Gravitational Tidal Matching

and Elliptic Curves

UCLA

UCLA Mani L. Bhaumik Institute for Theoretical Physics

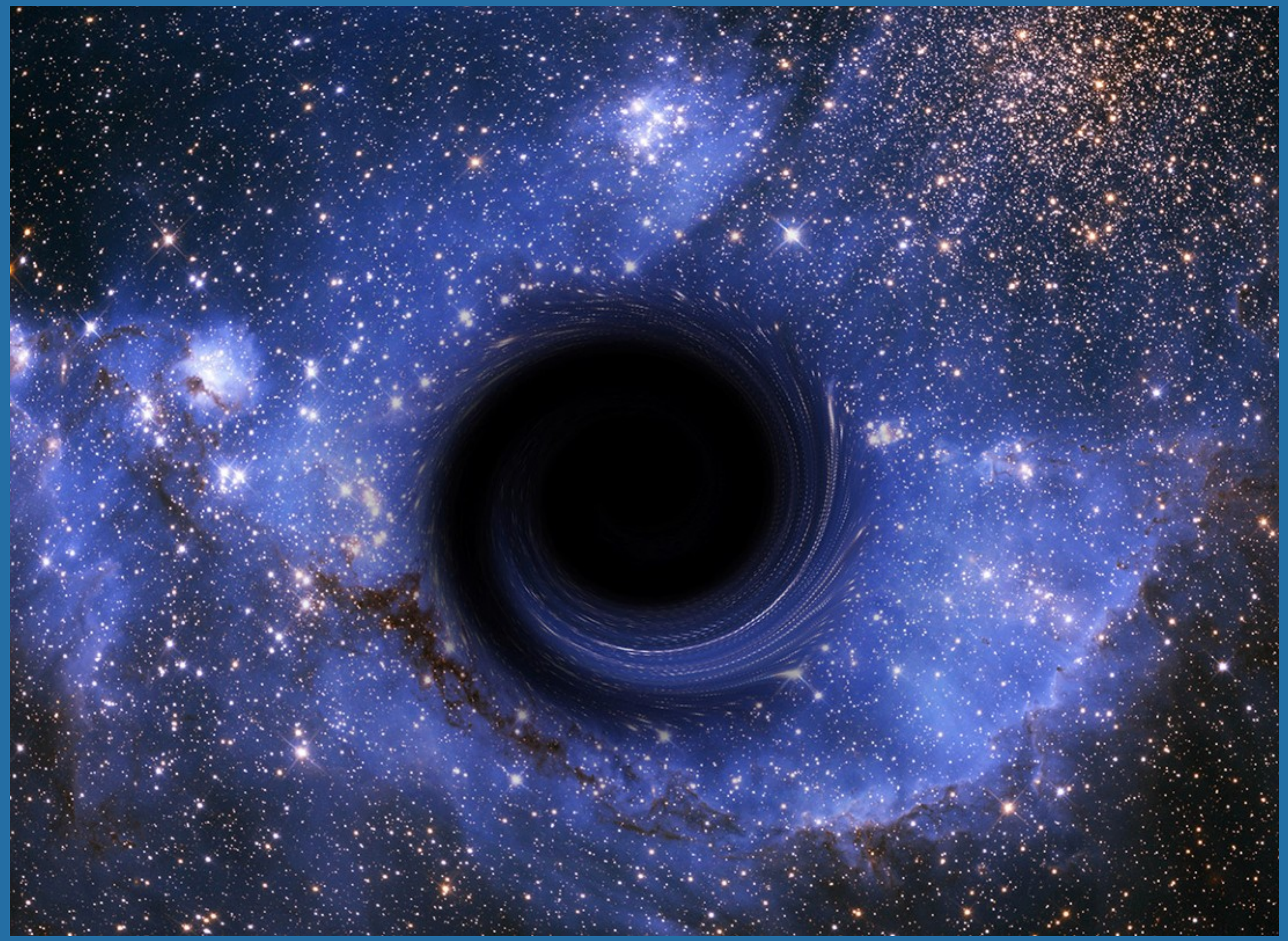
W.I.P. [S. Caron-Huot, M. Correia, GI, A. Wolz]

Giulia Isabella

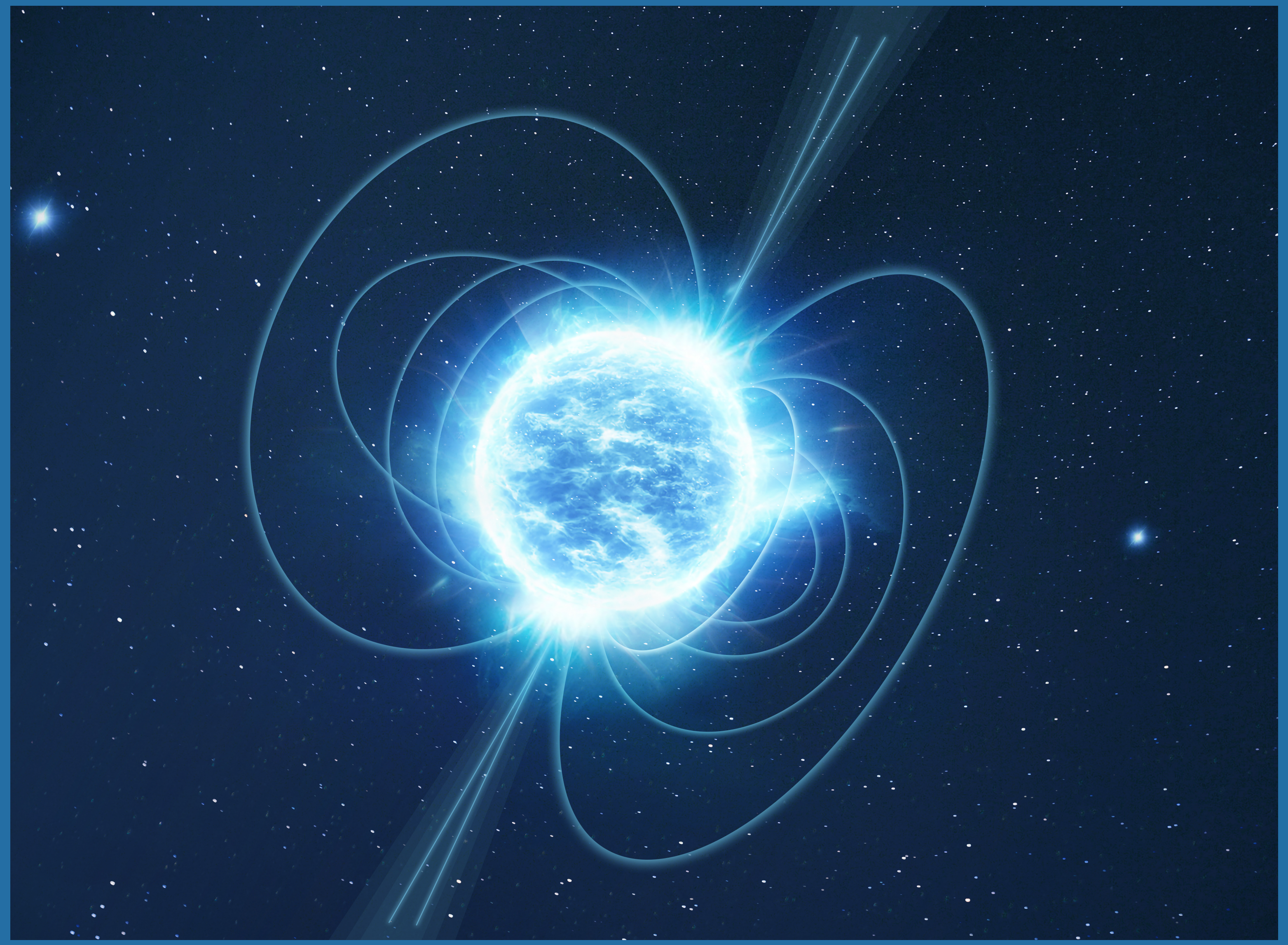
29th June 2026

Amplitudes 2026

Black holes are one of the most remarkable GR predictions

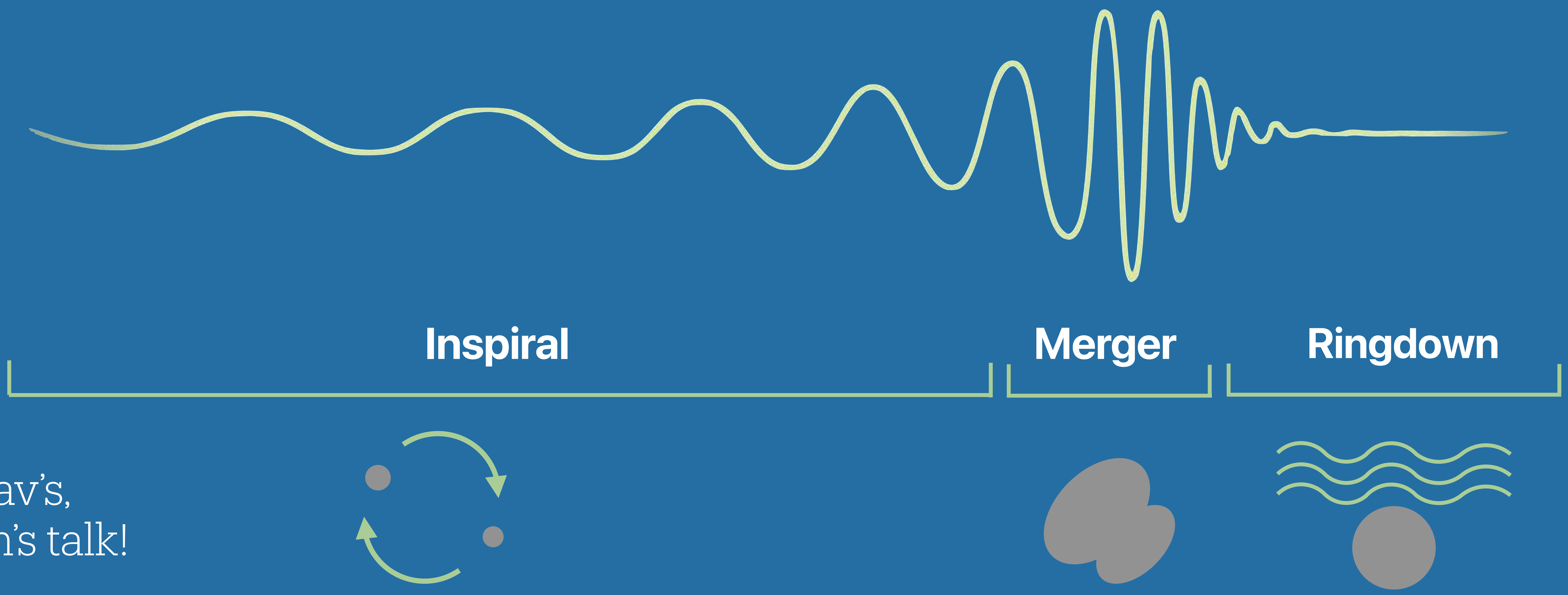


$\neq$



Finite size effects of compact objects are captured by *Love Numbers*

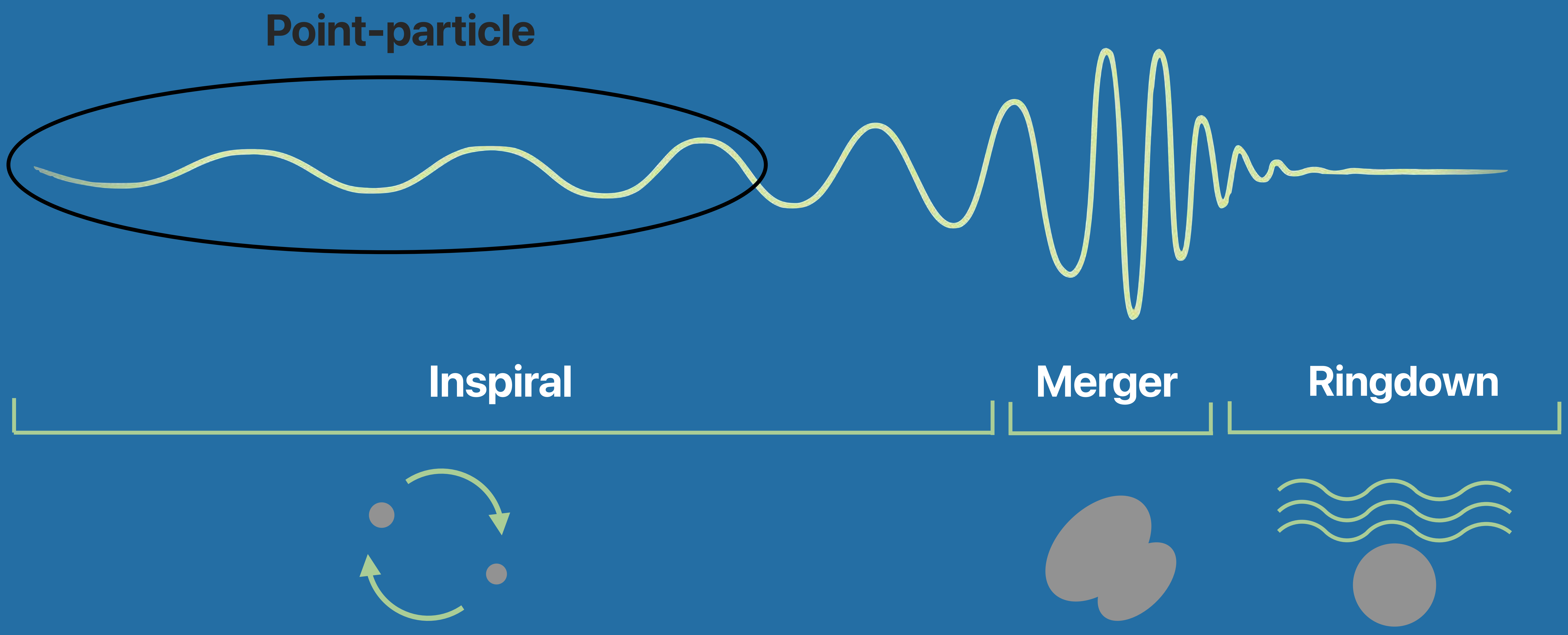
Gravitational Wave Signal



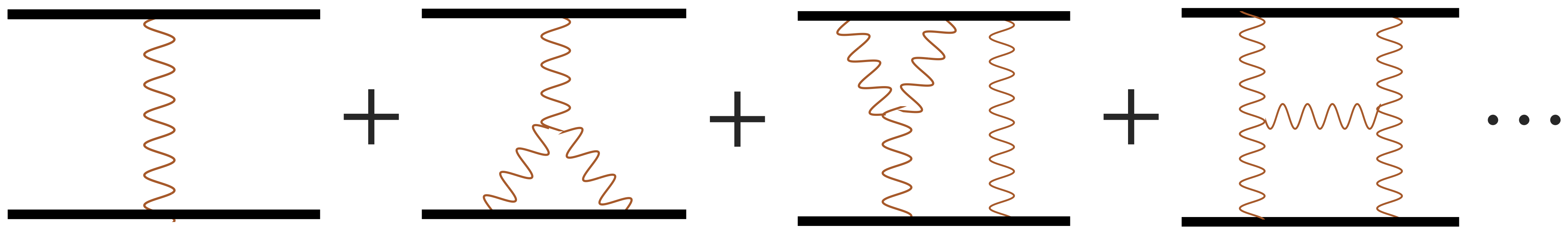
See David's, Gustav's, Canxin, and Graham's talk!

See Gustav's, Lara's and Gang's talk!

Gravitational Wave Signal

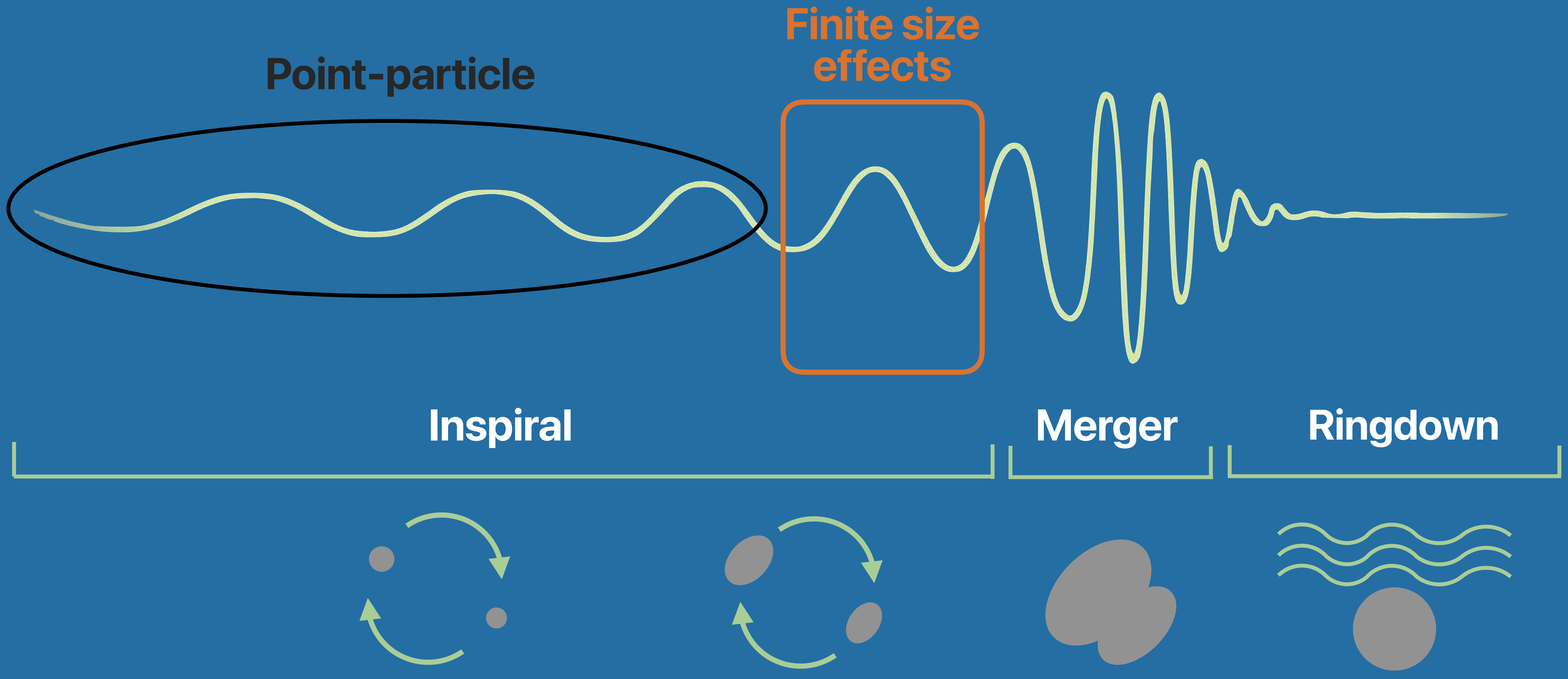


$$S = m_1 \int d\tau_1 \; + \; m_2 \int d\tau_2$$

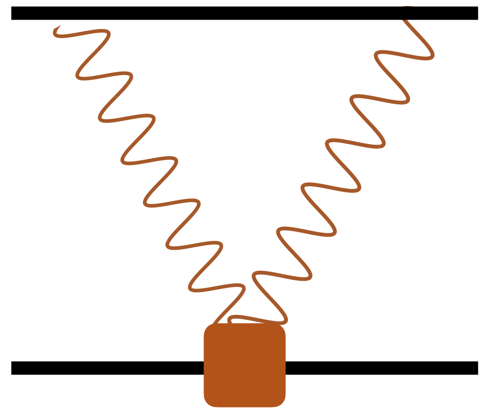


Inspiral: Point-particle approximation

Love Numbers in the Gravitational Wave Signal



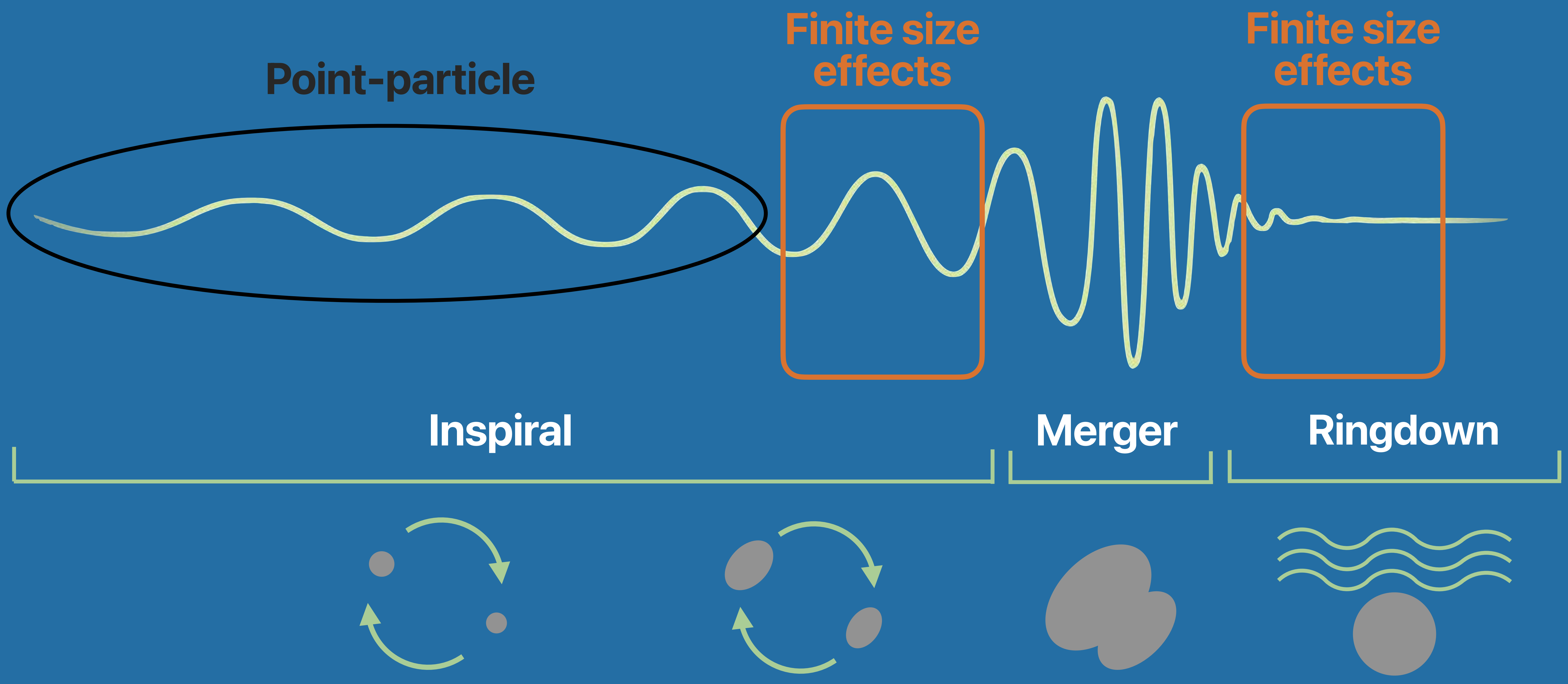
$$S = m_1 \int d\tau_1 \; + \; m_2 \int d\tau_2 \qquad + \lambda_E \int d\tau_1 \, E_{\mu\nu} E^{\mu\nu} \; + \; \ldots$$

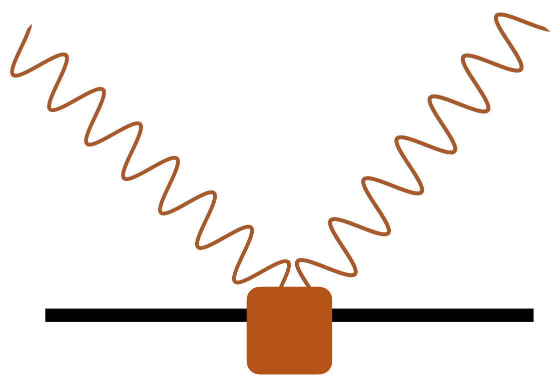


Love numbers: Tidal deformability, dissipation, ...

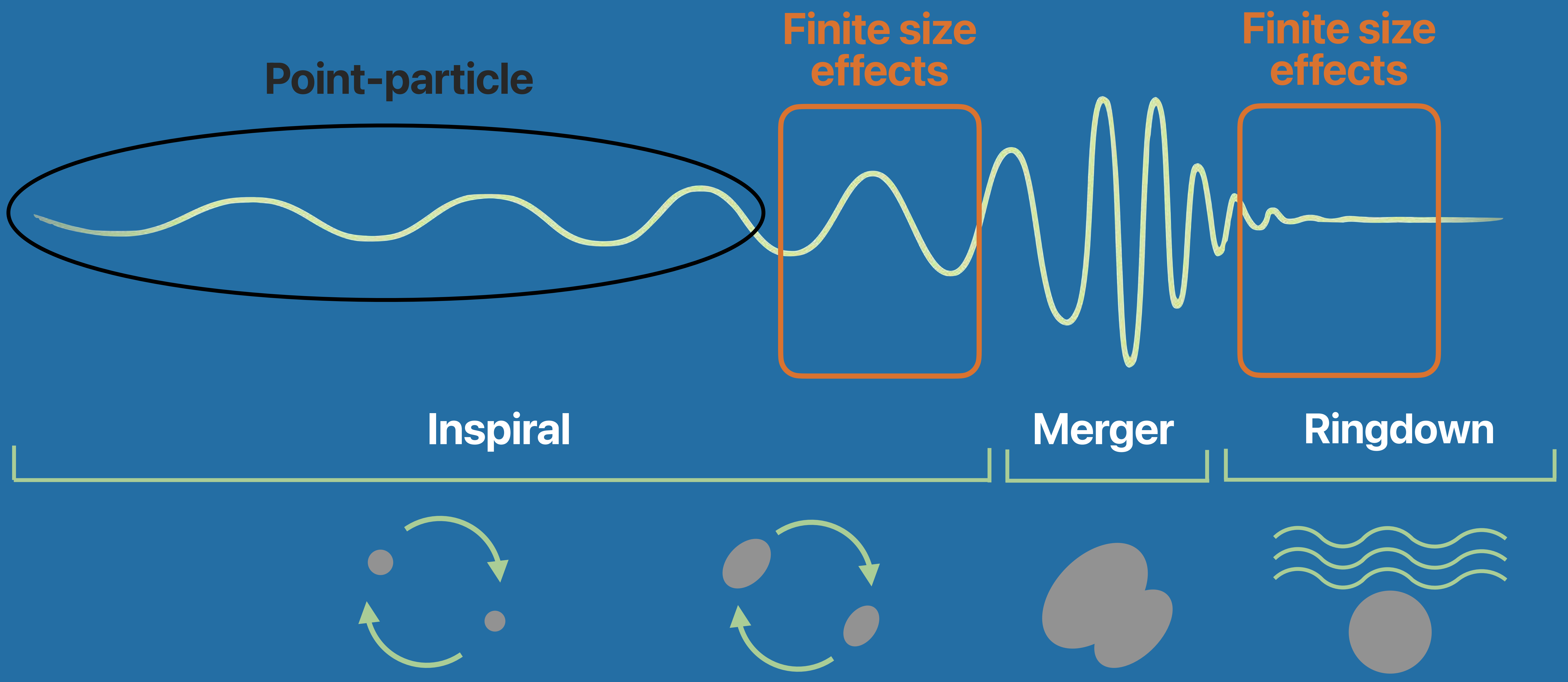
[Goldberger, Rothstein, 2006] [Goldberger, Ross, 2009]

Love Numbers in the Gravitational Wave Signal



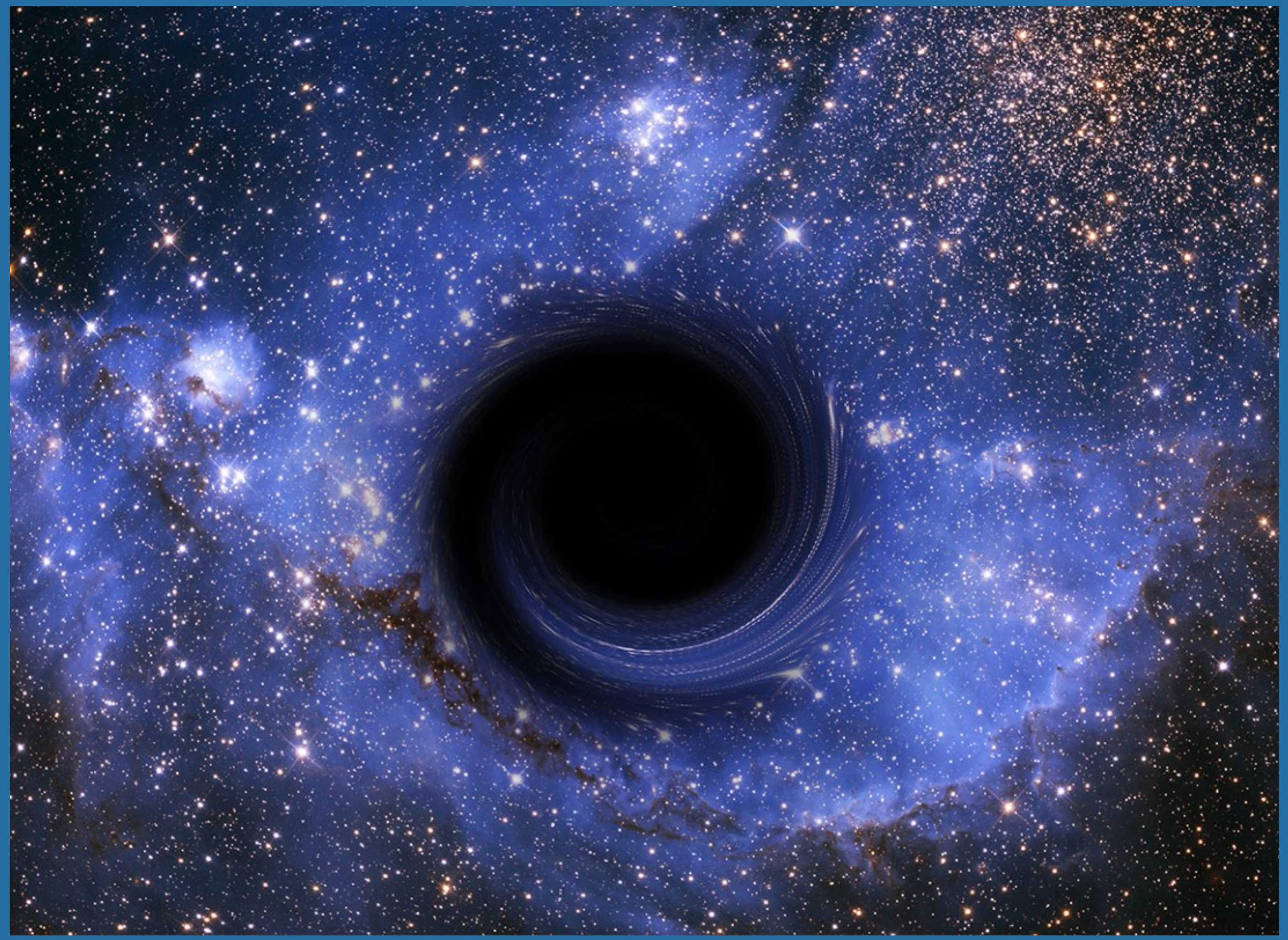


Love Numbers in the Gravitational Wave Signal



Feed finite size effects into the inspiral phase computations?

Matching Strategy

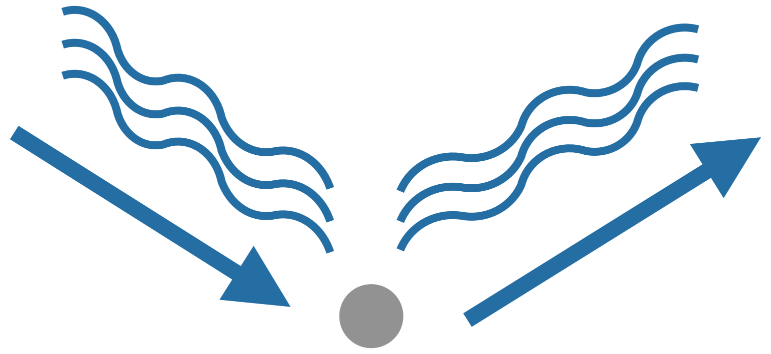


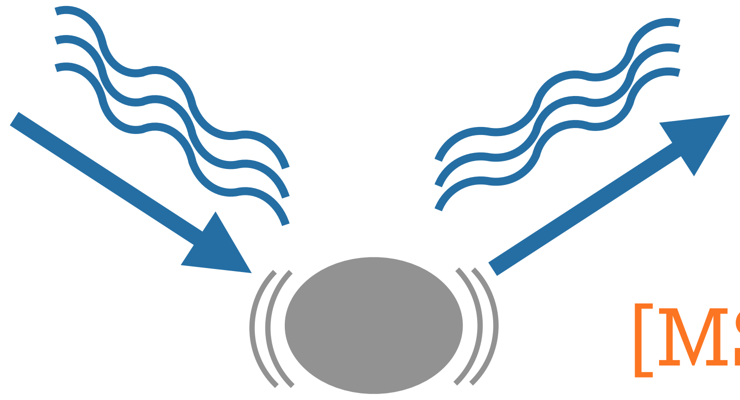
Can we compute frame independent Black Hole Love numbers?

Matching Strategy

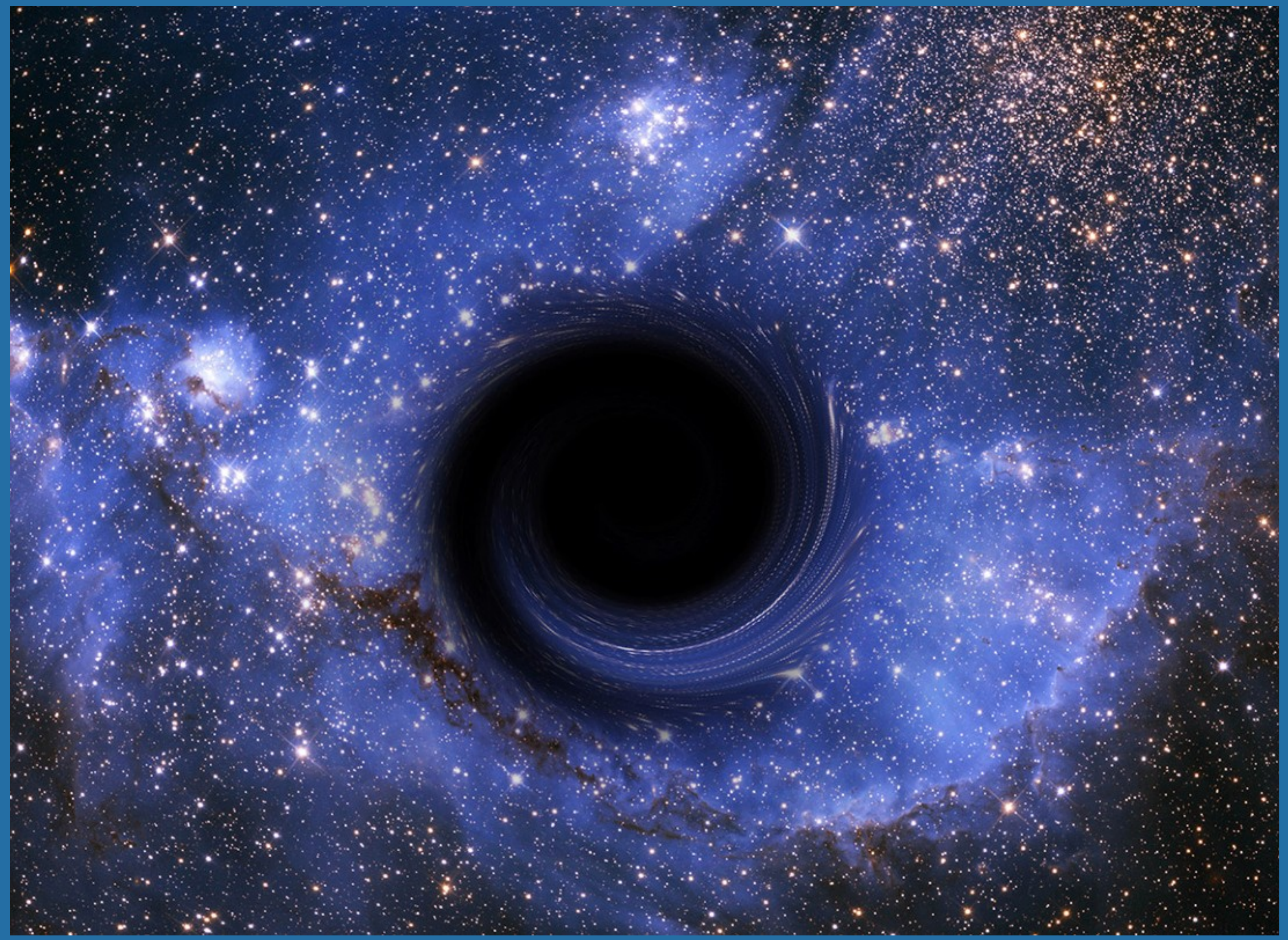
Compute the same gauge-independent object

EFT amplitude $\overset{!}{=}$ GR amplitude





[MST, 1996]



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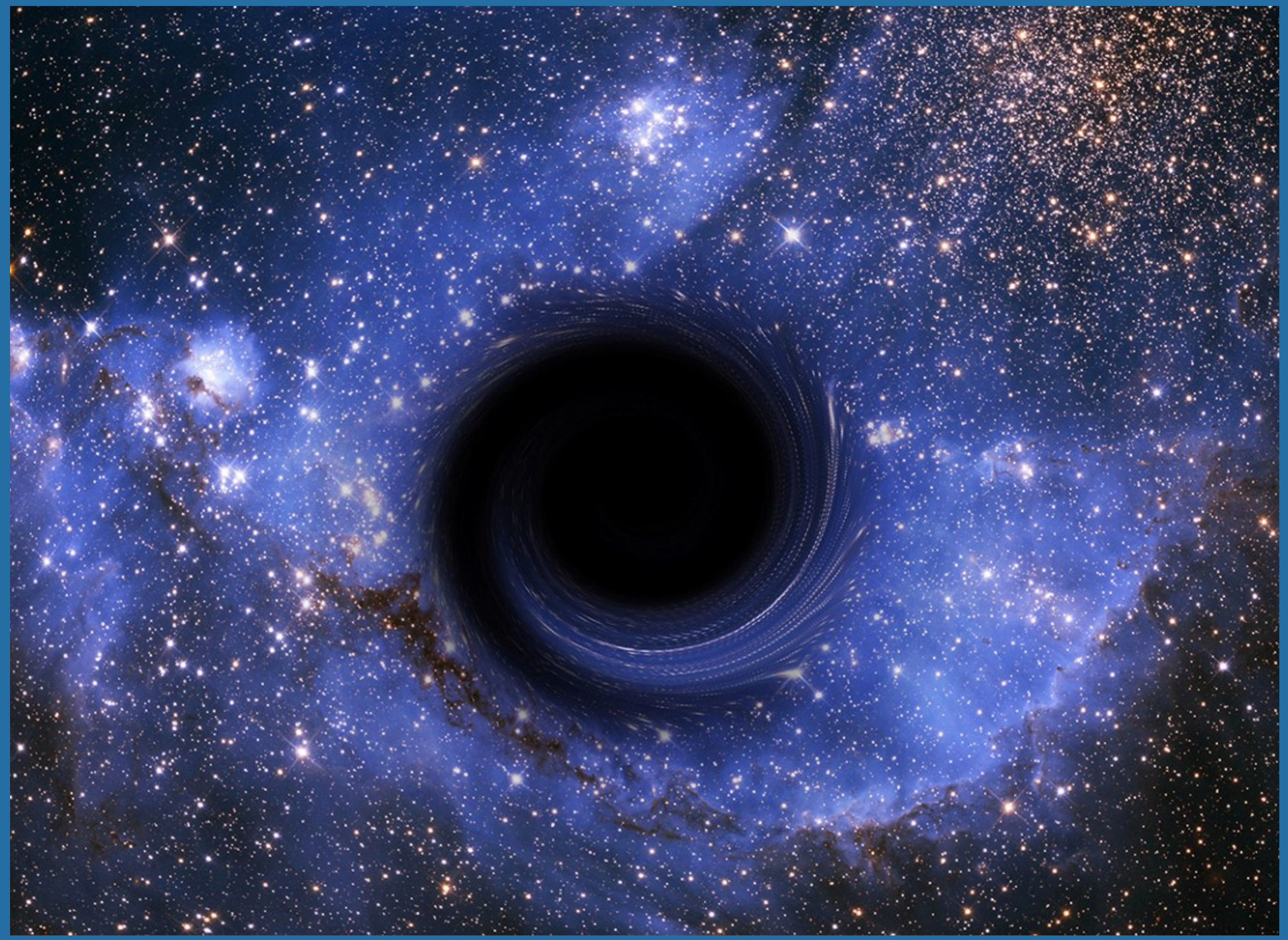

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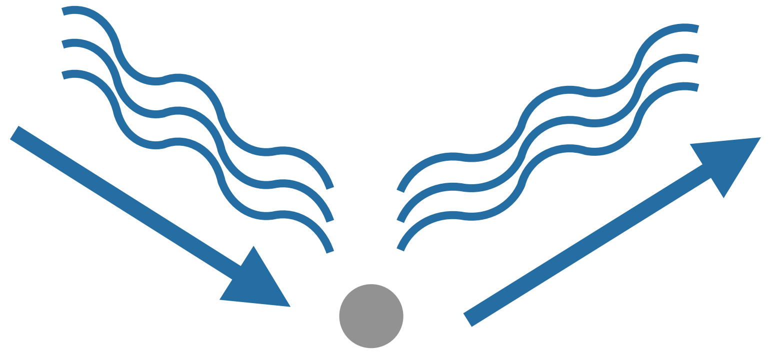
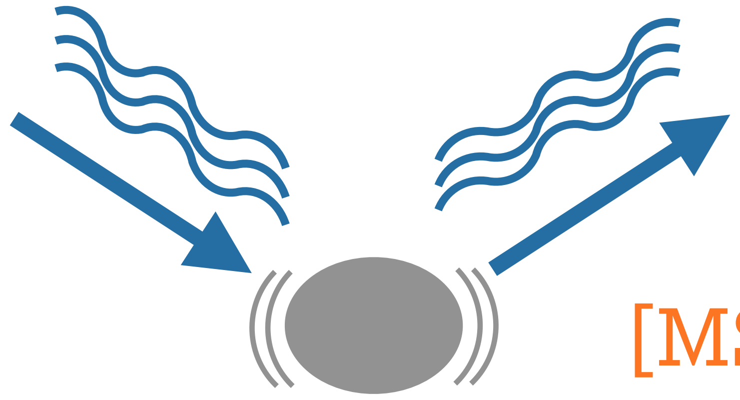




[MST, 1996]

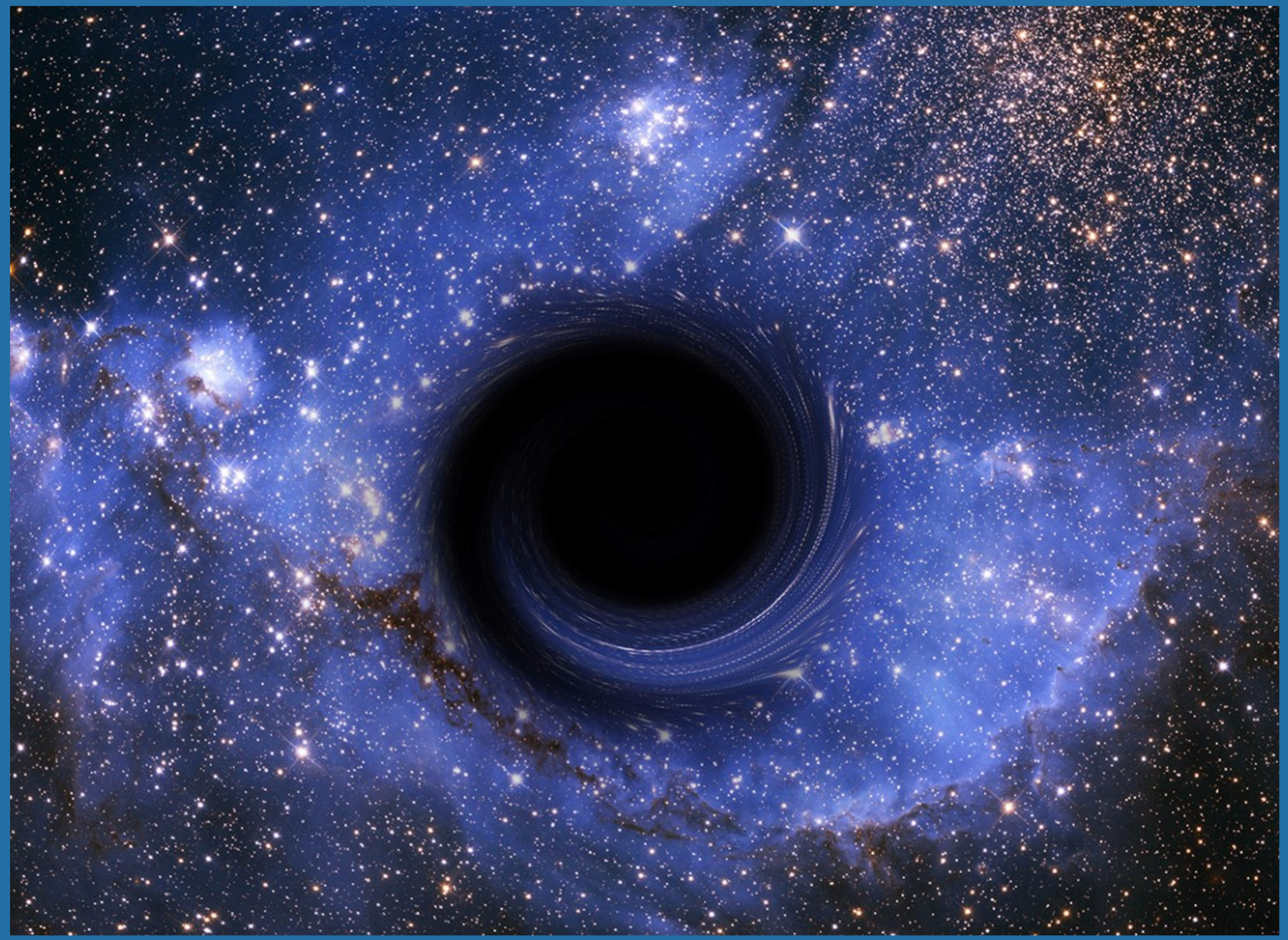
State of the art

$$\mathcal{O}(G^3)$$

2602.06947 [Bjerrum-Bohr, Chen, Eriksen, Shah]

2602.06951 [Ivanov, Li, Parra-Martinez, Zhou]

2602.06125 [Bautista, Driesse, Haddad, Jakobsen]



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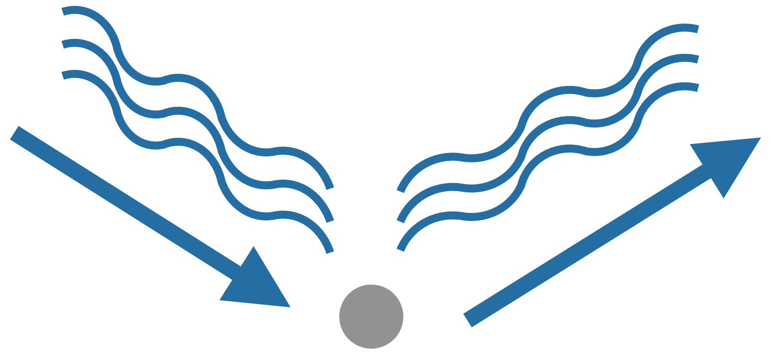

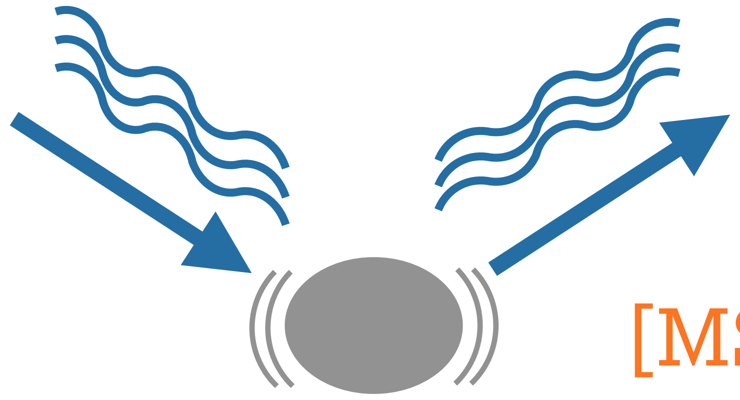

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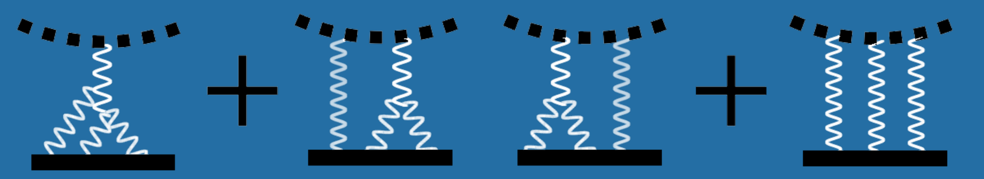
$\mathcal{O}(G^4)$

2606.28239 [Brunello, Meo, Smith]

2606.27544 [Bautista, Driesse, Haddad, Jakobsen]

Feynman diagrams



Amplitude

$T(\omega, t)$

2401.08752 [Ivanov, Li, Parra-Martinez, Zhou]

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Partial Wave

$S_\ell(\omega)$

Match to BHPT

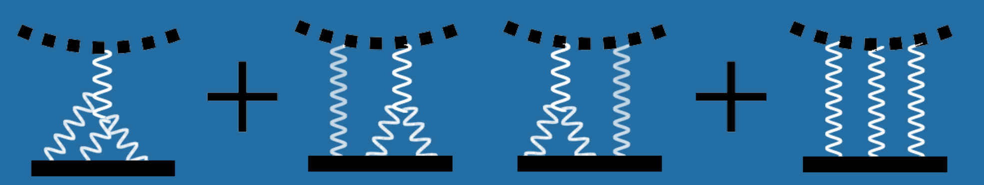
Black Hole Love numbers

Unfortunately, no new physics!

$\cancel{\mathcal{O}(G^3)}$ $\mathcal{O}(G^4)$

: Standard approach

Feynman diagrams



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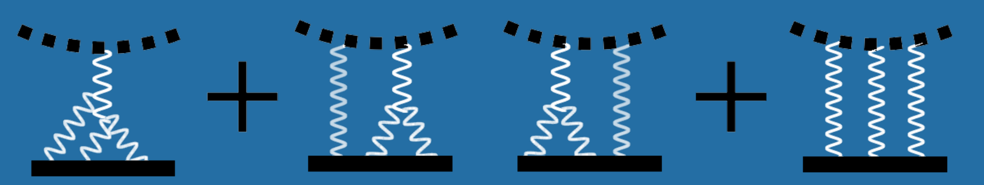
Unfortunately, no new physics!

Order needed

$\mathcal{O}(G^3) \longrightarrow \mathcal{O}(G^7)$

: Standard approach

Feynman diagrams



Amplitude

$T(\omega, t)$

Partial Wave

$S_\ell(\omega)$

Match to BHPT

Black Hole Love numbers

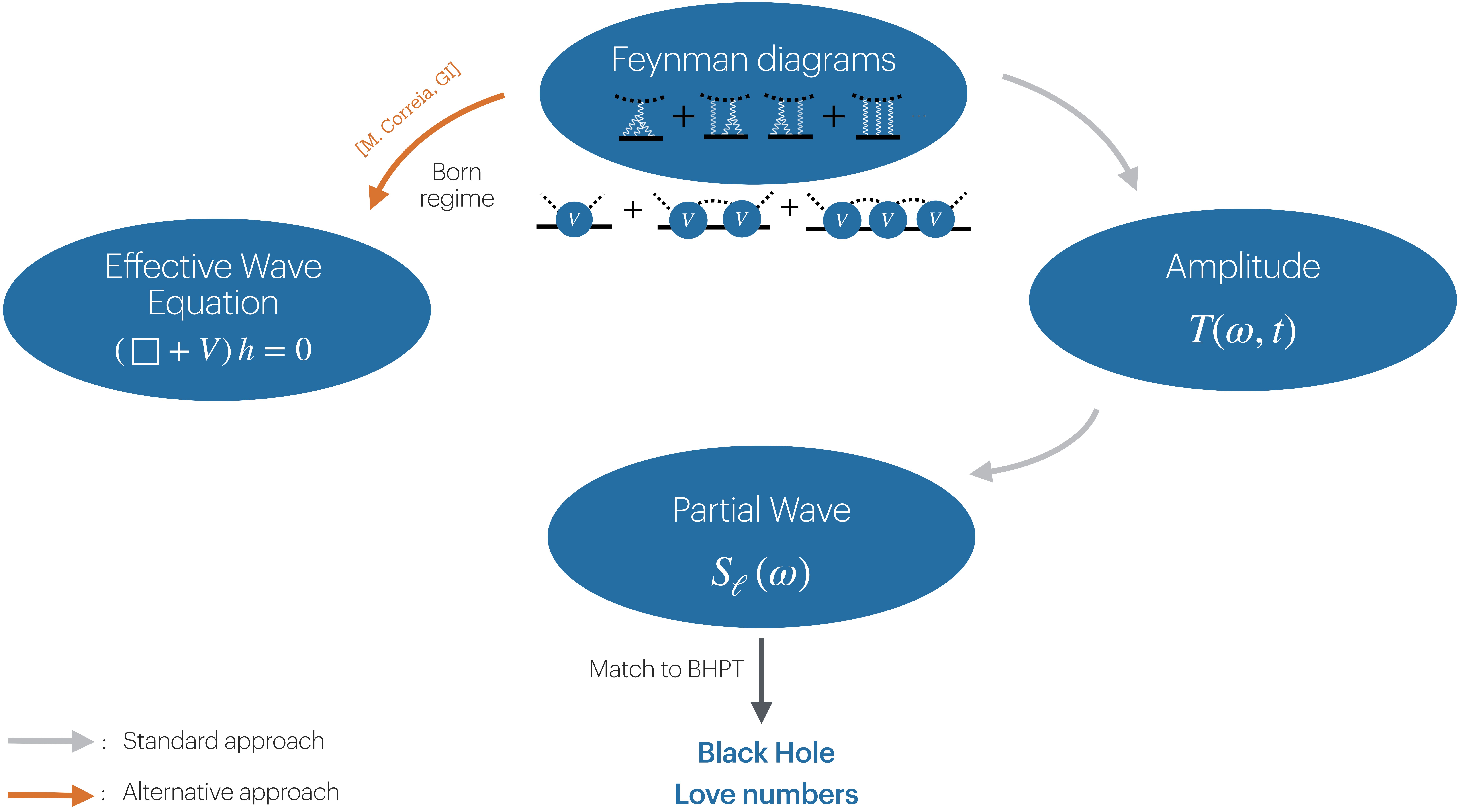
Unfortunately, no new physics!

Six loops!

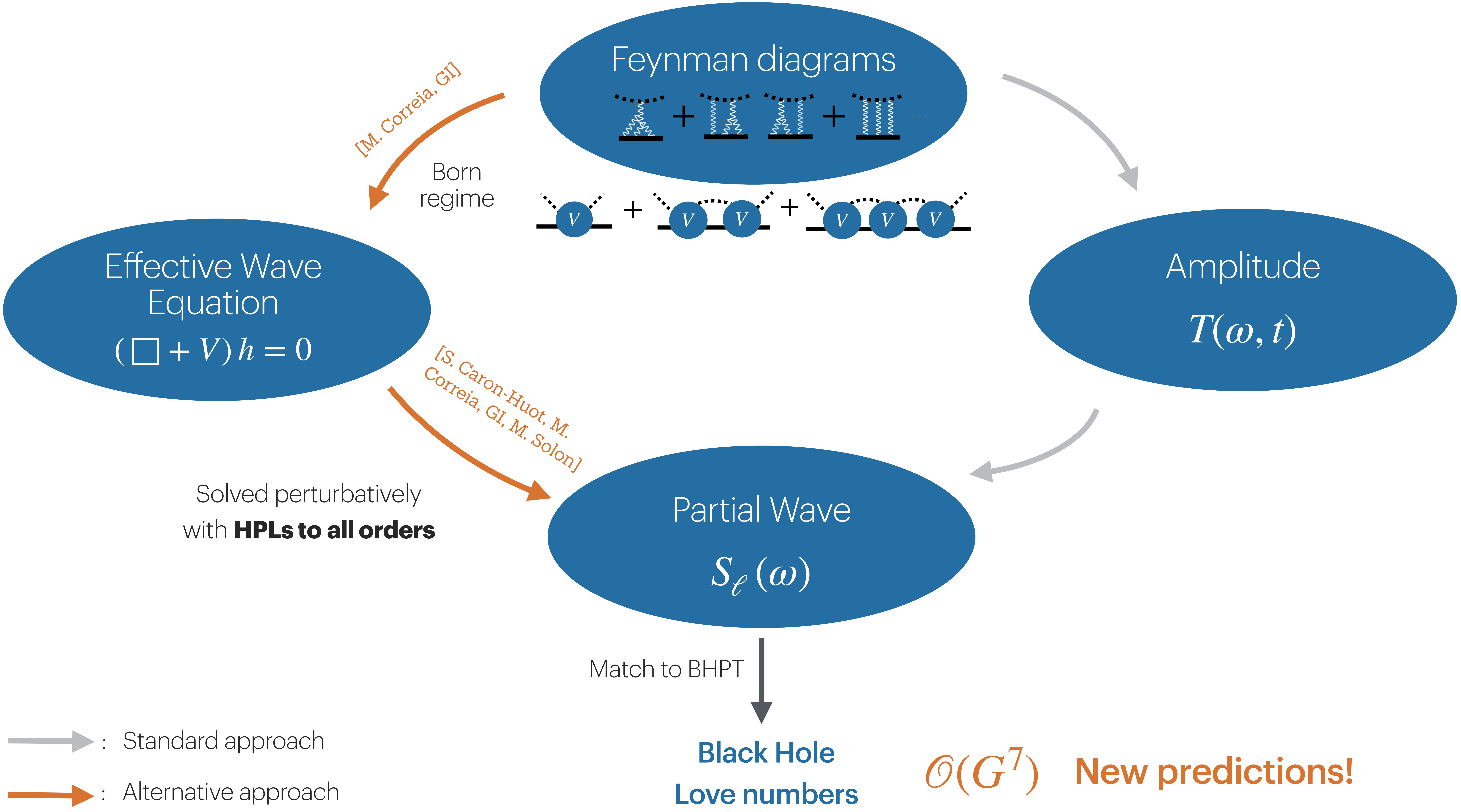
Order needed

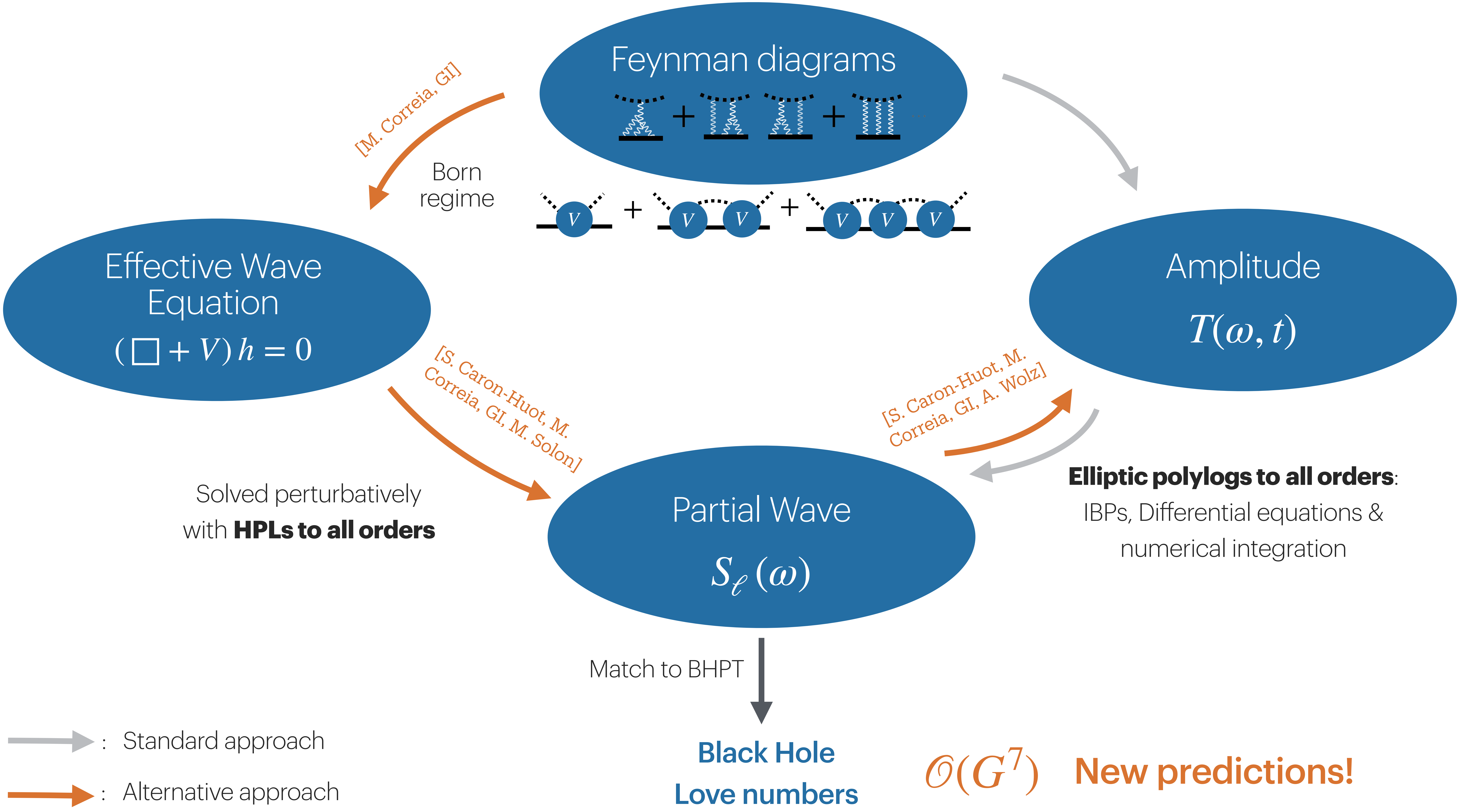
$\mathcal{O}(G^3) \longrightarrow \mathcal{O}(G^7)$

: Standard approach



Feynman diagrams
+
+
[M. Correia, GI]
Born
regime
+
+
V
Effective Wave
Equation
$(\Box + V)\,h = 0$
Amplitude
$T(\omega, t)$
Partial Wave
$S_\ell(\omega)$
Match to BHPT
Black Hole
Love numbers
: Standard approach
: Alternative approach


Feynman diagrams
[M. Correia, GI]
Born regime
Effective Wave Equation
$(\Box + V)\,h = 0$
Amplitude
$T(\omega, t)$
[S. Caron-Huot, M. Correia, GI, M. Solon]
Solved perturbatively with **HPLs to all orders**
Partial Wave
$S_\ell(\omega)$
Match to BHPT
Black Hole Love numbers
$\mathcal{O}(G^7)$ New predictions!
: Standard approach
: Alternative approach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Elliptic polylogs to all orders: IBPs, Differential equations & numerical integration
Solved perturbatively with HPLs to all orders
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: Standard approach
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The effective wave equation

W.I.P. [S. Caron-Huot, M. Correia, GI, A. Wolz]

2406.13737 [M. Correia, GI]

2503.13593 [S. Caron-Huot, M. Correia, GI, M. Solon]

The effective wave equation

Gravitational wave

W.I.P. [S. Caron-Huot, M. Correia, GI, A. Wolz]

2406.13737 [M. Correia, GI]

2503.13593 [S. Caron-Huot, M. Correia, GI, M. Solon]

Scalar wave

See Miguel's talk
Amplitudes 2025

Effective Action

$$S = m\int d\tau + \lambda_E \int d\tau\, E_{\mu\nu}E^{\mu\nu} + \lambda_B \int d\tau\, B_{\mu\nu}B^{\mu\nu} + \ldots$$

Short distance Love numbers

$$+\frac{1}{64\pi G}\int d^d x\, \nabla_\rho h_{\mu\nu} \nabla^\rho h^{\mu\nu} + \ldots$$

Long distance gravitational interactions

Effective Wave equation

$$\Big\downarrow \; \frac{\delta S}{\delta h} = 0$$

$$\left(\Box + V_{\text{Love}} + V_{\text{grav}} \right) h = 0$$

Effective Action

$$S = m\int d\tau \; + \lambda_E \int d\tau\, E_{\mu\nu}E^{\mu\nu} + \; \lambda_B \int d\tau\, B_{\mu\nu}B^{\mu\nu} + \; \dots$$

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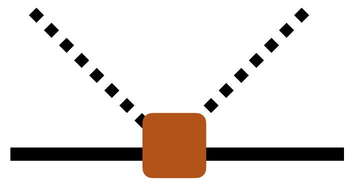
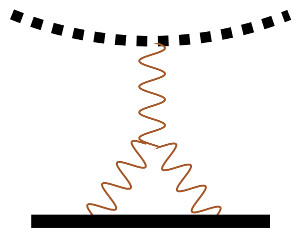
$V_{\text{Love}} =$ 

Love numbers = Boundary conditions

Regulate UV divergences
with $d = 4 - 2\epsilon$

$V_{\text{grav}} =$ 

Born series:

$$S_\ell(\omega) = V + \int VGV + \int VG \int VGV\dots$$

$=$ Harmonic Polylogarithms (HPLs)

Results and Matching to BHPT

EFT results to $\mathcal{O}(G^3)$

$$\delta_2^{(B)}(\omega) = GM\omega\left(\frac{1}{\epsilon} + \log\left(\frac{4\omega^2}{\mu_{\rm IR}^2}\right) + 2\gamma_E - \frac{10}{3}\right) + \frac{107\pi}{105}(GM\omega)^2 + (GM\omega)^3\left(-\frac{29}{81} + \frac{428}{105}\zeta_2 - \frac{8}{3}\zeta_3\right)$$

=

[MST, 1996]

BHPT

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Match with

2602.06947 [Bjerrum-Bohr, Chen, Eriksen, Shah]
2602.06951 [Ivanov, Li, Parra-Martinez, Zhou]
2602.06125 [Bautista, Driesse, Haddad, Jakobsen]

$$S_\ell(\omega) = e^{2i\delta_\ell(\omega)}$$

EFT results to $\mathcal{O}(G^4)$

=

BHPT

[MST, 1996]

Match with 2602.06947, 2602.06951,2602.06125

$$\boxed{\delta_2^{(B)}(\omega) = GM\omega\left(\frac{1}{\epsilon} + \log\left(\frac{4\omega^2}{\mu_{\rm IR}^2}\right) + 2\gamma_E - \frac{10}{3}\right) + \frac{107\pi}{105}(GM\omega)^2 + (GM\omega)^3\left(-\frac{29}{81} + \frac{428}{105}\zeta_2 - \frac{8}{3}\zeta_3\right)} + \frac{1695233\pi}{1157625}(GM\omega)^4$$

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$$S_\ell(\omega) = e^{2i\delta_\ell(\omega)}$$

EFT results to $\mathcal{O}(G^5)$

=

!

Match with 2602.06947, 2602.06951,2602.06125

$$\delta_2^{(B)}(\omega) = GM\omega\left(\frac{1}{\epsilon} + \log\left(\frac{4\omega^2}{\mu_{\rm IR}^2}\right) + 2\gamma_E - \frac{10}{3}\right) + \frac{107\pi}{105}(GM\omega)^2 + (GM\omega)^3\left(-\frac{29}{81} + \frac{428}{105}\zeta_2 - \frac{8}{3}\zeta_3\right) + \frac{1695233\pi}{1157625}(GM\omega)^4$$

$$+(GM\omega)^5\left[-\frac{25609}{66150} + \frac{6780932}{1157625}\zeta_2 - \frac{1}{525}3424\zeta_2^2 + \frac{91592}{11025}\zeta_3 + \frac{32}{5}\zeta_5 + \boxed{\frac{1}{20\pi M_{\rm Pl}^2}\frac{\lambda_{0,2}^B}{(GM)^5}}\right]$$

Static Magnetic Love number

$$\boxed{\lambda_{0,2}^B = 0}$$

First time shown in a gauge-independent way

[MST, 1996]

BHPT

$$\delta_2^{(B)}(\omega) = GM\omega\left(\frac{1}{\epsilon} + \log\left(\frac{4\omega^2}{\mu_{\rm IR}^2}\right) + 2\gamma_E - \frac{10}{3}\right) + \frac{107\pi}{105}(GM\omega)^2 + (GM\omega)^3\left(-\frac{29}{81} + \frac{428}{105}\zeta_2 - \frac{8}{3}\zeta_3\right) + \frac{1695233\pi}{1157625}(GM\omega)^4$$

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$$S_\ell(\omega) = e^{2i\delta_\ell(\omega)}$$

EFT results to $\mathcal{O}(G^6)$

$=$!

BHPT

[MST, 1996]

Match with 2602.06947, 2602.06951,2602.06125

$$\delta_2^{(B)}(\omega) = GM\omega\left(\frac{1}{\epsilon} + \log\left(\frac{4\omega^2}{\mu_{\rm IR}^2}\right) + 2\gamma_E - \frac{10}{3}\right) + \frac{107\pi}{105}(GM\omega)^2 + (GM\omega)^3\left(-\frac{29}{81} + \frac{428}{105}\zeta_2 - \frac{8}{3}\zeta_3\right) + \frac{1695233\pi}{1157625}(GM\omega)^4$$

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$$+(GM\omega)^6\left[\frac{756197385619\pi}{140390971875} + \boxed{\frac{1}{20\pi M_{\rm Pl}^2}\frac{\lambda_{1,2}^B}{(GM)^6}} + \frac{1}{10M_{\rm Pl}^2}\frac{\lambda_{0,2}^B}{(GM)^5}\right]$$

First Dissipative Magnetic Love number

Matching with BHPT:

$$\boxed{\frac{1}{M_{Pl}^2}\lambda_{1,2}^B = \frac{4i\pi}{135}(2GM)^6}$$

$$\delta_2^{(B)}(\omega) = GM\omega\left(\frac{1}{\epsilon} + \log\left(\frac{4\omega^2}{\mu_{\rm IR}^2}\right) + 2\gamma_E - \frac{10}{3}\right) + \frac{107\pi}{105}(GM\omega)^2 + (GM\omega)^3\left(-\frac{29}{81} + \frac{428}{105}\zeta_2 - \frac{8}{3}\zeta_3\right) + \frac{1695233\pi}{1157625}(GM\omega)^4$$

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$$+(GM\omega)^6\left[\frac{756197385619\,\pi}{140390971875} + \frac{64i}{225}\right]$$

$$S_\ell(\omega) = e^{2i\delta_\ell(\omega)}$$

EFT results to $\mathcal{O}(G^7)$

Match with 2602.06947, 2602.06951,2602.06125

$$\boxed{\delta_2^{(B)}(\omega) = GM\omega\left(\frac{1}{\epsilon} + \log\left(\frac{4\omega^2}{\mu_{\rm IR}^2}\right) + 2\gamma_E - \frac{10}{3}\right) + \frac{107\pi}{105}(GM\omega)^2 + (GM\omega)^3\left(-\frac{29}{81} + \frac{428}{105}\zeta_2 - \frac{8}{3}\zeta_3\right)} + \frac{1695233\pi}{1157625}(GM\omega)^4$$

$$+(GM\omega)^5\Big[-\frac{25609}{66150} + \frac{6780932}{1157625}\zeta_2 - \frac{1}{525}3424\zeta_2^2 + \frac{91592}{11025}\zeta_3 + \frac{32}{5}\zeta_5 + \frac{1}{20\pi M_{\rm Pl}^2}\frac{\lambda_{0,2}^B}{(GM)^5}\Big]$$

UV divergence!

$$+(GM\omega)^6\left[\frac{756197\ldots}{140390971875}\ \cdots\ \frac{\ldots}{\ldots M_{\rm Pl}}\frac{\ldots}{(GM)^6} + \frac{1}{10M_{\rm Pl}^2}\frac{\lambda_{0,2}^B}{(GM)^5}\right]$$

$$+(GM\omega)^7\Big[\boxed{\frac{1}{\epsilon}}\left(\frac{32}{675} + \frac{1}{20\pi M_{\rm Pl}^2}\frac{\lambda_{2,2}^B(\epsilon^{-1})}{(GM)^7} + \frac{107}{2100\pi M_{\rm Pl}^2}\frac{\lambda_{0,2}^B}{(GM)^5}\right)$$

$$-\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)\left(\frac{224}{675} + \frac{1}{20\pi M_{\rm Pl}^2}\frac{\lambda_{2,2}^B(\epsilon^{-1})}{(GM)^7} + \frac{321}{2100\pi M_{\rm Pl}^2}\frac{\lambda_{0,2}^B}{(GM)^5}\right)$$

$$+\frac{34189466369}{13127467500} + \frac{3024789542476}{140390971875}\zeta_2 - \frac{143296}{55125}\zeta_2^2 + \frac{54784}{3675}\zeta_2^3$$

$$+\frac{2902238896}{121550625}\zeta_3 - \frac{732736}{11025}\zeta_5 - \frac{128}{7}\zeta_7 + \frac{2\zeta_2}{5\pi M_{\rm Pl}^2}\frac{\lambda_{0,2}^B}{(GM)^5}$$

$$+\frac{77}{600\pi M_{\rm Pl}^2}\frac{\lambda_{2,2}^B(\epsilon^{-1})}{(GM)^7} + \frac{1}{20\pi M_{\rm Pl}^2}\frac{\lambda_{2,2}^B(\epsilon^0)}{(GM)^7} + \frac{1}{10M_{\rm Pl}^2}\frac{\lambda_{1,2}^B}{(GM)^6} + \frac{392717}{441000\pi M_{\rm Pl}^2}\frac{\lambda_{0,2}^B}{(GM)^5}\Big]$$

=
!

[MST, 1996]

BHPT

$$\delta_2^{(B)}(\omega) = GM\omega\left(\frac{1}{\epsilon} + \log\left(\frac{4\omega^2}{\mu_{\rm IR}^2}\right) + 2\gamma_E - \frac{10}{3}\right) + \frac{107\pi}{105}(GM\omega)^2$$

$$+(GM\omega)^3\left(-\frac{29}{81} + \frac{428}{105}\zeta_2 - \frac{8}{3}\zeta_3\right) + \frac{1695233\pi}{1157625}(GM\omega)^4$$

$$+(GM\omega)^5\Big[-\frac{25609}{66150} + \frac{6780932}{1157625}\zeta_2 - \frac{1}{525}3424\zeta_2^2 + \frac{91592}{11025}\zeta_3 + \frac{32}{5}\zeta_5\Big]$$

$$+(GM\omega)^6\left[\frac{756197385619\,\pi}{140390971875} + \frac{64i}{225}\right]$$

$$+(GM\omega)^7\Big[-\frac{128}{225}\log(GM\omega) + \frac{1185486913843}{420078960000} + \frac{128i\pi}{225}$$

$$-\frac{128\log(2)}{225} - \frac{128}{225}(\gamma + \log(2)) + \frac{54784\zeta_2^3}{3675} - \frac{143296\zeta_2^2}{55125}$$

$$+\frac{3024789542476\zeta_2}{140390971875} + \frac{2902238896\zeta_3}{121550625} - \frac{128\zeta_7}{7} - \frac{732736\zeta_5}{11025}\Big]$$

$$S_\ell(\omega) = e^{2i\delta_\ell(\omega)}$$

Black Hole Love number experimental predictions

Electric Love numbers

$\mathcal{O}(G^5)$ $$\lambda^E_{0,2} = 0$$

$\mathcal{O}(G^6)$ $$\frac{1}{M_{Pl}^2}\lambda^E_{1,2} = \frac{128 i\pi}{45}(GM)^6$$

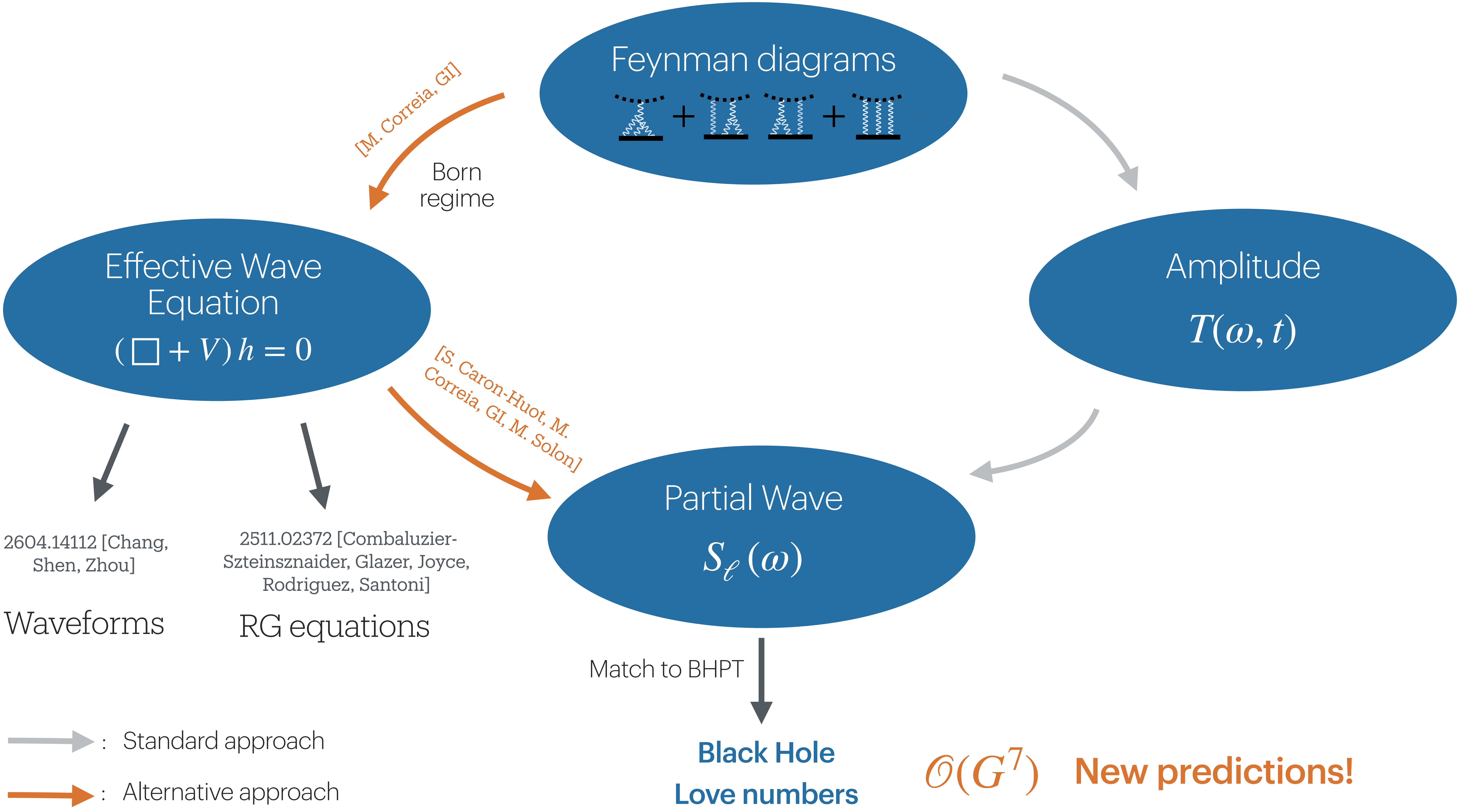
$\mathcal{O}(G^7)$ $$\frac{1}{M_{\rm Pl}^2}\lambda^E_{2,2} = -\frac{512\pi}{45}(GM)^7\left[\frac{1}{12\epsilon} + \log(2GM\mu) - \frac{131}{168} + \gamma_E\right]$$

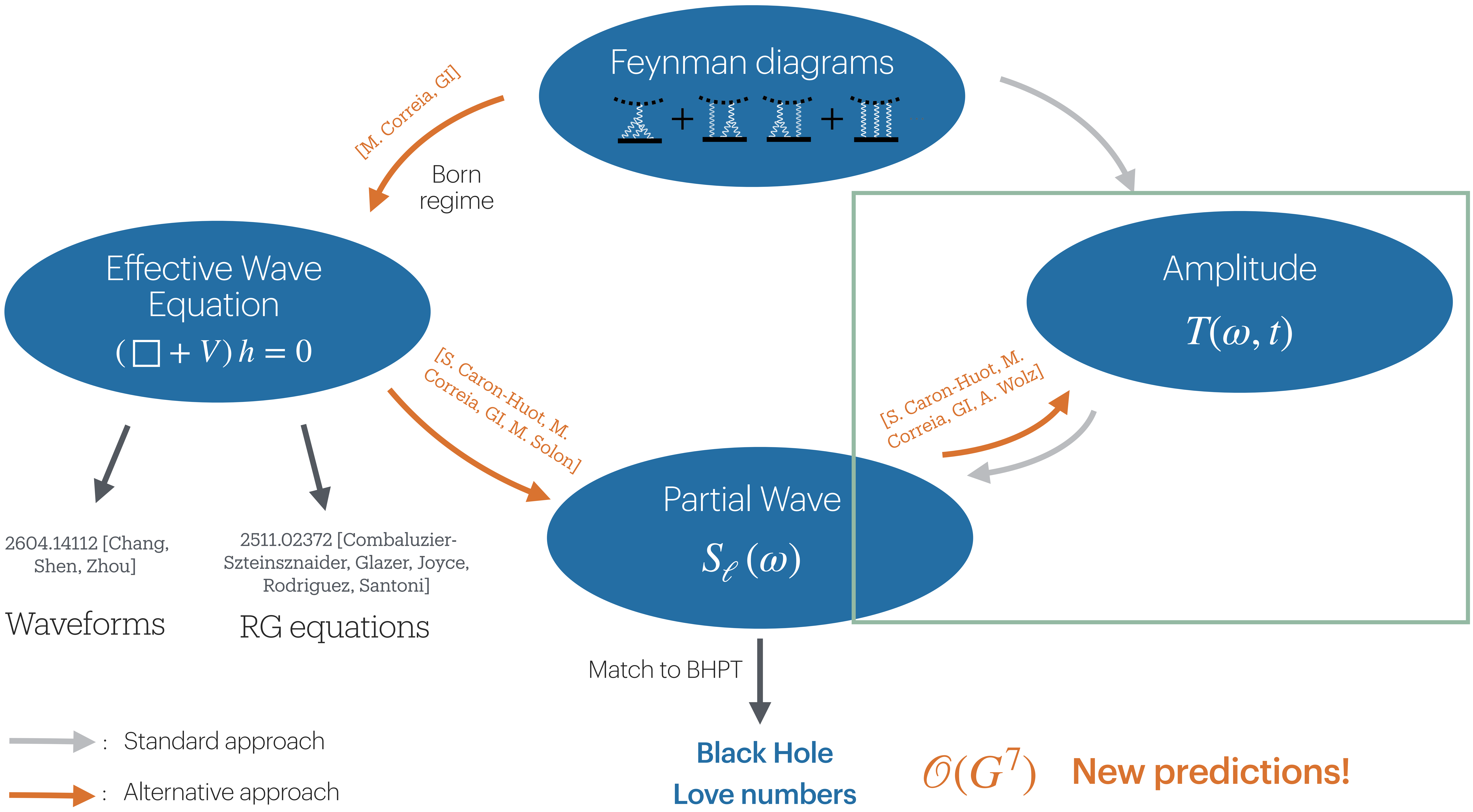
Magnetic Love numbers

$\mathcal{O}(G^5)$ $$\lambda^B_{0,2} = 0$$

$\mathcal{O}(G^6)$ $$\frac{1}{M_{Pl}^2}\lambda^B_{1,2} = \frac{256 i\pi}{135}(GM)^6$$

$\mathcal{O}(G^7)$ $$\frac{1}{M_{\rm Pl}^2}\lambda^B_{2,2} = -\frac{512\pi}{45}(GM)^7\left[\frac{1}{12\epsilon} + \log(2GM\mu) - \frac{377}{630} + \gamma_E\right]$$


Feynman diagrams
+
+
[M. Correia, GI]
Born regime
Amplitude
$T(\omega, t)$
Effective Wave Equation
$(\Box + V)\, h = 0$
[S. Caron-Huot, M. Correia, GI, M. Solon]
Partial Wave
$S_\ell(\omega)$
2604.14112 [Chang, Shen, Zhou]
Waveforms
2511.02372 [Combaluzier-Szteinsznaider, Glazer, Joyce, Rodriguez, Santoni]
RG equations
Match to BHPT
Black Hole Love numbers
$\mathcal{O}(G^7)$
New predictions!
: Standard approach
: Alternative approach



Feynman diagrams
[M. Correia, GI]
Born regime
Effective Wave Equation
$(\Box + V)\,h = 0$
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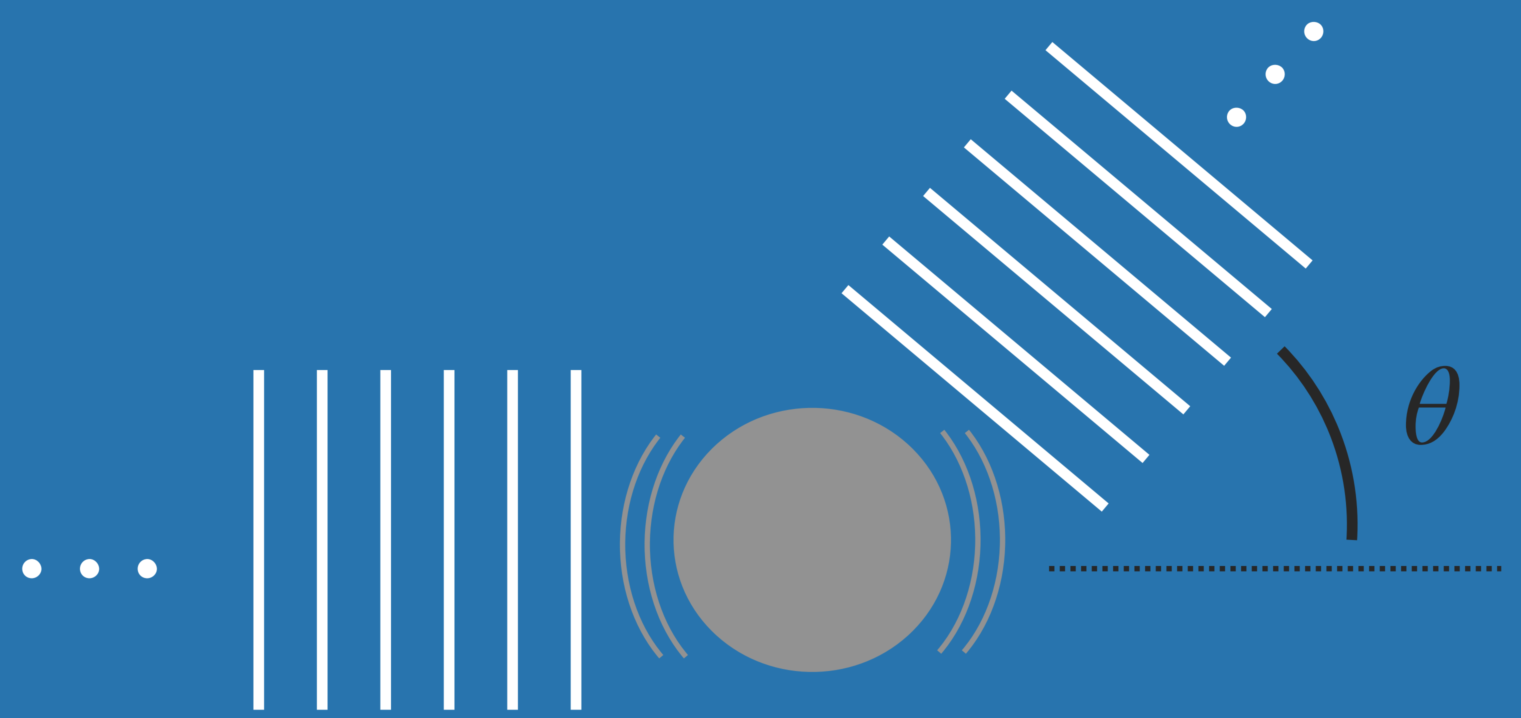
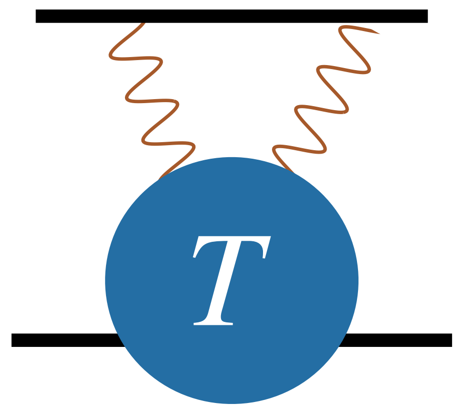
: Alternative approach

The scattering amplitude

$$T(\omega, t)$$



- We are at Amplitudes!
- Compton amplitude contains information about the two-body problem



W.I.P. [S. Caron-Huot, M. Correia, GI, A. Wolz]

Partial wave

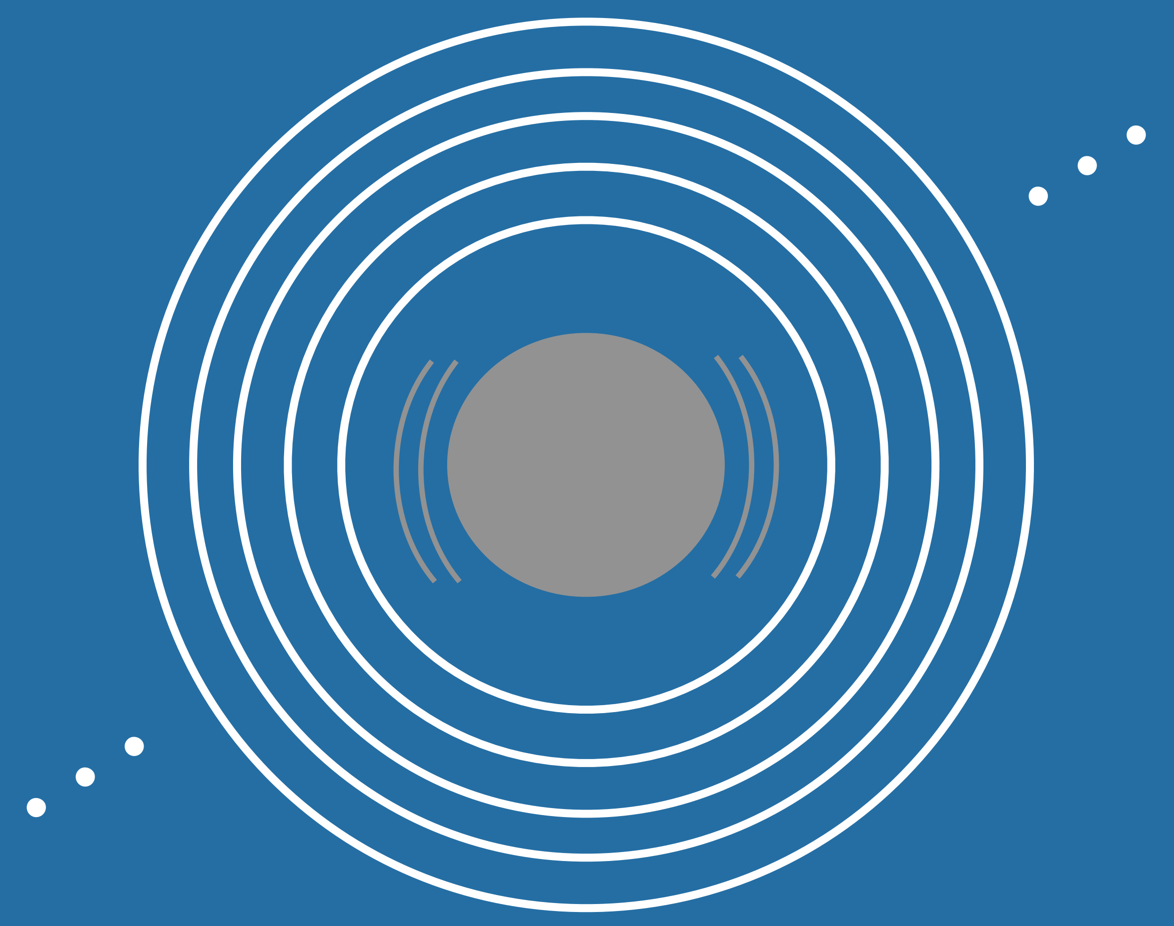
Amplitude

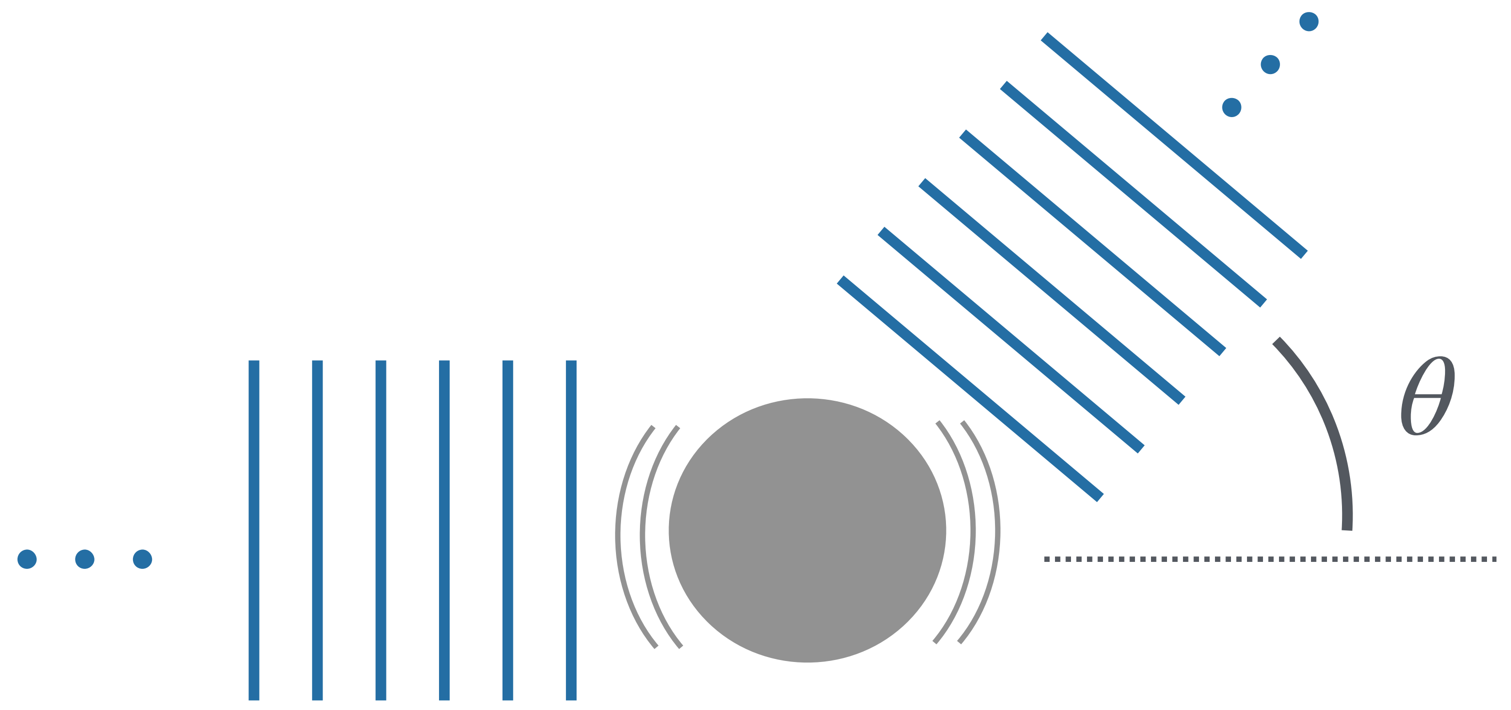
$$S_\ell(\omega) = e^{2i\delta_\ell(\omega)}$$

$$\sum_\ell$$

$$T(\omega, t)$$

$$t = (p_3 - p_1)^2 = \omega^2 x^2$$





We have the phase-shift
for this process!

Amplitudes/worldline
techniques

Partial wave

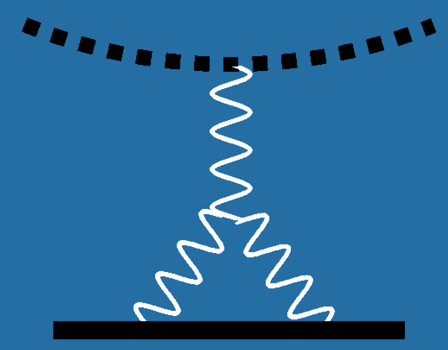
$\sum_{\ell}$

Amplitude

$t = (p_3 - p_1)^2 = \omega^2 x^2$

Rational functions

?



$$f_\ell(\omega) \sim \frac{1}{\ell + \frac{1}{2}}$$

$$\frac{1}{2x}$$

Partial wave

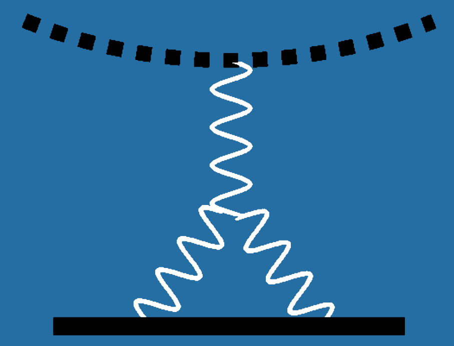
$\sum_{\ell}$

Amplitude

$$t = (p_3 - p_1)^2 = \omega^2 x^2$$

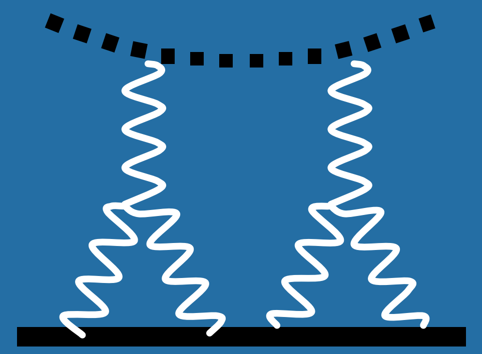
Rational functions

?



$$f_\ell(\omega) \sim \frac{1}{\ell + \frac{1}{2}}$$

$$\frac{1}{2x}$$



$$f_\ell(\omega) \sim \frac{1}{\left(\ell + \frac{1}{2}\right)^2}$$

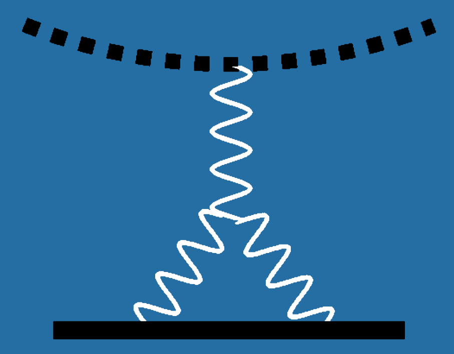
Partial wave

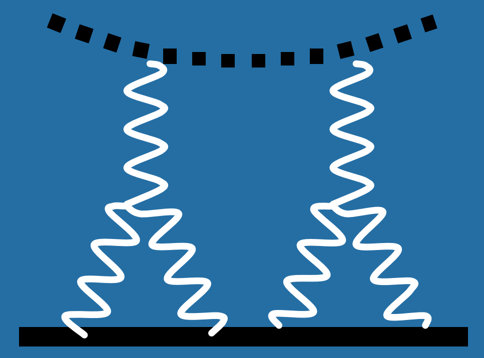
$\sum_{\ell}$

Amplitude

$$t = (p_3 - p_1)^2 = \omega^2 x^2$$

Rational functions	Elliptic HPLs
$f_\ell(\omega) \sim \dfrac{1}{\ell + \frac{1}{2}}$	$\dfrac{1}{2x}$
$f_\ell(\omega) \sim \dfrac{1}{\left(\ell + \frac{1}{2}\right)^2}$	$K(1 - x^2)$
$\cdots$	$\cdots$





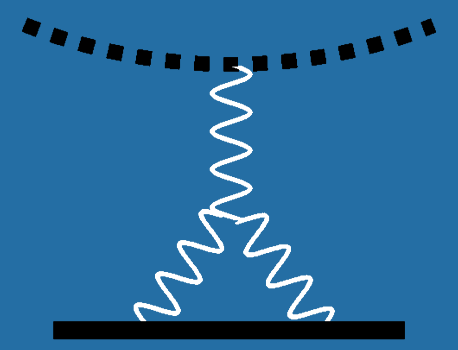
Partial wave

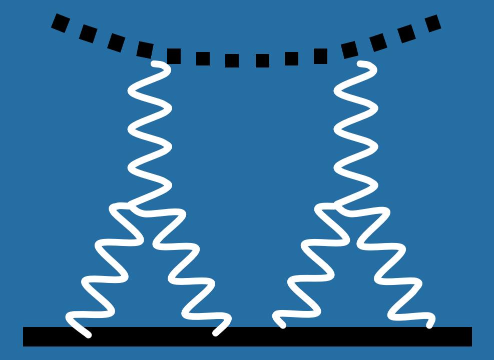
Amplitude

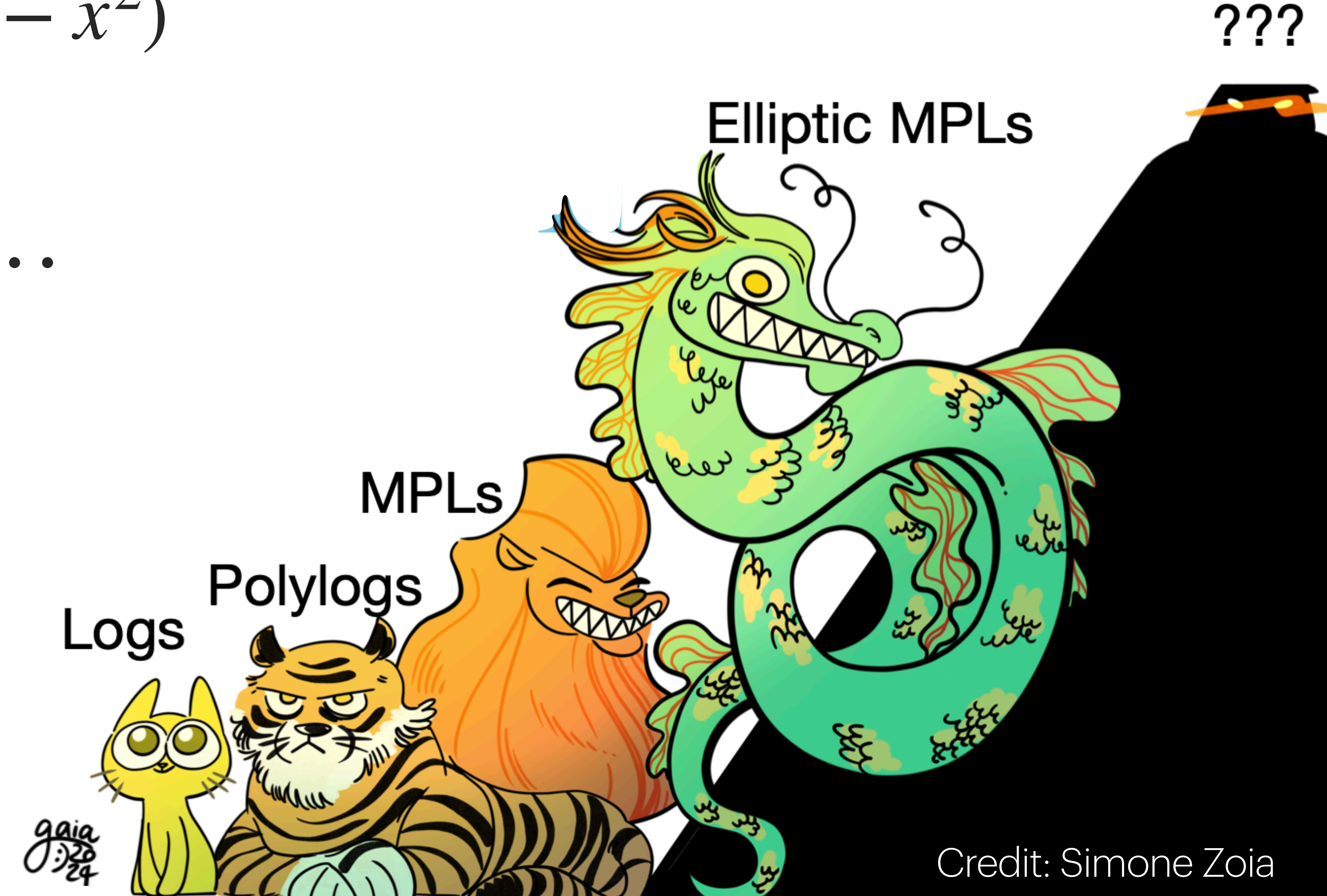
$$\sum_{\ell}$$

$$t = (p_3 - p_1)^2 = \omega^2 x^2$$

Rational functions	**Elliptic HPLs**
$f_\ell(\omega) \sim \dfrac{1}{\ell + \frac{1}{2}}$	$\dfrac{1}{2x}$
$f_\ell(\omega) \sim \dfrac{1}{\left(\ell + \frac{1}{2}\right)^2}$	$K(1 - x^2)$
$\cdots$	$\cdots$







Credit: Simone Zoia

Partial wave

$$\sum_{\ell}$$

Amplitude

$$t = (p_3 - p_1)^2 = \omega^2 x^2$$

Rational functions

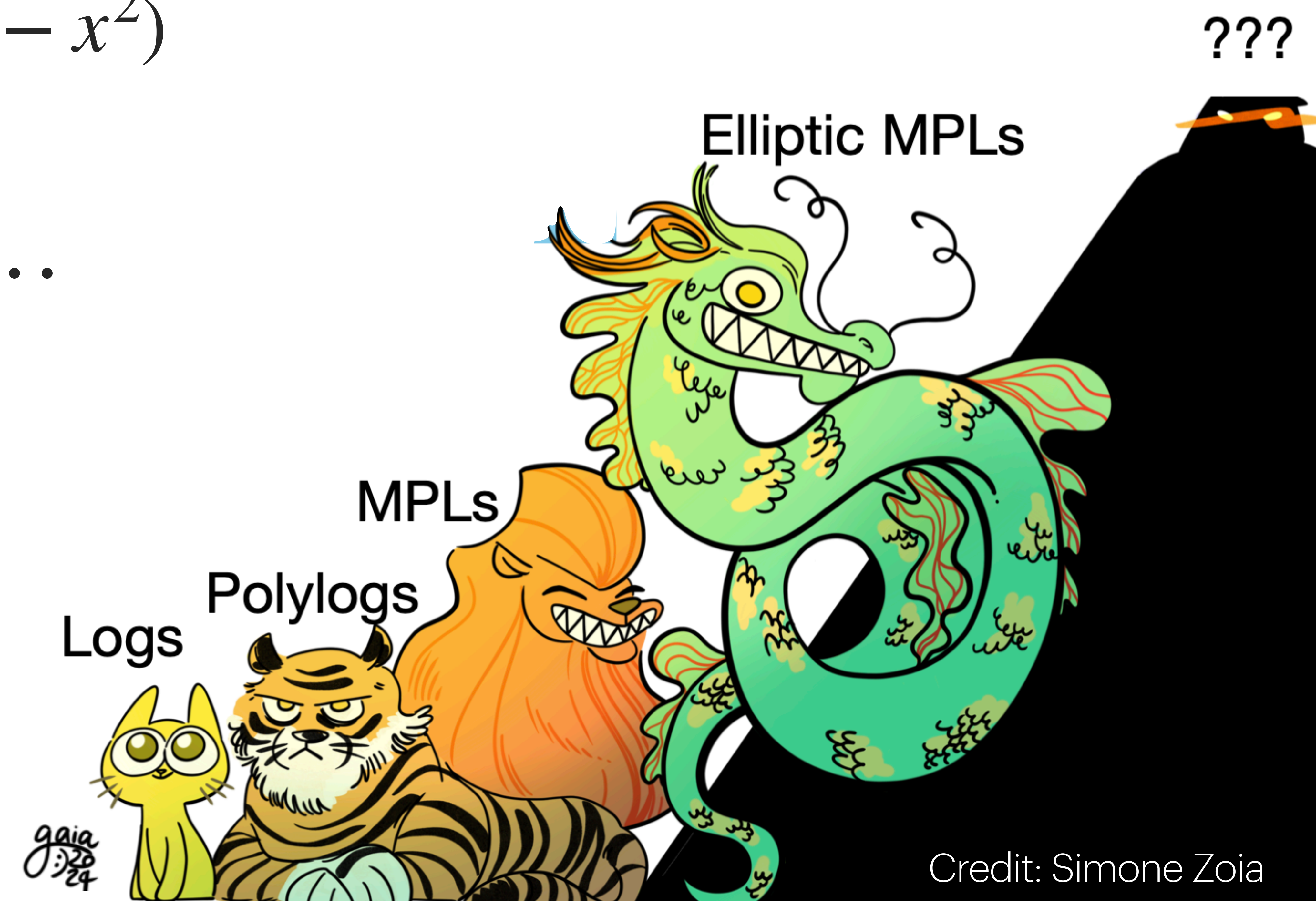
Elliptic HPLs

So SIMPLE in angular momentum space!

$$f_\ell(\omega) \sim \frac{1}{\ell + \frac{1}{2}} \qquad \frac{1}{2x}$$

$$f_\ell(\omega) \sim \frac{1}{\left(\ell + \frac{1}{2}\right)^2} \qquad K(1 - x^2)$$

$$\ldots \qquad \ldots$$



Credit: Simone Zoia

Scattering amplitude:
central object in QFT

Legendre
polynomials

$$\boxed{T(\omega, x)} = \mathcal{N} \sum_{\ell=0}^{\infty} \boxed{f_\ell(\omega)} \left(\ell + \frac{1}{2}\right) \boxed{P_\ell\left(1 - 2x^2\right)} = \sum_{i,\vec{g}} c_{i,\vec{g}}(\omega, x)\ I_{i;\vec{g}}(x)$$

Partial wave:
central object in QM

Scattering amplitude:
central object in QFT

Legendre
polynomials

Rational function in x

$$\boxed{T(\omega, x)} = \mathcal{N} \sum_{\ell=0}^{\infty} \boxed{f_\ell(\omega)} \left(\ell + \frac{1}{2}\right) \boxed{P_\ell\left(1 - 2x^2\right)} = \sum_{i,\vec{g}} \boxed{c_{i,\vec{g}}(\omega, x)} \boxed{I_{i;\vec{g}}(x)}$$

Partial wave:
central object in QM

Master Integrals

$$I_{i;\vec{g}}(x) = \oint_{\square} \omega_i(x, u) \, H_{\vec{g}}(u)$$

Scattering amplitude: central object in QFT

Legendre polynomials

Rational function in x

$$\boxed{T(\omega,x)} = \mathcal{N}\sum_{\ell=0}^{\infty}\boxed{f_\ell(\omega)}\left(\ell+\frac{1}{2}\right)\boxed{P_\ell\left(1-2x^2\right)} = \sum_{i,\vec{g}}\boxed{c_{i,\vec{g}}(\omega,x)}\,\boxed{I_{i;\vec{g}}(x)}$$

Partial wave: central object in QM

Master Integrals

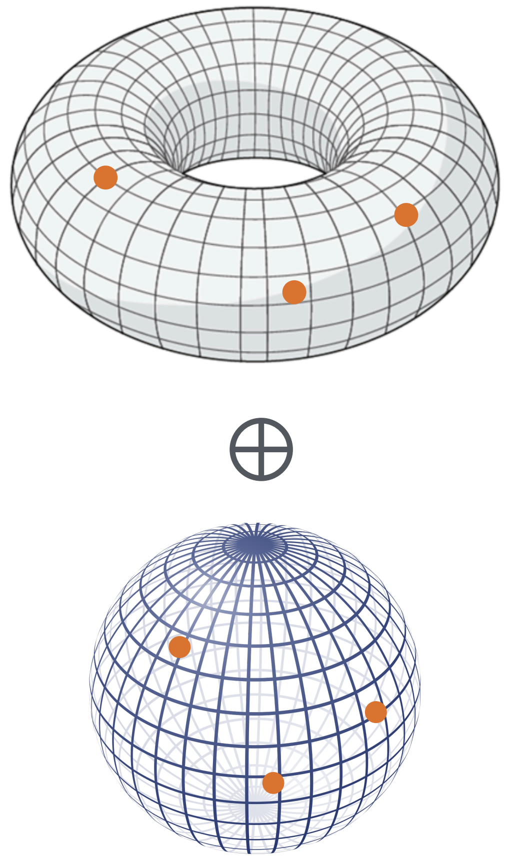
$$I_{i;\vec{g}}(x) = \oint_{\square}\omega_i(x,u)\,H_{\vec{g}}(u)$$

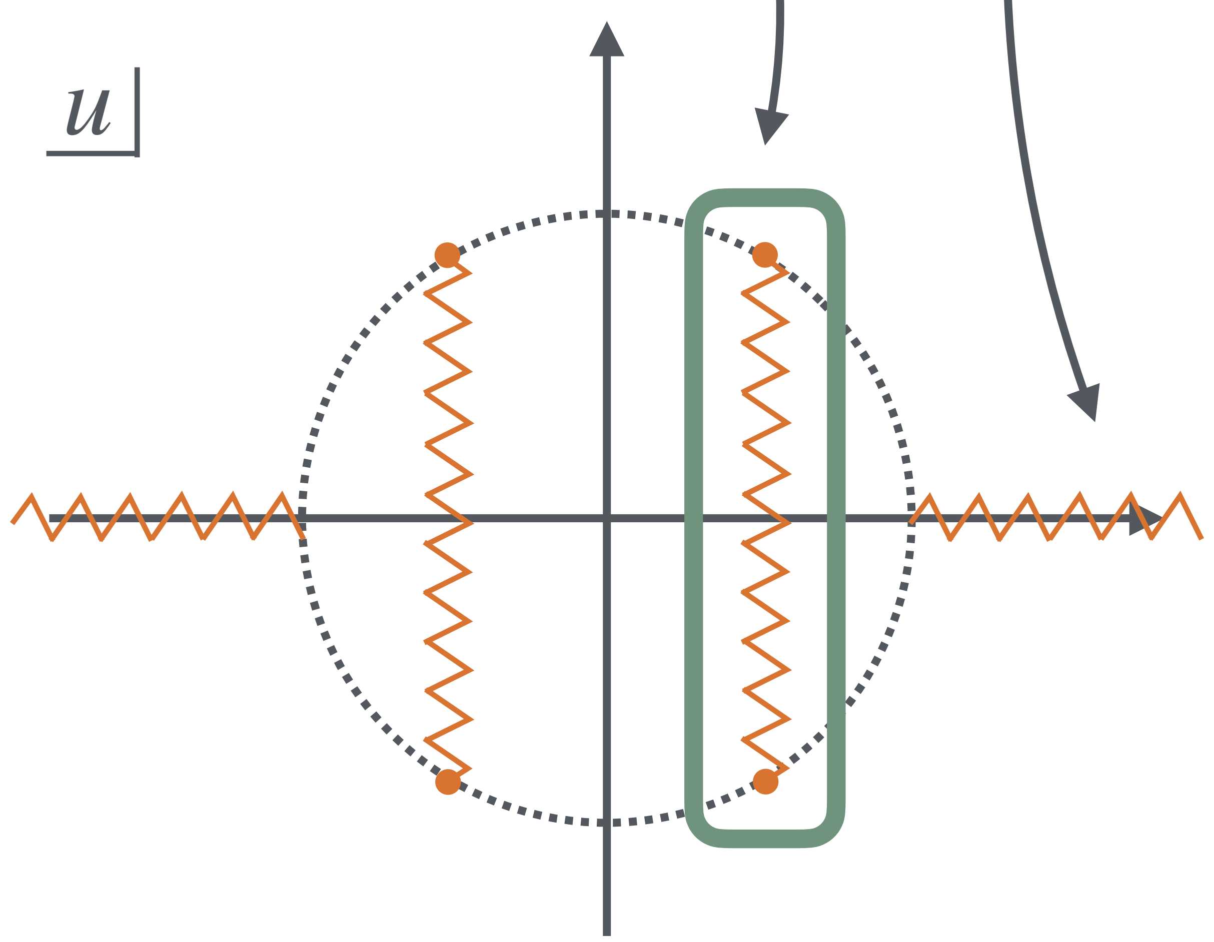
$$y^2 = u^4 - 2(1-2x^2)\,u^2 + 1$$

$$\left(\frac{du}{y}, \frac{u^2\,du}{y}, \frac{du}{y(u-1)} - \frac{du}{y(u+1)}\right)$$

$$\oplus$$

$$\left(\frac{u\,du}{y}, \frac{du}{u\,y}, \frac{du}{y(u-1)} + \frac{du}{y(u+1)}\right)$$



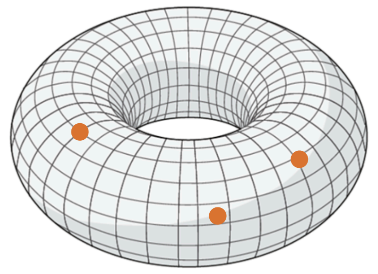


The master integrals: $I_{i;\vec{g}}(x) = \oint \omega_i(x,u)\, H_{\vec{g}}(u)$

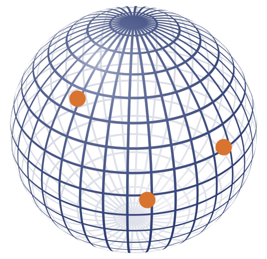
Differential Equations

Solved recursively weight by weight

$$(1-x^2)\frac{d}{dx} I_{i;h,\vec{g}}(x) = A_{ij} I_{j;h,\vec{g}}(x) + B_{ij} I_{j;\vec{g}}(x)$$



$$A = \begin{pmatrix} \boxed{\begin{array}{ccc} 1/(2x) - x & -1/(2x) & 0 \\ 1/(2x) & -1/(2x) + x & 0 \\ 1/(2x) & 1/(2x) & 1/x - x \end{array}} & \mathbf{0} \\ \mathbf{0} & \boxed{\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/x - x \end{array}} \end{pmatrix}$$

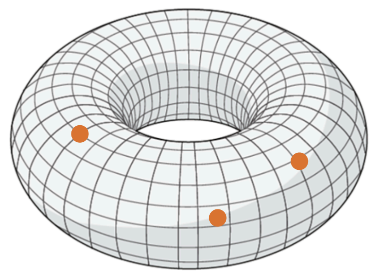


The master integrals: $I_{i;\vec{g}}(x) = \oint \omega_i(x,u)\, H_{\vec{g}}(u)$

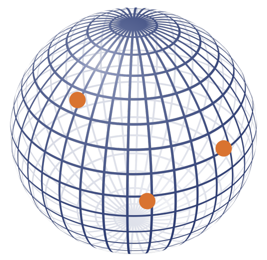
Differential Equations

Solved recursively weight by weight

$$(1-x^2)\frac{d}{dx} I_{i;h,\vec{g}}(x) = A_{ij} I_{j;h,\vec{g}}(x) + B_{ij} I_{j;\vec{g}}(x)$$



$$A = \begin{pmatrix} \begin{array}{|ccc|} \hline 1/(2x)-x & -1/(2x) & 0 \\ 1/(2x) & -1/(2x)+x & 0 \\ 1/(2x) & 1/(2x) & 1/x-x \\ \hline \end{array} & \mathbf{0} \\ \mathbf{0} & \begin{array}{|ccc|} \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/x-x \\ \hline \end{array} \end{pmatrix}$$

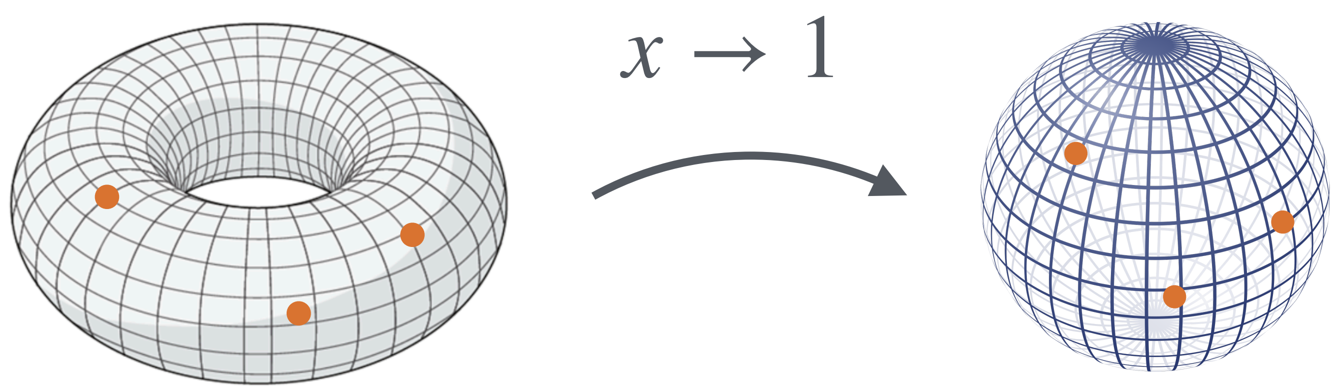


Boundary Conditions

Solved analytically in
the backward limit $x = 1$

$$(I_{1;\vec{g}}, I_{2;\vec{g}}, I_{3;\vec{g}})\Big|_{x=1} = (1,-1,-1)\times \operatorname{Im} H_{\vec{g}}(i)$$

$$(I_{4;\vec{g}}, I_{5;\vec{g}}, I_{6;\vec{g}})\Big|_{x=1} = (1,-1,-1)\times \operatorname{Re} H_{\vec{g}}(i)$$

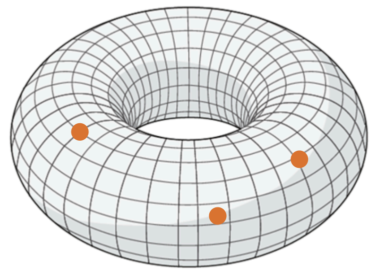


The master integrals: $I_{i;\vec{g}}(x) = \oint \omega_i(x,u)\, H_{\vec{g}}(u)$

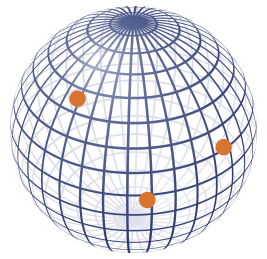
Differential Equations

Solved recursively weight by weight

$$(1-x^2)\frac{d}{dx} I_{i;h,\vec{g}}(x) = A_{ij} I_{j;h,\vec{g}}(x) + B_{ij} I_{j;\vec{g}}(x)$$



$$A = \begin{pmatrix} \boxed{\begin{matrix} 1/(2x) - x & -1/(2x) & 0 \\ 1/(2x) & -1/(2x) + x & 0 \\ 1/(2x) & 1/(2x) & 1/x - x \end{matrix}} & 0 \\ 0 & \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/x - x \end{matrix}} \end{pmatrix}$$

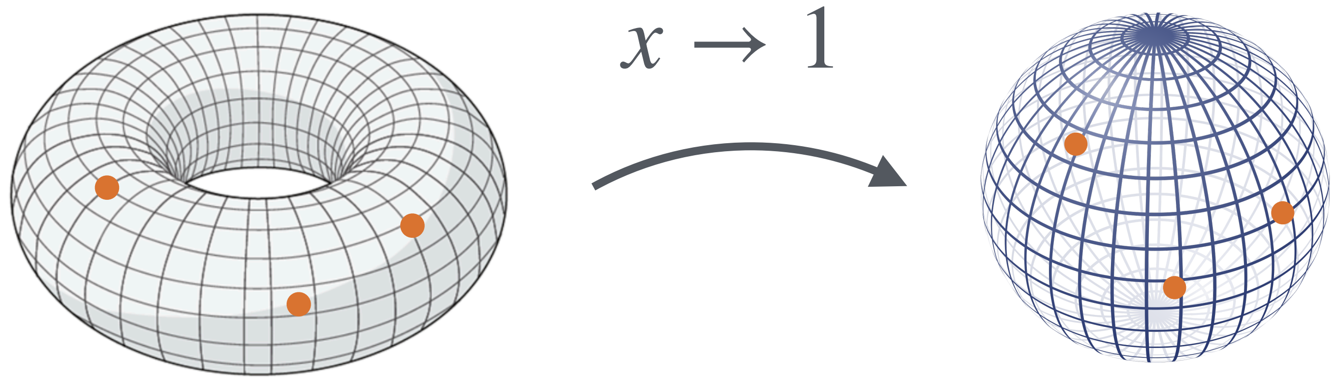


Boundary Conditions

Solved analytically in the backward limit $x = 1$

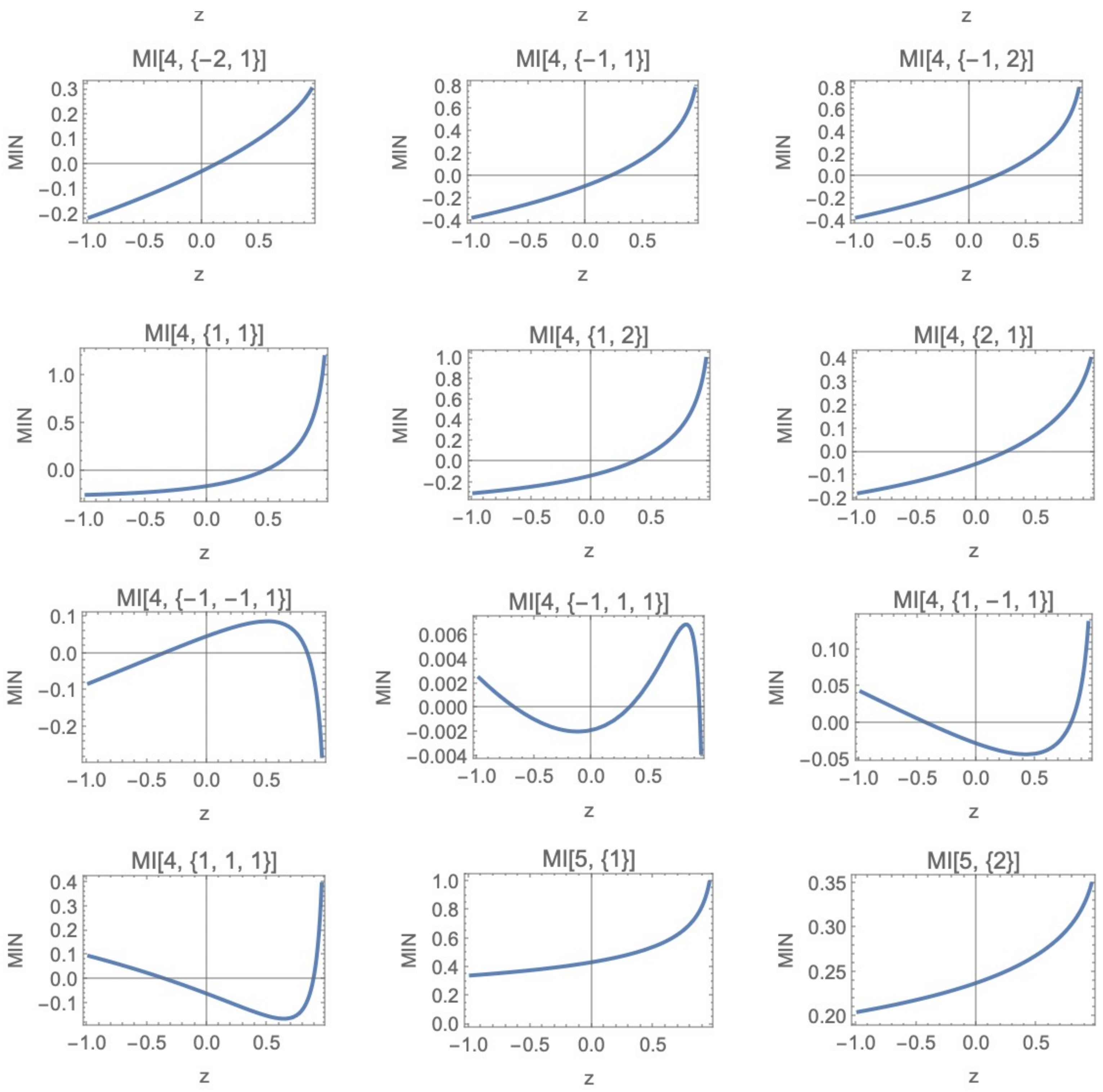
$$(I_{1;\vec{g}}, I_{2;\vec{g}}, I_{3;\vec{g}})\Big|_{x=1} = (1,-1,-1) \times \operatorname{Im} H_{\vec{g}}(i)$$

$$(I_{4;\vec{g}}, I_{5;\vec{g}}, I_{6;\vec{g}})\Big|_{x=1} = (1,-1,-1) \times \operatorname{Re} H_{\vec{g}}(i)$$



Numerical Evaluation

Single variable integration on analytic contour



1. Express Legendre Polynomials in integral form

$$T(\omega, x) = \sum_{\ell=0}^{\infty} f_\ell(\omega) \left(\ell + \frac{1}{2}\right) P_\ell\left(1 - 2x^2\right) = -\frac{1}{2\pi i} \oint \frac{u\, du\, (1 - u^4)}{2y^3} \sum_{\ell=0}^{\infty} f_\ell(\omega)\, u^{2\ell}$$

$$y^2 = u^4 - 2(1 - 2x^2)\, u^2 + 1$$

2. Resum partial wave Taylor series

$$\sum_{\ell=0}^{\infty} f_\ell(\omega)\, u^{2\ell} \propto H_{\vec{g}}(x)$$

Harmonic Polylogarithms

3. Integration By Part and projection in Master Integral basis

Elliptic Polylogarithms

1. Express Legendre Polynomials in integral form

$$T(\omega, x) = \sum_{\ell=0}^{\infty} f_\ell(\omega) \left(\ell + \frac{1}{2}\right) P_\ell\left(1 - 2x^2\right) = -\frac{1}{2\pi i} \oint_{\square} \frac{u\, du\, (1 - u^4)}{2y^3} \sum_{\ell=0}^{\infty} f_\ell(\omega)\, u^{2\ell}$$

$$y^2 = u^4 - 2(1 - 2x^2)\, u^2 + 1$$

2. Resum partial wave Taylor series

$$\sum_{\ell=0}^{\infty} f_\ell(\omega)\, u^{2\ell} \propto H_{\vec{g}}(x)$$

Harmonic Polylogarithms

3. Integration By Part and projection in Master Integral basis

Elliptic Polylogarithms

Loops	N of MIs	Time*
G^2	4	$2\,\mathrm{s}$
G^3	$+8$	$10\,\mathrm{s}$
G^4	$+29$	$50\,\mathrm{s}$
G^5	$+97$	$4\,\mathrm{min}$
G^6	$+319$	$16\,\mathrm{min}$
G^7	$+1022$	$39\,\mathrm{min}$

*On my laptop

Statistics of the computation: : $\mathcal{O}(G^6)$ example

```
G6amp = PWtoAmp[PWampG6, True, False, "Termwise"]
PrintSize[%]
```

1.
```
--- canonicalization took 108.683 s ---
--- extract finite part took 0.008 s ---
--- extract delta part took 0.004 s ---
--- canonical finite part took 6.117 s ---
```
2.
```
--- resummation to HPLs took 17.254 s ---
```
3.
```
--- finite projection: 10579 IBP terms ---
--- finite projection to MIs took 842.732 s ---
--- delta / UV part took 3.488 s ---
```
```
--- final expansion took 0.063 s ---
```

$\sim$ 15 min

$\sim$ 5.5 MB

$\sim$ 7800 terms

$$\frac{197024}{45}\,GM^6\,\pi\,\omega^5 - \frac{90680}{9}\,\text{EulerGamma}\,GM^6\,\pi\,\omega^5 - \frac{5884}{3}\,\text{EulerGamma}^2\,GM^6\,\pi\,\omega^5 + \frac{5240}{9}\,\text{EulerGamma}^3\,GM^6\,\pi\,\omega^5 + \frac{256}{3}\,\text{EulerGamma}^4\,GM^6\,\pi\,\omega^5 + \frac{7888}{3}\,i\,GM^6\,\pi^2\,\omega^5 + \cdots 11717 \cdots + \frac{1032\,GM^6\,\pi\,w^2\,\omega^5\,\text{MI}[6,\{1\}]\times Z[4]}{-1+w^2} - \frac{64\,GM^6\,\pi\,w^4\,\omega^5\,\text{MI}[6,\{1\}]\times Z[4]}{-1+w^2} - \frac{256\,GM^6\,\pi\,\omega^5\,\text{MI}[4]\times Z[5]}{5\,(-1+w^2)} + \frac{256\,GM^6\,\pi\,\omega^5\,\text{MI}[4]\times Z[5]}{5\,w^2\,(-1+w^2)}$$

large output | show less | show more | show all | set size limit...

```
output size: 5.55 MB
```

Exponentiated amplitude to $\mathcal{O}(G^2)$

$$i\hat{T} = e^{i\hat{\Delta}} - 1$$

$$t = \omega^2 x^2$$

$$\Delta(x, \omega) = \frac{4\pi GM}{x^2} I_4 + \pi^2 (GM)^2 \omega \left(\frac{x^2}{2} - \frac{15}{2} \right) I_6$$

$$\boxed{I_4 = 1 \qquad I_6 = -\frac{1}{x}}$$

Exponentiated amplitude to $\mathcal{O}(G^3)$

$$i\hat{T} = e^{i\hat{\Delta}} - 1$$

$$t = \omega^2 x^2$$

Match with 2602.06951 [Ivanov, Li, Parra-Martinez, Zhou]

$$\Delta(x,\omega) = \frac{4\pi GM}{x^2} I_4 + \pi^2 (GM)^2 \omega \left(\frac{x^2}{2} - \frac{15}{2}\right) I_6$$
$$+ 2\pi (GM)^3 \omega^2 \Big[\frac{32}{3x^2} I_{4;2} + \left(60 - 4x^2\right) I_{6;2} - \frac{32}{3x^2} I_{5;2} + 18 I_{4;1} + 18 I_{5;1} + \zeta_2 \left(x^2 - 15\right) I_6$$
$$- 10 I_4 + 12 \log\left(\frac{\bar{\mu}^2}{4\omega^2}\right) + 8\log(\omega) + \frac{4}{\epsilon} + \frac{104}{3} + 4\left(\gamma_{\mathrm{E}} - \log(\pi)\right) \Big]$$

HPLs

$$I_{4;1} = -\left[\log(x) + \log(x+1)\right]/2 \qquad I_{5;1} = -\left[\log(x) - \log(x+1)\right]/2$$

$$I_{4;2} = \frac{1}{4}\left[\mathrm{Li}_2(-x) - \mathrm{Li}_2(x) - \log(1-x)\,\log(x) - \log(x+1)\,\log(x) + \zeta_2\right]$$

$$I_{5;2} = \frac{1}{4}\left[3\mathrm{Li}_2(-x) + \mathrm{Li}_2(x) + \log(1-x)\,\log(x) + \log(x+1)\,\log(x) + \zeta_2\right]$$

$$I_{6;2} = -\frac{1}{4x}\left[\mathrm{Li}_2(-x) - \mathrm{Li}_2(x) - \log(1-x)\,\log(x) + \log(x+1)\,\log(x) + \zeta_2\right]$$

Exponentiated amplitude to $\mathcal{O}(G^4)$

$$i\hat{T} = e^{i\hat{\Delta}} - 1$$

$$t = \omega^2 x^2$$

Match with 2602.06951 [Ivanov, Li, Parra-Martinez, Zhou]

$$\begin{aligned}\Delta(x,\omega) = \boxed{\begin{aligned}&\frac{4\pi GM}{x^2} I_4 + \pi^2 (GM)^2 \omega \left(\frac{x^2}{2} - \frac{15}{2}\right) I_6 \\ &+ 2\pi (GM)^3 \omega^2 \left[\frac{32}{3x^2} I_{4;2} + \left(60 - 4x^2\right) I_{6;2} - \frac{32}{3x^2} I_{5;2} + 18 I_{4;1} + 18 I_{5;1} + \zeta_2 \left(x^2 - 15\right) I_6 \right. \\ &\left. \quad - 10 I_4 + 12 \log\left(\frac{\bar{\mu}^2}{4\omega^2}\right) + 8\log(\omega) + \frac{4}{\epsilon} + \frac{104}{3} + 4\left(\gamma_{\mathrm{E}} - \log(\pi)\right)\right]\end{aligned}} \\ &+ 2\pi^2 (GM)^4 \omega^3 \left[-24 I_4 - \frac{3}{128} x^2 (-218 + x^2) I_6 + \frac{59}{64} (2x^2 - 1) I_{1;1} + \frac{1}{64} (1741 - x^2) I_{1;2} + \frac{59}{64} \left(I_{2;1} + I_{2;2}\right) \right. \\ &\left. \quad + 16 \left(I_{4;1} - I_{5;1}\right) + 16 \log\left(\frac{\bar{\mu}^2}{4\omega^2}\right) + 8 \log(\omega) + \frac{4}{\epsilon} + \frac{152}{3} + 4\left(\gamma_E - \log(\pi)\right)\right]\end{aligned}$$

Appearance of elliptic functions

$$\boxed{\begin{aligned}I_{1;1} &= K(1-x^2)/2 \\ I_{2;1} &= K(1-x^2)/2 - E(1-x^2) - x + 1\end{aligned}}$$

Exponentiated amplitude to $\mathcal{O}(G^5)$

$$i\hat{T} = e^{i\hat{\Delta}} - 1$$

$$t = \omega^2 x^2$$

Match with 2602.06951 [Ivanov, Li, Parra-Martinez, Zhou]

$$
\begin{aligned}
\Delta(x,\omega) = \; & \boxed{\begin{aligned} & \frac{4\pi GM}{x^2} I_4 + \pi^2 (GM)^2 \omega \left(\frac{x^2}{2} - \frac{15}{2} \right) I_6 \; + 2\pi (GM)^3 \omega^2 \Big[\frac{32}{3x^2} I_{4;2} + \left(60 - 4x^2\right) I_{6;2} - \frac{32}{3x^2} I_{5;2} + 18 I_{4;1} + 18 I_{5;1} + \zeta_2 \left(x^2 - 15\right) I_6 \\ & \quad - 10 I_4 + 12 \log\left(\frac{\bar{\mu}^2}{4\omega^2} \right) + 8 \log(\omega) + \frac{4}{\epsilon} + \frac{104}{3} + 4\Big(\gamma_{\mathrm{E}} - \log(\pi) \Big) \Big] \end{aligned}} \\
& + 2\pi^2 (GM)^4 \omega^3 \Big[-24 I_4 - \frac{3}{128} x^2 (-218 + x^2) I_6 + \frac{59}{64} (2x^2 - 1) I_{1;1} + \frac{1}{64} (1741 - x^2) I_{1;2} + \frac{59}{64} \left(I_{2;1} + I_{2;2} \right) \\
& \qquad + 16 \left(I_{4;1} - I_{5;1} \right) + 16 \log\left(\frac{\bar{\mu}^2}{4\omega^2} \right) + 8 \log(\omega) + \frac{4}{\epsilon} + \frac{152}{3} + 4 \left(\gamma_E - \log(\pi) \right) \Big] \\
& + 2\pi (GM)^5 \omega^4 \Big[\frac{x^2 - 1741}{4} \left(I_{1;-2,2} + 2 I_{1;-1,3} + 2 I_{1;1,3} + I_{1;2,2} \right) - \frac{59}{4} \left(I_{2;-2,2} + I_{2;-1,2} + 2 I_{2;-1,3} + I_{2;1,2} + 2 I_{2;1,3} + I_{2;2,2} \right) \\
& \quad + \frac{2783}{12} \left(I_{4;2} - I_{5;2} \right) - 485 I_{4;3} - 32 I_{5;3} + \frac{512}{5x^2} \left(I_{5;4} - I_{4;4} \right) + 256 \left(I_{4;-1,2} - I_{4;1,2} - I_{5;-1,2} + I_{5;1,2} \right) \\
& \quad + \frac{3}{4} x^2 (x^2 - 218) I_{6;2} + 59 x^2 I_{6;3} + 64 (x^2 - 15) I_{6;4} - \frac{265 x^2}{24} \left(I_{4;1} + I_{5;1} \right) + 128 \zeta_2 \left(I_{4;1} - I_{5;1} \right) \\
& \quad - \frac{x^2 - 1741}{8} \zeta_2 I_{1;2} + \frac{59}{8} \zeta_2 I_{2;2} + I_4 \left(\frac{352}{9} + \frac{539 x^2}{72} - 192 \zeta_2 - \frac{59}{8} \zeta_3 \right) + \frac{59}{8} I_{2;1} \left(\zeta_2 + \zeta_3 \right) \\
& \quad + \frac{1}{8} I_{1;1} \left[59 (2x^2 - 1) \zeta_2 - (x^2 - 1741) \zeta_3 \right] + I_6 \Big[-\frac{3}{16} x^4 \zeta_2 + \frac{x^2}{8} \left(327 \zeta_2 - 59 \zeta_3 - 32 \zeta_4 \right) + 60 \zeta_4 \Big] - \frac{59}{4} (2x^2 - 1) \left(I_{1;-1,2} + I_{1;1,2} \right)
\end{aligned}
$$

+ weight 0

Exponentiated amplitude to $\mathcal{O}(G^6)$

$$i\hat{T} = e^{i\hat{\Delta}} - 1$$

Match with 2602.06951 [Ivanov, Li, Parra-Martinez, Zhou]

$$t = \omega^2 x^2$$

$$\Delta(x,\omega) = \boxed{\frac{4\pi GM}{x^2} I_4 + \pi^2 (GM)^2 \omega \left(\frac{x^2}{2} - \frac{15}{2}\right) I_6 \; + 2\pi (GM)^3 \omega^2 \Big[\frac{32}{3x^2} I_{4;2} + \left(60 - 4x^2\right) I_{6;2} - \frac{32}{3x^2} I_{5;2} + 18 I_{4;1} + 18 I_{5;1} + \zeta_2 \left(x^2 - 15\right) I_6 - 10 I_4 + 12 \log\left(\frac{\bar{\mu}^2}{4\omega^2}\right) + 8\log(\omega) + \frac{4}{\epsilon} + \frac{104}{3} + 4\left(\gamma_{\mathrm{E}} - \log(\pi)\right)\Big]}$$

$$+ 2\pi^2 (GM)^4 \omega^3 \Big[-24 I_4 - \frac{3}{128} x^2 (-218 + x^2) I_6 + \frac{59}{64}(2x^2 - 1) I_{1;1} + \frac{1}{64}(1741 - x^2) I_{1;2} + \frac{59}{64}\left(I_{2;1} + I_{2;2}\right)$$

$$+ 16\left(I_{4;1} - I_{5;1}\right) + 16 \log\left(\frac{\bar{\mu}^2}{4\omega^2}\right) + 8 \log(\omega) + \frac{4}{\epsilon} + \frac{152}{3} + 4\left(\gamma_E - \log(\pi)\right)\Big] + 2\pi (GM)^5 \omega^4 \Big[\frac{x^2 - 1741}{4}\left(I_{1;-2,2} + 2 I_{1;-1,3} + 2 I_{1;1,3} + I_{1;2,2}\right)$$

$$- \frac{59}{4}\left(I_{2;-2,2} + I_{2;-1,2} + 2 I_{2;-1,3} + I_{2;1,2} + 2 I_{2;1,3} + I_{2;2,2}\right) + \frac{2783}{12}\left(I_{4;2} - I_{5;2}\right) - 485 I_{4;3} - 32 I_{5;3} + \frac{512}{5x^2}\left(I_{5;4} - I_{4;4}\right)$$

$$+ 256\left(I_{4;-1,2} - I_{4;1,2} - I_{5;-1,2} + I_{5;1,2}\right) + \frac{3}{4} x^2 (x^2 - 218) I_{6;2} + 59 x^2 I_{6;3} + 64 (x^2 - 15) I_{6;4} - \frac{265 x^2}{24}\left(I_{4;1} + I_{5;1}\right)$$

$$+ 128 \zeta_2 \left(I_{4;1} - I_{5;1}\right) - \frac{x^2 - 1741}{8} \zeta_2 I_{1;2} + \frac{59}{8} \zeta_2 I_{2;2} + I_4 \left(\frac{352}{9} + \frac{539 x^2}{72} - 192 \zeta_2 - \frac{59}{8}\zeta_3\right) + \frac{59}{8} I_{2;1}\left(\zeta_2 + \zeta_3\right)$$

$$+ \frac{1}{8} I_{1;1}\left[59(2x^2 - 1)\zeta_2 - (x^2 - 1741)\zeta_3\right] + I_6 \Big[-\frac{3}{16} x^4 \zeta_2 + \frac{x^2}{8}\left(327\zeta_2 - 59\zeta_3 - 32\zeta_4\right) + 60\zeta_4 \Big] - \frac{59}{4}(2x^2 - 1)\left(I_{1;-1,2} + I_{1;1,2}\right)$$

+ weight 0

$$+ (GM)^6 \pi^2 \omega^5 \Big[\frac{16}{9}\left(24x^2 + 1261\right) I_4 + \frac{x^2}{12288}\left(60x^4 - 57847x^2 + 2855161\right) I_6 + \frac{(2x^2 - 1)(1863431 - 4344x^2)}{12288} I_{1;1} - \frac{243638x^2 + 8310701}{4096} I_{1;2}$$

$$+ \frac{709}{1024}(2x^2 - 1) I_{1;3} + \frac{410759 - 2x^2}{256} I_{1;4} + \frac{1863431 - 4344x^2}{12288}\left(I_{2;1} + I_{2;2}\right) + \frac{128}{3}(x^2 + 9)\left(I_{5;1} - I_{4;1}\right) + \frac{2816}{3}\left(I_{4;2} + I_{5;2}\right) + \frac{709}{1024}\left(I_{2;3} + I_{2;4}\right)\Big]$$

+ weight 0

Exponentiated amplitude to $\mathcal{O}(G^7)$

$$i\hat{T} = e^{i\hat{\Delta}} - 1$$

$$t = \omega^2 x^2$$

Match with 2602.06951 [Ivanov, Li, Parra-Martinez, Zhou]

$$\boxed{\begin{aligned}\Delta(x,\omega) = \frac{4\pi GM}{x^2} I_4 &+ \pi^2 (GM)^2 \omega \left(\frac{x^2}{2} - \frac{15}{2}\right) I_6 + 2\pi (GM)^3 \omega^2 \Big[\frac{32}{3x^2} I_{4;2} + \left(60 - 4x^2\right) I_{6;2} - \frac{32}{3x^2} I_{5;2} + 18 I_{4;1} + 18 I_{5;1} + \zeta_2 \left(x^2 - 15\right) I_6 \\ &- 10 I_4 + 12 \log\left(\frac{\bar{\mu}^2}{4\omega^2}\right) + 8 \log(\omega) + \frac{4}{\epsilon} + \frac{104}{3} + 4\left(\gamma_{\mathrm{E}} - \log(\pi)\right)\Big]\end{aligned}}$$

$$\begin{aligned}
&+2\pi^2 (GM)^4 \omega^3 \Big[-24 I_4 - \frac{3}{128} x^2 (-218 + x^2) I_6 + \frac{59}{64} (2x^2 - 1) I_{1;1} + \frac{1}{64} (1741 - x^2) I_{1;2} + \frac{59}{64} \left(I_{2;1} + I_{2;2}\right) \\
&+16 \left(I_{4;1} - I_{5;1}\right) + 16 \log\left(\frac{\bar{\mu}^2}{4\omega^2}\right) + 8 \log(\omega) + \frac{4}{\epsilon} + \frac{152}{3} + 4 \left(\gamma_E - \log(\pi)\right)\Big] + 2\pi (GM)^5 \omega^4 \Big[\frac{x^2 - 1741}{4} \left(I_{1;-2,2} + 2 I_{1;-1,3} + 2 I_{1;1,3} + I_{1;2,2}\right) \\
&- \frac{59}{4} \left(I_{2;-2,2} + I_{2;-1,2} + 2 I_{2;-1,3} + I_{2;1,2} + 2 I_{2;1,3} + I_{2;2,2}\right) + \frac{2783}{12} \left(I_{4;2} - I_{5;2}\right) - 485 I_{4;3} - 32 I_{5;3} + \frac{512}{5x^2} \left(I_{5;4} - I_{4;4}\right) \\
&+256 \left(I_{4;-1,2} - I_{4;1,2} - I_{5;-1,2} + I_{5;1,2}\right) + \frac{3}{4} x^2 (x^2 - 218) I_{6;2} + 59 x^2 I_{6;3} + 64 (x^2 - 15) I_{6;4} - \frac{265 x^2}{24} \left(I_{4;1} + I_{5;1}\right) \\
&+128 \zeta_2 \left(I_{4;1} - I_{5;1}\right) - \frac{x^2 - 1741}{8} \zeta_2 I_{1;2} + \frac{59}{8} \zeta_2 I_{2;2} + I_4 \left(\frac{352}{9} + \frac{539 x^2}{72} - 192 \zeta_2 - \frac{59}{8} \zeta_3\right) + \frac{59}{8} I_{2;1} \left(\zeta_2 + \zeta_3\right) \\
&+\frac{1}{8} I_{1;1} \left[59 (2x^2 - 1) \zeta_2 - (x^2 - 1741) \zeta_3\right] + I_6 \left[-\frac{3}{16} x^4 \zeta_2 + \frac{x^2}{8} \left(327 \zeta_2 - 59 \zeta_3 - 32 \zeta_4\right) + 60 \zeta_4\right] - \frac{59}{4} (2x^2 - 1) \left(I_{1;-1,2} + I_{1;1,2}\right)
\end{aligned}$$

+ weight 0

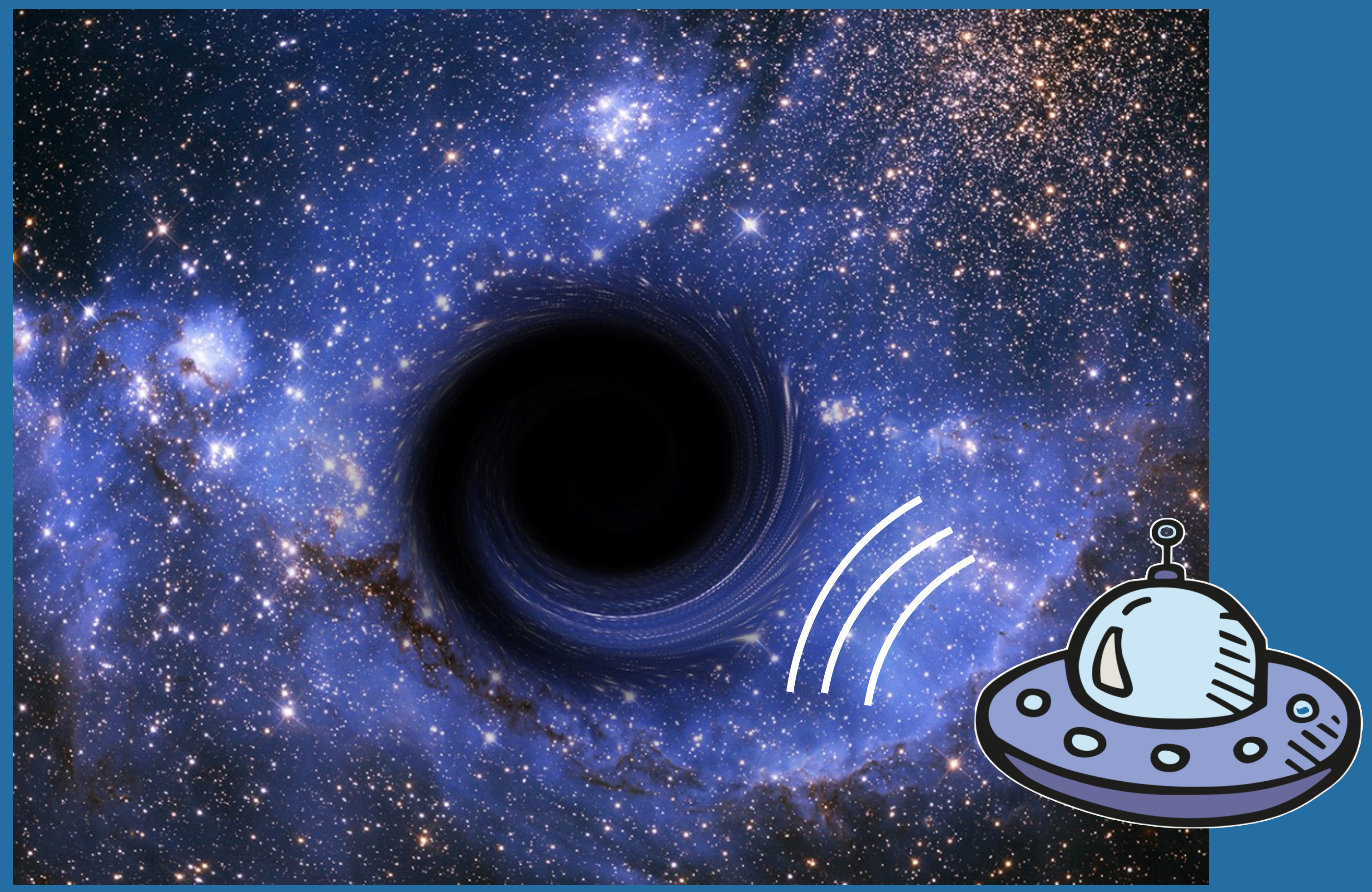
$$\begin{aligned}
&+(GM)^6 \pi^2 \omega^5 \Big[\frac{16}{9} \left(24 x^2 + 1261\right) I_4 + \frac{x^2}{12288} \left(60 x^4 - 57847 x^2 + 2855161\right) I_6 + \frac{(2x^2 - 1)(1863431 - 4344 x^2)}{12288} I_{1;1} - \frac{243638 x^2 + 8310701}{4096} I_{1;2} \\
&+\frac{709}{1024} (2x^2 - 1) I_{1;3} + \frac{410759 - 2x^2}{256} I_{1;4} + \frac{1863431 - 4344 x^2}{12288} \left(I_{2;1} + I_{2;2}\right) + \frac{128}{3} (x^2 + 9) \left(I_{5;1} - I_{4;1}\right) + \frac{2816}{3} \left(I_{4;2} + I_{5;2}\right) + \frac{709}{1024} \left(I_{2;3} + I_{2;4}\right)\Big]
\end{aligned}$$

+ weight 0

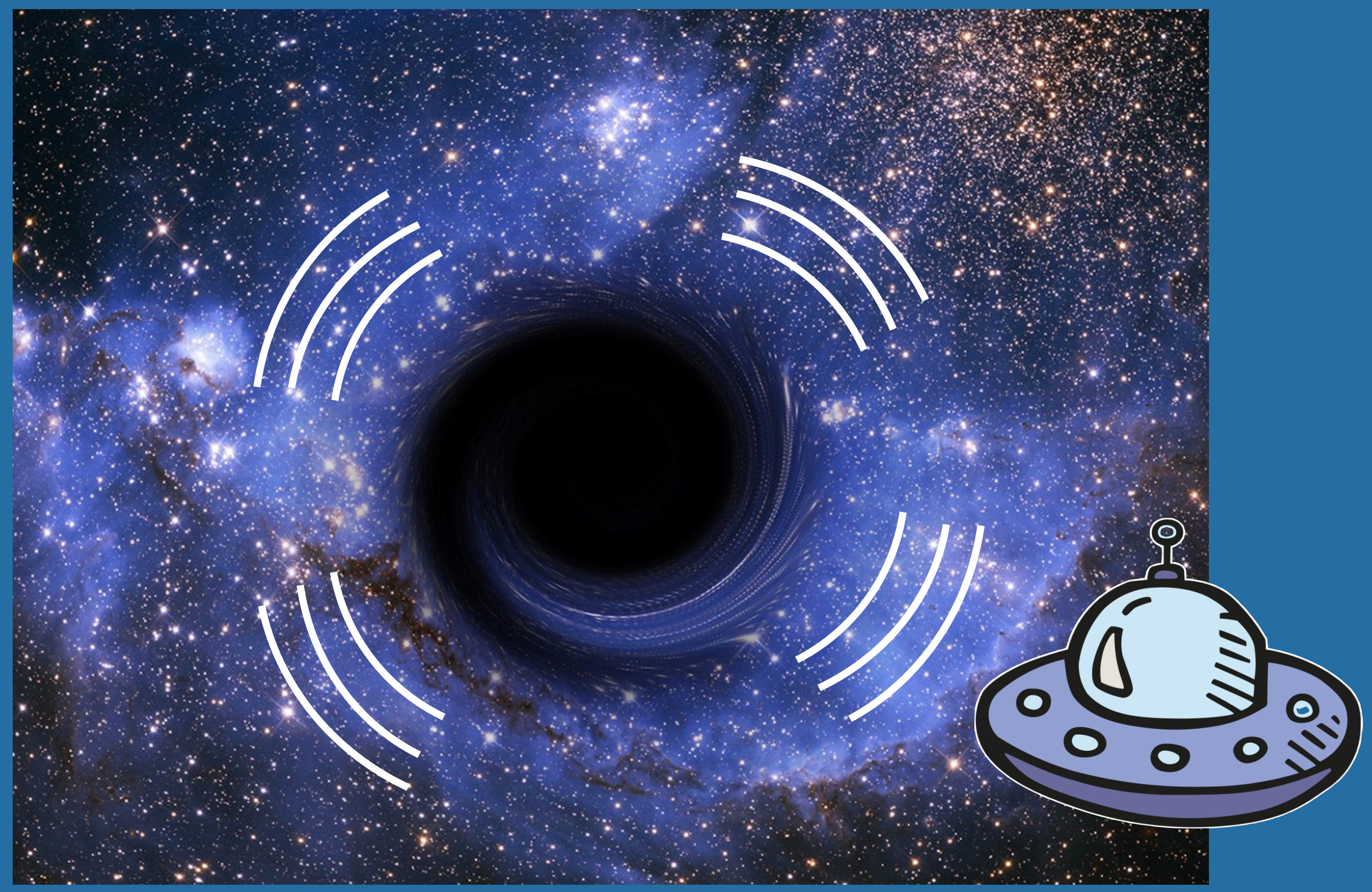
$$\begin{aligned}
&+(GM)^7 \pi \omega^6 \Big[\Big(-\frac{345997 x^4}{259200} + \frac{512 x^2 \zeta_2}{3} + \frac{181 x^2 \zeta_3}{128} + \frac{96341051 x^2}{345600} + \frac{80704 \zeta_2}{9} - \frac{22367509 \zeta_3}{9216} + 1536 \zeta_4 + 118 \zeta_5 - \frac{8244638059}{3110400}\Big) I_4 \\
&+\left(\frac{x^2}{16} - \frac{410759}{32}\right) I_{1;-4,2} + \left(\frac{709}{128} - \frac{709 x^2}{64}\right) I_{1;-3,2} + \left(\frac{x^2}{8} - \frac{410759}{16}\right) I_{1;-3,3} + \left(\frac{121819 x^2}{256} + \frac{8310701}{512}\right) I_{1;-2,2} + \left(\frac{709}{64} - \frac{709 x^2}{32}\right) I_{1;-2,3} + \left(-\frac{63 x^2}{8} - \frac{187911}{16}\right) I_{1;-2,4}
\end{aligned}$$

+ ...

Black Hole Differential Cross-section

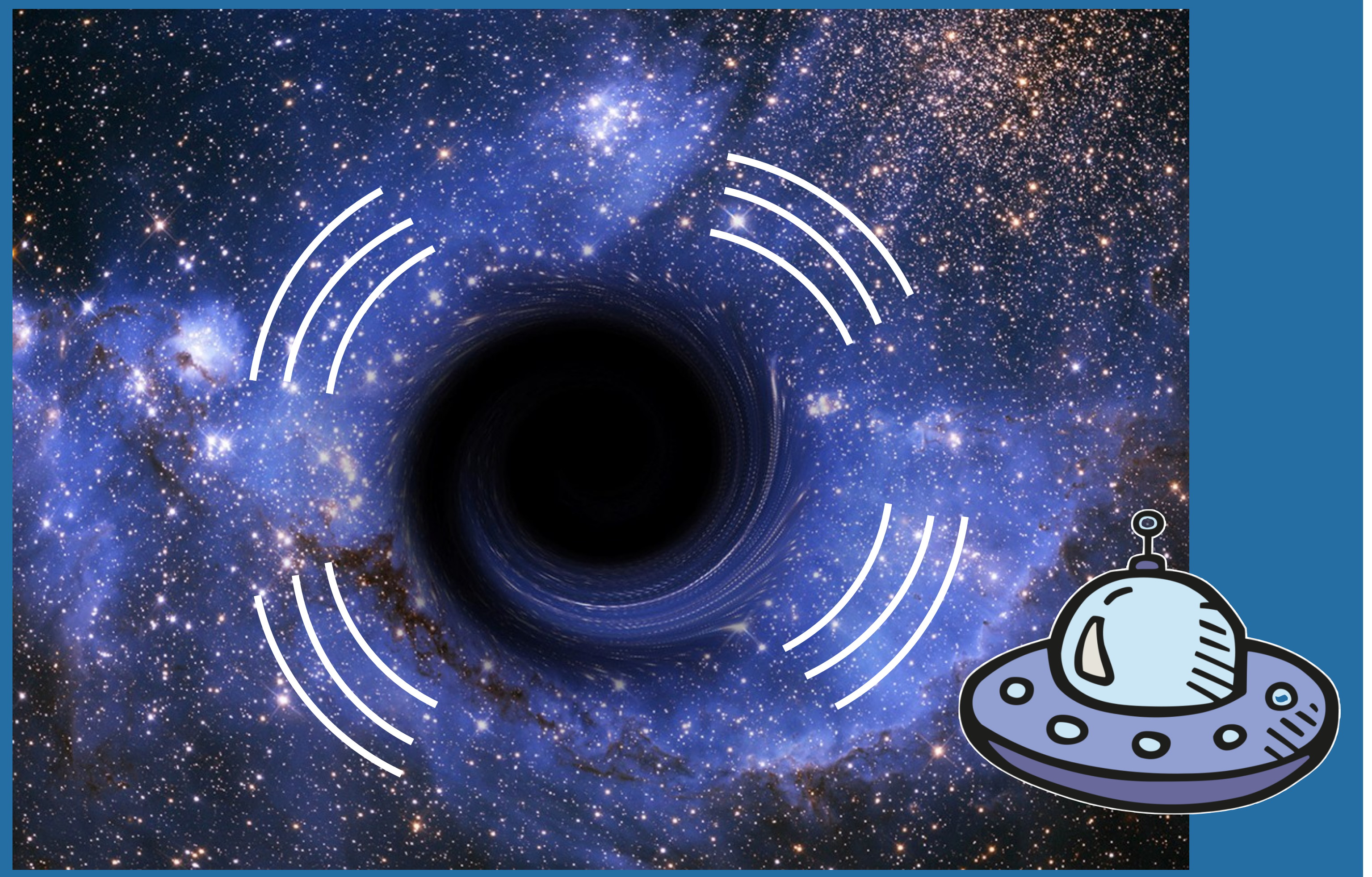


Black Hole Differential Cross-section

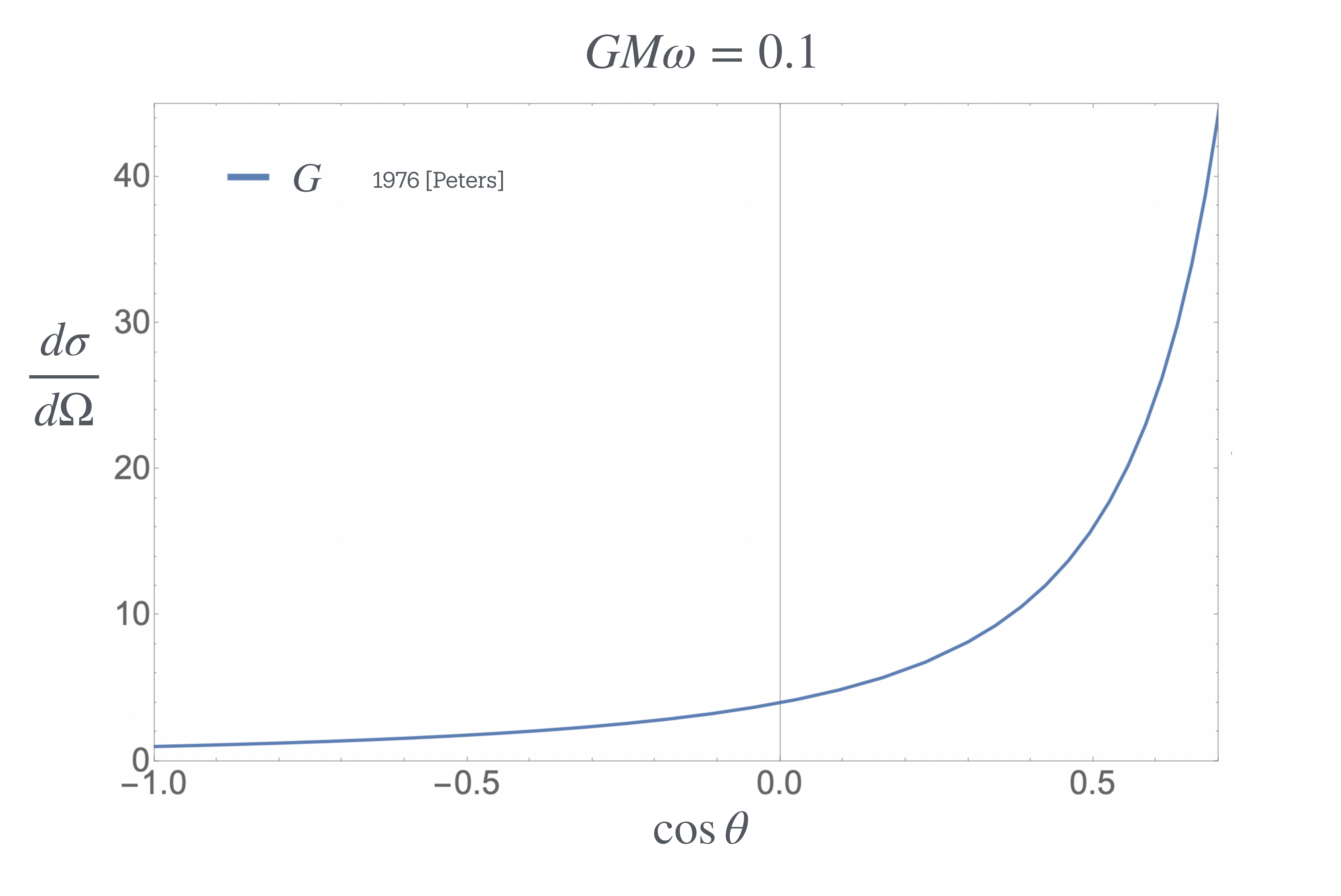


$$\frac{d\sigma}{d\Omega} = \frac{|T(\omega, x)|^2}{64\pi^2}$$

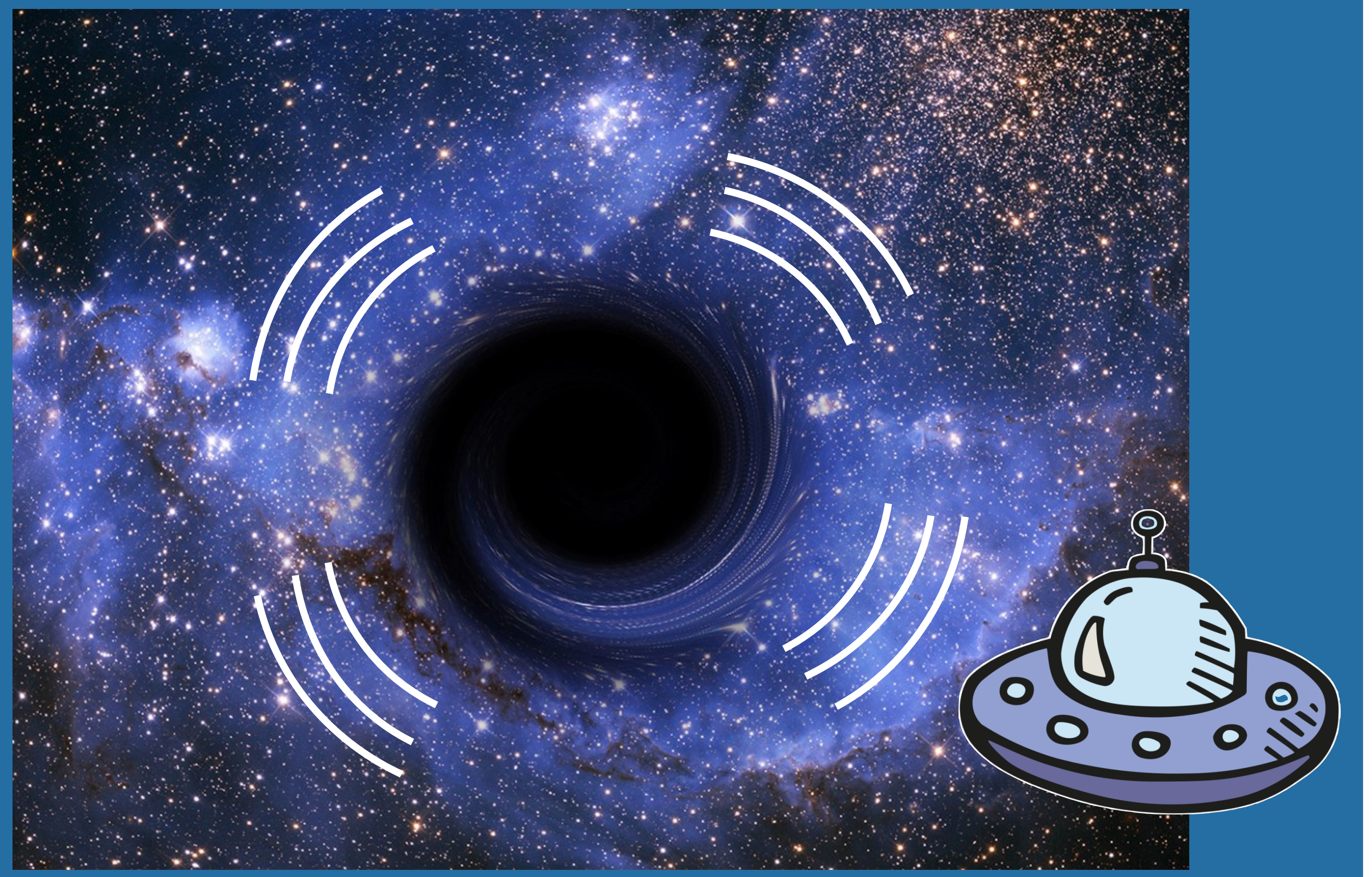
Black Hole Differential Cross-section



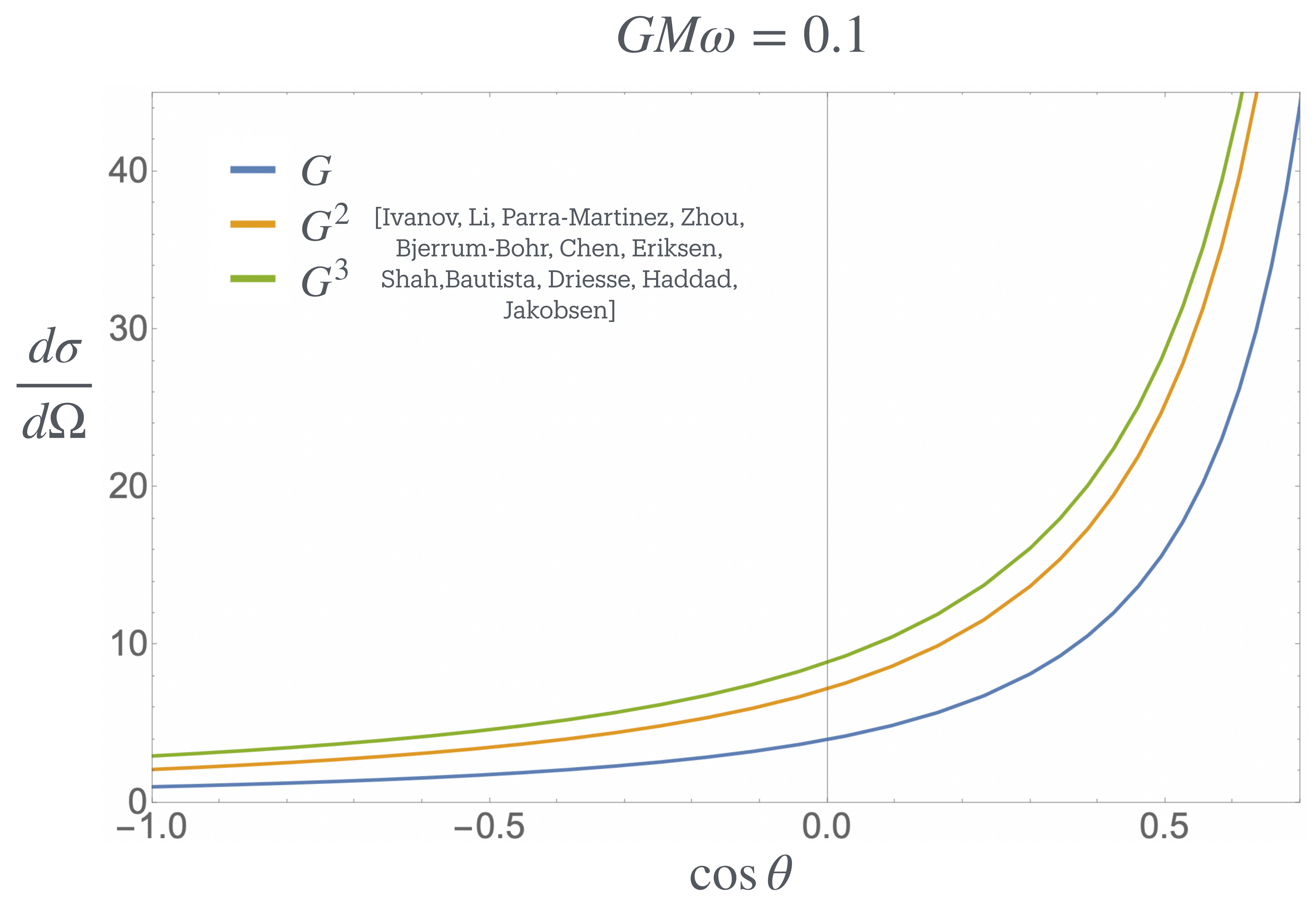
$$\frac{d\sigma}{d\Omega} = \frac{|T(\omega, x)|^2}{64\pi^2}$$



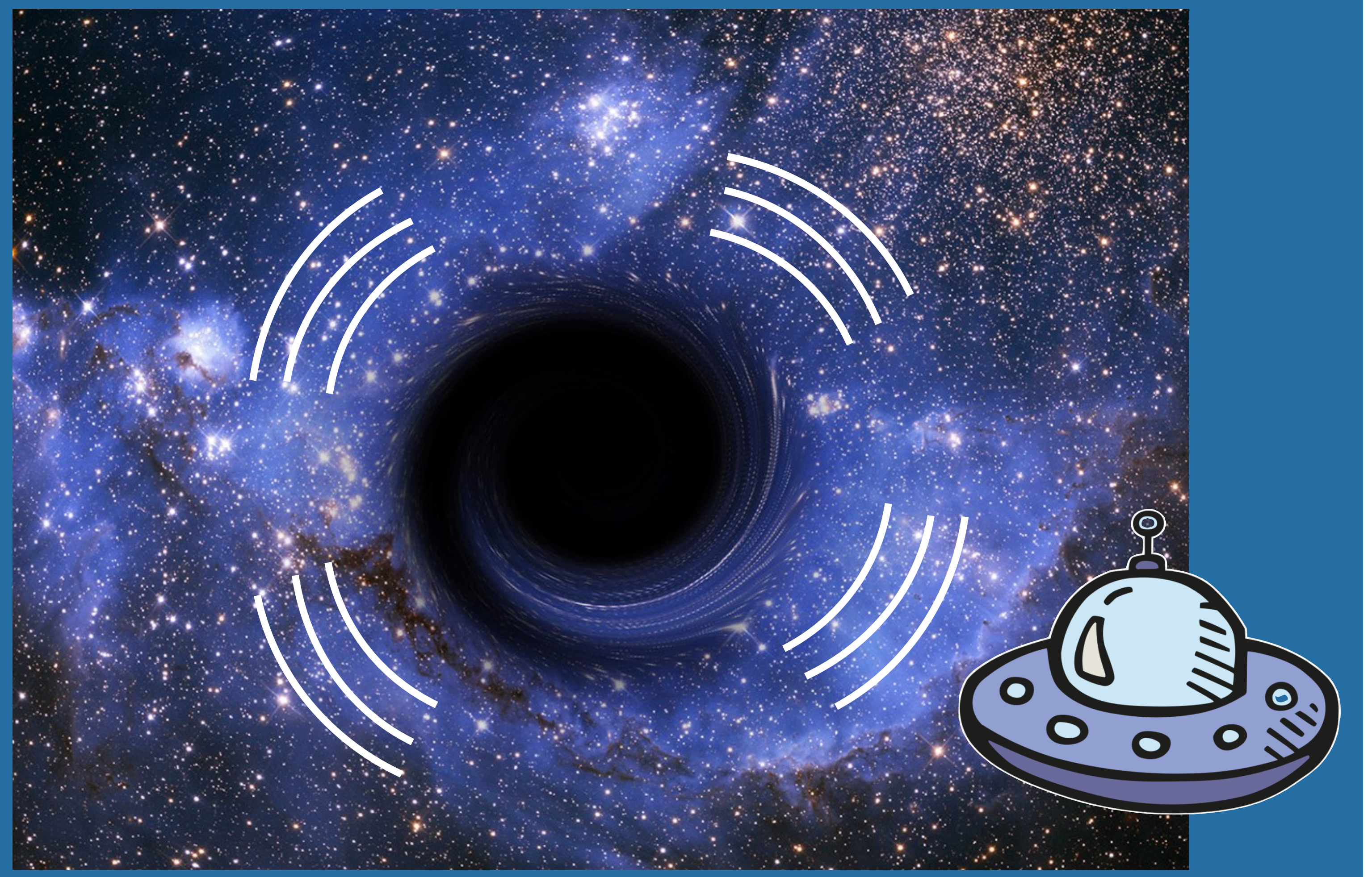
Black Hole Differential Cross-section



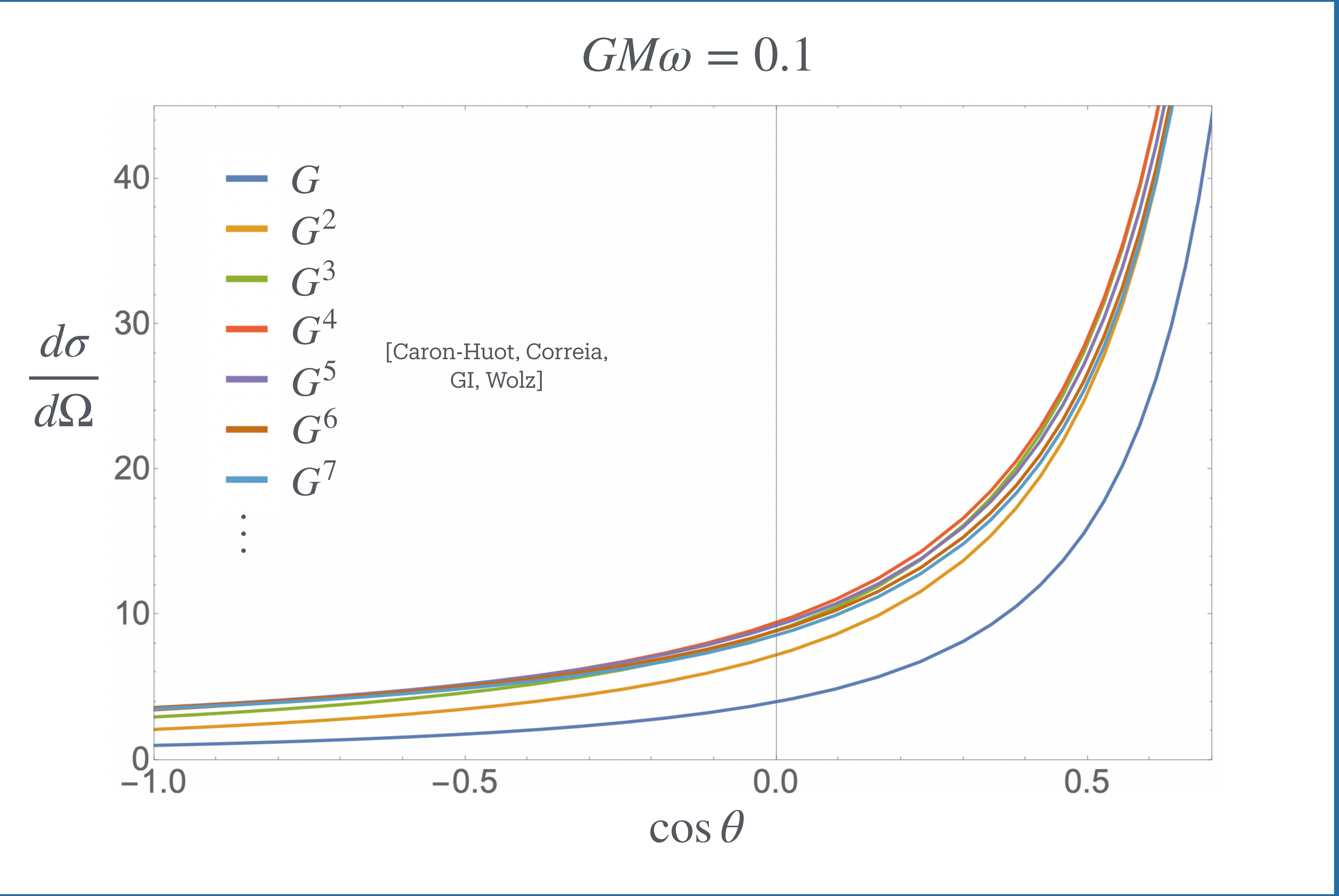
$$\frac{d\sigma}{d\Omega} = \frac{|T(\omega, x)|^2}{64\pi^2}$$



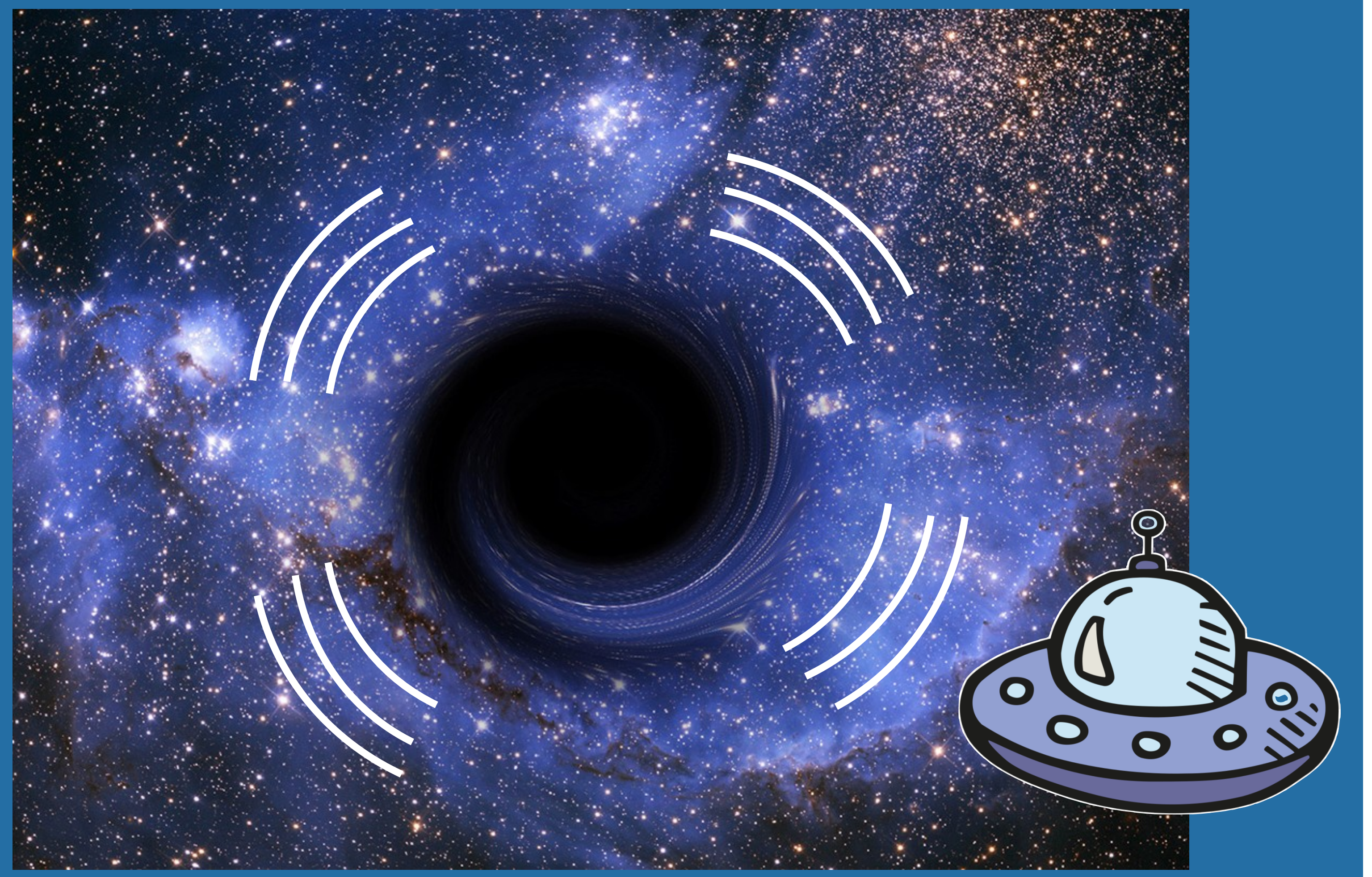
Black Hole Differential Cross-section



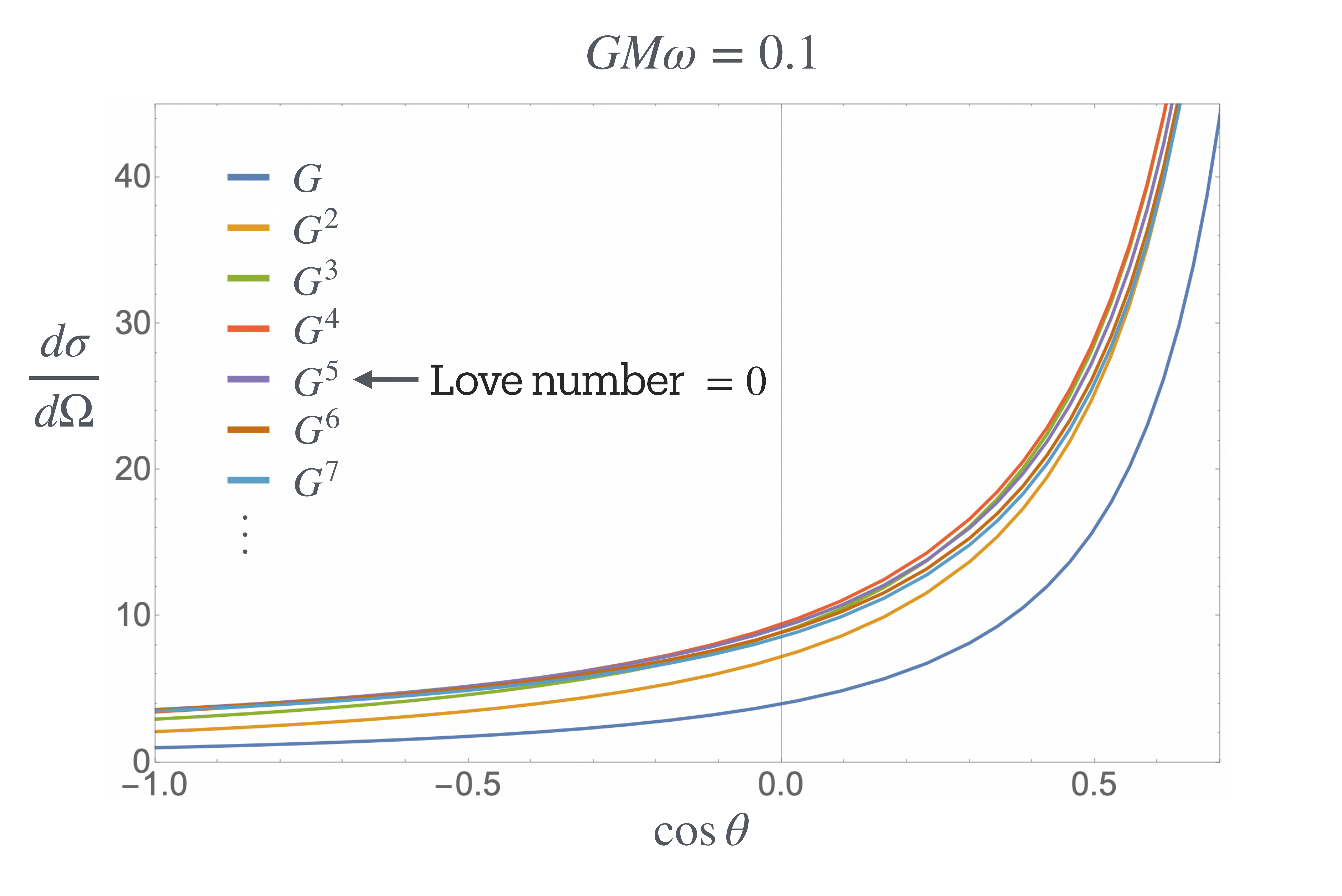
$$\frac{d\sigma}{d\Omega} = \frac{|T(\omega, x)|^2}{64\pi^2}$$



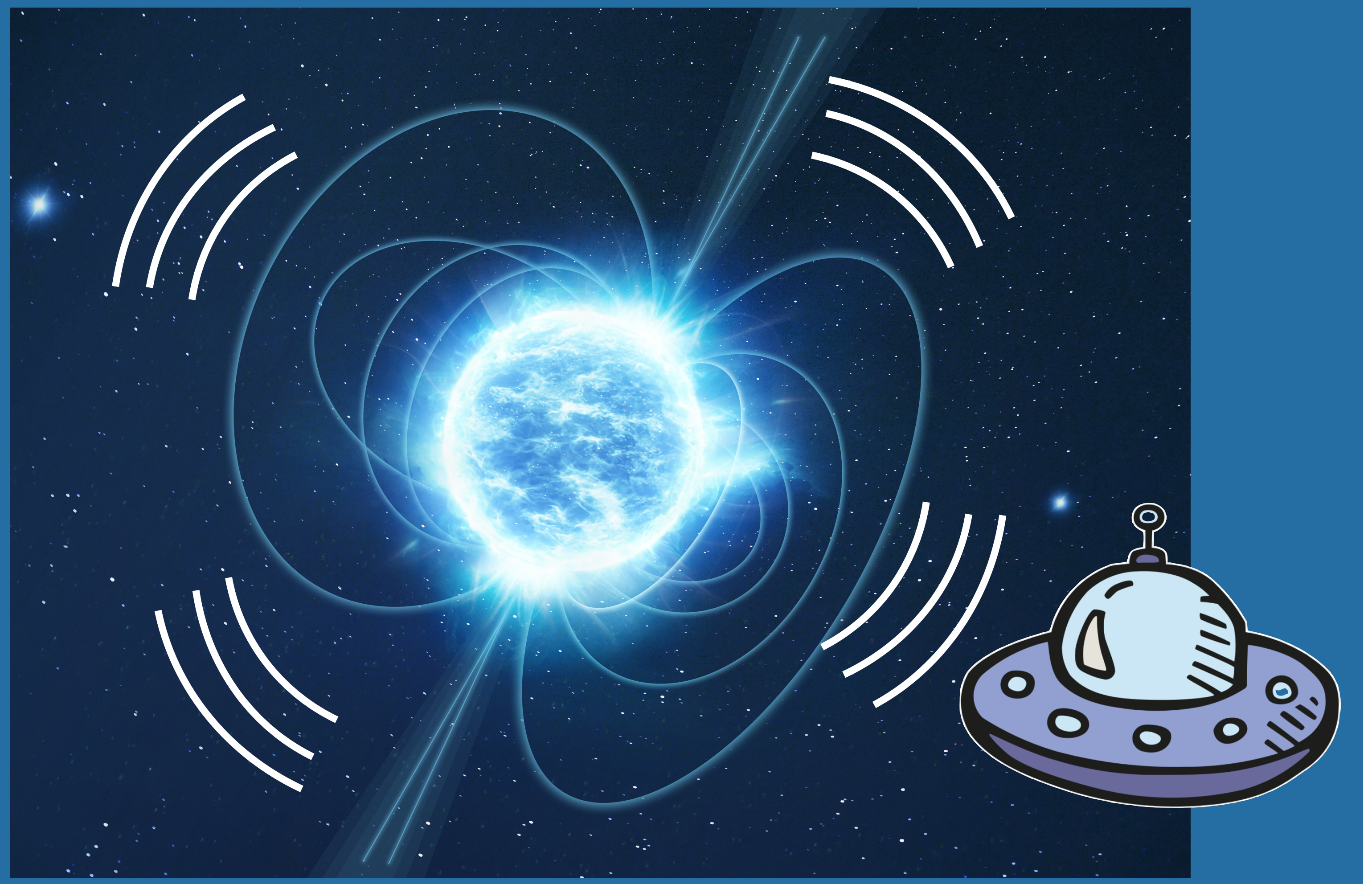
Black Hole Differential Cross-section



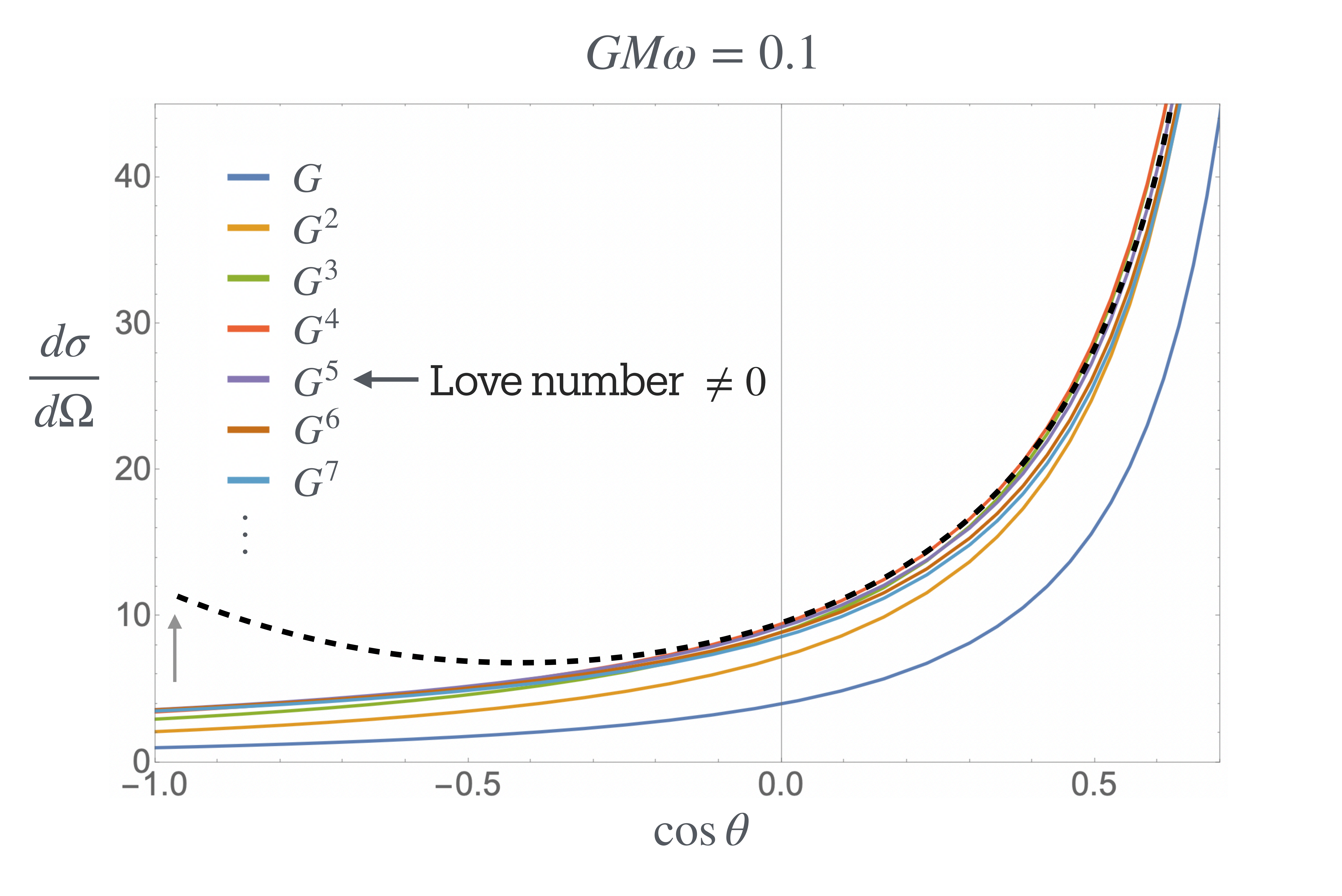
$$\frac{d\sigma}{d\Omega} = \frac{|\,T(\omega, x)\,|^2}{64\pi^2}$$



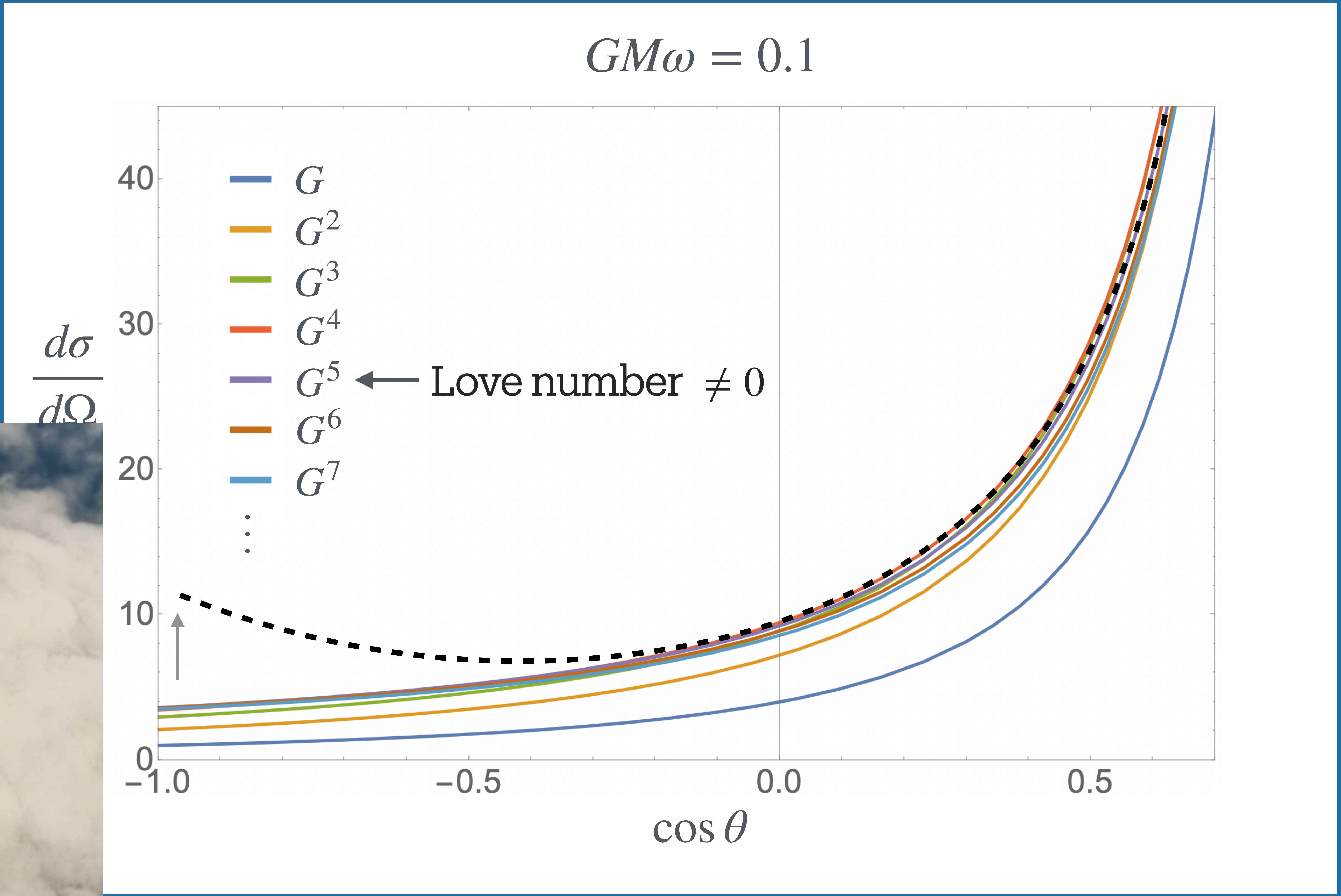
Neutron Star Differential Cross-section?



$$\frac{d\sigma}{d\Omega} = \frac{|T(\omega, x)|^2}{64\pi^2}$$

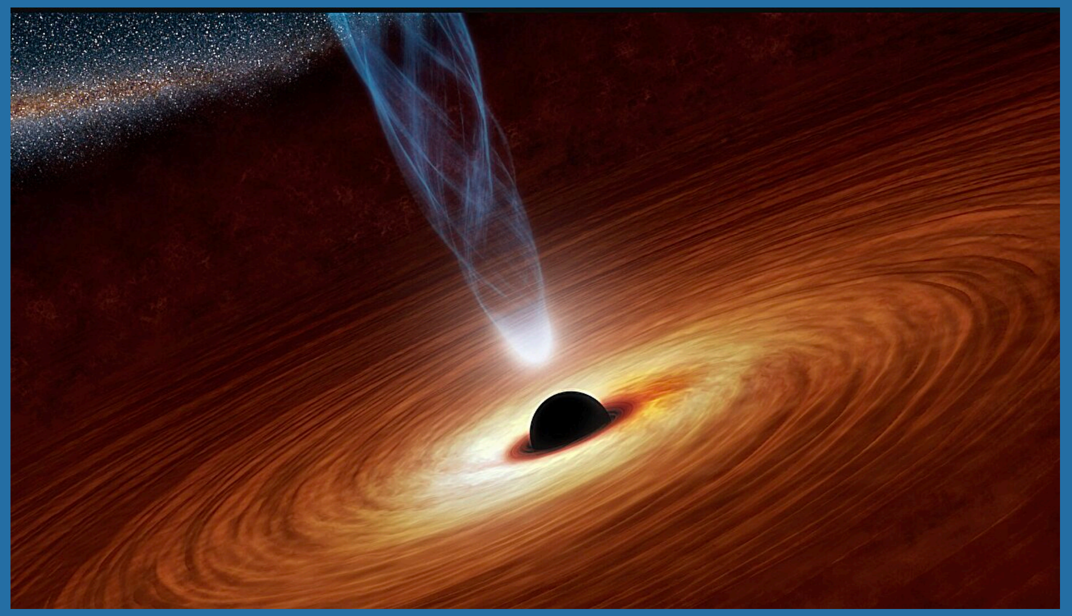


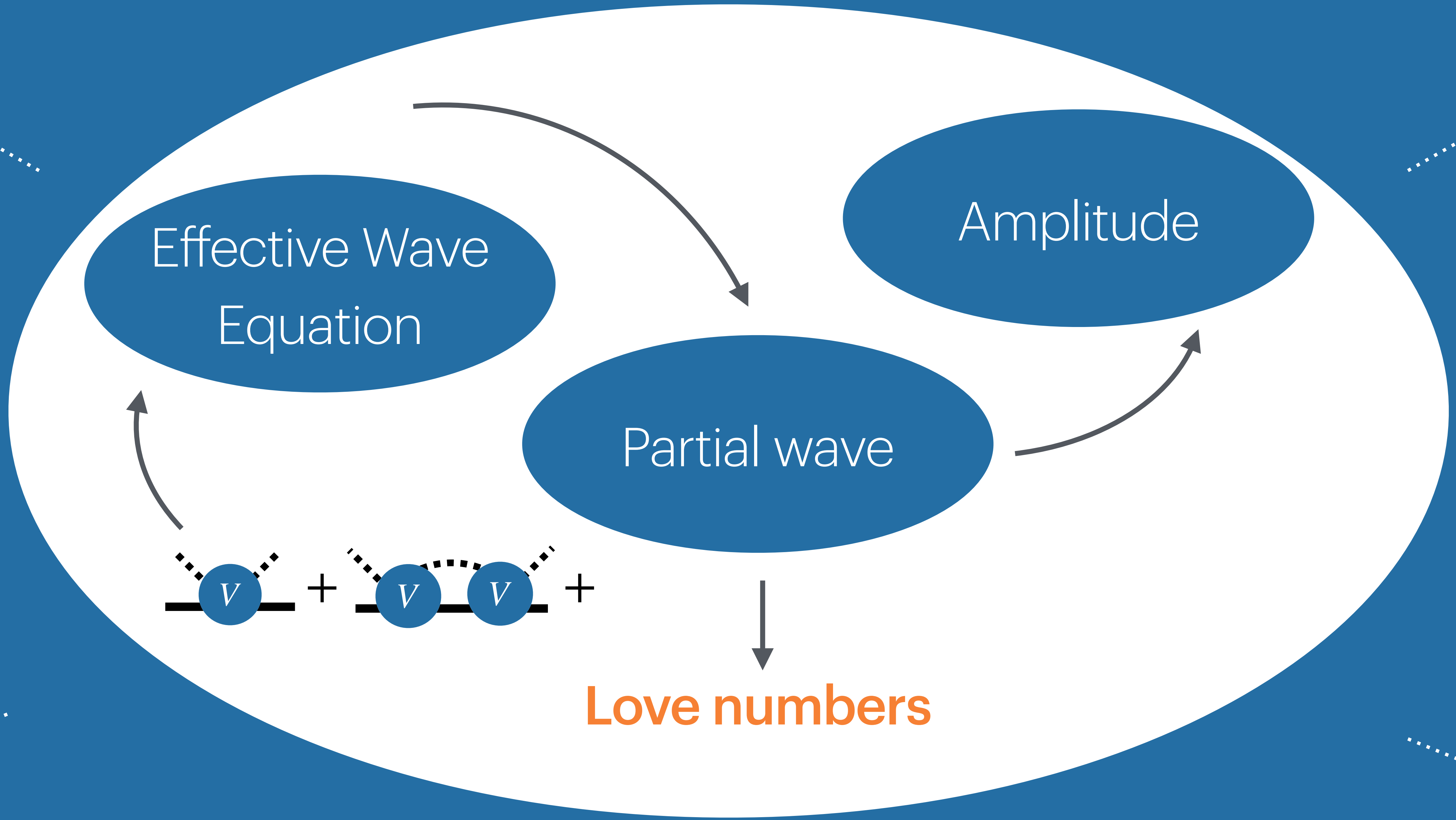
Neutron Star Differential Cross-section?



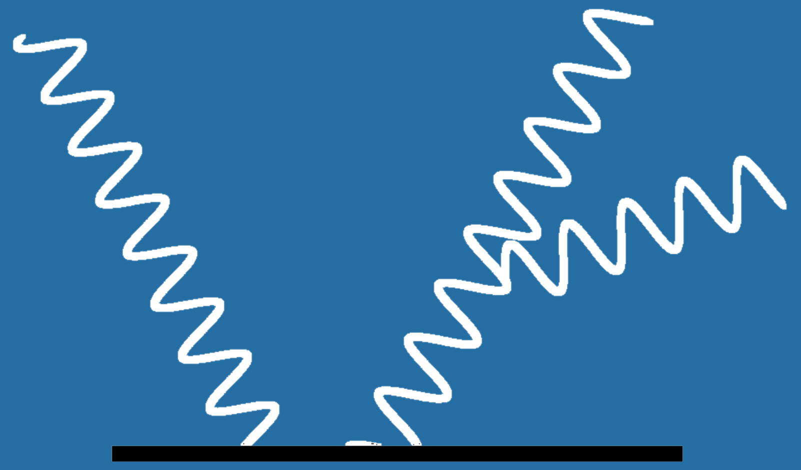
What next?

Kerr-Compton

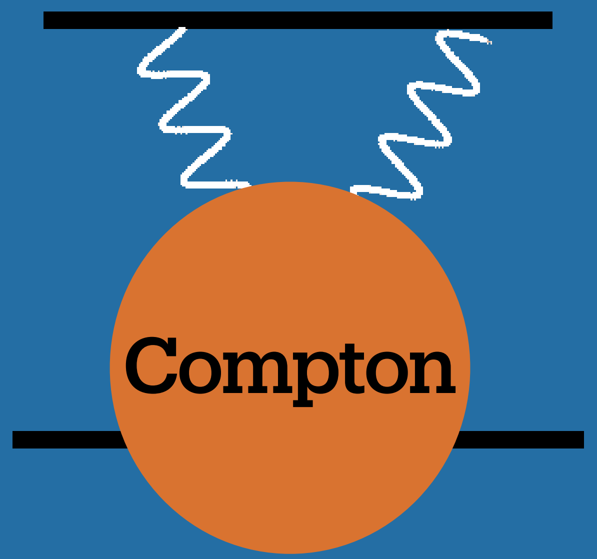




Non-linear
Love numbers



2-body 1SF



Regge Limit
and QNMs

$t^{\alpha(s)}$