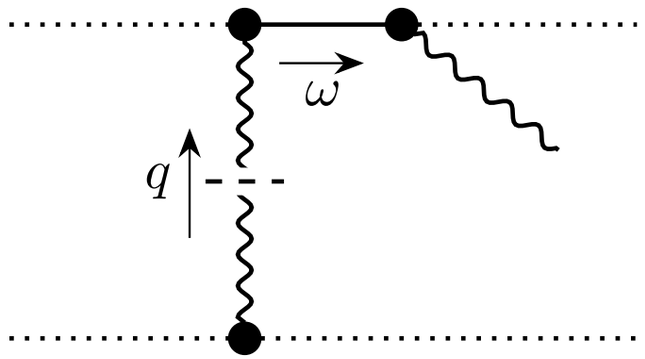
Gravity: double copy

Integrands

NLO

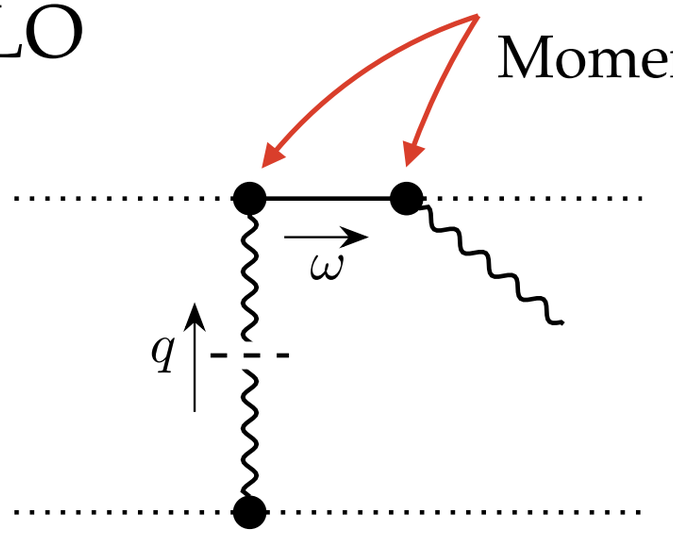


$$= \int e^3 Q_1^2 Q_2 \, \delta(q^2) \left(p_2 \cdot \epsilon^* + \frac{1}{m_1^2 \omega^2} \left(p_1^a \cdot p_2 \, p_1^b \cdot \epsilon^* \, q \cdot k + \dots \right) \right)$$

Integrands

NLO

Each vertex: τ integral
Momenta depend on τ



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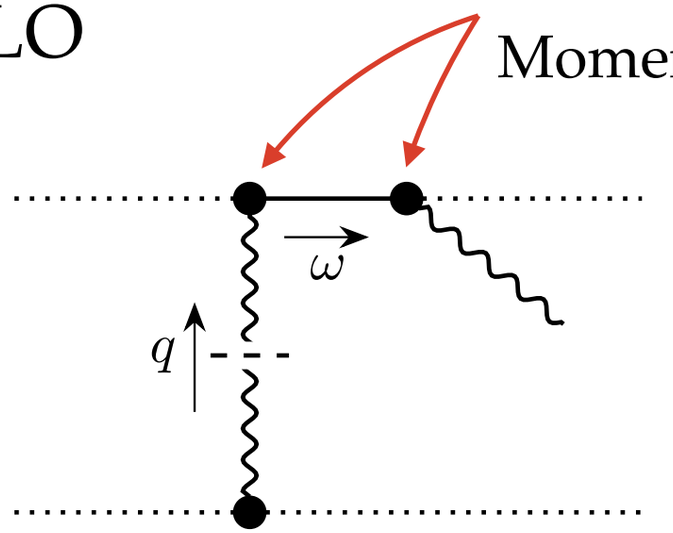

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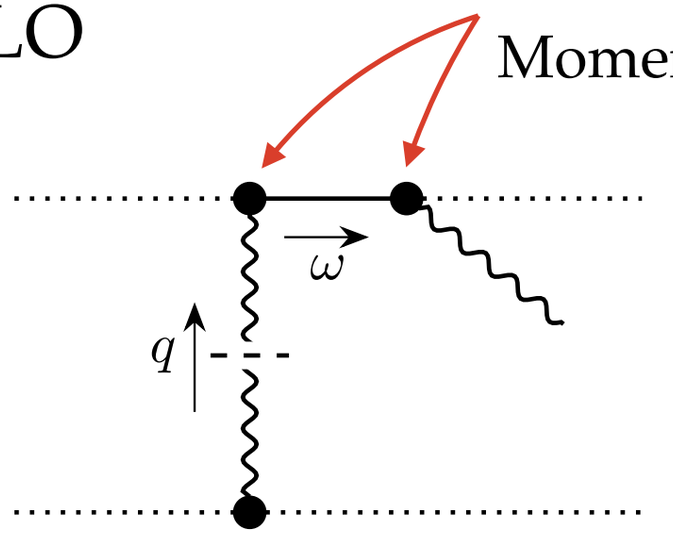

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Rational integrand familiar from scattering but now “decorated”

Integrands

NLO

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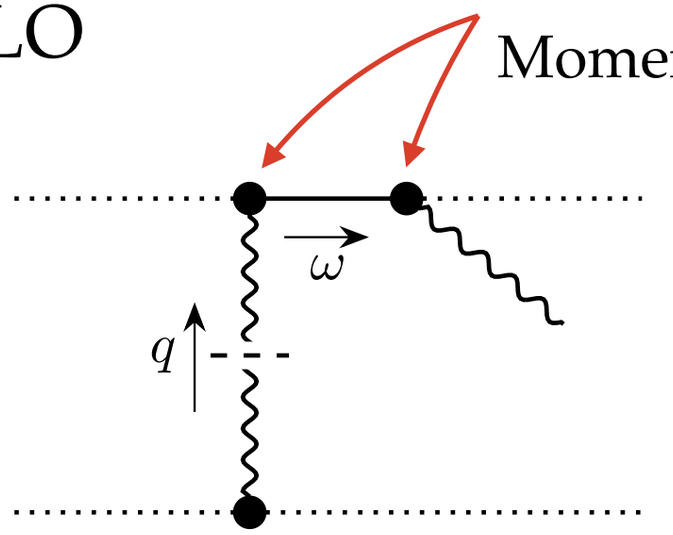

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Rational integrand familiar from scattering but now “decorated”

BCFW-like construction of decorated integrands?

Conclusions

- Perturbative expansion for circular binaries in weak field
  - Observables: integrals of rational functions, decorated cousins of scattering amplitudes
  - BCFW for decorated integrands?

- Schild solution appropriate zeroth order background
  - Secular drift away from Schild, slow in weak field