Classical scattering from QFT via phase space dequantization

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Introduction

- Scattering-amplitude techniques play an important role in gravitational-wave physics.
  [See talks: Kosower, Jakobsen, Bohnenblust, Chen, Isabella, Brown]
- Several formalisms to extract observables: KMOC, WQFT, WEFT, HEFT, eikonal methods, Magnus amplitudes, ...
- Evidence suggests that scattering observables are unified by generating functions.

$$\Delta O \sim \{\chi, O\} + \{\chi, \{\chi, O\}\} + \text{radiation} + \dots$$

[Gonzo, **CS**, Alessio, Kim, Lee, Zeng, Bjerrum-Bohr, Damgaard, Plante, Vanhove...]

Question for this talk

Why do classical scattering observables organize into this generating-function structure?

Outline

Main Results

- Stratonovich-Weyl (SW) quantization establishes a connection between classical observables and phase-space functions.
- The algebraic simplicity of the SW transform explains the origin of the generating functions.

Outline:

1. Review of SW phase-space quantization on Lorentz-covariant phase space
2. Extension to the massless sector, incorporating radiative effects
3. Observables as SW transforms
4. Hermitian scattering operator $\hat{K} \sim \log(\hat{S})$ and the master formula for observables

Phase-space quantization

No creation/annihilation of massive states $\Rightarrow$ quantum mechanics of massive states.
Given a quantum-mechanical system whose phase space $\mathcal{P}$ is parametrized by $\xi \in \mathcal{P}$,

Operator/function correspondence

$$A_{\mathrm{W}}(\xi) = \mathrm{Tr}[\hat{A}\,\hat{\Omega}(\xi)] \qquad \Leftrightarrow \qquad \hat{A} = \int \mathrm{d}\mu(\xi)\, A_{\mathrm{W}}(\xi)\hat{\Omega}(\xi)$$

- $\hat{A} \quad \Leftrightarrow \quad A_{\mathrm{W}}(\xi)$ is one-to-one.
- $\hat{\Omega}(\xi)$ is called a quantizer.
- $\mathrm{d}\mu(\xi)$ is the phase-space measure.

Flat phase space

For a flat phase space with coordinates $(\vec{x}, \vec{p})$, the quantizer is

$$\hat{\Omega}_0(\vec{x}, \vec{p}) = \int đ^3 q \; e^{\frac{i}{\hbar}\vec{x}\cdot\vec{q}} \left|\vec{p} - \frac{\vec{q}}{2}\right\rangle\left\langle\vec{p} + \frac{\vec{q}}{2}\right|$$

which gives the usual Wigner-Weyl transform

$$A_{\mathrm{W}}(\vec{x}, \vec{p}) = \int đ^3 q \; e^{\frac{i}{\hbar}\vec{x}\cdot\vec{q}} \left\langle\vec{p} + \frac{\vec{q}}{2}\middle|\hat{A}\middle|\vec{p} - \frac{\vec{q}}{2}\right\rangle$$

How to generalize this Wigner-Weyl transform to more general phase spaces?

Stratonovich-Weyl axioms

Given a phase space $\mathcal{P}$ with a symmetry group G, Stratonovich proposed that $\hat{\Omega}(\xi)$ should satisfy

- Bijection: $\hat{A} \Leftrightarrow A_{\mathrm{W}}$ is one-to-one.
- Covariance: for $g \in G$, $\hat{U}(g)\hat{\Omega}(\xi)\hat{U}^{\dagger}(g) = \hat{\Omega}(g(\xi))$
- Traciality: $\int_{\mathcal{P}} d\mu(\eta)\, \mathrm{Tr}[\hat{\Omega}(\xi)\hat{\Omega}(\eta)]\hat{\Omega}(\eta) = \hat{\Omega}(\xi)$
- Hermiticity: $\hat{\Omega}^{\dagger}(\xi) = \hat{\Omega}(\xi)$
- Normalization: $\mathrm{Tr}\,\hat{\Omega}(\xi) = 1$

This works for generic phase spaces with non-trivial geometry and constraints.

[Stratonovich, Gracia-Bondia, J. C. Varilly]

Lorentz-covariant phase space

Covariant, relativistic SW transform

$$\hat{\Omega}(x,p) = \int \hat{d}^4 q\, \hat{\delta}(2p\cdot q)\sqrt{m_q}\, e^{-iq\cdot x/\hbar}\left|m_q p - \frac{q}{2}\right\rangle\left\langle m_q p + \frac{q}{2}\right|$$

$$\Updownarrow$$

$$A_{\mathrm{W}}(x,p) = \int \hat{d}^4 q\, \hat{\delta}(2p\cdot q)\, \sqrt{m_q}\, e^{-iq\cdot x/\hbar}\left\langle m_q p + \frac{q}{2}\middle|\hat{A}\middle|m_q p - \frac{q}{2}\right\rangle$$

[Carinena, Gracia-Bondia, Varilly; Grossmann, Royer]

- The covariant phase space is parametrized by (x^μ, p_μ).
- The factor $m_q = \sqrt{1-\frac{q^2}{4m^2}}$ ensures on-shellness and Stratonovich's axioms.
- This is reminiscent of the KMOC formalism [Kosower, Maybee, O'Connell, Gonzo, Cristofoli].

Lorentz-covariant phase space

Shift symmetry

$$\hat{\Omega}(x, p) = \hat{\Omega}(x + \alpha p, p) \quad \Rightarrow \quad A_{\mathrm{W}}(x, p) = A_{\mathrm{W}}(x + \alpha p, p)$$

- (x^μ, p_μ) is naively 8-dimensional, with only 6 physical degrees of freedom.
- One degree of freedom is removed by the on-shell condition $p^2 = m^2$.
- Translation along the free worldline is a gauge redundancy.

Phase-space count

$$\begin{aligned} (x^\mu, p_\mu) : \ 8 \ &\to p^2 = m^2 : \ 7 \\ &\to x \sim x + \alpha p : \ 6 \end{aligned}$$

Groenewold product and Moyal bracket

The advantage of the SW transform is that multiplication of operators is directly translated into the *Groenewold* product.

Groenewold product

$$(A_{\mathrm{W}} \star B_{\mathrm{W}})(\xi) := (\hat{A}\hat{B})_{\mathrm{W}}(\xi) = \int_{\mathcal{P}} \mathrm{d}\mu(\eta) \int_{\mathcal{P}} \mathrm{d}\mu(\zeta)\ \mathsf{Tr}\big[\hat{\Omega}(\xi)\hat{\Omega}(\eta)\hat{\Omega}(\zeta)\big]\ A_{\mathrm{W}}(\eta) B_{\mathrm{W}}(\zeta)$$

Thus the quantum commutator is mapped to the *Moyal* bracket:

Moyal bracket

$$[\hat{A}, \hat{B}] \qquad \Longrightarrow \qquad \{A_{\mathrm{W}}, B_{\mathrm{W}}\}_{\mathrm{Moyal}} := A_{\mathrm{W}} \star B_{\mathrm{W}} - B_{\mathrm{W}} \star A_{\mathrm{W}}$$

[Groenewold, Moyal]

Flat phase space: Moyal $\Rightarrow$ Poisson

Revisit the flat-space example: the Groenewold product in differential form

Groenewold product of flat phase space

$$(A_{\mathrm{W}} \star B_{\mathrm{W}})(\vec{x}, \vec{p}) = A_{\mathrm{W}} \exp\left[\frac{i\hbar}{2}\left(\overleftarrow{\partial}_x \cdot \vec{\partial}_p - \overleftarrow{\partial}_p \cdot \vec{\partial}_x\right)\right] B_{\mathrm{W}}$$

The small-$\hbar$ expansion of the exponential factor gives

Moyal bracket of flat phase space

$$\{A_{\mathrm{W}}, B_{\mathrm{W}}\}_{\mathrm{Moyal}} = i\hbar\{A_{\mathrm{W}}, B_{\mathrm{W}}\}_{\mathrm{Poisson}} + \mathcal{O}(\hbar^2)$$

How about covariant phase space?

Covariant phase space: Moyal $\Rightarrow$ Dirac

We need to gauge-fix the shift symmetry G_s, e.g.

$$g(x,p) = x \cdot p$$

Covariant measure

$$\mathrm{d}\mu(x,p) = \frac{\mathrm{d}^4 x \mathrm{d}^4 p}{(2\pi\hbar)^4} \frac{\delta^{(+)}(p^2 - m^2)}{\mathrm{Vol}(G_s)}$$

The Dirac bracket automatically emerges in the small-$\hbar$ limit.

$$\{A_{\mathrm{W}}, B_{\mathrm{W}}\}_{\mathrm{Moyal}} = i\hbar\{A_{\mathrm{W}}, B_{\mathrm{W}}\}_{\mathrm{Dirac}} + \mathcal{O}(\hbar^2)$$

This reflects the fact that the physical phase space is constrained.

SW transform with radiation

So far we have only discussed massive states. If massless particles (e.g. photons or gravitons) are involved, we perform the SW transform only on the massive sector.

Partial SW transform with radiation

$$\mathcal{H} = \mathcal{H}_{\text{massive}} \otimes \mathcal{H}_{\text{rad}} \quad \Rightarrow \quad \hat{A}_{\mathrm{W}}(x,p) := \mathrm{Tr}_{\mathcal{H}_{\text{massive}}}\left[\hat{A}\left(\hat{\Omega}(x,p) \otimes \mathbb{1}_{\text{rad}}\right)\right]$$

- $\hat{A}_{\mathrm{W}}$ is a phase-space function in the massive variables.
- It remains an operator in the massless radiative sector.

SW transform with radiation

Radiative mode expansion

$$\hat{A}_{\mathrm{W}}(x,p) \sim \sum_{m,n=0}^{\infty} \int (a^{\dagger}_{k_i})^m (a_{k'_j})^n \underbrace{\left\langle m_q p + \frac{q}{2}, k_1, \ldots, k_m \middle| \hat{A} \middle| m_q p - \frac{q}{2}, k'_1, \ldots, k'_n \right\rangle}_{A^{(m,n)}_{\mathrm{W}}}$$

If $\hat{A}_{\mathrm{W}}$ and $\hat{B}_{\mathrm{W}}$ are finite in the small-$\hbar$ limit,

Mixed Moyal/commutator bracket

$$[\hat{A}, \hat{B}] \quad \Rightarrow \quad \underbrace{\{\hat{A}_{\mathrm{W}}, \hat{B}_{\mathrm{W}}\}_{\mathrm{Moyal}}}_{\text{mixed Moyal/commutator}} = i\hbar \underbrace{\{\hat{A}_{\mathrm{W,cl}}, \hat{B}_{\mathrm{W,cl}}\}}_{\text{mixed Dirac/commutator}} + \mathcal{O}(\hbar^2)$$

Observables for point-particle states

For a state $\rho = |\psi\rangle\langle\psi|$, the expectation value of a quantum observable $\hat{O}$ is

$$\langle O\rangle = \mathrm{Tr}[\rho\hat{O}] = \int_{\mathcal{P}} d\mu(\eta)\rho_{\mathrm{W}}(\eta)O_{\mathrm{W}}(\eta)$$

$$\mathrm{Tr}[\hat{A}\hat{B}] = \int_{\mathcal{P}} d\mu(\xi)A_{\mathrm{W}}(\xi)B_{\mathrm{W}}(\xi)$$

ρ_{W} is the *Wigner* function.

Consider a state localized around a phase-space point ξ in the classical limit:

$$\lim_{\hbar\to 0}\rho_{\mathrm{W},\xi}(\eta) = \delta_{\mathcal{P}}(\eta,\xi)$$

The SW transform directly gives the corresponding classical observable.

Classical observables as SW transforms

$$\langle O\rangle \quad \xrightarrow{\hbar\to 0} \quad \lim_{\hbar\to 0} O_{\mathrm{W}}(\xi)$$

Classical scattering and Magnus amplitudes

Given an incoming state parametrized by (b_1, p_1, b_2, p_2), the change in an observable O is

$$\Delta\hat{O} := \hat{S}^\dagger \hat{O}\hat{S} - \hat{O} \quad \Rightarrow \quad \langle \Delta\hat{O}\rangle = \lim_{\hbar\to 0} \Delta O_{\mathrm{W}}$$

It is convenient to introduce the Hermitian scattering operator

Hermitian scattering operator

$$\hat{S} = e^{\frac{i}{\hbar}\hat{K}}$$

- The matrix elements of $\hat{K}$ are also called Magnus amplitudes.
- The classical limit is taken as the multi-soft limit according to the correspondence principle $\hbar \ll p \cdot b$.

[Gonzo, **CS**, Alessio, Kim, Lee, Guo, Scheopner, Patil, Haddad,Plefka, Mogull, Brandenhuber, Travaglini, Pichini, . . .]

Hermitian scattering operator

The SW transform of $\hat{K}$ is finite in the classical limit.

$$\hat{K}_{\mathrm{W}} \quad \xrightarrow{\hbar\to 0} \quad \hat{K}_{\mathrm{W,cl}}$$

Master formula for classical observables

Baker-Campbell-Hausdorff formula: $\Delta\hat{O} = \sum_{r=1}^{\infty} \frac{(-i)^r}{\hbar^r\, r!} \underbrace{[\hat{K},[\hat{K},\ldots,[\hat{K},\hat{O}]]]}_{r\text{-fold commutator}}$

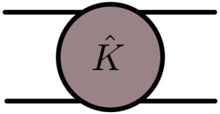
$$\Rightarrow \quad \langle\Delta\hat{O}\rangle = \sum_{r=1}^{\infty} \frac{1}{r!} \langle 0| \underbrace{\left\{\hat{K}_{\mathrm{W,cl}}, \{\hat{K}_{\mathrm{W,cl}}, \ldots \{\hat{K}_{\mathrm{W,cl}}, \hat{O}_{\mathrm{W}}\}\ldots\}\right\}}_{r\text{-fold mixed Dirac/commutator bracket}} |0\rangle_{\mathrm{rad}}$$

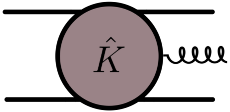
Massive variables: Dirac bracket   Radiative modes: commutator

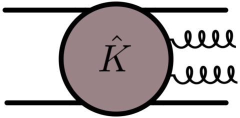
Magnus amplitudes

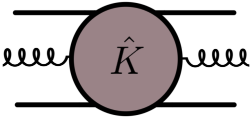
Apply the previous radiative SW expansion to the Hermitian generator $\hat{K}$:

$$\hat{K}_{\mathrm{W,cl}}(x,p) \sim \sum_{m,n=0}^{\infty} \int (\bar{a}^{\dagger}_{\bar{k}_i})^m (\bar{a}_{\bar{k}'_j})^n \underbrace{\left\langle m_q p + \frac{q}{2}, \bar{k}_1, \dots, \bar{k}_m \middle| \hat{K} \middle| m_q p - \frac{q}{2}, \bar{k}'_1, \dots, \bar{k}'_n \right\rangle}_{K^{(m,n)}_{\mathrm{W,cl}}}$$

$K^{(0,0)}_{\mathrm{W,cl}} \sim$ 

$K^{(0,1)}_{\mathrm{W,cl}} \sim$ 

$K^{(0,2)}_{\mathrm{W,cl}} \sim$ 

$K^{(1,1)}_{\mathrm{W,cl}} \sim$ 

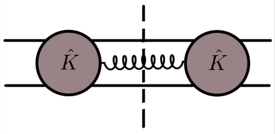
Example: impulse $O = p_1^\mu$

Conservative sector

$$\Delta p_{1,\mathrm{con}} = \sum_{k=1}^{\infty} \frac{1}{k!} \left\{ K^{(0,0)}_{\mathrm{W,cl}}, \left\{ K^{(0,0)}_{\mathrm{W,cl}}, \ldots \left\{ K^{(0,0)}_{\mathrm{W,cl}}, p_1 \right\} \ldots \right\} \right\}_{\mathrm{Dirac}}$$

Radiative sector

$$\Delta p_{1,\mathrm{rad}} \sim \int_{\bar{k}} \left(K^{(0,1)}_{\mathrm{W,cl}}(\bar{k}) \left\{ K^{(1,0)}_{\mathrm{W,cl}}(\bar{k}), p_1 \right\}_{\mathrm{Dirac}} - K^{(1,0)}_{\mathrm{W,cl}}(\bar{k}) \left\{ K^{(0,1)}_{\mathrm{W,cl}}(\bar{k}), p_1 \right\}_{\mathrm{Dirac}} \right) + \ldots$$



[Alessio, Gonzo, **CS**]

Example: waveform

For the waveform observable $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, it suffices to compute $\langle h_{\mu\nu} \rangle \sim \langle a_{\bar{q}} \rangle$.

Radiative mode $\rightarrow$ waveform

$$\langle a_{\bar{q}} \rangle^{(1)} = \langle 0 | \{ \hat{K}_{\mathrm{W,cl}}, a_{\bar{q}} \} | 0 \rangle = i\, K^{(1,0)}_{\mathrm{W,cl}}(\bar{q})$$

$$\langle \bar{a}_{\bar{q}} \rangle^{(2)} = \frac{i}{2} \{ K_{\mathrm{W,cl}}, K^{(1,0)}_{\mathrm{W,cl}}(\bar{q}) \}_{\mathrm{D}} + \frac{1}{2} \int_{\bar{k}} \Big(K^{(0,1)}_{\mathrm{W,cl}}(\bar{k})\, K^{(2,0)}_{\mathrm{W,cl}}(\bar{q}, \bar{k}) - K^{(1,0)}_{\mathrm{W,cl}}(\bar{k})\, K^{(1,1)}_{\mathrm{W,cl}}(\bar{q}; \bar{k}) \Big) + \mathrm{c.c.}$$

- $a_{\bar{q}}$ annihilates a radiative mode of momentum $\hbar \bar{q}$.
- Its expectation value reconstructs the classical far-zone waveform order by order.

[Alessio, Gonzo, **CS**]

Conclusion and Outlook

Conclusion:

- Stratonovich-Weyl quantization provides a covariant phase-space language for the classical scattering problem.
- Classical observables follow from the kernels $K_{\mathrm{W,cl}}^{(m,n)}$ in the radiative expansion of $\hat{K}_{\mathrm{W}}$.
- The small-$\hbar$ limit of the relativistic Moyal bracket gives the Dirac bracket.

Outlook:

- Phase spaces with spin?
- Resummed form of the master formula?
- Bound states?
- Scattering of waves?

Backup: star product

$$A_{\mathrm{W}} \star B_{\mathrm{W}}(x,p) = \int \left(\prod_{j=1,2} d\mu(y_j, q_j) \right) L(x,p,y_1,q_1,y_2,q_2) A_{\mathrm{W}}(y_1,q_1) B_{\mathrm{W}}(y_2,q_2),$$

$$\begin{aligned} L(x,p,y_1,q_1,y_2,q_2) &= \frac{2^6 \mathcal{C}^{13/2}}{m^9} \Big[(p\cdot q_1)(q_1\cdot q_2)(q_2\cdot p)\Big]^{3/2} \\ &\times \exp\left[\frac{2i\mathcal{C}}{\hbar m^2}\Big((p\cdot q_1)\, q_2\cdot(x-y_1) + (q_1\cdot q_2)\, p\cdot(y_1-y_2) + (q_2\cdot p)\, q_1\cdot(y_2-x)\Big)\right] \end{aligned}$$

$$\mathcal{C} := m^2 \Big[(p\cdot q_1)^2 + (q_1\cdot q_2)^2 + (q_2\cdot p)^2 + 2m^{-2}(p\cdot q_1)(q_1\cdot q_2)(q_2\cdot p)\Big]^{-1/2}$$

Backup: two-body gauge

$$p_1 \cdot (x_2 - x_1) = 0, \qquad p_1^2 - m_1^2 = 0, \qquad p_2 \cdot (x_2 - x_1) = 0, \qquad p_2^2 - m_2^2 = 0$$

- The shift symmetry requires gauge fixing on covariant phase space.
- On the reduced two-body phase space, the bracket extracted from the $\star$-product is the standard Dirac bracket.
- This is the constrained relativistic analogue of the Poisson bracket in ordinary Wigner quantization.

Backup: Dirac brackets

$$\begin{aligned}
\{p^\mu, p^\nu\}_{\mathrm{Dirac}} =& 0\\
\{x^\mu, p^\nu\}_{\mathrm{Dirac}} =& \eta^{\mu\nu} - \frac{p^\mu p^\nu}{m^2}\\
\{x^\mu, x^\nu\}_{\mathrm{Dirac}} =& \frac{1}{m^2}(p^\mu x^\nu - p^\nu x^\mu),
\end{aligned}$$