



THE UNIVERSITY
of EDINBURGH



The Magnus Expansion in QFT

Based on [2512.05017](#), with
Andreas Brandhuber, Paolo Pichini,
Gabriele Travaglini and Pablo Vives Matasan
+ Sebastien MacDonald (to appear)

Graham R. Brown
29.05.2026-Amplitudes 2026-QMUL

Why we are here: The S-Matrix

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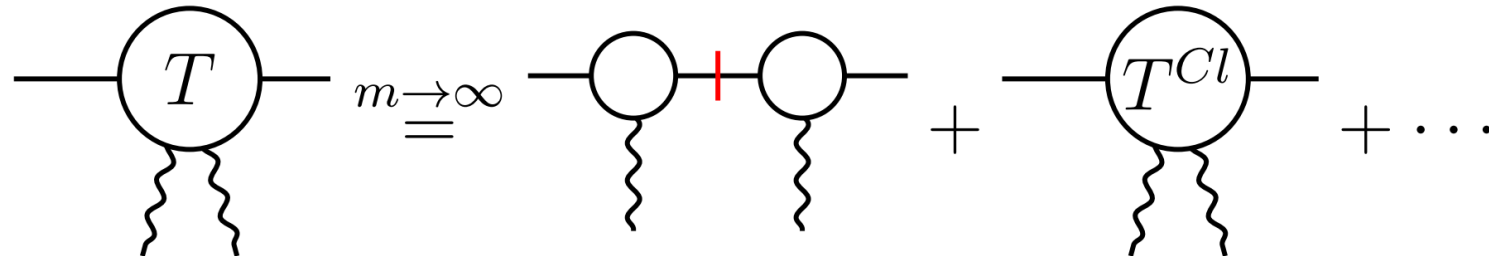
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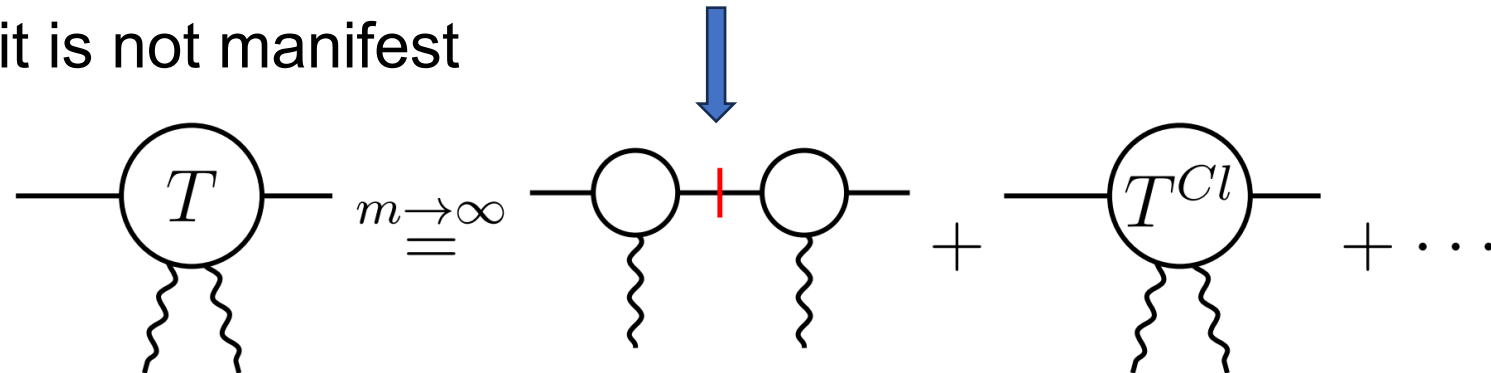
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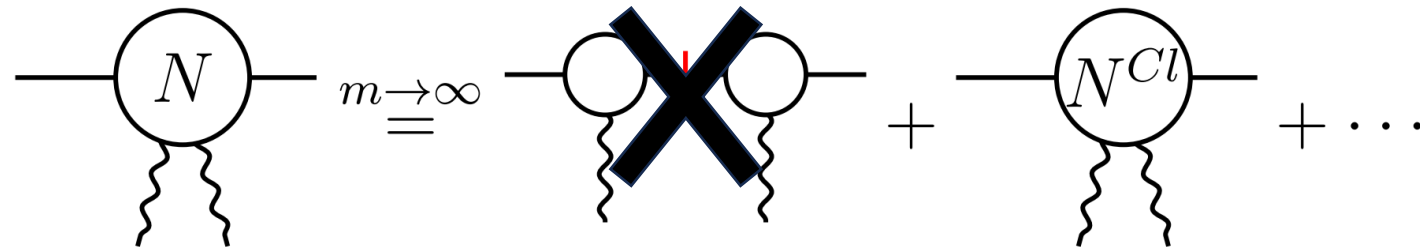
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The classical limit of N can be used directly



How to compute N ?

Just take the log [Damgaard, Plante, Vanhove '21 +Hansen '23] ... :

$$iN = \log(1 + iT) = iT - \frac{(iT)^2}{2} + \frac{(iT)^3}{3} - \dots ,$$

Is there a better/more fundamental way?

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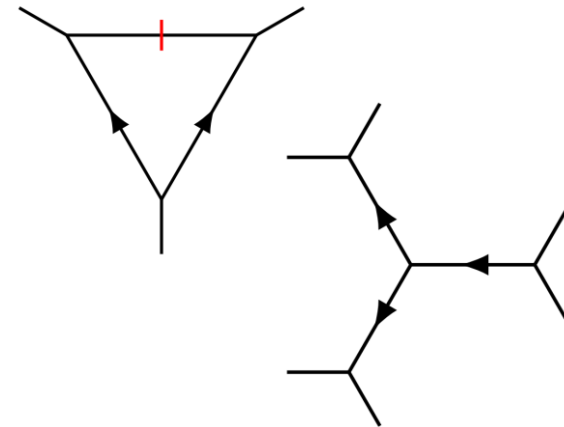
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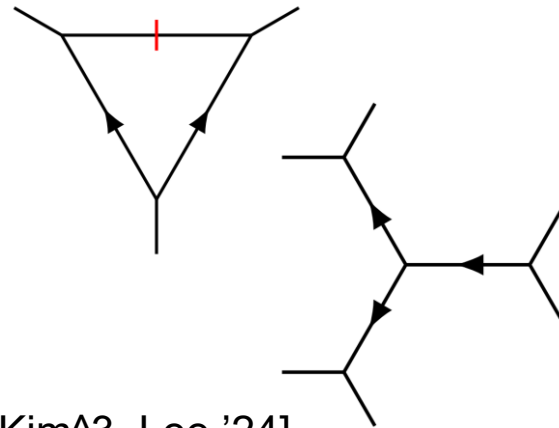
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Building on previous work at tree-level [Kim³, Lee '24]

Wick contractions review

$$S = \mathcal{T} \exp \left(-i \int d^4x \mathcal{H}_I(x) \right) \quad \text{Simple model: } \mathcal{H}_I = \frac{:\phi^3:}{3!}$$

Wick contractions review

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$$\begin{aligned} \phi_i \phi_j &= :\phi_i \phi_j: + \overline{\phi_i \phi_j} \\ &= :\phi_i \phi_j: + i \Delta_{ij}^{(+)} \end{aligned}$$

$$i \Delta_{ij}^{(+)} \sim (2\pi) \delta^{(+)}(p^2 - m^2)$$

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$$\mathcal{T}(\phi_i \phi_j) = \theta_{ij} \phi_i \phi_j + \theta_{ji} \phi_j \phi_i ,$$



$$\mathcal{T}(\phi_i \phi_j) = :\phi_i \phi_j: + i\Delta_{ij}^F$$

$$i\Delta_{ij}^F \sim \frac{i}{p^2 - m^2 + i\epsilon}$$

$$iT^{(2)} = \frac{(-i)^2}{2!} \int d^4x_1 d^4x_2 \mathcal{T} \left(\frac{:\phi_1^3:}{3!} \frac{:\phi_2^3:}{3!} \right)$$

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$$\mathcal{T} \left(\frac{:\phi_1^3:}{3!} \frac{:\phi_2^3:}{3!} \right) = \quad :\frac{\phi_1^3}{3!} \frac{\phi_2^3}{3!}: \quad + i \Delta_{12}^F :\frac{\phi_1^2}{2!} \frac{\phi_2^2}{2!}: + \frac{1}{2!} (i \Delta_{12}^F)^2 :\phi_1 \phi_2: \quad + \frac{1}{3!} (i \Delta_{12}^F)^3$$

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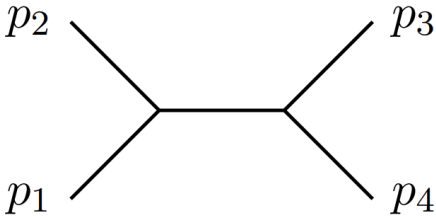
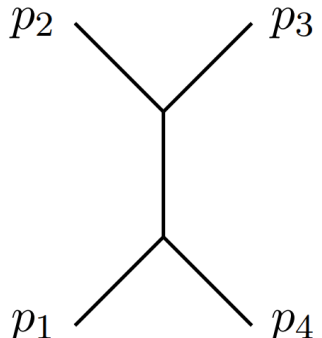
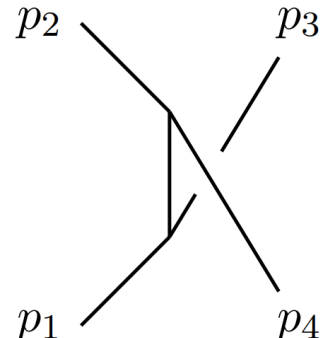
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$$= \begin{array}{c} x_1 \\ \bigcirc \end{array} \begin{array}{c} x_2 \\ \bigcirc \end{array} + \begin{array}{c} \bigcirc \text{---} \bigcirc \end{array} + \frac{1}{2!} \begin{array}{c} \bigcirc \text{---} \bigcirc \end{array} + \frac{1}{3!} \begin{array}{c} \bigcirc \text{---} \bigcirc \end{array}$$



$$\langle p_3 p_4 | iT^{(2)} | p_1 p_2 \rangle_{Tree} =$$


+

+


Magnus Wick Contractions

What is the equivalent of the Dyson expansion for N ?

$$iN^{(1)} = -i \int d^4x_1 \mathcal{H}_1 ,$$

$$iN^{(2)} = \frac{(-i)^2}{2} \int d^4x_1 d^4x_2 \theta_{12} [\mathcal{H}_1, \mathcal{H}_2] ,$$

$$iN^{(3)} = \frac{(-i)^3}{6} \int d^4x_1 d^4x_2 d^4x_3 \theta_{12} \theta_{23} ([\mathcal{H}_1, [\mathcal{H}_2, \mathcal{H}_3]] + [\mathcal{H}_3, [\mathcal{H}_2, \mathcal{H}_1]])$$

$$iN^{(4)} = \frac{(-i)^4}{12} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \theta_{12} \theta_{23} \theta_{34} ([\mathcal{H}_1, [\mathcal{H}_2, [\mathcal{H}_3, \mathcal{H}_4]]] \\ + [\mathcal{H}_4, [\mathcal{H}_3, [\mathcal{H}_1, \mathcal{H}_2]]] + [\mathcal{H}_2, [\mathcal{H}_3, [\mathcal{H}_4, \mathcal{H}_1]]] + [\mathcal{H}_1, [\mathcal{H}_4, [\mathcal{H}_3, \mathcal{H}_2]]])$$

$\vdots$

General formula see [Chen, Strichartz '87] ...

Commutator Wick Contractions

$$iN^{(n)} \sim \int d^4x_1 \cdots d^4x_n \underbrace{\theta_{12}\theta_{34} \cdots \theta_{n-1,n}}_{\text{Deal with later}} \underbrace{\left[\mathcal{H}_1, \left[\mathcal{H}_2, \cdots \left[\mathcal{H}_{n-1}, \mathcal{H}_n \right] \cdots \right] \right]}_{\text{Contract this}}$$

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$$\begin{aligned} \overline{[\mathcal{H}_i, \mathcal{H}_j]} &= \overline{\mathcal{H}_i \mathcal{H}_j} - \overline{\mathcal{H}_j \mathcal{H}_i} \\ &= i\Delta_{ij}^{(+)} \mathcal{H}'_i \mathcal{H}'_j - i\Delta_{ji}^{(+)} \mathcal{H}'_j \mathcal{H}'_i \end{aligned}$$

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$$\begin{aligned} i\Delta_{ij} &= i\Delta_{ij}^{(+)} - i\Delta_{ji}^{(+)} \\ \Delta_{ij}^{(1)} &= i\Delta_{ij}^{(+)} + i\Delta_{ji}^{(+)} \end{aligned}$$

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$$:[A, B] := 0, \quad :\{A, B\} := 2 :AB:$$

Dyson iT

$$i\Delta_{ij}^{(+)}$$



$\mathcal{T}(\dots)$

$$i\Delta_{ij}^F = \overset{j}{\circ} \text{---} \overset{i}{\circ} \sim \frac{i}{p^2 - m^2 + i\epsilon}$$

Magnus iN

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Magnus iN

$$i\Delta_{ij}, \quad i\Delta_{ij}^{(1)}$$

Dyson iT

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θ_{ij}

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$$i\Delta_{ij}^A = -i\Delta_{ij}\theta_{ji} = \begin{array}{c} j \quad i \\ \circ \leftarrow \circ \end{array} \sim \frac{i}{p^2 - m^2 - i\epsilon p^0}$$

$$\Delta_{ij}^{(1)} = \begin{array}{c} j \quad i \\ \circ \text{---} \textcolor{red}{|} \text{---} \circ \end{array} \sim 2\pi\delta(p^2 - m^2)$$

4-points tree-level

$$iN^{(2)} = \frac{(-i)^2}{2} \int d^4x_1 d^4x_2 \theta_{12} [\overline{\mathcal{H}_1}, \mathcal{H}_2]$$

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 iN^{(2)} &= \frac{(-i)^2}{2} \int d^4x_1 d^4x_2 \theta_{12} [\overline{\mathcal{H}_1}, \mathcal{H}_2] \\
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 \end{aligned}$$

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$$= \frac{(-i)^2}{2} \int d^4x_1 d^4x_2 i \Delta_{12}^R : \frac{\phi_1^2}{2!} \frac{\phi_2^2}{2!} : = \frac{1}{2} \begin{array}{c} x_2 \qquad x_1 \\ \circ \longrightarrow \circ \end{array}$$

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$$\langle p_3 p_4 | iN^{(2)} | p_1 p_2 \rangle_{Tree} = \frac{1}{2} \begin{array}{c} p_2 \quad p_3 \\ \diagdown \quad \diagup \\ \text{---} \longrightarrow \text{---} \\ \diagup \quad \diagdown \\ p_1 \quad p_4 \end{array} + \frac{1}{2} \begin{array}{c} p_2 \quad p_3 \\ \diagdown \quad \diagup \\ \text{---} \longleftarrow \text{---} \\ \diagup \quad \diagdown \\ p_1 \quad p_4 \end{array} + (t/u\text{-channel})$$

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 &= \frac{(-i)^2}{2} \int d^4x_1 d^4x_2 i \Delta_{12}^R : \frac{\phi_1^2}{2!} \frac{\phi_2^2}{2!} : = \frac{1}{2} \overset{x_2}{\circ} \rightarrow \overset{x_1}{\circ}
 \end{aligned}$$

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Murua coefficient: $\omega(\circ \rightarrow \circ) = \frac{1}{2}$

Matches
[Murua'06]
[Kim³, Lee '24]

5-points tree-level

$$iN^{(3)} = \frac{(-i)^3}{6} \int_{x_1, x_2, x_3} \theta_{12} \theta_{23} \theta_{13} \left([\mathcal{H}_1, \overline{[\mathcal{H}_2, \mathcal{H}_3]}] + \cdots \right)$$

5-points tree-level

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 &= \frac{(-i)^3}{6} \int_{x_1, x_2, x_3} (i\Delta_{23}^R) \frac{1}{(2!)^2} \left[(i\Delta_{12}^R) : \phi_1^2 \phi_2 \phi_3^2 : + \theta_{12} (i\Delta_{13}^R) : \phi_1^2 \phi_2^2 \phi_3 : \right] + \dots
 \end{aligned}$$

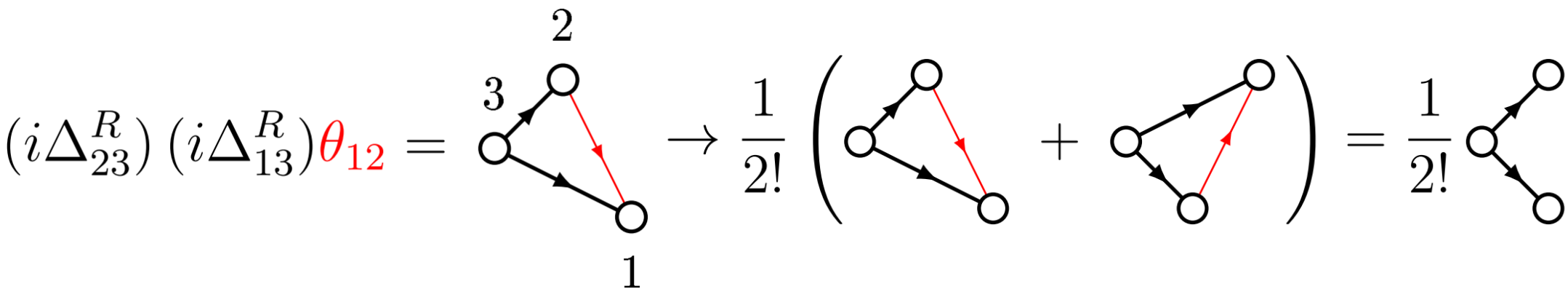
5-points tree-level

$$\begin{aligned}
 iN^{(3)} &= \frac{(-i)^3}{6} \int_{x_1, x_2, x_3} \theta_{12} \theta_{23} \theta_{13} \left([\mathcal{H}_1, \overbrace{[\mathcal{H}_2, \mathcal{H}_3]}^{\text{green bracket}}] + \dots \right) \\
 &= \frac{(-i)^3}{6} \int_{x_1, x_2, x_3} (i\Delta_{23}^R) \frac{1}{(2!)^2} \left[(i\Delta_{12}^R) : \phi_1^2 \phi_2 \phi_3^2 : + \underbrace{\theta_{12}}_{\text{blue arrow}} (i\Delta_{13}^R) : \phi_1^2 \phi_2^2 \phi_3 : \right] + \dots
 \end{aligned}$$

$$(i\Delta_{23}^R) (i\Delta_{13}^R) \theta_{12} = \text{diagram}$$

5-points tree-level

$$\begin{aligned}
 iN^{(3)} &= \frac{(-i)^3}{6} \int_{x_1, x_2, x_3} \theta_{12} \theta_{23} \theta_{13} \left([\mathcal{H}_1, [\mathcal{H}_2, \mathcal{H}_3]] + \dots \right) \\
 &= \frac{(-i)^3}{6} \int_{x_1, x_2, x_3} (i\Delta_{23}^R) \frac{1}{(2!)^2} \left[(i\Delta_{12}^R) : \phi_1^2 \phi_2 \phi_3 : + \theta_{12} (i\Delta_{13}^R) : \phi_1^2 \phi_2^2 \phi_3 : \right] + \dots
 \end{aligned}$$



$$\theta_{12} + \theta_{21} = 1$$

5-points tree-level

$$\langle p_3 p_4 p_5 | iN^{(3)} | p_1 p_2 \rangle_{Tree} =$$

$$\sum_{\text{Permutations}} \left[\frac{1}{3} \text{ (diagram 1)} + \frac{1}{6} \text{ (diagram 2)} + \frac{1}{6} \text{ (diagram 3)} + \frac{1}{3} \text{ (diagram 4)} \right]$$

$$\omega(\text{linear chain of 3 nodes}) = \frac{1}{3}$$

$$\omega(\text{triangle}) = \omega(\text{inverted triangle}) = \frac{1}{6}$$

Matches
[Murua'06]
[Kim³, Lee '24]

Sum of Murua coefficients at tree-level is 1

6-points tree-level

$$\langle 456 | iN^{(4)} | 123 \rangle$$

$$= \frac{1}{16} \sum_{S_6} \left(\frac{1}{6} \left(\text{diagram 1} + \text{diagram 2} \right) + \frac{1}{4} \sum_{S_6} \left(\frac{1}{4} \left(\text{diagram 3} + \text{diagram 4} \right) + \frac{1}{12} \left(\text{diagram 5} + \text{diagram 6} \right) \right) \right)$$

The diagrams are tree-level Feynman diagrams for 6-point interactions. Diagram 1 and 2 are s-channel exchange diagrams with 3-point vertices. Diagram 3 and 4 are t-channel exchange diagrams with 3-point vertices. Diagram 5 and 6 are contact diagrams with 4-point vertices.

$$\omega \left(\text{diagram 7} \right) = \frac{1}{4}$$

$$\omega \left(\text{diagram 8} \right) = \omega \left(\text{diagram 9} \right) = \omega \left(\text{diagram 10} \right) = \frac{1}{12}$$

$$\omega \left(\text{diagram 11} \right) = \omega \left(\text{diagram 12} \right) = \frac{1}{6}$$

$$\omega \left(\text{diagram 13} \right) = \omega \left(\text{diagram 14} \right) = 0$$

Matches
[Murua'06]
[Kim^3, Lee '24]

6-points tree-level

$$\langle 456 | iN^{(4)} | 123 \rangle$$

$$= \frac{1}{16} \sum_{S_6} \left(\frac{1}{6} \left(\text{diagram 1} + \text{diagram 2} \right) + \frac{1}{4} \sum_{S_6} \left(\frac{1}{4} \left(\text{diagram 3} + \text{diagram 4} \right) + \frac{1}{12} \left(\text{diagram 5} + \text{diagram 6} \right) \right) \right)$$

$$\omega \left(\text{diagram 7} \right) = \frac{1}{4}$$

$$\omega \left(\text{diagram 8} \right) = \omega \left(\text{diagram 9} \right) = \omega \left(\text{diagram 10} \right) = \frac{1}{12}$$

$$\omega \left(\text{diagram 11} \right) = \omega \left(\text{diagram 12} \right) = \frac{1}{6}$$

$$\omega \left(\text{diagram 13} \right) = \omega \left(\text{diagram 14} \right) = 0$$

Matches
[Murua'06]
[Kim^3, Lee '24]

General Structure: Tree-level

For n vertices n-1 Wick contractions:

$$\left[\overbrace{\mathcal{H}_1, \left[\overbrace{\mathcal{H}_2, \dots \left[\overbrace{\mathcal{H}_{n-1}, \mathcal{H}_n} \right] \dots} \right]} \right] \xrightarrow[n-1]{\text{contractions}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \left\{ \mathcal{H}'_1, \left\{ \mathcal{H}'_2, \dots \left\{ \mathcal{H}'_{n-1}, \mathcal{H}'_n \right\} \dots \right\} \right\}$$

General Structure: Tree-level

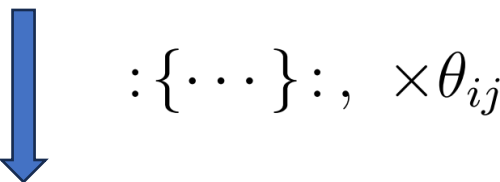
For n vertices $n-1$ Wick contractions:

$$\begin{aligned} \left[\overbrace{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_{n-1}, \mathcal{H}_n}^{n-1} \right] &\xrightarrow{\text{contractions}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \left\{ \mathcal{H}'_1, \{ \mathcal{H}'_2, \dots \{ \mathcal{H}'_{n-1}, \mathcal{H}'_n \} \dots \} \right\} \\ &\quad \downarrow \quad \quad \quad : \{ \dots \} :, \times \theta_{ij} \\ &\quad \quad \quad (i\Delta_{1a_1}^{R/A}) \cdots (i\Delta_{n-1n}^{R/A}) : \mathcal{H}'_1 \mathcal{H}'_2 \cdots \mathcal{H}'_{n-1} \mathcal{H}'_n : \end{aligned}$$

General Structure: Tree-level

For n vertices n-1 Wick contractions:

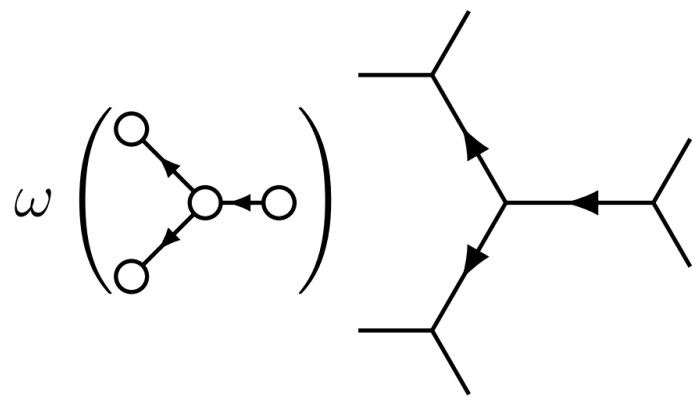
$$\left[\overbrace{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_{n-1}, \mathcal{H}_n}^{n-1 \text{ contractions}} \right] \xrightarrow{\quad} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \left\{ \mathcal{H}'_1, \{ \mathcal{H}'_2, \dots \{ \mathcal{H}'_{n-1}, \mathcal{H}'_n \} \dots \} \right\}$$



$$(i\Delta_{1a_1}^{R/A}) \cdots (i\Delta_{n-1n}^{R/A}) : \mathcal{H}'_1 \mathcal{H}'_2 \cdots \mathcal{H}'_{n-1} \mathcal{H}'_n :$$

Only retarded or advanced props at tree-level.

Coefficients of each diagram given by [Murua'06]
[Kim^3, Lee '24]



1-loop

One extra Wick contraction beyond tree

$$\left[\overbrace{\mathcal{H}_1, \left[\overbrace{\mathcal{H}_2, \dots \left[\overbrace{\mathcal{H}_{n-1}, \mathcal{H}_n} \right] \dots} \right]} \right] \xrightarrow[n-1]{\text{contractions}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \left\{ \mathcal{H}'_1, \left\{ \mathcal{H}'_2, \dots \left\{ \mathcal{H}'_{n-1}, \mathcal{H}'_n \right\} \dots \right\} \right\}$$

1-loop

One extra Wick contraction beyond tree

$$\left[\overbrace{\mathcal{H}_1, \overbrace{[\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots}] } \right] \xrightarrow[n-1]{\text{contractions}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \left\{ \mathcal{H}'_1, \overbrace{\{\mathcal{H}'_2, \cdots \{\mathcal{H}'_{n-1}, \mathcal{H}'_n\} \cdots\}} \right\}$$

$$\overbrace{\{\mathcal{H}_i, \mathcal{H}_j\}} = \frac{1}{2} \Delta_{ij}^{(1)} \{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2} i\Delta_{ij} [\mathcal{H}'_i, \mathcal{H}'_j]$$

One more contraction 

1-loop

One extra Wick contraction beyond tree

$$\left[\overbrace{\mathcal{H}_1, [\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots]} \right] \xrightarrow[n-1]{\text{contractions}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \left\{ \mathcal{H}'_1, \overbrace{\{\mathcal{H}'_2, \cdots \{\mathcal{H}'_{n-1}, \mathcal{H}'_n\} \cdots\}} \right\}$$

$$\{\overbrace{\mathcal{H}_i, \mathcal{H}_j}\} = \frac{1}{2} \Delta_{ij}^{(1)} \{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2} i\Delta_{ij} [\cancel{\mathcal{H}'_i} \cancel{\mathcal{H}'_j}] \quad \text{One more contraction} \quad \downarrow \quad : \{ \cdots \} :, \times \theta_{ij}$$

$$(i\Delta_{2n}^{(1)}) (i\Delta_{1a_1}^{R/A}) \cdots (i\Delta_{n-1n}^{R/A}) : \mathcal{H}'_1 \mathcal{H}''_2 \cdots \mathcal{H}'_{n-1} \mathcal{H}''_n :,$$

1-loop

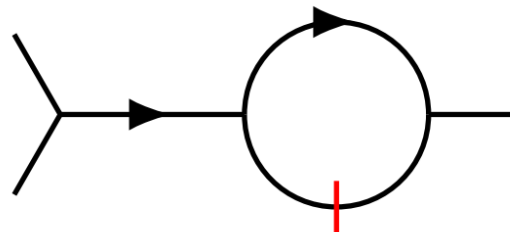
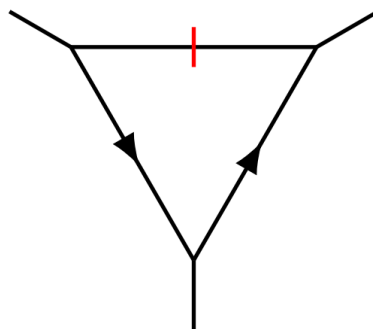
One extra Wick contraction beyond tree

$$\left[\overbrace{\mathcal{H}_1, [\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots]}^{n-1 \text{ contractions}} \right] \rightarrow (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \left\{ \mathcal{H}'_1, \overbrace{\{\mathcal{H}'_2, \cdots \{\mathcal{H}'_{n-1}, \mathcal{H}'_n\} \cdots\}} \right\}$$

$$\{\overbrace{\mathcal{H}_i, \mathcal{H}_j}\} = \frac{1}{2} \Delta_{ij}^{(1)} \{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2} i\Delta_{ij} [\cancel{\mathcal{H}'_i} \mathcal{H}'_j] \quad \text{One more contraction} \quad \downarrow \quad : \{ \cdots \} :, \times \theta_{ij}$$

Exactly one cut propagator at 1-loop!

$$(i\Delta_{2n}^{(1)}) (i\Delta_{1a_1}^{R/A}) \cdots (i\Delta_{n-1n}^{R/A}) : \mathcal{H}'_1 \mathcal{H}''_2 \cdots \mathcal{H}'_{n-1} \mathcal{H}''_n :,$$



$$\begin{aligned}
\langle 23 | iN^{(3)} | 1 \rangle_{1-loop} = & \sum_{C_3} \left[\frac{1}{6} \text{Diagram 1} + \frac{1}{12} \text{Diagram 2} + \frac{1}{12} \text{Diagram 3} + \frac{1}{6} \text{Diagram 4} \right] \\
& + \left[\frac{1}{6} \text{Diagram 5} + \frac{1}{12} \text{Diagram 6} + \frac{1}{12} \text{Diagram 7} + \frac{1}{6} \text{Diagram 8} \right]
\end{aligned}$$

The diagrams are as follows:

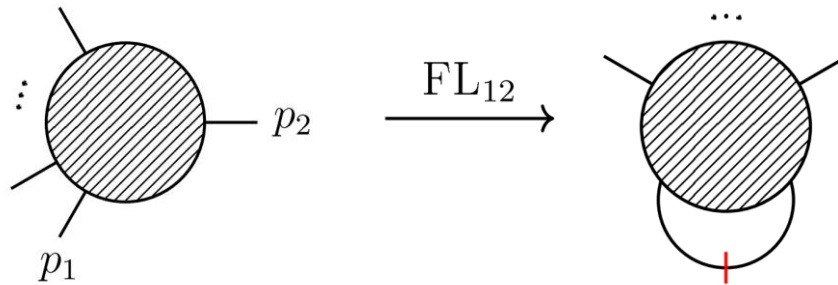
- Diagram 1:** A triangle loop with a red vertical tick on the top horizontal edge. Arrows on the two slanted edges point towards the bottom vertex.
- Diagram 2:** A triangle loop with a red vertical tick on the top horizontal edge. Arrows on the two slanted edges point away from the bottom vertex.
- Diagram 3:** A triangle loop with a red vertical tick on the top horizontal edge. Arrows on the two slanted edges point towards the bottom vertex.
- Diagram 4:** A triangle loop with a red vertical tick on the top horizontal edge. Arrows on the two slanted edges point away from the bottom vertex.
- Diagram 5:** A circular loop with a red vertical tick on the bottom edge. An incoming line from the left splits into two lines labeled p_1 and p_2 . An outgoing line to the right is labeled p_3 . The arrow on the horizontal line points into the loop.
- Diagram 6:** A circular loop with a red vertical tick on the bottom edge. An incoming line from the left splits into two lines labeled p_1 and p_2 . An outgoing line to the right is labeled p_3 . The arrow on the horizontal line points into the loop.
- Diagram 7:** A circular loop with a red vertical tick on the bottom edge. An incoming line from the left splits into two lines labeled p_1 and p_2 . An outgoing line to the right is labeled p_3 . The arrow on the horizontal line points out of the loop.
- Diagram 8:** A circular loop with a red vertical tick on the bottom edge. An incoming line from the left splits into two lines labeled p_1 and p_2 . An outgoing line to the right is labeled p_3 . The arrow on the horizontal line points out of the loop.

$$\begin{aligned}
\langle 23 | iN^{(3)} | 1 \rangle_{1-loop} = & \sum_{C_3} \left[\frac{1}{6} \text{triangle}_1 + \frac{1}{12} \text{triangle}_2 + \frac{1}{12} \text{triangle}_3 + \frac{1}{6} \text{triangle}_4 \right] \\
& + \left[\frac{1}{6} \text{bubble}_1 + \frac{1}{12} \text{bubble}_2 + \frac{1}{12} \text{bubble}_3 + \frac{1}{6} \text{bubble}_4 \right]
\end{aligned}$$

The Feynman diagrams are as follows:

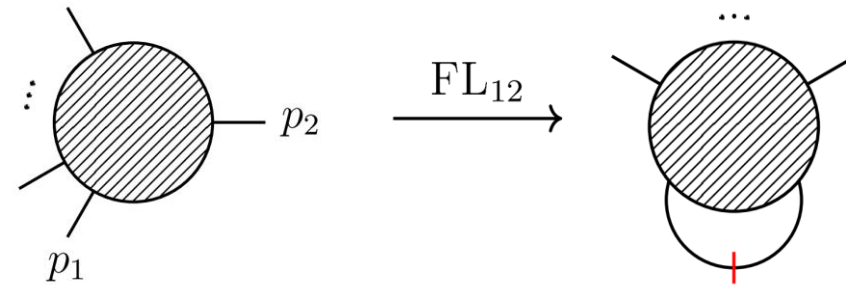
- Triangle diagrams:** Four diagrams with a triangle loop. The top horizontal edge has a red tick mark. The bottom vertex has an incoming line. The top two vertices have outgoing lines. Arrows indicate a clockwise loop flow.
- Bubble diagrams:** Four diagrams with a bubble loop. The bottom vertex has an incoming line labeled p_1 . The top vertex has an outgoing line labeled p_2 . The right vertex has an outgoing line labeled p_3 . The bubble loop has a red tick mark on its bottom edge. Arrows indicate a clockwise loop flow.

Forward limit:



$$\begin{aligned}
 \langle 23 | iN^{(3)} | 1 \rangle_{1-loop} = & \sum_{C_3} \left[\frac{1}{6} \text{triangle} + \frac{1}{12} \text{triangle} + \frac{1}{12} \text{triangle} + \frac{1}{6} \text{triangle} \right] \\
 & + \left[\frac{1}{6} \text{bubble} + \frac{1}{12} \text{bubble} + \frac{1}{12} \text{bubble} + \frac{1}{6} \text{bubble} \right]
 \end{aligned}$$

Forward limit:



$$\omega \left(\text{triangle} \right) = \frac{1}{2} \omega \left(\text{M-shape} \right) = \frac{1}{2} \omega \left(\text{3-loop} \right) = \frac{1}{12}$$

2-loops

Two extra Wick contraction beyond tree

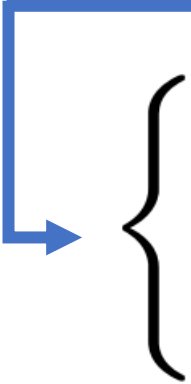
$$\left[\overbrace{\mathcal{H}_1, \overbrace{[\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots}] } \right] \xrightarrow[n-1]{\text{contractions}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \left\{ \mathcal{H}'_1, \left\{ \mathcal{H}'_2, \cdots \left\{ \mathcal{H}'_{n-1}, \mathcal{H}'_n \right\} \cdots \right\} \right\}$$

2-loops

Two extra Wick contraction beyond tree

$$\begin{aligned}\overline{[\mathcal{H}_i, \mathcal{H}_j]} &= \frac{1}{2}i\Delta_{ij}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2}\Delta_{ij}^{(1)}[\mathcal{H}'_i, \mathcal{H}'_j] \\ \overline{\{\mathcal{H}_i, \mathcal{H}_j\}} &= \frac{1}{2}\Delta_{ij}^{(1)}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2}i\Delta_{ij}[\mathcal{H}'_i, \mathcal{H}'_j]\end{aligned}$$

$$\left[\overbrace{\mathcal{H}_1, [\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots]} \right] \xrightarrow[n-1]{\text{contractions}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \left\{ \mathcal{H}'_1, \left\{ \mathcal{H}'_2, \cdots \left\{ \mathcal{H}'_{n-1}, \mathcal{H}'_n \right\} \cdots \right\} \right\}$$

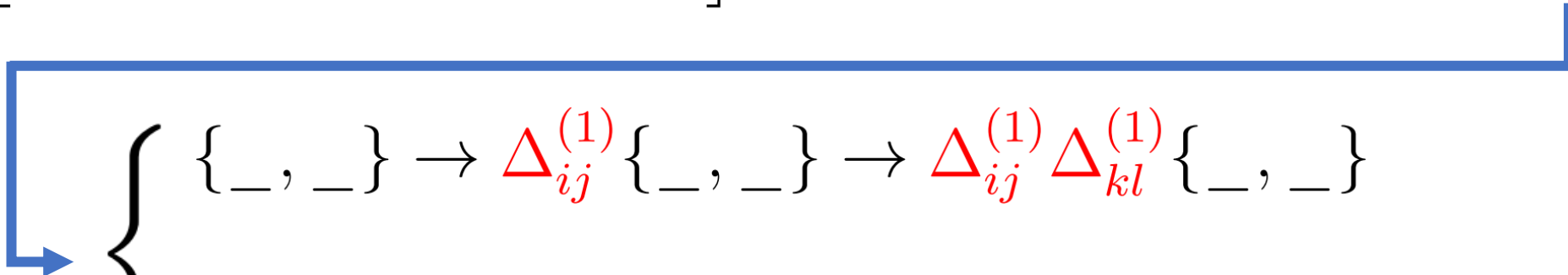


2-loops

Two extra Wick contraction beyond tree

$$\begin{aligned}\overline{[\mathcal{H}_i, \mathcal{H}_j]} &= \frac{1}{2}i\Delta_{ij}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2}\Delta_{ij}^{(1)}[\mathcal{H}'_i, \mathcal{H}'_j] \\ \overline{\{\mathcal{H}_i, \mathcal{H}_j\}} &= \frac{1}{2}\Delta_{ij}^{(1)}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2}i\Delta_{ij}[\mathcal{H}'_i, \mathcal{H}'_j]\end{aligned}$$

$$\left[\mathcal{H}_1, \overbrace{[\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots]} \right] \xrightarrow[n-1]{\text{contractions}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \left\{ \mathcal{H}'_1, \left\{ \mathcal{H}'_2, \cdots \left\{ \mathcal{H}'_{n-1}, \mathcal{H}'_n \right\} \cdots \right\} \right\}$$



$$\left\{ \{ _, _ \} \rightarrow \Delta_{ij}^{(1)} \{ _, _ \} \rightarrow \Delta_{ij}^{(1)} \Delta_{kl}^{(1)} \{ _, _ \} \right.$$

2-loops

Two extra Wick contraction beyond tree

$$\begin{aligned}\overline{[\mathcal{H}_i, \mathcal{H}_j]} &= \frac{1}{2}i\Delta_{ij}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2}\Delta_{ij}^{(1)}[\mathcal{H}'_i, \mathcal{H}'_j] \\ \overline{\{\mathcal{H}_i, \mathcal{H}_j\}} &= \frac{1}{2}\Delta_{ij}^{(1)}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2}i\Delta_{ij}[\mathcal{H}'_i, \mathcal{H}'_j]\end{aligned}$$

$$\left[\mathcal{H}_1, \overbrace{[\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots]} \right] \xrightarrow[n-1]{\text{contractions}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \left\{ \mathcal{H}'_1, \left\{ \mathcal{H}'_2, \cdots \left\{ \mathcal{H}'_{n-1}, \mathcal{H}'_n \right\} \cdots \right\} \right\}$$

$$\left\{ \begin{aligned} \{ _, _ \} &\rightarrow \Delta_{ij}^{(1)} \{ _, _ \} \rightarrow \Delta_{ij}^{(1)} \Delta_{kl}^{(1)} \{ _, _ \} \\ \{ _, _ \} &\rightarrow \Delta_{ij} [_, _] \rightarrow \Delta_{ij} \Delta_{ik} \{ _, _ \} \end{aligned} \right.$$

2-loops

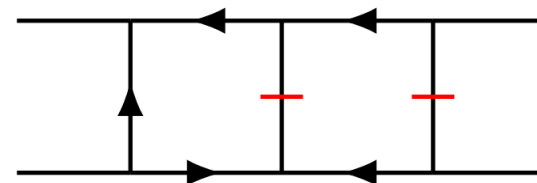
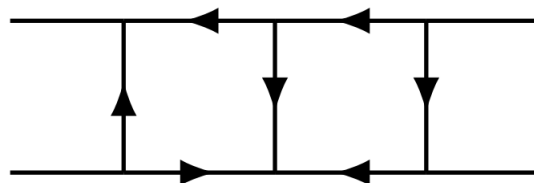
Two extra Wick contraction beyond tree

$$\begin{aligned}\overline{[\mathcal{H}_i, \mathcal{H}_j]} &= \frac{1}{2}i\Delta_{ij}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2}\Delta_{ij}^{(1)}[\mathcal{H}'_i, \mathcal{H}'_j] \\ \{\overline{\mathcal{H}_i}, \mathcal{H}_j\} &= \frac{1}{2}\Delta_{ij}^{(1)}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2}i\Delta_{ij}[\mathcal{H}'_i, \mathcal{H}'_j]\end{aligned}$$

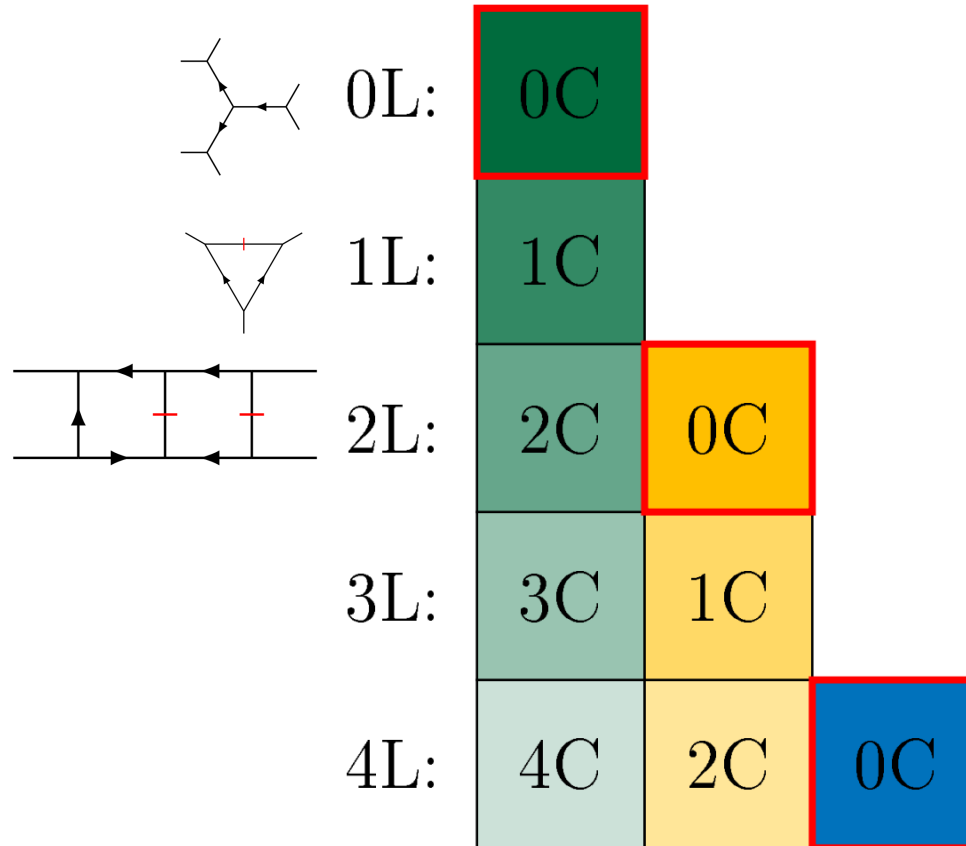
$$\left[\mathcal{H}_1, \overbrace{[\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots]} \right] \xrightarrow[n-1]{\text{contractions}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \left\{ \mathcal{H}'_1, \left\{ \mathcal{H}'_2, \cdots \left\{ \mathcal{H}'_{n-1}, \mathcal{H}'_n \right\} \cdots \right\} \right\}$$

$$\left\{ \begin{aligned} \{ _, _ \} &\rightarrow \Delta_{ij}^{(1)} \{ _, _ \} \rightarrow \Delta_{ij}^{(1)} \Delta_{kl}^{(1)} \{ _, _ \} \\ \{ _, _ \} &\rightarrow \Delta_{ij} [_, _] \rightarrow \Delta_{ij} \Delta_{ik} \{ _, _ \} \end{aligned} \right.$$

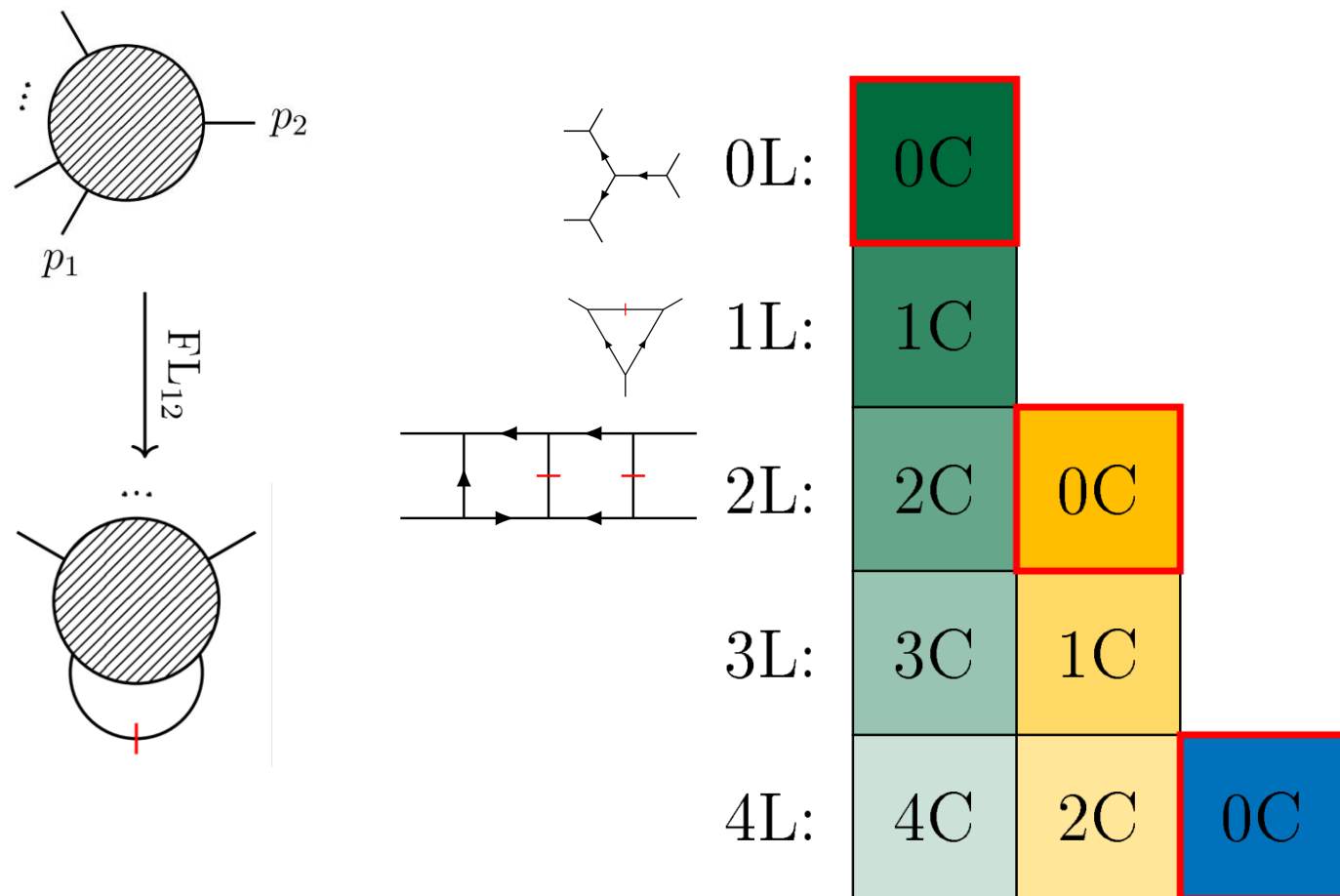
Exactly 0 cuts or 2 cuts
at 2-loop!



General loop structure

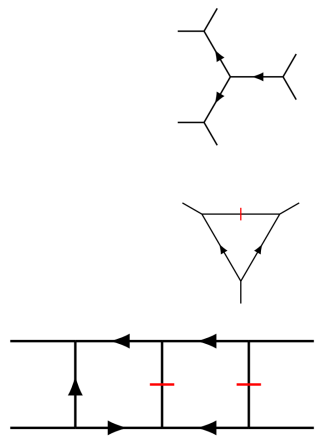
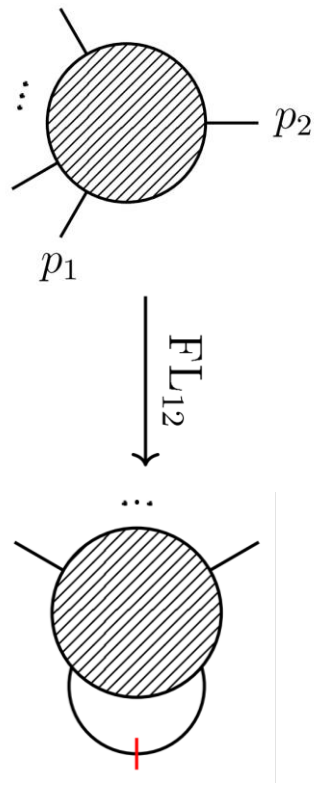


General loop structure



General loop structure

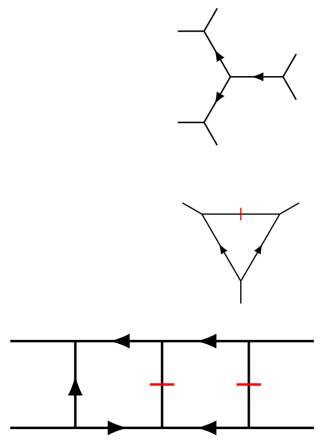
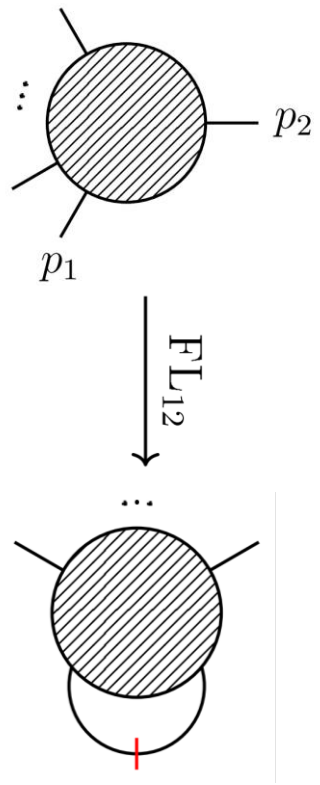
Tree-level Murua coeff:
[Kim³, Lee '24]



0L:	0C		
1L:	1C		
2L:	2C	0C	
3L:	3C	1C	
4L:	4C	2C	0C

General loop structure

Tree-level Murua coeff:
[Kim³, Lee '24]

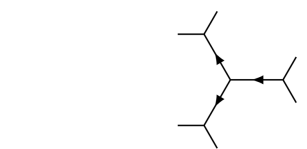
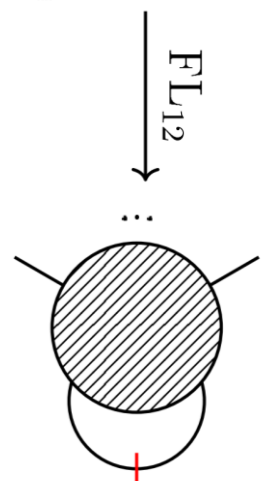
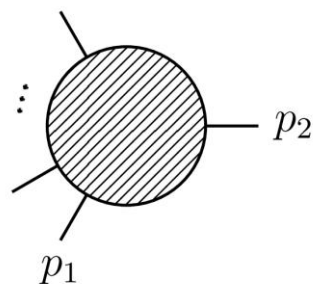


0L:	0C		
1L:	1C		
2L:	2C	0C	
3L:	3C	1C	
4L:	4C	2C	0C

Cut Murua coeffs from:

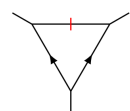
$$\omega \left(\begin{array}{c} \text{A} \rightarrow \text{B} \end{array} \right) = \frac{1}{2} \omega \left(\begin{array}{c} \text{A} \rightarrow \text{B} \end{array} \right)$$

General loop structure



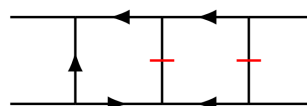
0L:

0C



1L:

1C



2L:

2C

0C

3L:

3C

1C

4L:

4C

2C

0C

Tree-level Murua coeff:
[Kim³, Lee '24]

Cut Murua coeffs from:

$$\omega \left(\begin{array}{c} \text{A} \rightarrow \text{B} \end{array} \right) = \frac{1}{2} \omega \left(\begin{array}{c} \text{A} \rightarrow \text{B} \end{array} \right)$$

Higher-loop zero cut coeffs:
[Kim '25][Guo, Kim³, Lee '26]

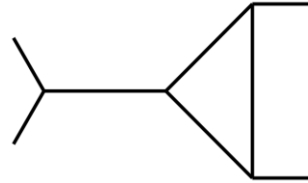
Magnus Feynman Rules

Magnus Feynman Rules

Step 1:

Draw all diagrams
without arrows

...



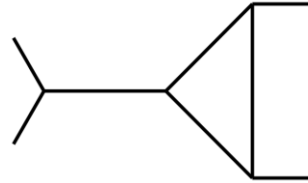
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Magnus Feynman Rules

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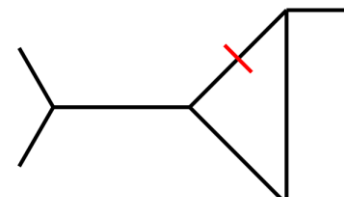
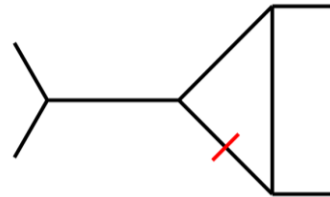
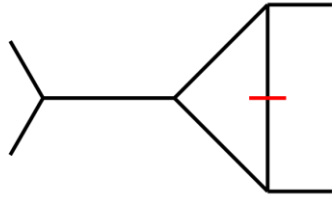
...



...

Step 2:

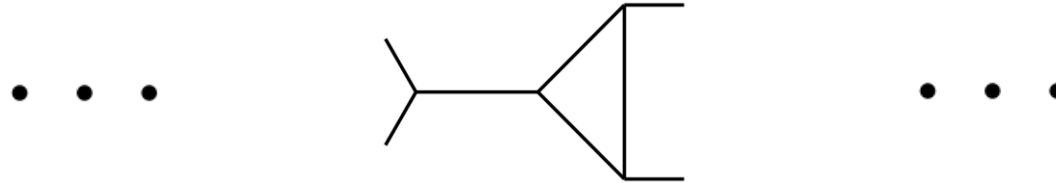
Add cuts in all possible
numbers and ways



Magnus Feynman Rules

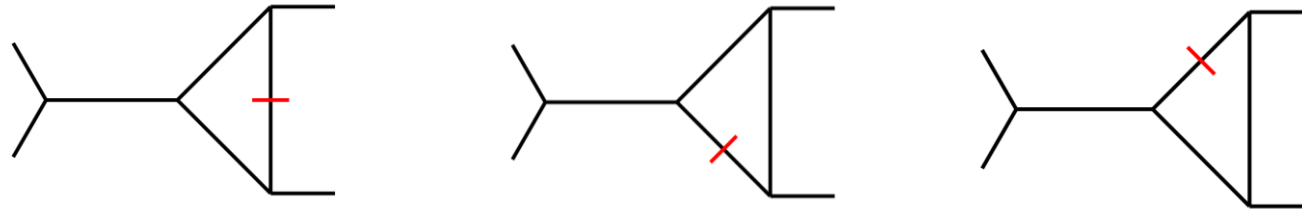
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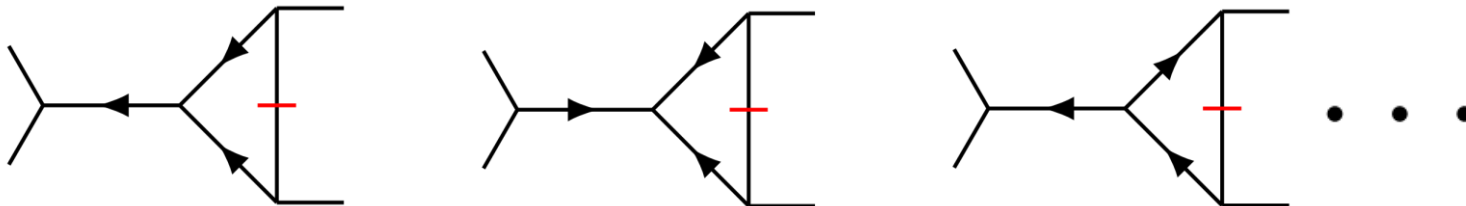
Step 2:

Add cuts in all possible
numbers and ways



Step 3:

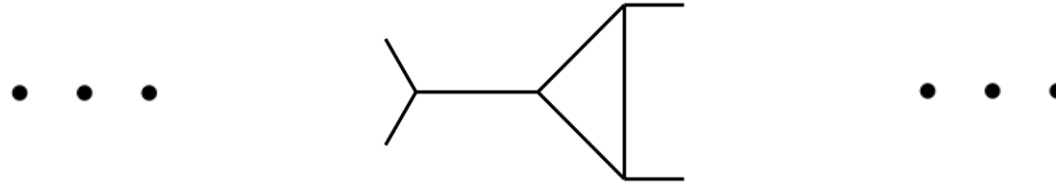
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Magnus Feynman Rules

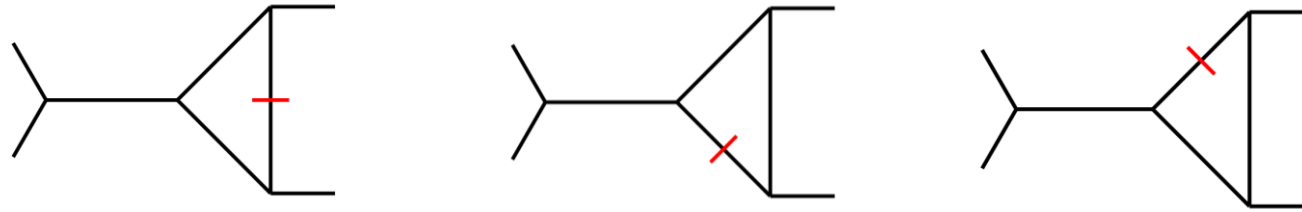
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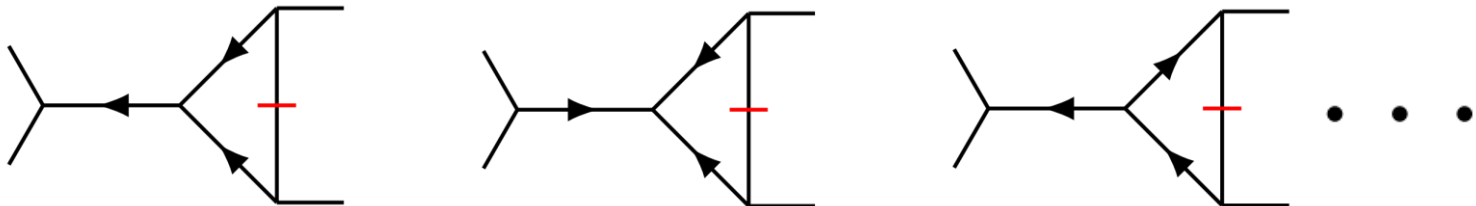
Step 2:

Add cuts in all possible numbers and ways



Step 3:

Add arrows in all possible ways



Step 4:

Multiply each graph by its Mura Coefficient

$$\frac{1}{2}\omega \left(\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \end{array} \right) \text{ (with arrows) } + \frac{1}{2}\omega \left(\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \end{array} \right) \text{ (with arrows) } + \dots$$

Murua properties

$$\omega(\bigcirc) = 1,$$

Swapped Graphs have the same Murua coefficients

$$\omega\left(\begin{array}{c} \circ \rightarrow \circ \\ \circ \rightarrow \circ \end{array}\right) = \omega\left(\begin{array}{c} \circ \rightarrow \circ \\ \circ \rightarrow \circ \end{array}\right)$$

[Kim³, Lee '24]

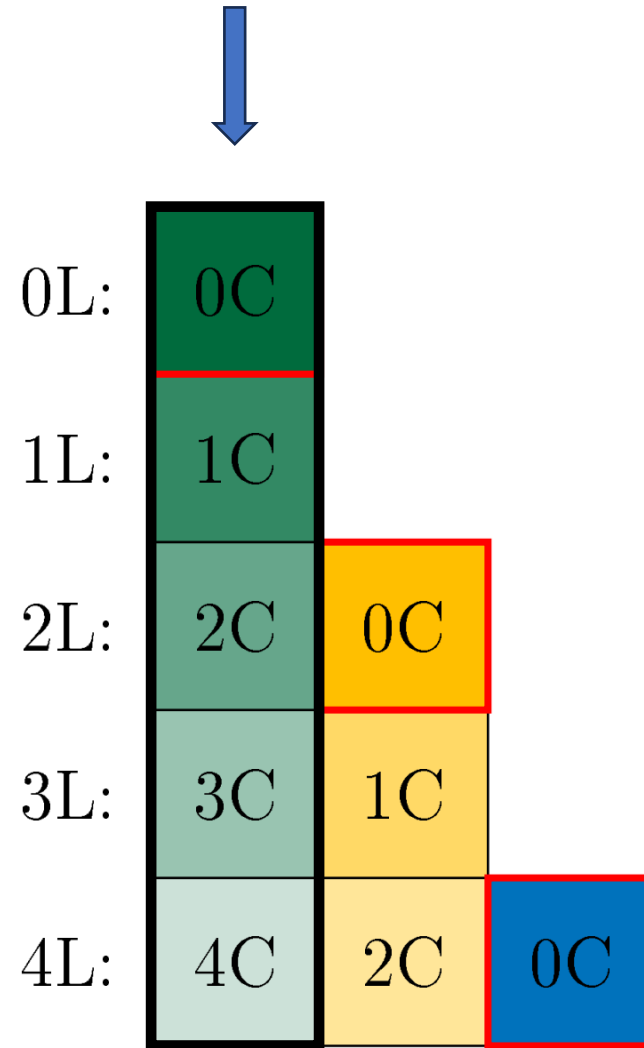
$$\omega\left(\begin{array}{c} \text{A} \rightarrow \circ \rightarrow \circ \rightarrow \text{B} \\ \text{A} \rightarrow \circ \rightarrow \circ \rightarrow \text{B} \end{array}\right) + \omega\left(\begin{array}{c} \text{A} \rightarrow \circ \leftarrow \circ \leftarrow \text{B} \\ \text{A} \rightarrow \circ \leftarrow \circ \leftarrow \text{B} \end{array}\right) = \omega\left(\begin{array}{c} \text{A} \rightarrow \circ \rightarrow \text{B} \\ \text{A} \rightarrow \circ \rightarrow \text{B} \end{array}\right)$$

$$\omega\left(\begin{array}{c} \text{A} \rightarrow \text{B} \\ \text{A} \rightarrow \text{B} \end{array}\right) = \frac{1}{2} \omega\left(\begin{array}{c} \text{A} \rightarrow \text{B} \\ \text{A} \rightarrow \text{B} \end{array}\right)$$

Classical limit

Conjecture:

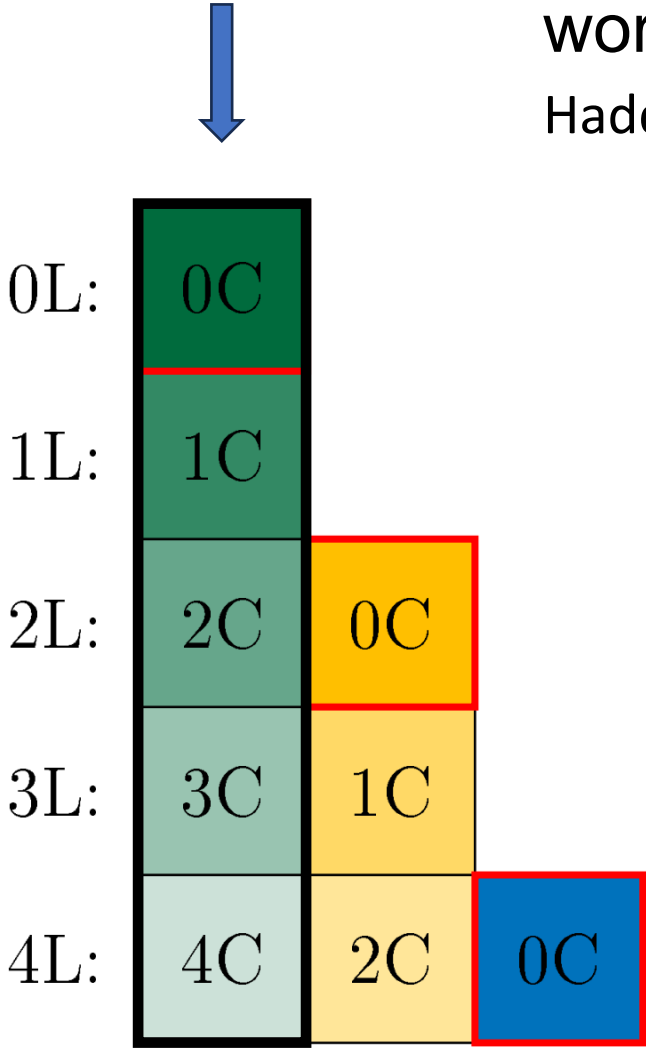
Only the maximal cut pieces contribute in the classical limit of N



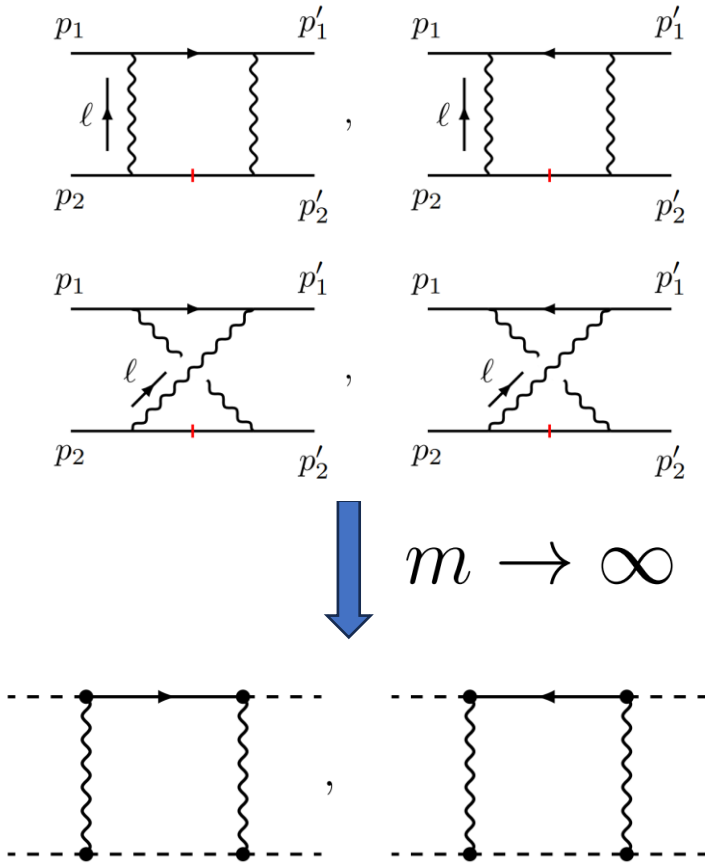
Classical limit

Conjecture:

Only the maximal cut pieces contribute in the classical limit of N



Matches intuition from worldlines [Mogull, Plefka, Steinhoff; Haddad, Jakobsen, Mogull, Plefka]



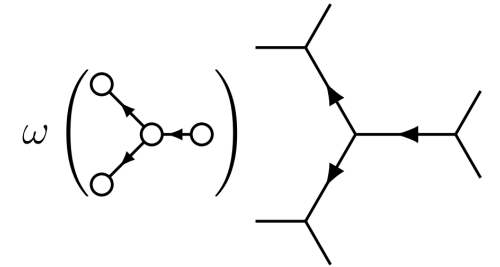
Conclusions

$$S = \exp(iN)$$

- The Magnus expansion has three types of propagators:

$$i\Delta_{ij}^R = \overset{j}{\circ} \rightarrow \overset{i}{\circ} \quad i\Delta_{ij}^A = \overset{j}{\circ} \leftarrow \overset{i}{\circ} \quad \Delta_{ij}^{(1)} = \overset{j}{\circ} \text{---} \overset{i}{\circ}$$

- Only very specific numbers of cuts can appear.



- Proved the all-loop structure of the Magnus expansion.

What now?

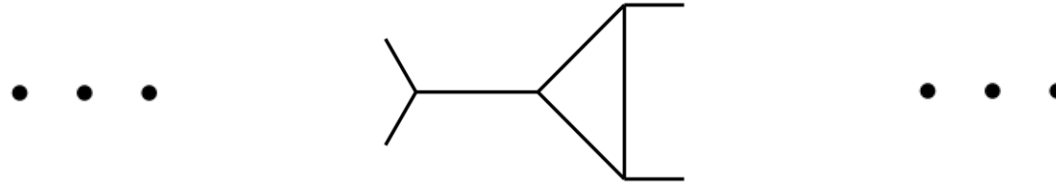
- Proof of classical limit conjecture
- Soft limits, infrared divergences, exponentiation...
- Doing the integrals, generalised unitarity ...

Magnus Feynman Rules

Thank you!

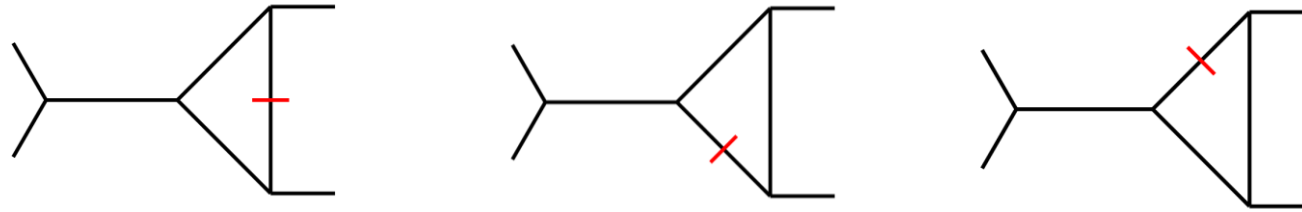
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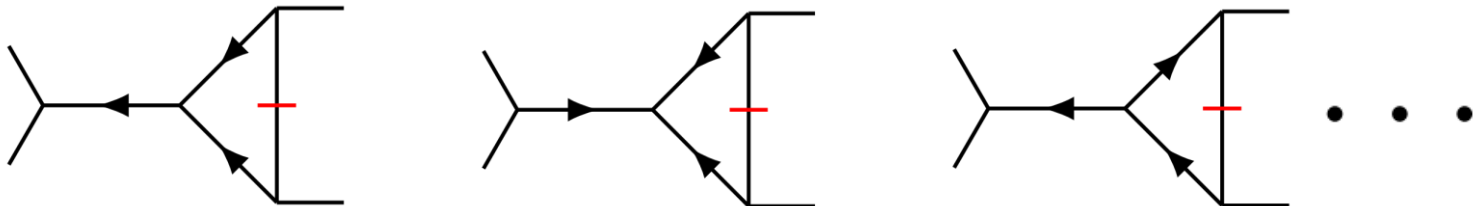
Step 2:

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$$\frac{1}{2}\omega \left(\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \end{array} \right) \text{ (diagram) } + \frac{1}{2}\omega \left(\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \end{array} \right) \text{ (diagram) } + \dots$$