The Magnus Expansion in QFT

Based on 2512.05017, with
Andreas Brandhuber, Paolo Pichini,
Gabriele Travaglini and Pablo Vives Matasan
+ Sebastien MacDonald (to appear)

Graham R. Brown

29.05.2026-Amplitudes 2026-QMUL

Why we are here: The S-Matrix

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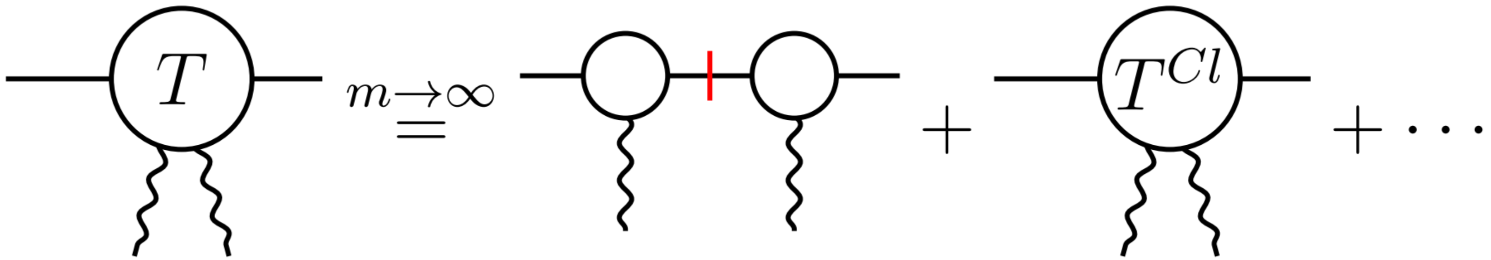

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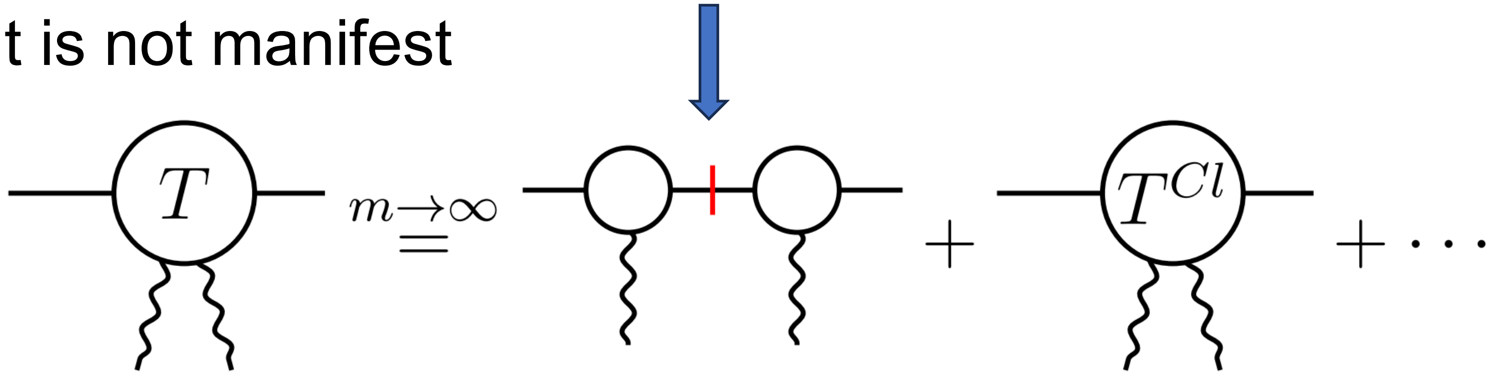

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[Magnus '54],
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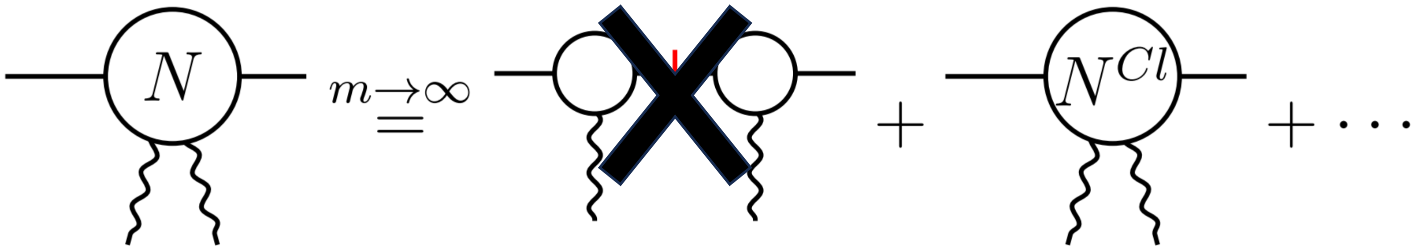
The classical limit of N can be used directly



How to compute N?

Just take the log [Damgaard, Plante, Vanhove ’21 +Hansen ‘23] … :

$$iN = \log(1 + iT) = iT - \frac{(iT)^2}{2} + \frac{(iT)^3}{3} - \dots ,$$

Is there a better/more fundamental way?

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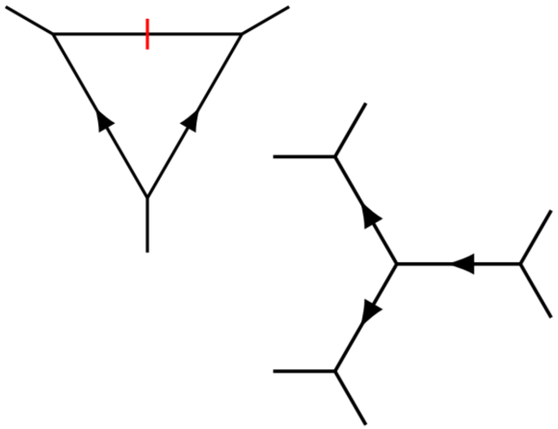

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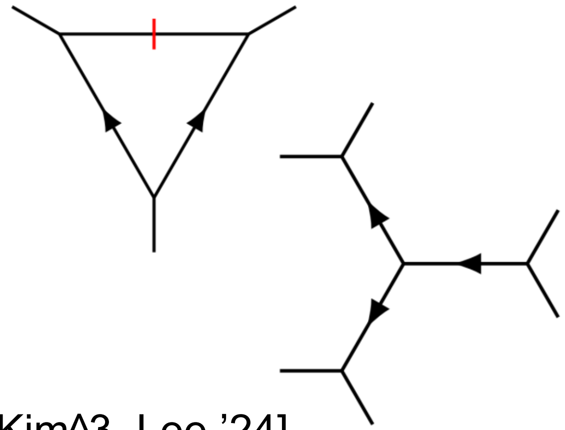

Building on previous work at tree-level [Kim^3, Lee ’24]

Wick contractions review

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Simple model: $\mathcal{H}_I = \dfrac{:\phi^3:}{3!}$

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$$\boxed{\begin{aligned} \phi_i\phi_j &= :\phi_i\phi_j: + \contraction{}{\phi}{_i}{\phi}\phi_i\phi_j \\ &= :\phi_i\phi_j: + i\Delta_{ij}^{(+)} \end{aligned}}$$

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$$\mathcal{T}(\phi_i\phi_j) = \theta_{ij}\phi_i\phi_j + \theta_{ji}\phi_j\phi_i\,,$$

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$$i\Delta_{ij}^F \sim \frac{i}{p^2 - m^2 + i\epsilon}$$

$$iT^{(2)} = \frac{(-i)^2}{2!} \int d^4x_1 d^4x_2 \mathcal{T}\left(\frac{:\phi_1^3:}{3!} \frac{:\phi_2^3:}{3!} \right)$$

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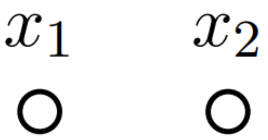
$=$ 

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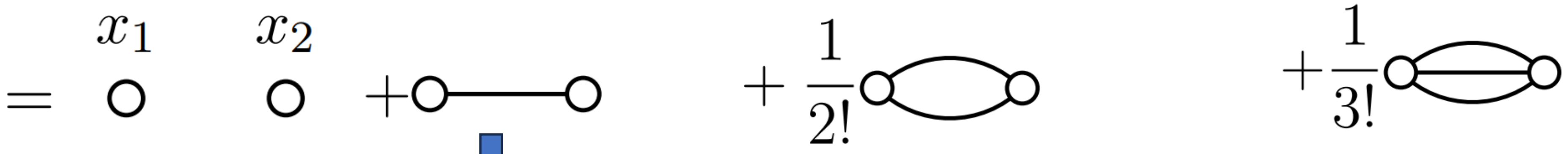
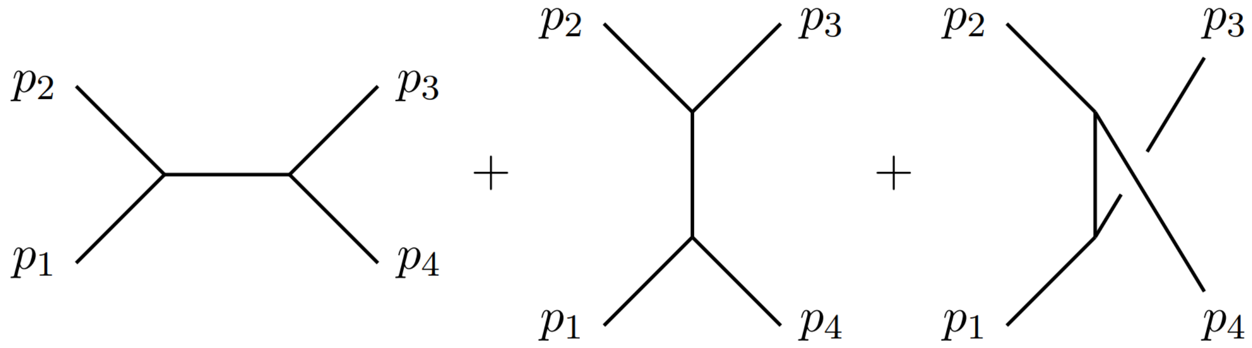


$$\langle p_3 p_4 | iT^{(2)} | p_1 p_2 \rangle_{Tree} =$$



Magnus Wick Contractions

What is the equivalent of the Dyson expansion for N ?

$$iN^{(1)} = -i \int d^4x_1 \; \mathcal{H}_1 \,,$$

$$iN^{(2)} = \frac{(-i)^2}{2} \int d^4x_1 d^4x_2 \; \theta_{12} \; [\mathcal{H}_1, \mathcal{H}_2] \,,$$

$$iN^{(3)} = \frac{(-i)^3}{6} \int d^4x_1 d^4x_2 d^4x_3 \; \theta_{12}\theta_{23} \; ([\mathcal{H}_1, [\mathcal{H}_2, \mathcal{H}_3]] + [\mathcal{H}_3, [\mathcal{H}_2, \mathcal{H}_1]])$$

$$iN^{(4)} = \frac{(-i)^4}{12} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \; \theta_{12}\theta_{23}\theta_{34} \Big([\mathcal{H}_1, [\mathcal{H}_2, [\mathcal{H}_3, \mathcal{H}_4]]] + [\mathcal{H}_4, [\mathcal{H}_3, [\mathcal{H}_1, \mathcal{H}_2]]] + [\mathcal{H}_2, [\mathcal{H}_3, [\mathcal{H}_4, \mathcal{H}_1]]] + [\mathcal{H}_1, [\mathcal{H}_4, [\mathcal{H}_3, \mathcal{H}_2]]] \Big)$$

$$\vdots$$

General formula see [Chen, Strichartz ‘87] …

Commutator Wick Contractions

$$iN^{(n)} \sim \int d^4x_1 \cdots d^4x_n \underbrace{\theta_{12}\theta_{34} \ldots \theta_{n-1,n}}_{\text{Deal with later}} \underbrace{\left[\mathcal{H}_1, \left[\mathcal{H}_2, \cdots \left[\mathcal{H}_{n-1}, \mathcal{H}_n\right] \cdots \right]\right]}_{\text{Contract this}}$$

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$$\begin{aligned} [\mathcal{H}_i, \mathcal{H}_j] &= \mathcal{H}_i \mathcal{H}_j - \mathcal{H}_j \mathcal{H}_i \\ &= i\Delta^{(+)}_{ij} \mathcal{H}'_i \mathcal{H}'_j - i\Delta^{(+)}_{ji} \mathcal{H}'_j \mathcal{H}'_i \end{aligned}$$

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$$iN^{(n)} \sim \int d^4x_1 \cdots d^4x_n \, \underbrace{\theta_{12}\theta_{34}\ldots\theta_{n-1,n}}_{\text{Deal with later}} \, \underbrace{\left[\mathcal{H}_1, \left[\mathcal{H}_2, \cdots \left[\mathcal{H}_{n-1}, \mathcal{H}_n\right] \cdots\right]\right]}_{\text{Contract this}}$$

$$\begin{aligned}
[\mathcal{H}_i, \mathcal{H}_j] &= \mathcal{H}_i\mathcal{H}_j - \mathcal{H}_j\mathcal{H}_i \\
&= i\Delta_{ij}^{(+)}\mathcal{H}_i'\mathcal{H}_j' - i\Delta_{ji}^{(+)}\mathcal{H}_j'\mathcal{H}_i' \\
&= \frac{1}{2} i\Delta_{ij}\{\mathcal{H}_i', \mathcal{H}_j'\} + \frac{1}{2}\Delta_{ij}^{(1)}[\mathcal{H}_i', \mathcal{H}_j']
\end{aligned}$$

$$\begin{aligned}
i\Delta_{ij} &= i\Delta_{ij}^{(+)} - i\Delta_{ji}^{(+)} \\
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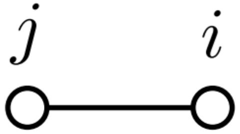
$$\boxed{:[A, B]: = 0\,, \quad :\{A, B\}: = 2:AB:}$$

Dyson iT

$$i\Delta_{ij}^{(+)}$$

$$\Big\downarrow \mathcal{T}(\cdots)$$

$$i\Delta_{ij}^{F} =$$



$$\sim \frac{i}{p^2 - m^2 + i\epsilon}$$

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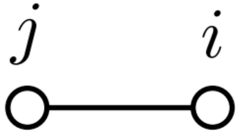
$$i\Delta_{ij}^{F} =$$  $$\sim \frac{i}{p^2 - m^2 + i\epsilon}$$

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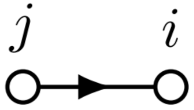
Magnus iN

$$i\Delta_{ij}\,, \qquad i\Delta_{ij}^{(1)}$$

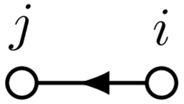
$$\theta_{ij}$$

$$i\Delta^R_{ij} = i\Delta_{ij}\theta_{ij} =$$



$$\sim \frac{i}{p^2 - m^2 + i\epsilon p^0}$$

$$i\Delta^A_{ij} = -i\Delta_{ij}\theta_{ji} =$$



$$\sim \frac{i}{p^2 - m^2 - i\epsilon p^0}$$

$$\Delta^{(1)}_{ij} =$$

$$\sim 2\pi\delta(p^2 - m^2)$$

4-points tree-level

$$iN^{(2)} = \frac{(-i)^2}{2} \int d^4x_1 d^4x_2 \theta_{12} [\mathcal{H}_1, \mathcal{H}_2]$$

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4-points tree-level

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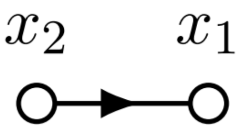
&= \frac{(-i)^2}{2}\int d^4x_1 d^4x_2\, i\Delta^R_{12} : \frac{\phi_1^2}{2!}\frac{\phi_2^2}{2!} : \quad = \frac{1}{2}
\end{aligned}
$$



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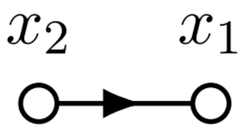
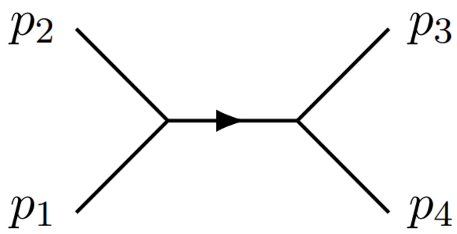
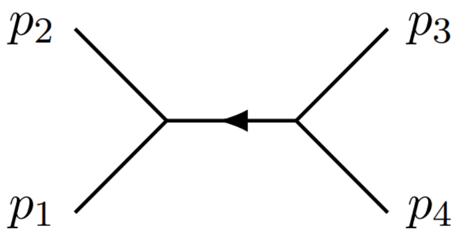
$$= \frac{(-i)^2}{2}\int d^4x_1 d^4x_2 \, i\Delta_{12}^R : \frac{\phi_1^2}{2!}\frac{\phi_2^2}{2!} : \quad = \frac{1}{2}$$



$$\langle p_3 p_4 | iN^{(2)} | p_1 p_2 \rangle_{Tree} = \frac{1}{2}$$



$+\frac{1}{2}$



$+(t/u\text{-channel})$

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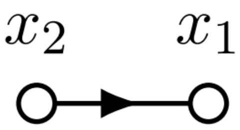
$$= \frac{(-i)^2}{2}\int d^4x_1 d^4x_2\, i\Delta_{12}^{R} : \frac{\phi_1^2}{2!}\frac{\phi_2^2}{2!} : \quad = \frac{1}{2}$$



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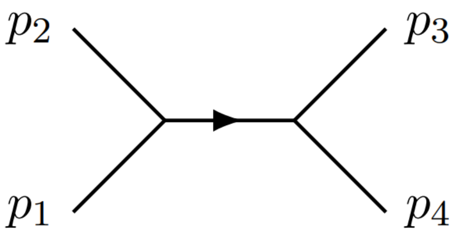
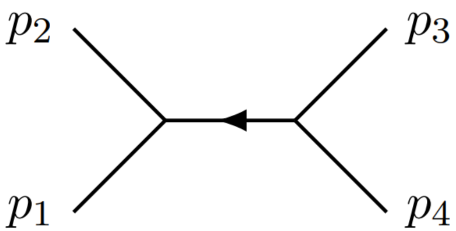


$$+\frac{1}{2}$$



$+(t/u\text{-channel})$

Murua coefficient: $\omega\,($  $) = \frac{1}{2}$

Matches
[Murua'06]
[Kim^3, Lee '24]

5-points tree-level

$$iN^{(3)} = \frac{(-i)^3}{6} \int_{x_1, x_2, x_3} \theta_{12}\theta_{23}\theta_{13} \left([\mathcal{H}_1, [\mathcal{H}_2, \mathcal{H}_3]] + \cdots \right)$$

5-points tree-level

$$iN^{(3)} = \frac{(-i)^3}{6} \int_{x_1,x_2,x_3} \theta_{12}\theta_{23}\theta_{13} \left([\mathcal{H}_1, [\mathcal{H}_2, \mathcal{H}_3]] + \cdots \right)$$

5-points tree-level

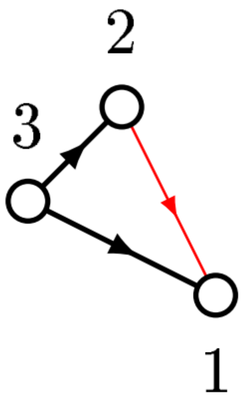
$$
\begin{aligned}
iN^{(3)} &= \frac{(-i)^3}{6}\int_{x_1,x_2,x_3}\theta_{12}\theta_{23}\theta_{13}\left([\mathcal{H}_1,[\mathcal{H}_2,\mathcal{H}_3]]+\cdots\right)\\
&= \frac{(-i)^3}{6}\int_{x_1,x_2,x_3}(i\Delta^R_{23})\frac{1}{(2!)^2}\left[(i\Delta^R_{12})\,:\phi_1^2\phi_2\phi_3^2:+\theta_{12}(i\Delta^R_{13})\,:\phi_1^2\phi_2^2\phi_3:\right]+\cdots
\end{aligned}
$$

5-points tree-level

$$iN^{(3)} = \frac{(-i)^3}{6}\int_{x_1,x_2,x_3}\theta_{12}\theta_{23}\theta_{13}\left([\mathcal{H}_1,[\mathcal{H}_2,\mathcal{H}_3]]+\cdots\right)$$

$$= \frac{(-i)^3}{6}\int_{x_1,x_2,x_3}(i\Delta^R_{23})\frac{1}{(2!)^2}\left[(i\Delta^R_{12}):\phi_1^2\phi_2\phi_3^2: +\theta_{12}(i\Delta^R_{13}):\phi_1^2\phi_2^2\phi_3:\right]+\cdots$$

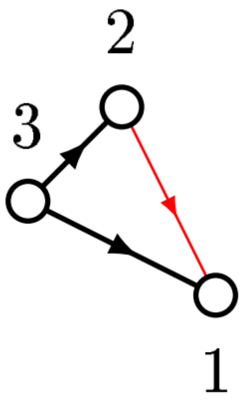
$$(i\Delta^R_{23})\,(i\Delta^R_{13}){\color{red}\theta_{12}} =$$

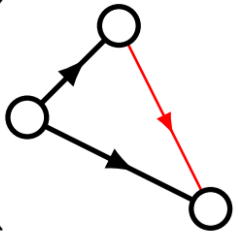
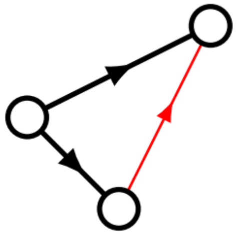
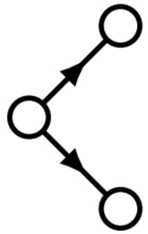


5-points tree-level

$$iN^{(3)} = \frac{(-i)^3}{6}\int_{x_1,x_2,x_3} \theta_{12}\theta_{23}\theta_{13}\left([\mathcal{H}_1,[\mathcal{H}_2,\mathcal{H}_3]] + \cdots\right)$$

$$= \frac{(-i)^3}{6}\int_{x_1,x_2,x_3} (i\Delta^R_{23})\frac{1}{(2!)^2}\left[(i\Delta^R_{12}) :\phi_1^2\phi_2\phi_3^2: +\theta_{12}(i\Delta^R_{13}) :\phi_1^2\phi_2^2\phi_3:\right] + \cdots$$

$(i\Delta^R_{23})\,(i\Delta^R_{13})\theta_{12} =$ 

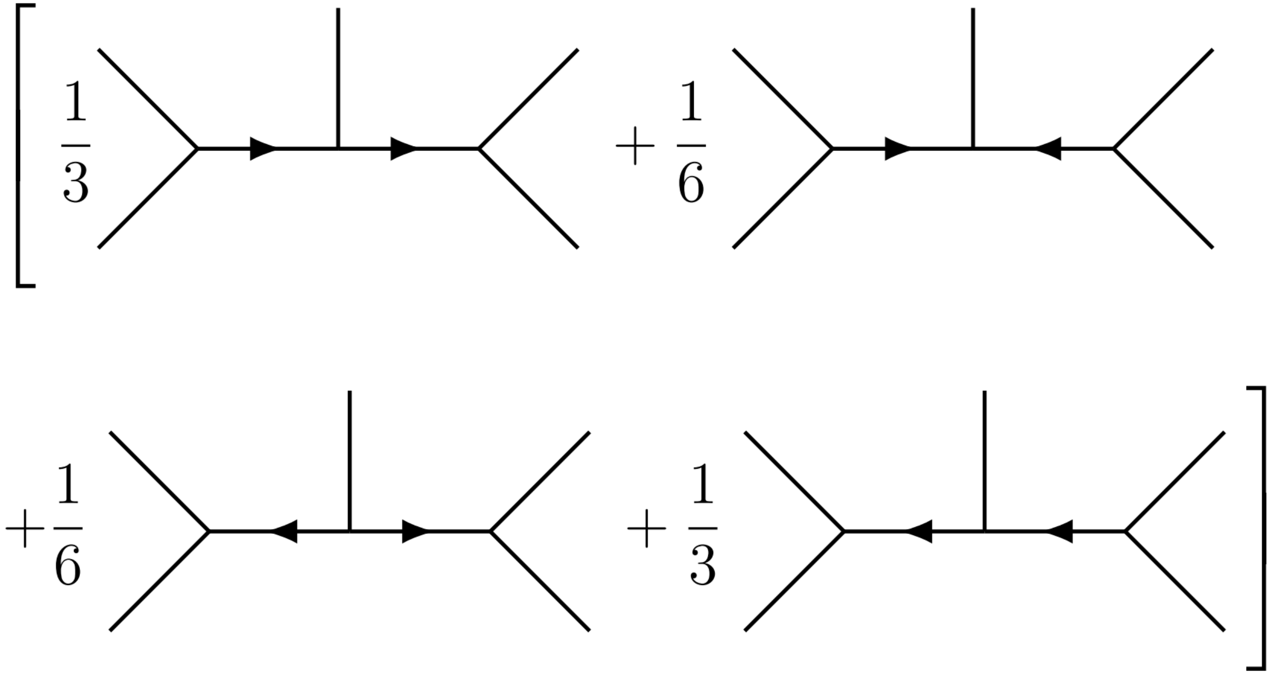
$\rightarrow \frac{1}{2!}\Big($  $+$  $\Big) = \frac{1}{2!}$ 

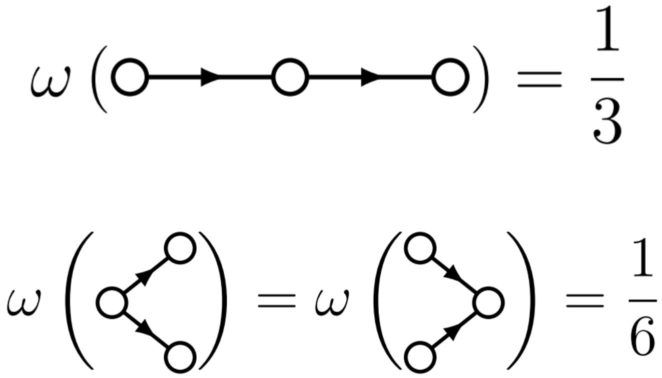
$$\theta_{12} + \theta_{21} = 1$$

5-points tree-level

$$\langle p_3 p_4 p_5 | i N^{(3)} | p_1 p_2 \rangle_{Tree} =$$

$$\sum_{\text{Permutations}}$$



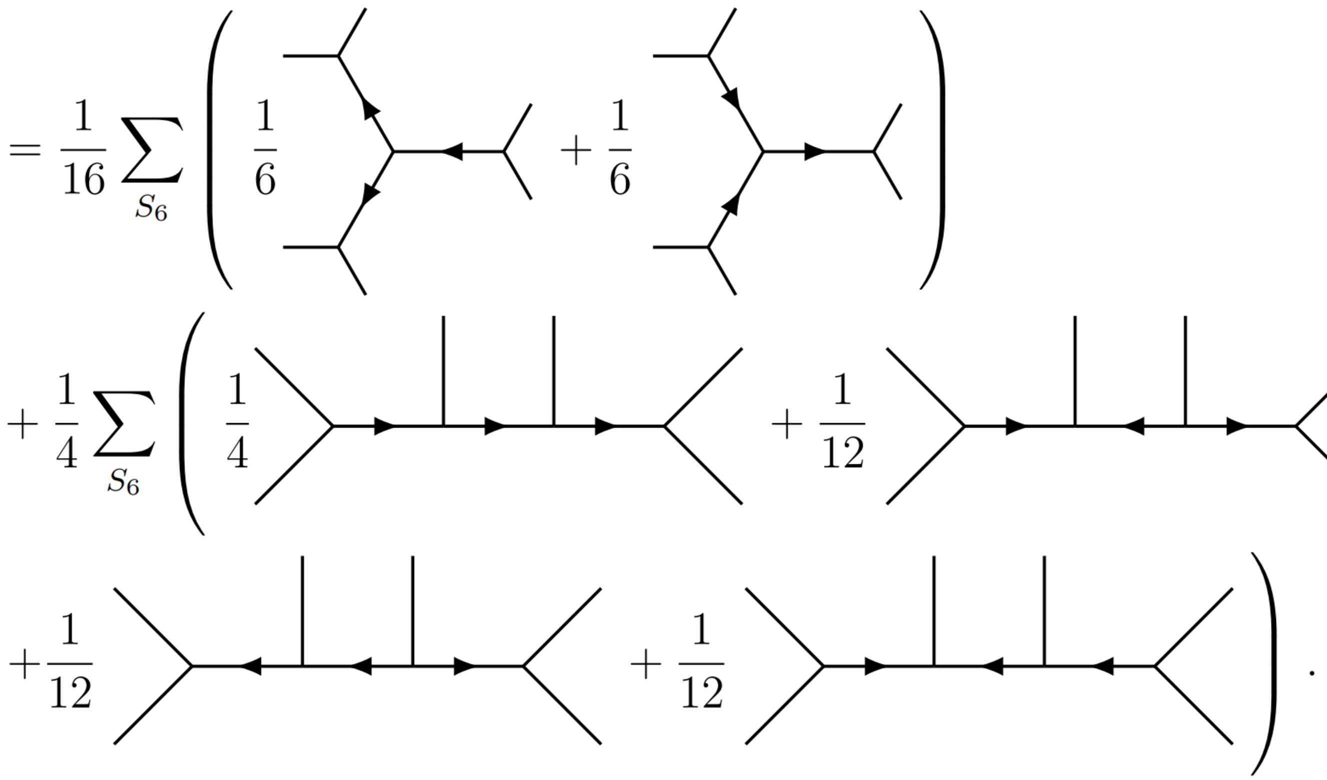


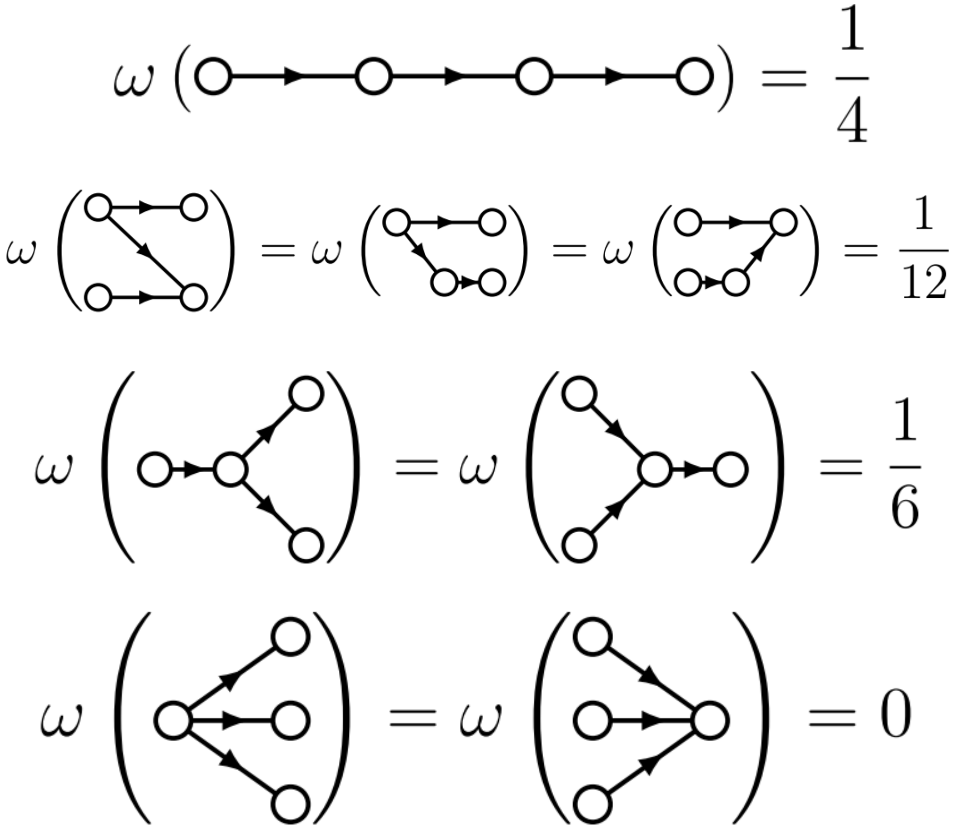
Matches
[Murua’06]
[Kim^3, Lee ’24]

Sum of Murua coefficients at tree-level is 1

6-points tree-level

$$\langle 456|iN^{(4)}|123\rangle$$

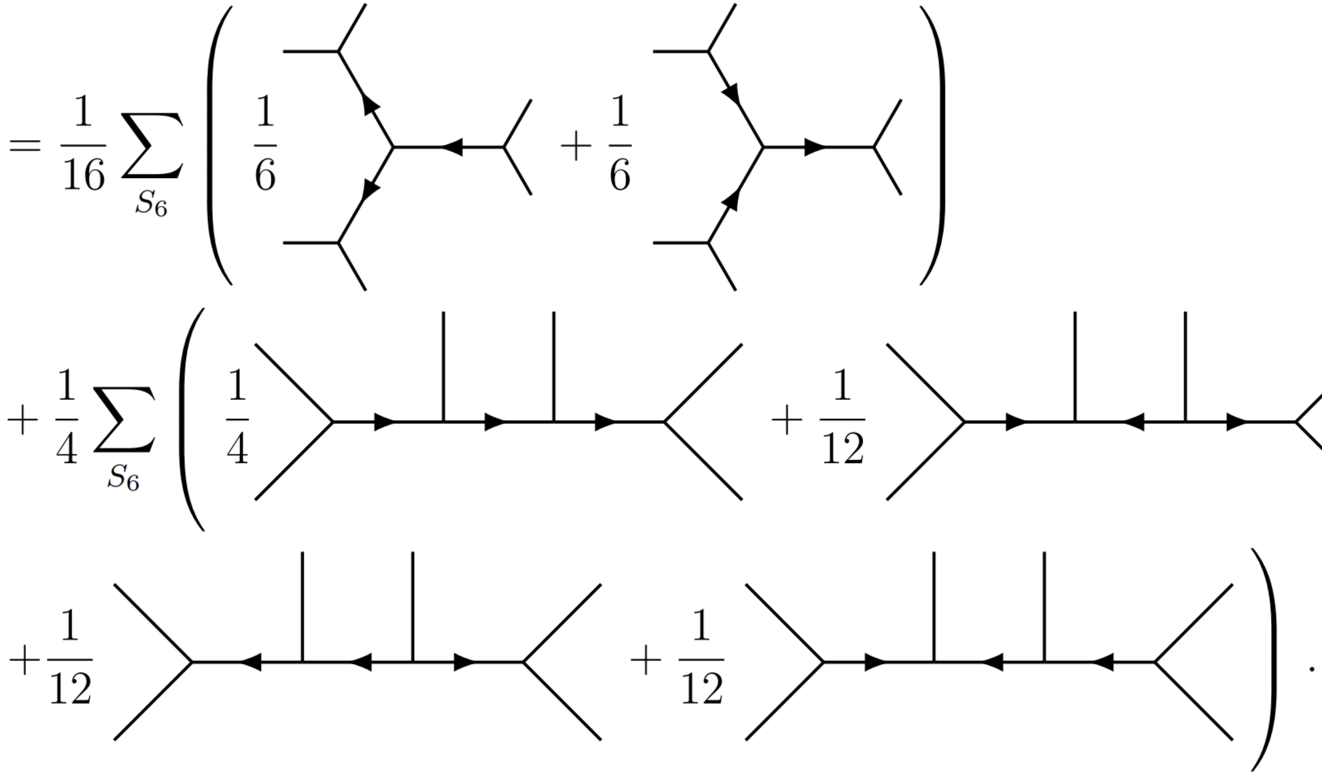


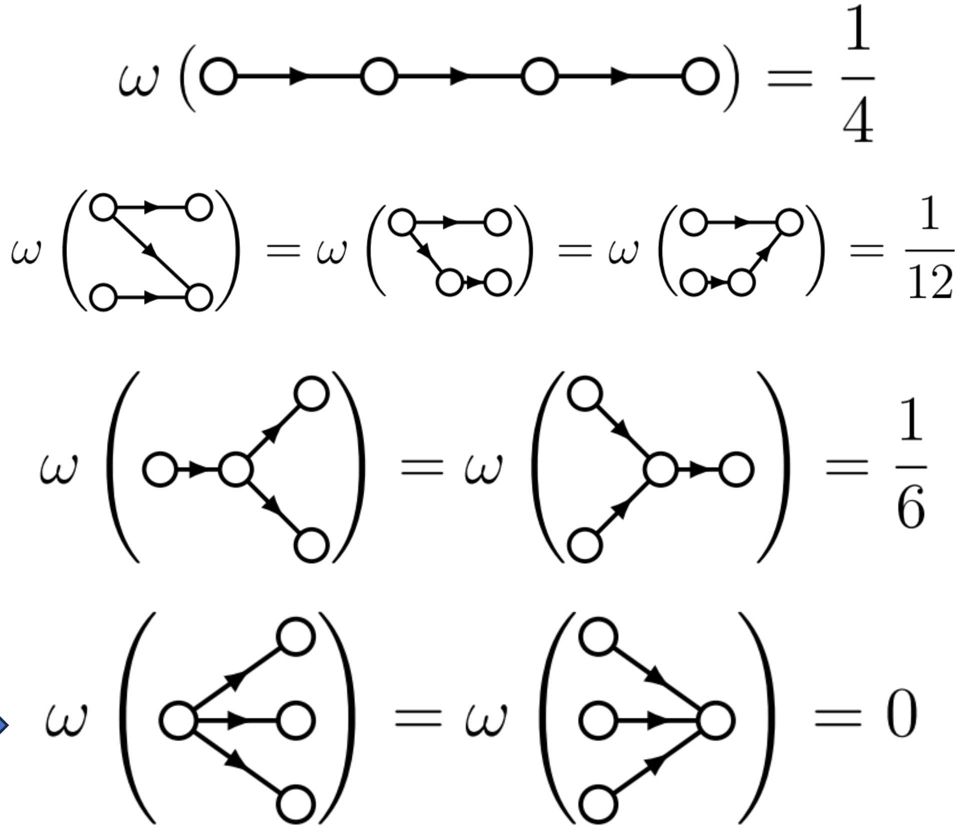


Matches
[Murua'06]
[Kim^3, Lee '24]

6-points tree-level

$$\langle 456|iN^{(4)}|123\rangle$$





Matches
[Murua'06]
[Kim^3, Lee '24]

General Structure: Tree-level

For n vertices n-1 Wick contractions:

$$\Big[\mathcal{H}_1, \big[\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots \big]\Big] \xrightarrow{\substack{n-1 \\ \text{contractions}}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \Big\{\mathcal{H}'_1, \{\mathcal{H}'_2, \cdots \{\mathcal{H}'_{n-1}, \mathcal{H}'_n\} \cdots \}\Big\}$$

General Structure: Tree-level

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$$\Big\downarrow \quad :\{\cdots\}:, \ \times\theta_{ij}$$

$$(i\Delta^{R/A}_{1a_1}) \cdots (i\Delta^{R/A}_{n-1n}) :\mathcal{H}'_1\mathcal{H}'_2 \cdots \mathcal{H}'_{n-1}\mathcal{H}'_n:$$

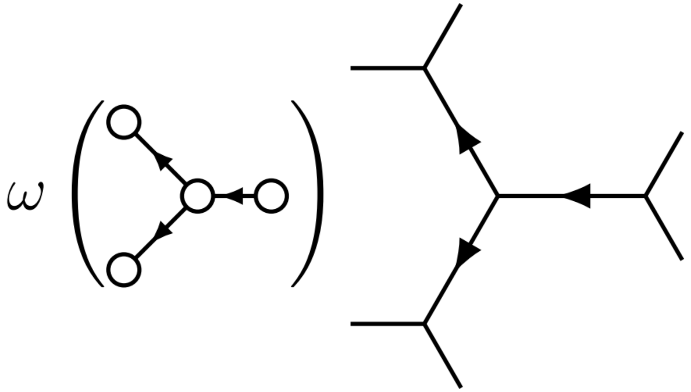
General Structure: Tree-level

For n vertices n-1 Wick contractions:

$$\left[\mathcal{H}_1, \left[\mathcal{H}_2, \cdots \left[\mathcal{H}_{n-1}, \mathcal{H}_n\right] \cdots\right]\right] \xrightarrow{\substack{n-1 \\ \text{contractions}}} (i\Delta_{1a_1})\cdots(i\Delta_{n-1n})\left\{\mathcal{H}'_1, \{\mathcal{H}'_2, \cdots \{\mathcal{H}'_{n-1}, \mathcal{H}'_n\}\cdots\}\right\}$$

$$\Big\downarrow \; :\{\cdots\}:, \; \times\theta_{ij}$$

$$(i\Delta^{R/A}_{1a_1})\cdots(i\Delta^{R/A}_{n-1n}) :\mathcal{H}'_1\mathcal{H}'_2\cdots\mathcal{H}'_{n-1}\mathcal{H}'_n:$$



Only retarded or advanced props at tree-level.

Coefficients of each diagram given by [Murua'06] [Kim^3, Lee '24]

1-loop

One extra Wick contraction beyond tree

$$\Big[\mathcal{H}_1, \big[\mathcal{H}_2, \cdots \big[\mathcal{H}_{n-1}, \mathcal{H}_n\big] \cdots \big]\Big] \xrightarrow{\substack{n-1 \\ \text{contractions}}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1 n}) \Big\{\mathcal{H}'_1, \big\{\mathcal{H}'_2, \cdots \big\{\mathcal{H}'_{n-1}, \mathcal{H}'_n\big\} \cdots \big\}\Big\}$$

1-loop

One extra Wick contraction beyond tree

$$\Big[\mathcal{H}_1, \big[\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots \big]\Big] \xrightarrow{\substack{n-1 \\ \text{contractions}}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \Big\{\mathcal{H}'_1, \{\mathcal{H}'_2, \cdots \{\mathcal{H}'_{n-1}, \mathcal{H}'_n\} \cdots \}\Big\}$$

$$\{\mathcal{H}_i, \mathcal{H}_j\} = \frac{1}{2}\Delta^{(1)}_{ij}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2} i\Delta_{ij}[\mathcal{H}'_i, \mathcal{H}'_j]$$

One more contraction

1-loop

One extra Wick contraction beyond tree

$$\left[\mathcal{H}_1, \left[\mathcal{H}_2, \cdots \left[\mathcal{H}_{n-1}, \mathcal{H}_n\right] \cdots \right]\right] \xrightarrow[\text{contractions}]{n-1} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \left\{\mathcal{H}'_1, \left\{\mathcal{H}'_2, \cdots \left\{\mathcal{H}'_{n-1}, \mathcal{H}'_n\right\} \cdots \right\}\right\}$$

$$\{\mathcal{H}_i, \mathcal{H}_j\} = \frac{1}{2}\Delta^{(1)}_{ij}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2} i\Delta_{ij}[\cancel{\mathcal{H}'_i,} \mathcal{H}'_j]$$

One more contraction

$$:\{\cdots\}:, \quad \times\theta_{ij}$$

$$(i\Delta^{(1)}_{2n})(i\Delta^{R/A}_{1a_1}) \cdots (i\Delta^{R/A}_{n-1n}) : \mathcal{H}'_1\mathcal{H}''_2 \cdots \mathcal{H}'_{n-1}\mathcal{H}''_n :,$$

1-loop

One extra Wick contraction beyond tree

$$\left[\mathcal{H}_1, \left[\mathcal{H}_2, \cdots \left[\mathcal{H}_{n-1}, \mathcal{H}_n\right]\cdots\right]\right] \xrightarrow{\substack{n-1 \\ \text{contractions}}} (i\Delta_{1a_1})\cdots(i\Delta_{n-1n})\left\{\mathcal{H}'_1, \{\mathcal{H}'_2, \cdots\{\mathcal{H}'_{n-1},\mathcal{H}'_n\}\cdots\}\right\}$$

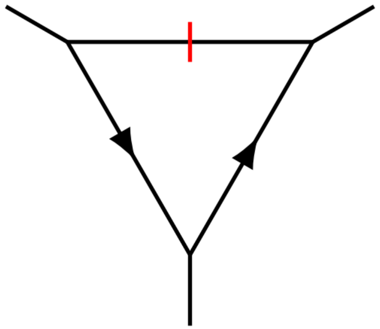
$$\{\mathcal{H}_i,\mathcal{H}_j\} = \frac{1}{2}\Delta^{(1)}_{ij}\{\mathcal{H}'_i,\mathcal{H}'_j\} + \frac{1}{2}i\Delta_{ij}[\cancel{\mathcal{H}'_i,}\mathcal{H}'_j]$$

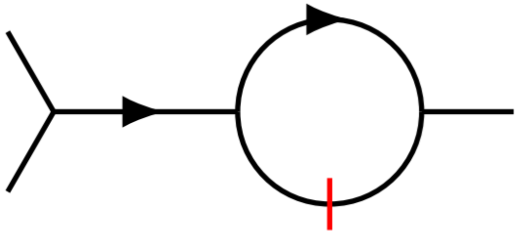
One more contraction

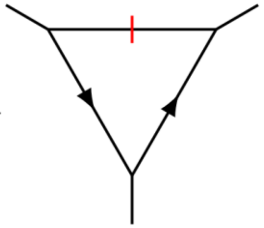
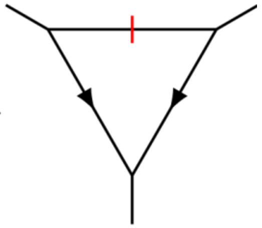
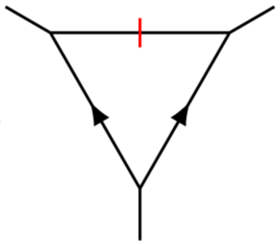
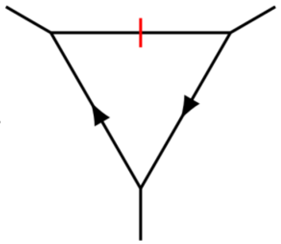
$:\{\cdots\}:, \quad \times\theta_{ij}$

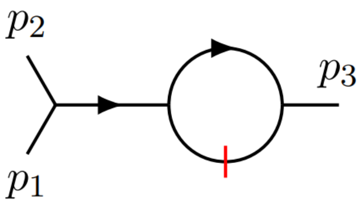
Exactly one cut propagator at 1-loop!

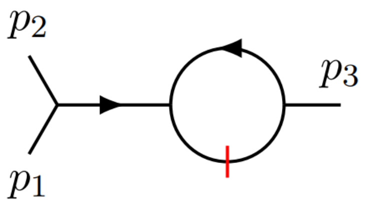
$$(i\Delta^{(1)}_{2n})(i\Delta^{R/A}_{1a_1})\cdots(i\Delta^{R/A}_{n-1n}) : \mathcal{H}'_1\mathcal{H}''_2\cdots\mathcal{H}'_{n-1}\mathcal{H}''_n :,$$

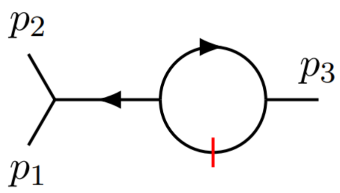


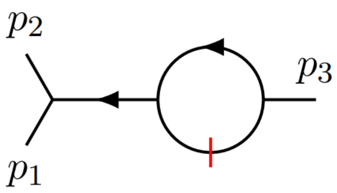


$\langle 23|iN^{(3)}|1\rangle_{1-loop} = \sum_{C_3} \Big[\frac{1}{6}$  $+ \frac{1}{12}$  $+ \frac{1}{12}$  $+ \frac{1}{6}$  $\Big]$

$+ \Big[\frac{1}{6}$ 

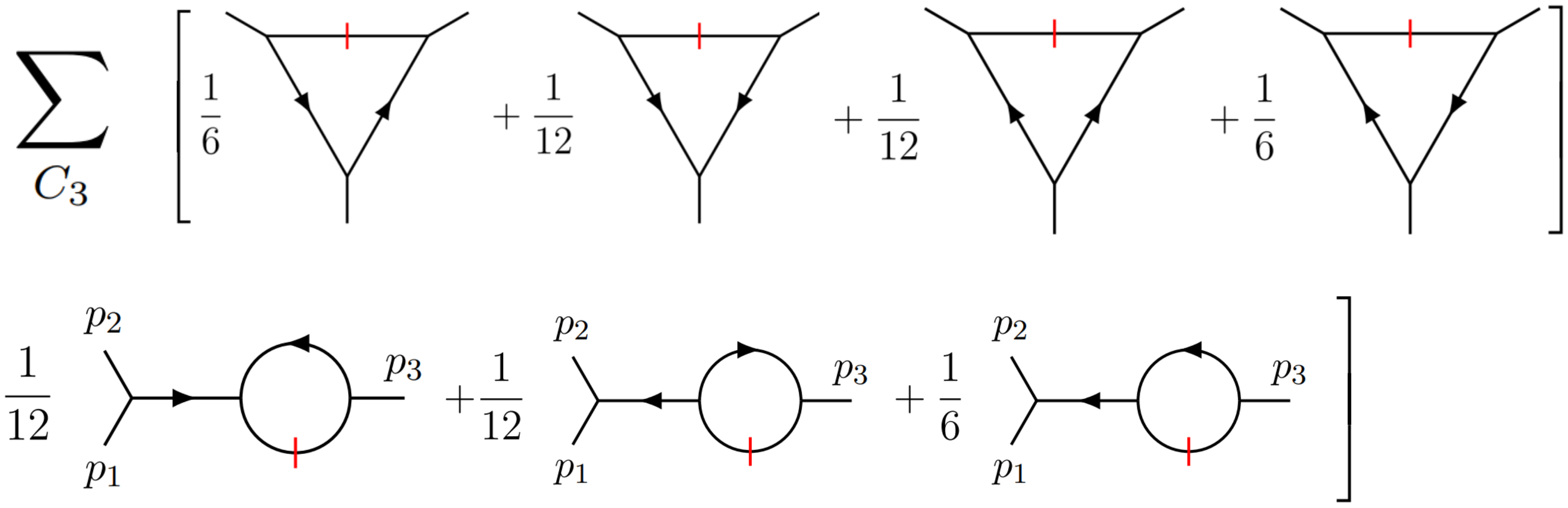
$+ \frac{1}{12}$ 

$+ \frac{1}{12}$ 

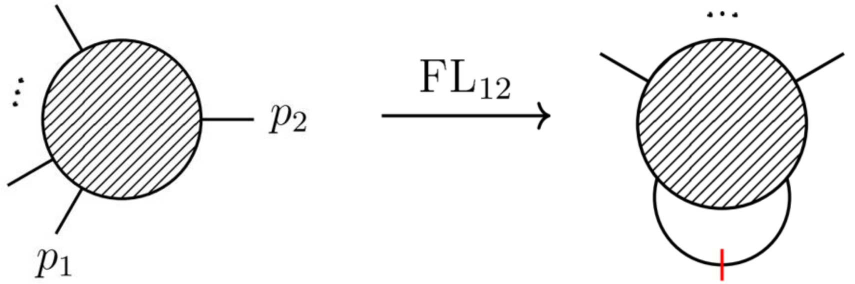
$+ \frac{1}{6}$ 

$\Big]$

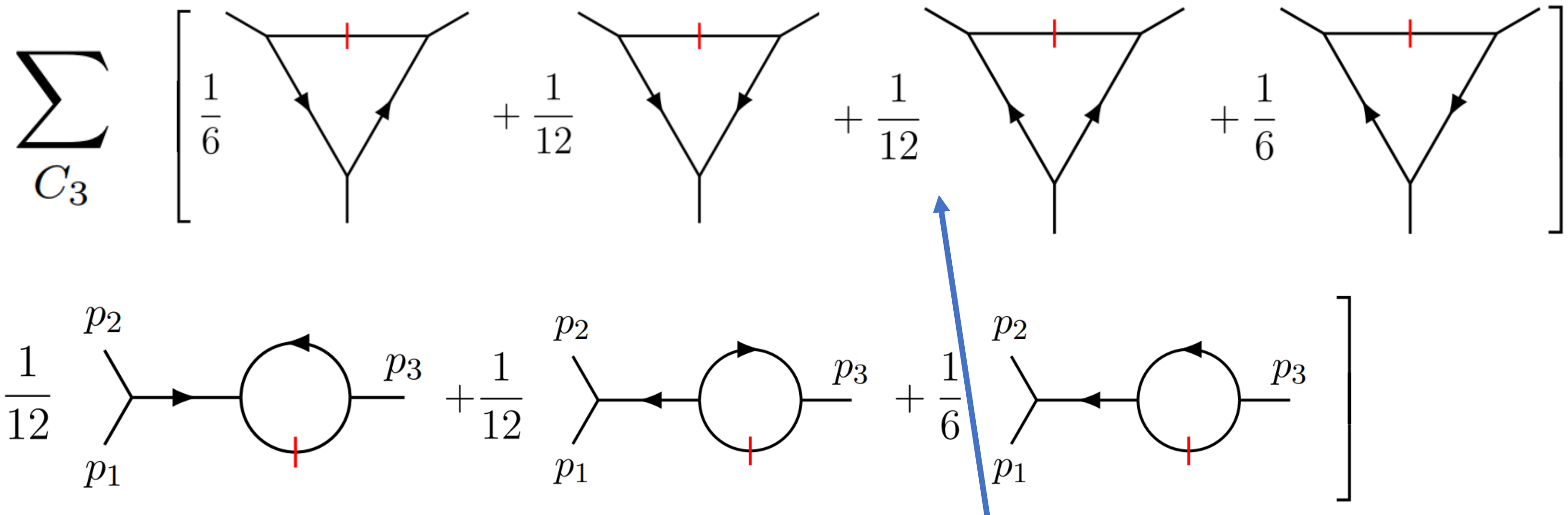
$$\langle 23|iN^{(3)}|1\rangle_{1-loop} =$$



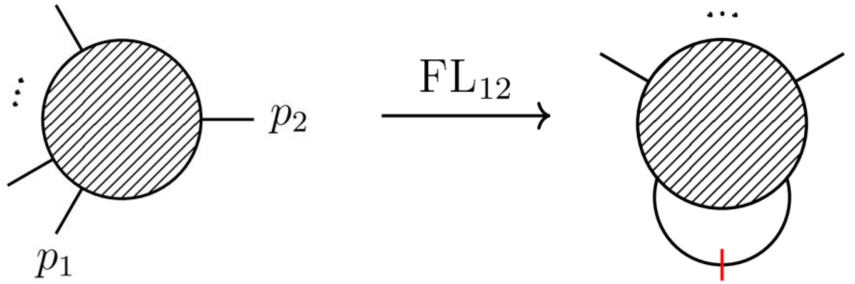
Forward limit:

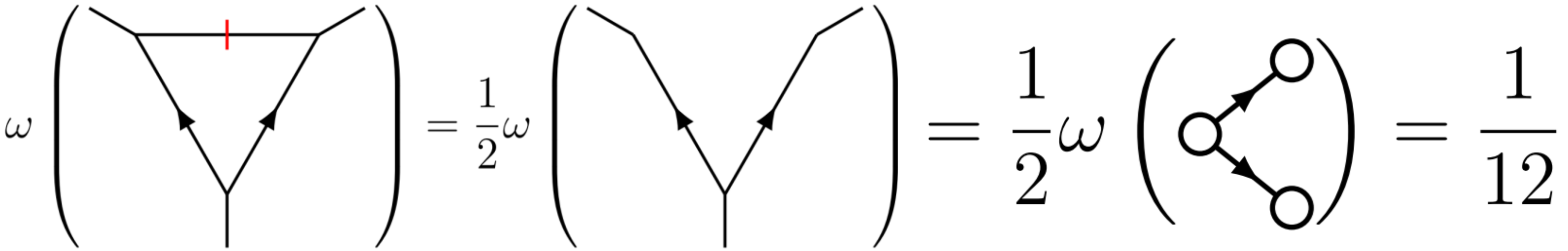


$$\langle 23|iN^{(3)}|1\rangle_{1-loop} = \sum_{C_3}$$



Forward limit:





2-loops

Two extra Wick contraction beyond tree

$$\Big[\mathcal{H}_1, \big[\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots \big]\Big] \xrightarrow{\substack{n-1 \\ \text{contractions}}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \Big\{\mathcal{H}'_1, \{\mathcal{H}'_2, \cdots \{\mathcal{H}'_{n-1}, \mathcal{H}'_n\} \cdots \}\Big\}$$

2-loops

$$[\overline{\mathcal{H}_i, \mathcal{H}_j}] = \frac{1}{2} i\Delta_{ij}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2}\Delta^{(1)}_{ij}[\mathcal{H}'_i, \mathcal{H}'_j]$$

$$\{\overline{\mathcal{H}_i, \mathcal{H}_j}\} = \frac{1}{2}\Delta^{(1)}_{ij}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2} i\Delta_{ij}[\mathcal{H}'_i, \mathcal{H}'_j]$$

Two extra Wick contraction beyond tree

$$\Big[\mathcal{H}_1, [\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots]\Big] \xrightarrow[\text{contractions}]{n-1} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \Big\{\mathcal{H}'_1, \{\mathcal{H}'_2, \cdots \{\mathcal{H}'_{n-1}, \mathcal{H}'_n\} \cdots \}\Big\}$$

$$\Bigg\{$$

2-loops

$$[\overbrace{\mathcal{H}_i,\mathcal{H}_j}] = \frac{1}{2} i\Delta_{ij}\{\mathcal{H}_i',\mathcal{H}_j'\} + \frac{1}{2}\Delta_{ij}^{(1)}[\mathcal{H}_i',\mathcal{H}_j']$$

$$\{\overbrace{\mathcal{H}_i,\mathcal{H}_j}\} = \frac{1}{2}\Delta_{ij}^{(1)}\{\mathcal{H}_i',\mathcal{H}_j'\} + \frac{1}{2} i\Delta_{ij}[\mathcal{H}_i',\mathcal{H}_j']$$

Two extra Wick contraction beyond tree

$$\left[\mathcal{H}_1, \left[\mathcal{H}_2, \cdots \left[\mathcal{H}_{n-1}, \mathcal{H}_n\right] \cdots \right]\right] \xrightarrow[\text{contractions}]{n-1} (i\Delta_{1a_1})\cdots(i\Delta_{n-1n})\left\{\mathcal{H}_1', \left\{\mathcal{H}_2', \cdots \left\{\mathcal{H}_{n-1}', \mathcal{H}_n'\right\} \cdots \right\}\right\}$$

$$\left\{ \begin{array}{l} \{\_\,,\,\_\} \to \Delta_{ij}^{(1)}\{\_\,,\,\_\} \to \Delta_{ij}^{(1)}\Delta_{kl}^{(1)}\{\_\,,\,\_\} \\ \\ \end{array} \right.$$

2-loops

$$[\mathcal{H}_i, \mathcal{H}_j] = \frac{1}{2} i\Delta_{ij}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2}\Delta^{(1)}_{ij}[\mathcal{H}'_i, \mathcal{H}'_j]$$

$$\{\mathcal{H}_i, \mathcal{H}_j\} = \frac{1}{2}\Delta^{(1)}_{ij}\{\mathcal{H}'_i, \mathcal{H}'_j\} + \frac{1}{2} i\Delta_{ij}[\mathcal{H}'_i, \mathcal{H}'_j]$$

Two extra Wick contraction beyond tree

$$\Big[\mathcal{H}_1, \big[\mathcal{H}_2, \cdots [\mathcal{H}_{n-1}, \mathcal{H}_n] \cdots\big]\Big] \xrightarrow[\text{contractions}]{n-1} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \Big\{\mathcal{H}'_1, \{\mathcal{H}'_2, \cdots \{\mathcal{H}'_{n-1}, \mathcal{H}'_n\} \cdots\}\Big\}$$

$$\begin{cases} \{\_, \_\} \to \Delta^{(1)}_{ij}\{\_, \_\} \to \Delta^{(1)}_{ij}\Delta^{(1)}_{kl}\{\_, \_\} \\ \{\_, \_\} \to \Delta_{ij}[\_, \_] \to \Delta_{ij}\Delta_{ik}\{\_, \_\} \end{cases}$$

2-loops

$$[\overline{\mathcal{H}_i, \mathcal{H}_j}] = \frac{1}{2} i\Delta_{ij}\{\mathcal{H}_i', \mathcal{H}_j'\} + \frac{1}{2}\Delta_{ij}^{(1)}[\mathcal{H}_i', \mathcal{H}_j']$$

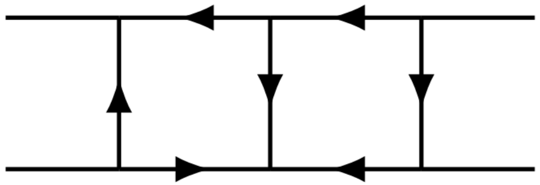
$$\{\overline{\mathcal{H}_i, \mathcal{H}_j}\} = \frac{1}{2}\Delta_{ij}^{(1)}\{\mathcal{H}_i', \mathcal{H}_j'\} + \frac{1}{2} i\Delta_{ij}[\mathcal{H}_i', \mathcal{H}_j']$$

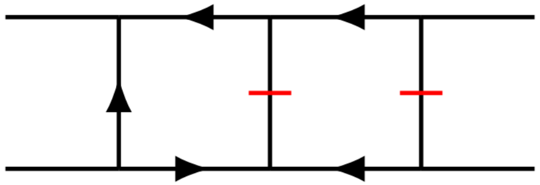
Two extra Wick contraction beyond tree

$$\Big[\mathcal{H}_1, \big[\mathcal{H}_2, \cdots \big[\mathcal{H}_{n-1}, \mathcal{H}_n\big] \cdots \big]\Big] \xrightarrow{\substack{n-1 \\ \text{contractions}}} (i\Delta_{1a_1}) \cdots (i\Delta_{n-1n}) \Big\{\mathcal{H}_1', \big\{\mathcal{H}_2', \cdots \big\{\mathcal{H}_{n-1}', \mathcal{H}_n'\big\} \cdots \big\}\Big\}$$

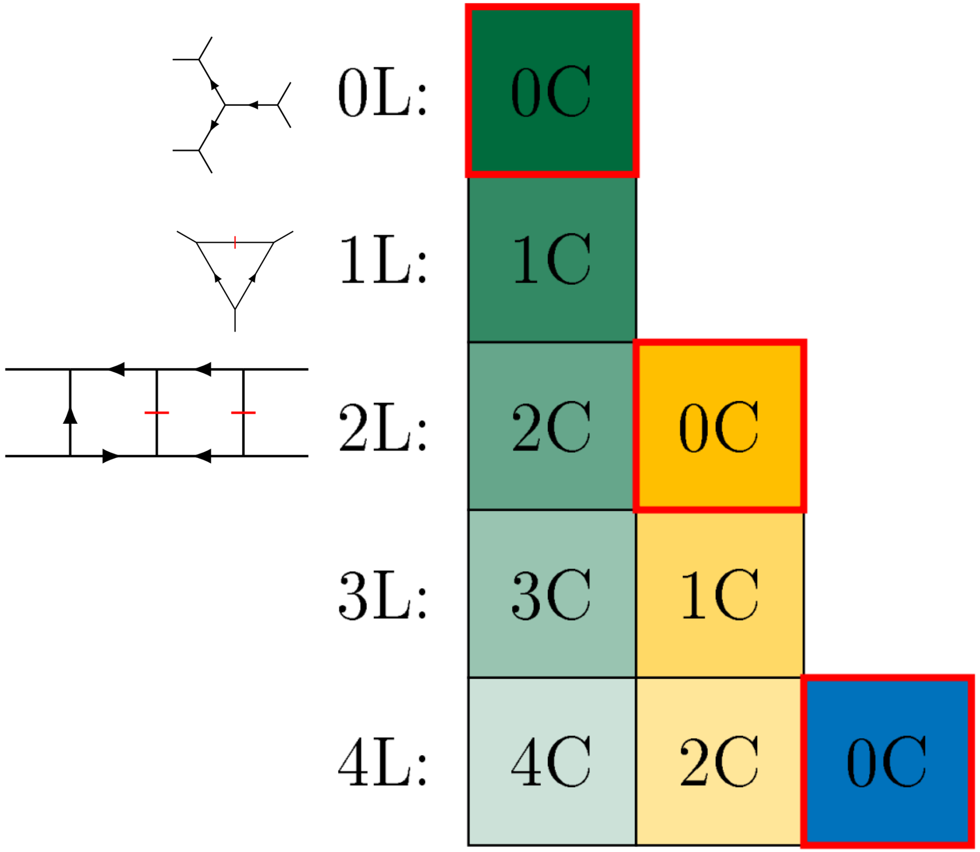
$$\begin{cases} \{\_, \_\} \to \Delta_{ij}^{(1)}\{\_, \_\} \to \Delta_{ij}^{(1)}\Delta_{kl}^{(1)}\{\_, \_\} \\ \{\_, \_\} \to \Delta_{ij}[\_, \_] \to \Delta_{ij}\Delta_{ik}\{\_, \_\} \end{cases}$$

Exactly 0 cuts or 2 cuts at 2-loop!

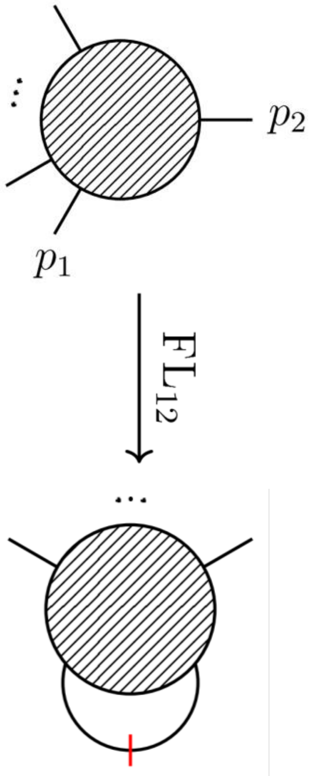


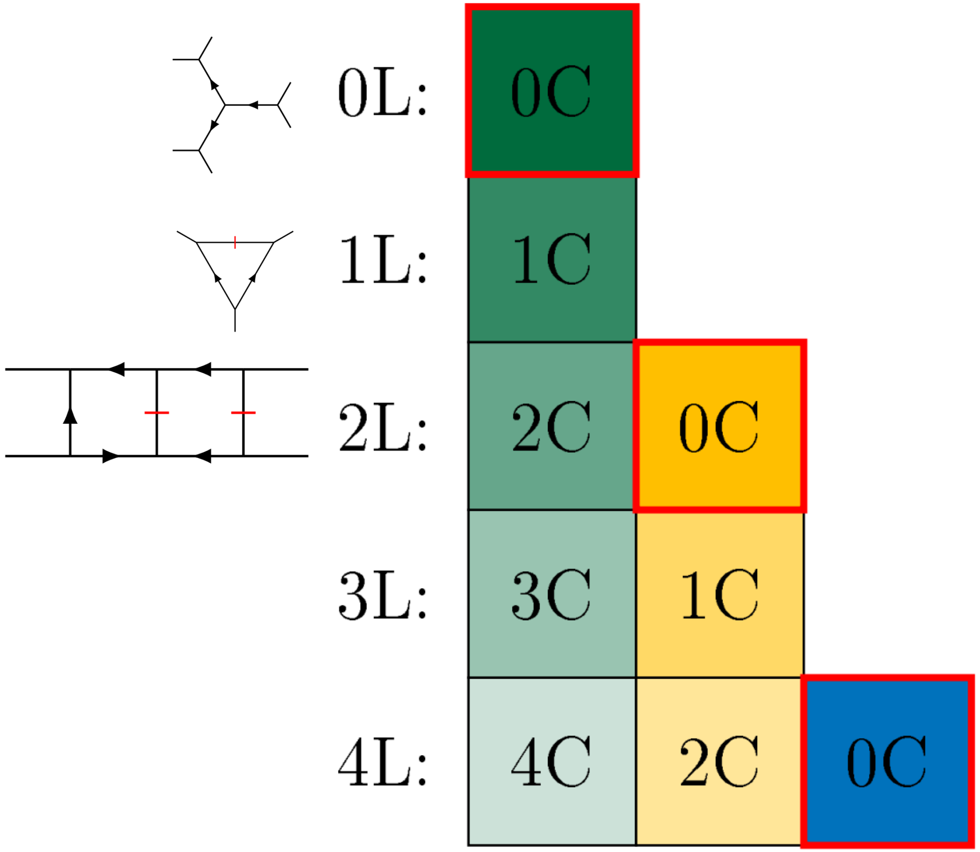


General loop structure



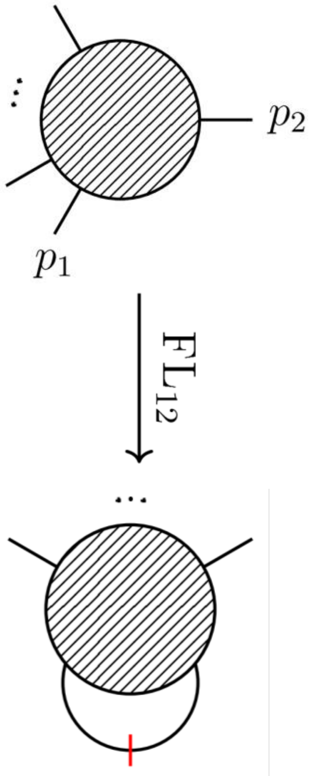
General loop structure

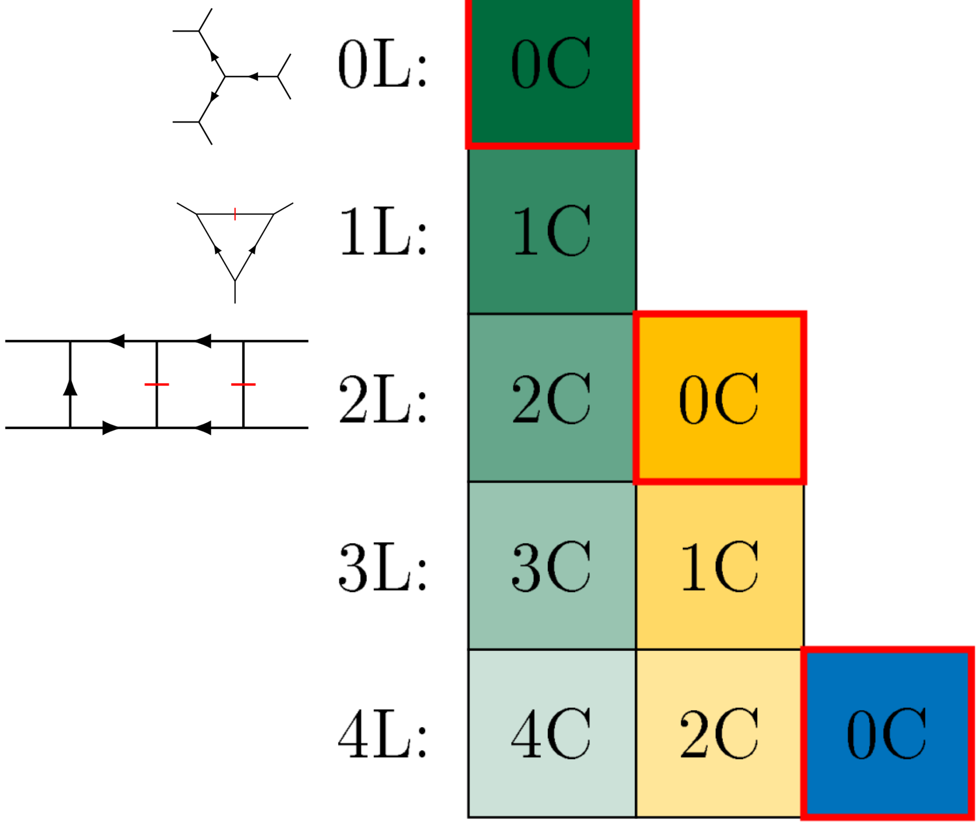




General loop structure

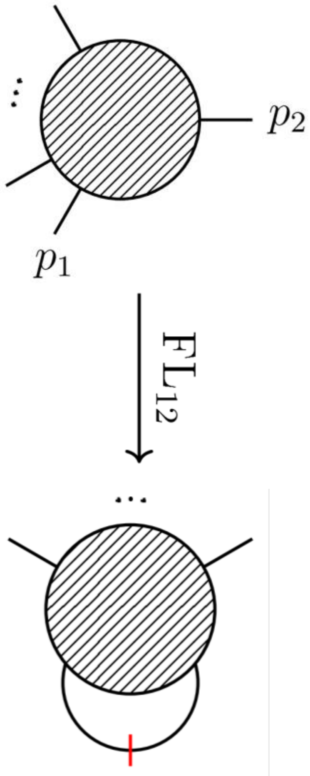
Tree-level Murua coeff:
[Kim^3, Lee ’24]

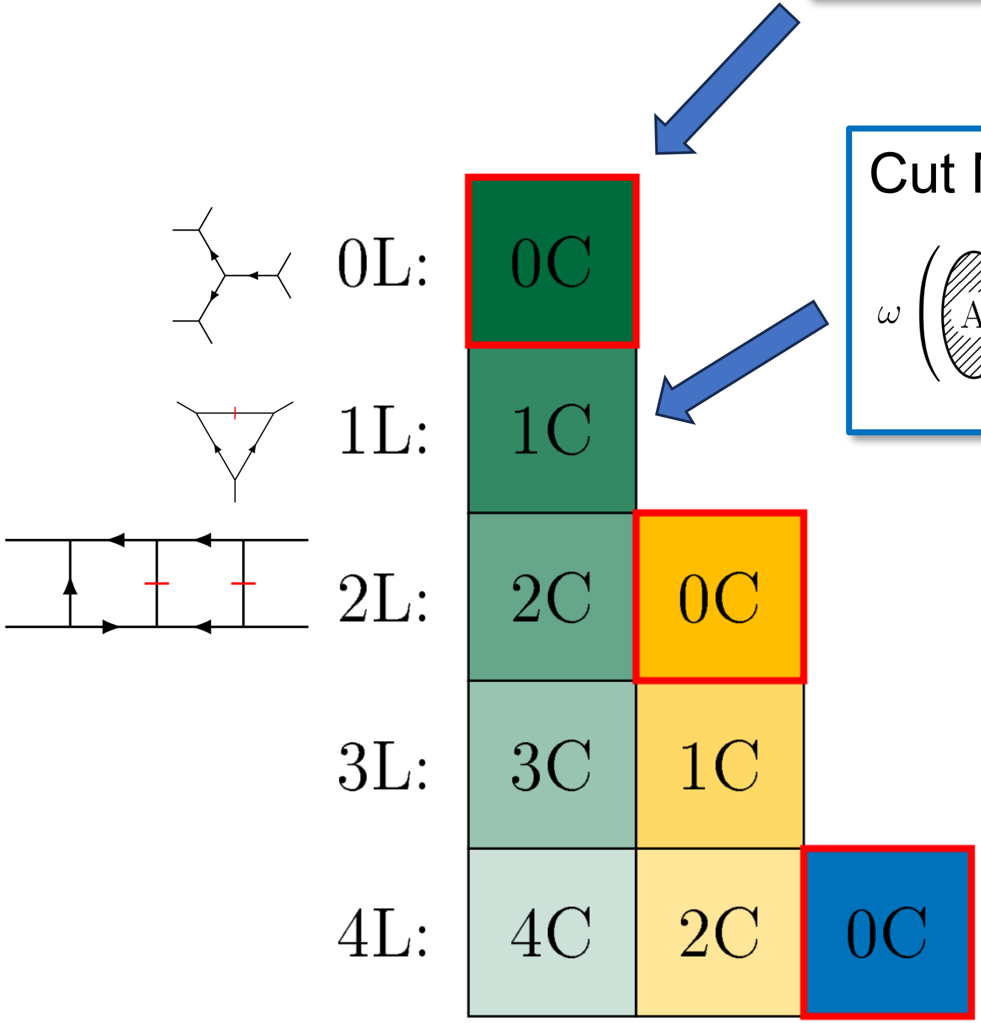




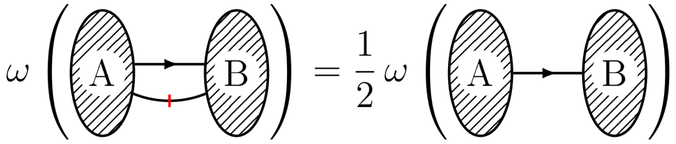
General loop structure

Tree-level Murua coeff:
[Kim^3, Lee '24]





Cut Murua coeffs from:



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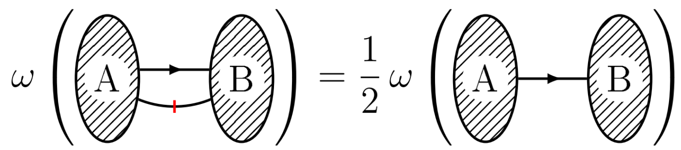
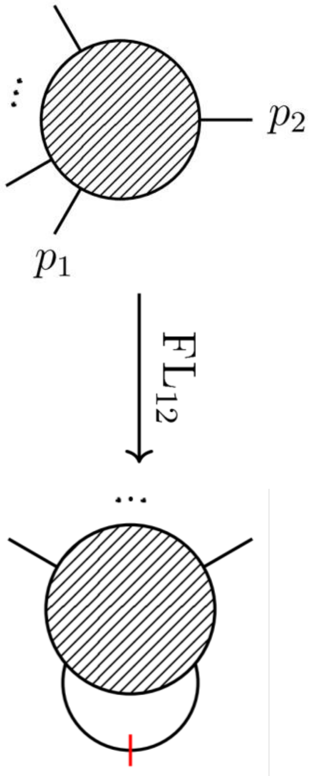
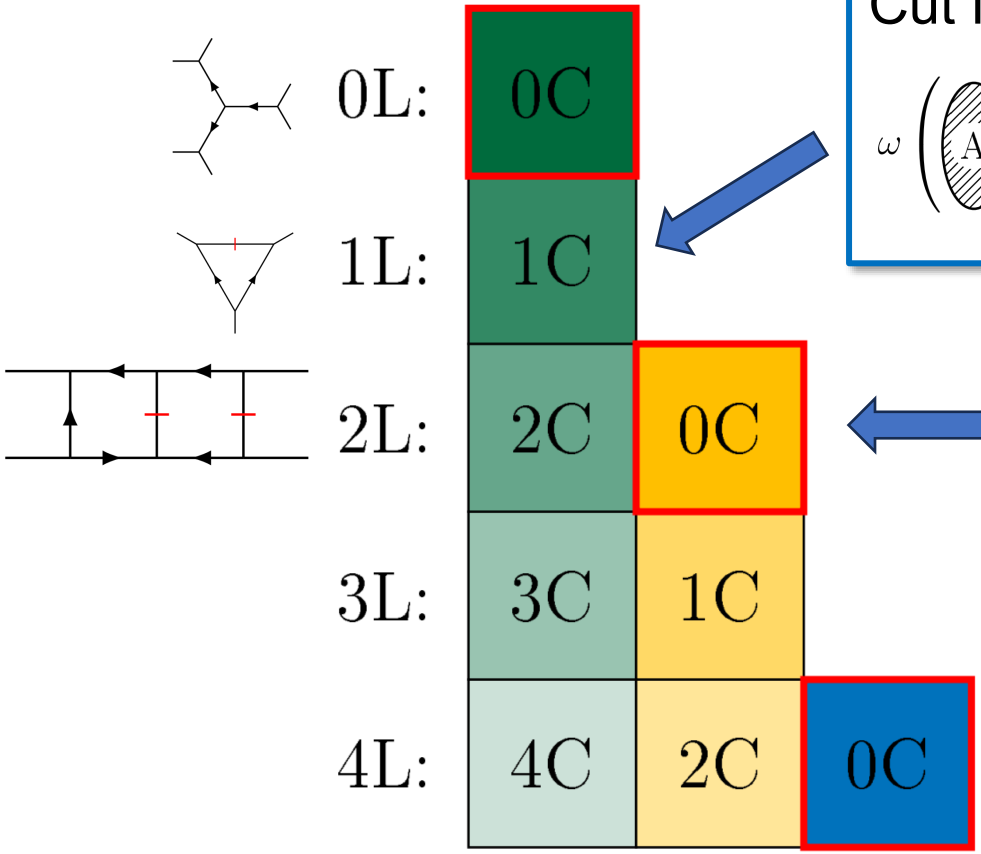


Higher-loop zero cut coeffs:
[Kim ‘25][Guo, Kim^3, Lee ‘26]

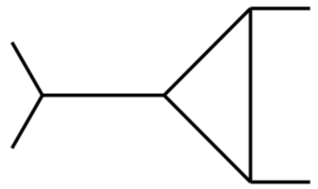




Magnus Feynman Rules

Magnus Feynman Rules

Step 1:
Draw all diagrams without arrows

. . .  . . .

Magnus Feynman Rules

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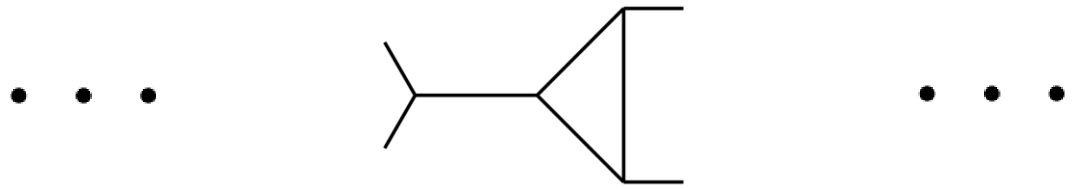
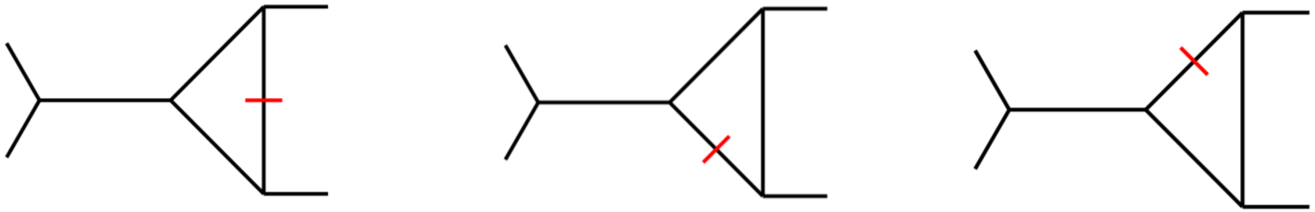


Step 2:
Add cuts in all possible numbers and ways



Magnus Feynman Rules

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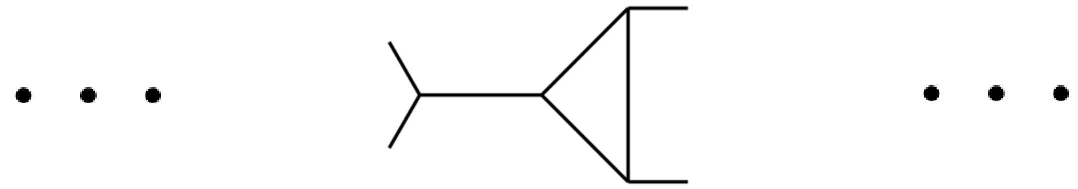

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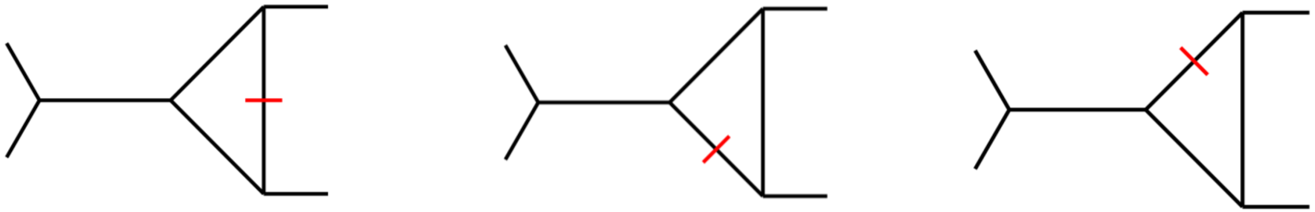
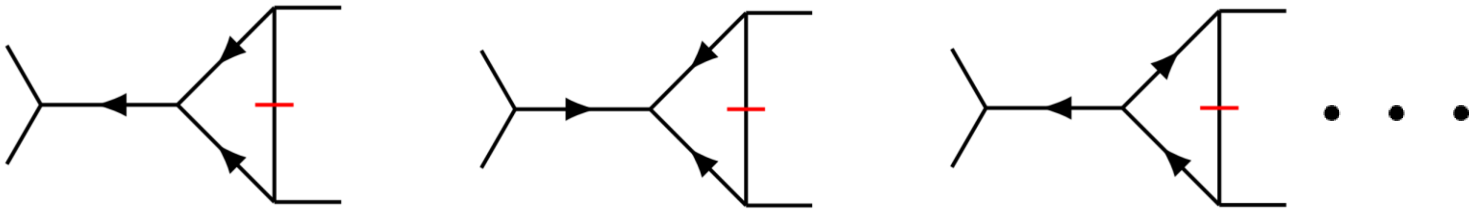


<u>Step 3:</u>
Add arrows in all possible ways



Magnus Feynman Rules

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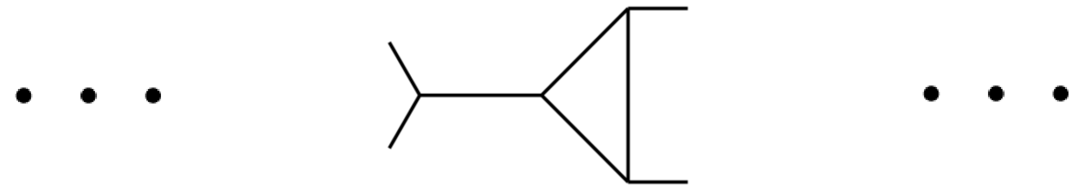

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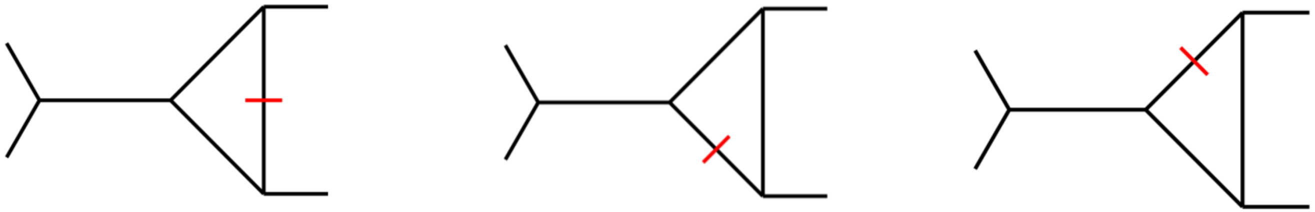

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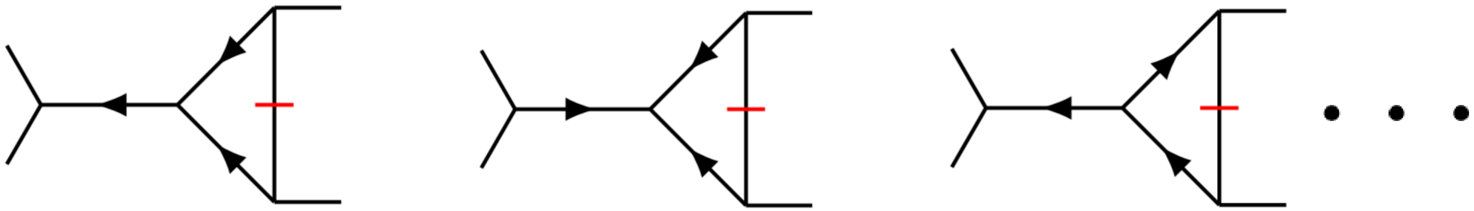
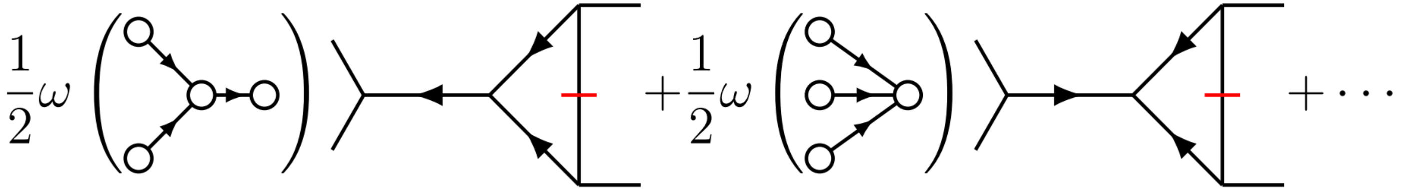


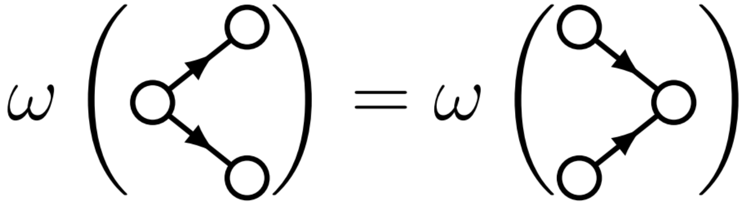
Step 4:
Multiply each graph by its Mura Coefficient



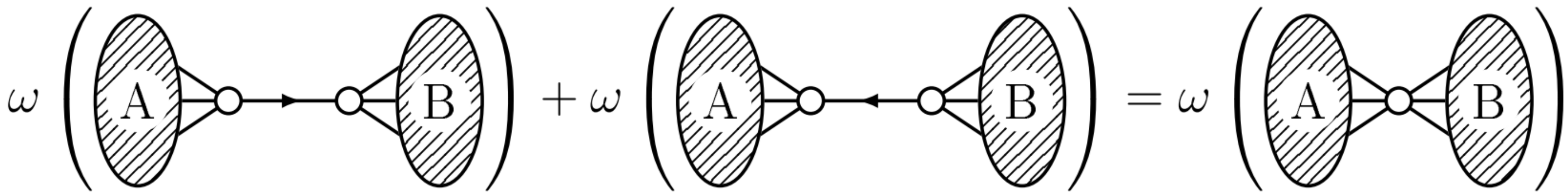
Murua properties

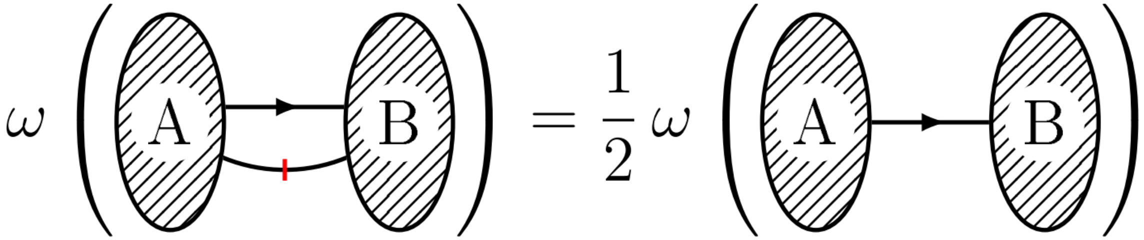
$$\omega\left(\circ\right) = 1\,,$$

Swapped Graphs have the same Murua coefficients



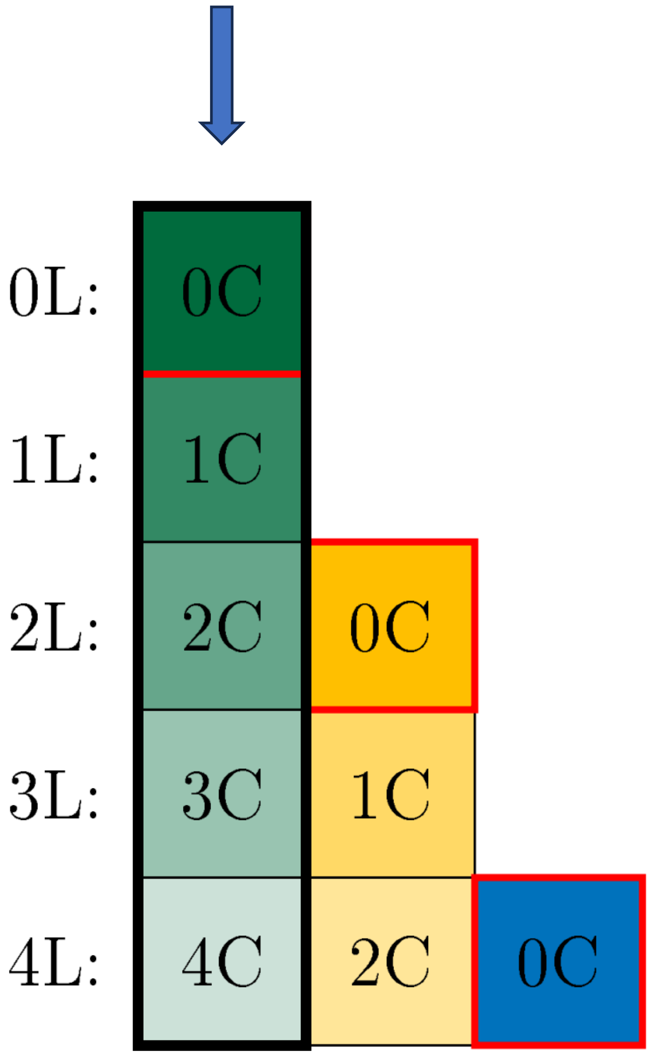
[Kim^3, Lee ’24]





Classical limit

<u>Conjecture:</u>
Only the maximal cut pieces contribute in the classical limit of N



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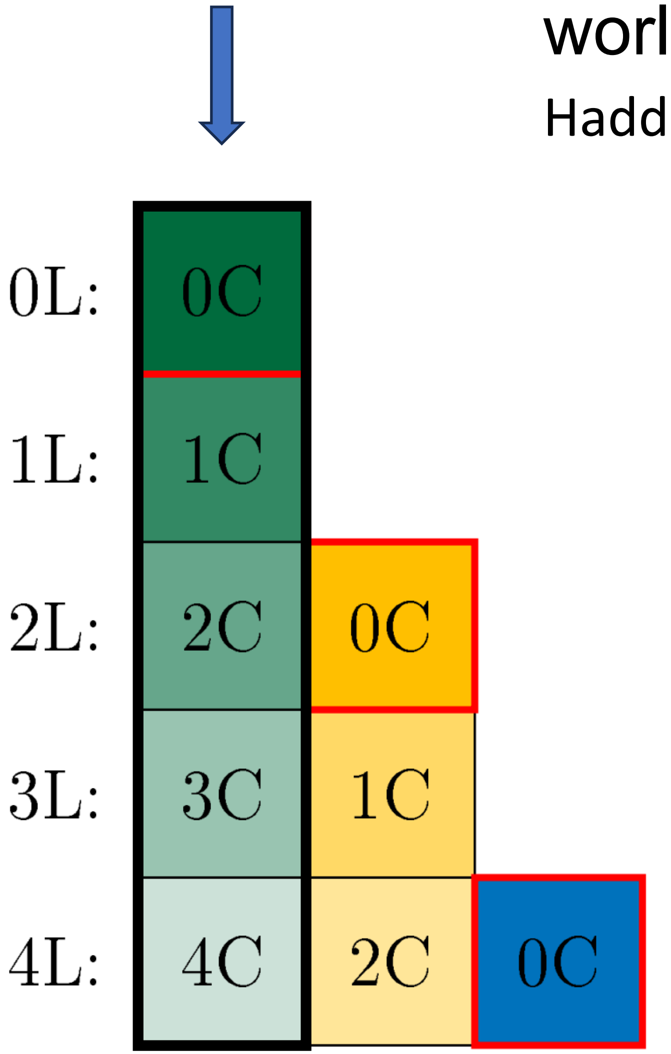
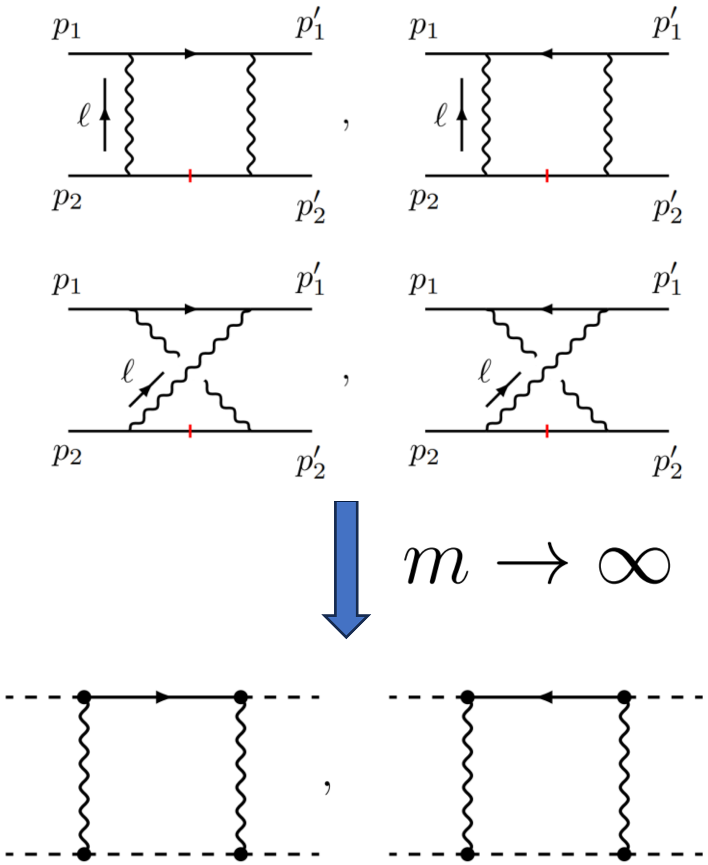


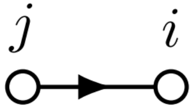
Matches intuition from worldlines [Mogull, Plefka, Steinhoff; Haddad, Jakobsen, Mogull, Plefka]

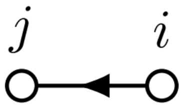


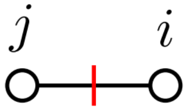
Conclusions

$$\boxed{S = \exp(iN)}$$

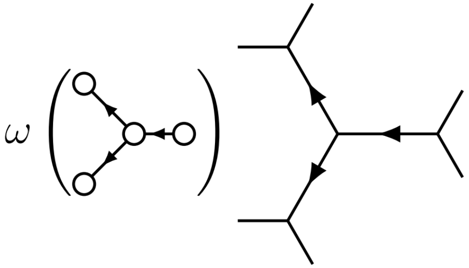
- The Magnus expansion has three types of propagators:

$i\Delta^R_{ij} =$ 

$i\Delta^A_{ij} =$ 

$\Delta^{(1)}_{ij} =$ 

- Only very specific numbers of cuts can appear.



- Proved the all-loop structure of the Magnus expansion.

What now?

- Proof of classical limit conjecture

- Soft limits, infrared divergences, exponentiation…

- Doing the integrals, generalised unitarity …

Magnus Feynman Rules

Thank you!

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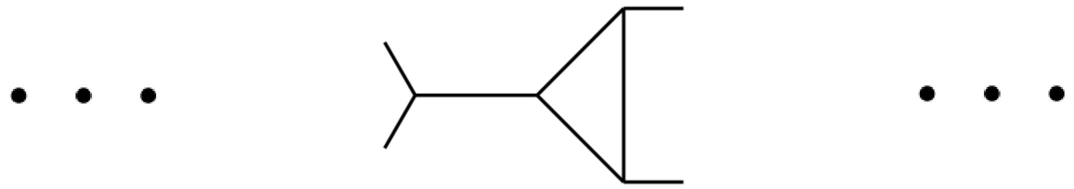

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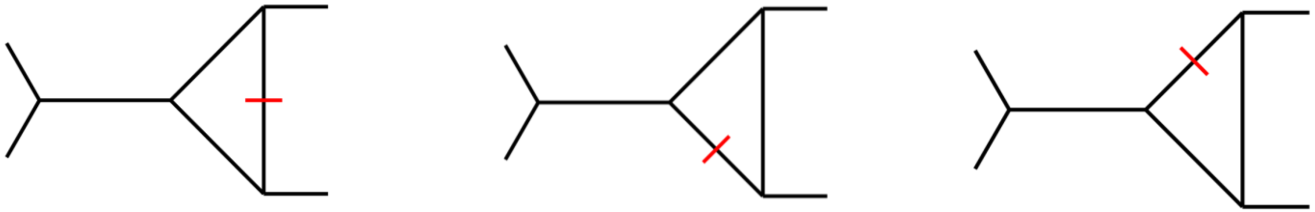

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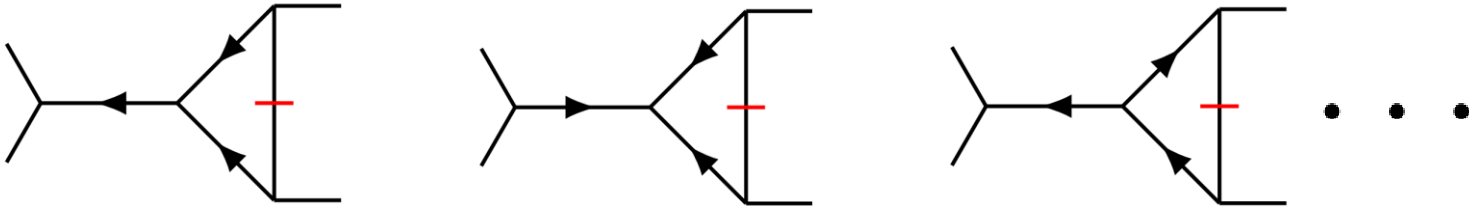

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