PRINCETON UNIVERSITY

AMPLITUDES 29 JUNE -03 JULY 2026

The Hydrotope: An AI-Discovered Closed Formula for Surface Water-Wave Amplitudes

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Based on 2606.28280 with

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Motivation

- Fluid mechanics is one of the most successful effective field theories
  - ◆ Symmetries + conservation laws
  - ◆ Transport coefficients, like viscosity, are Wilson coefficients

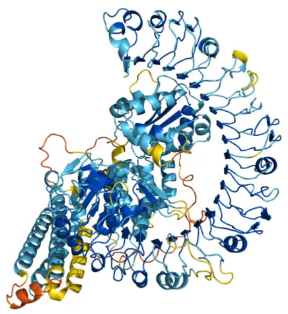
- Fluid systems are notoriously hard to study

Turbulence

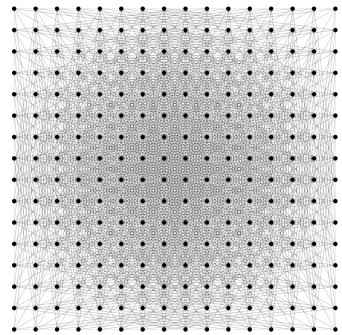
Surface Waves

Motivation

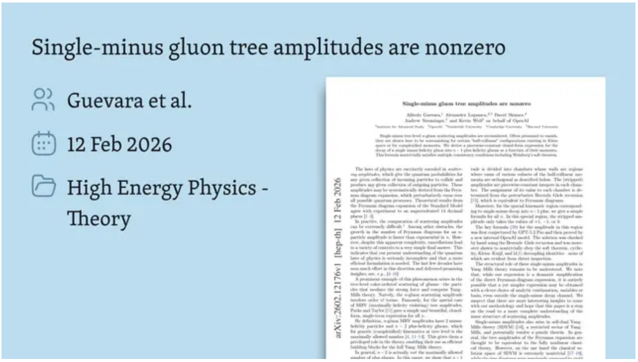
- AI and machine learning have been extremely useful for scientific discoveries



AlphaFold (DeepMind)



Erdős problem (OpenAI)



Single-minus amplitude

[A. Guevara et al. (2026)]

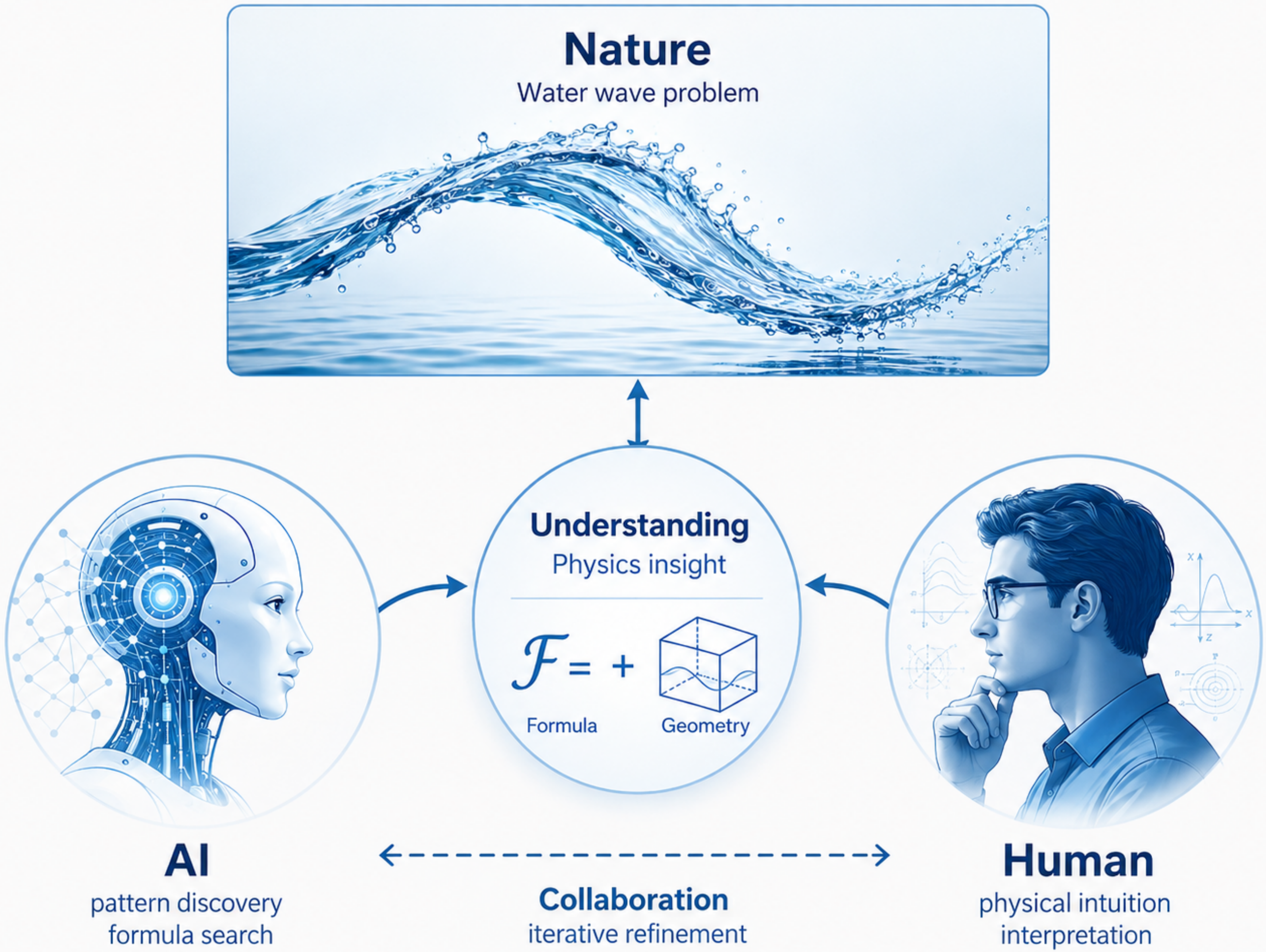
- It is extremely hard to understand why and how AI works

  ◆ Memorization-generalization phase transition

  ◆ In-context learning

  ◆ Scaling laws

  ◆ ....

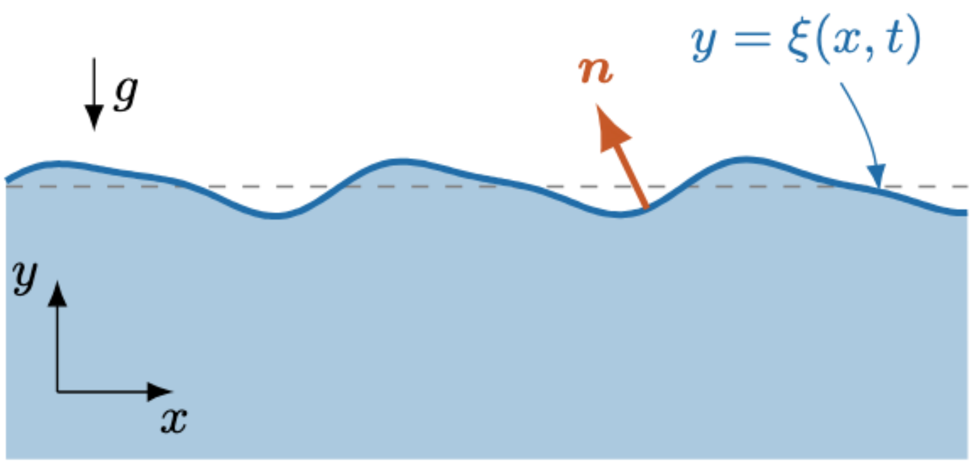

Nature
Water wave problem
Understanding
Physics insight
$\mathcal{F}$ = +
Formula
Geometry
AI
pattern discovery
formula search
Collaboration
iterative refinement
Human
physical intuition
interpretation

The Surface Water-Wave Problem

One of The Oldest Problems

"Now, the next waves of interest, that are easily seen by everyone and which are usually used as an example of waves in elementary courses, are water waves. As we shall soon see, they are the worst possible example, because they are in no respects like sound and light; they have all the complications that waves can have."

Richard Feynman

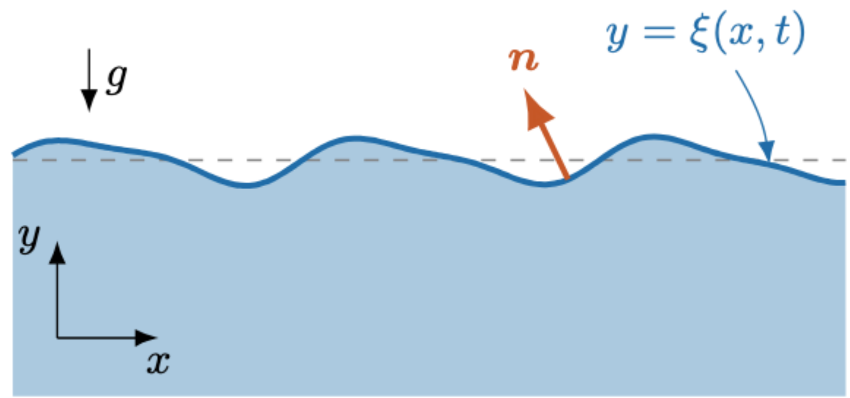


- Basic assumptions:

$$\nabla \times \vec{v} = 0 \qquad \vec{v} = \nabla \phi$$

$$\nabla \cdot \vec{v} = 0 \qquad \nabla^2 \phi = 0$$

One of The Oldest Problem



- Surface Bernoulli equation:

$$(\dot{\phi} + \frac{1}{2}(\partial_x \phi)^2 + \frac{1}{2}(\partial_y \phi)^2 + g y)\Big|_{y=\xi(x,t)} = 0$$

Nonlinearly realized symmetries
(inherited from free-falling particles)

$$K_x = mx - tp_x \qquad K_y = m(y - \frac{1}{2}gt^2) - tp_y \qquad \longrightarrow \qquad m \to \partial_\phi \;,\; p_i \to \partial_i$$

$$K_x : \phi(x, y, t) \to \phi(x - \epsilon t, y, t) + \epsilon x - \frac{1}{2}\epsilon^2 t$$

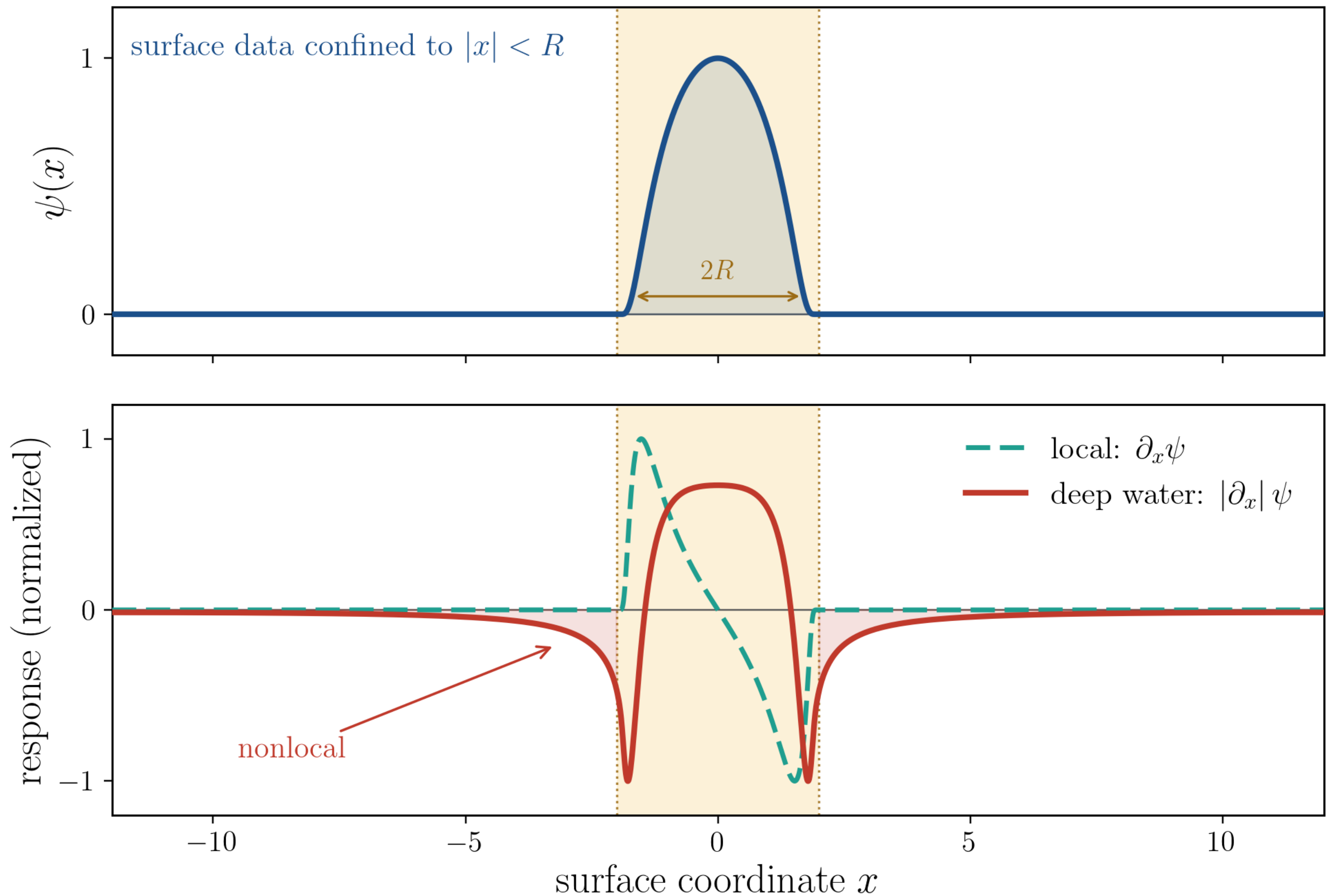
- Surface kinematic constraint (SKC): $\dot{\xi} = \partial_y \phi - (\partial_x \xi)(\partial_x \phi)$

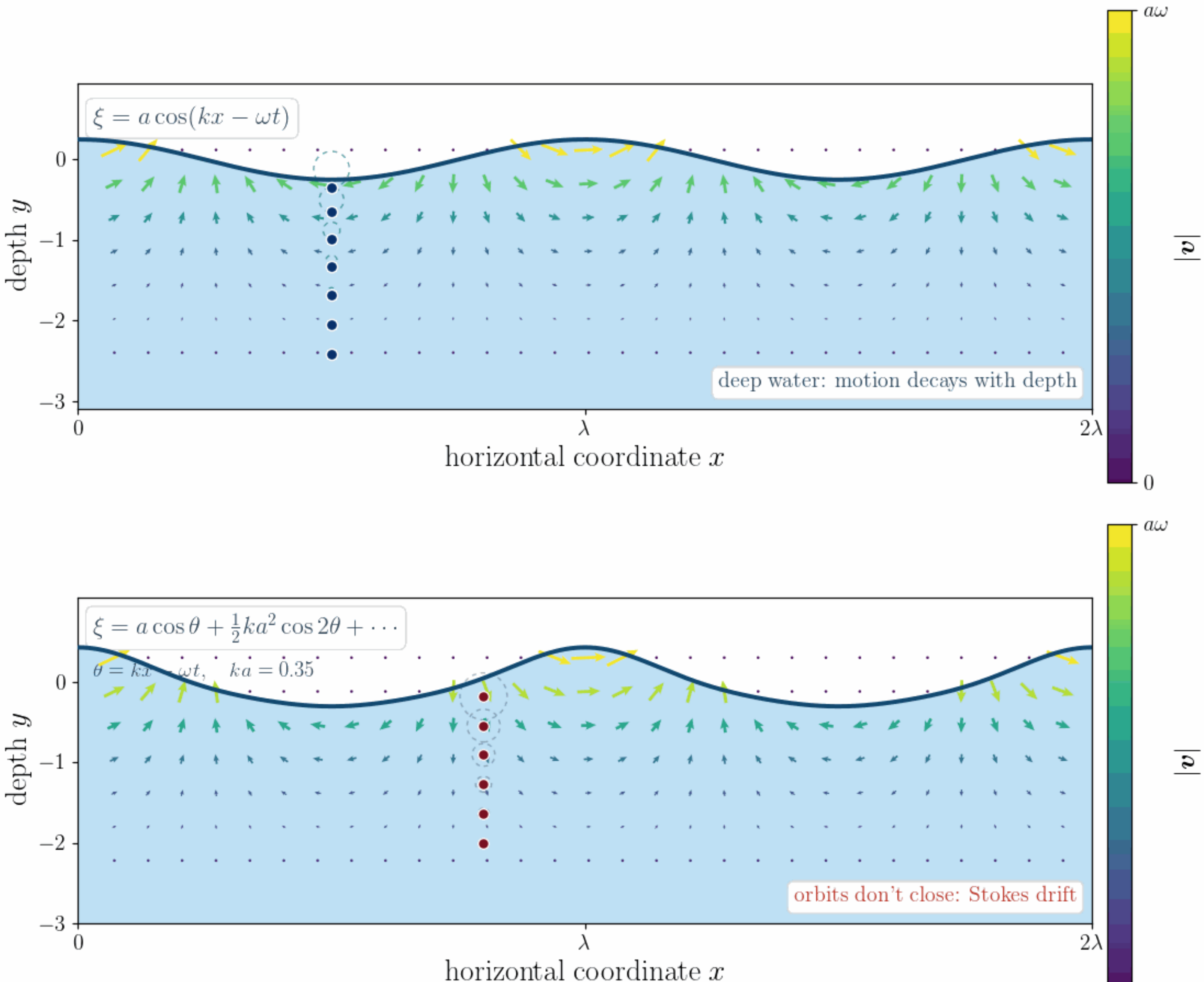
Matter current comoving with the surface

- Nonlocal response: SKC + Laplace

$$\psi = \phi\,|_{y=\xi} \qquad \dot{\xi} = G(\xi)\psi = \underline{|\nabla|}\psi + \cdots$$

Spatial Nonlocality





$\xi = a\cos(kx - \omega t)$
deep water: motion decays with depth
depth y
horizontal coordinate x
0
-1
-2
-3
λ
2λ
$a\omega$
$|\boldsymbol{v}|$
$\xi = a\cos\theta + \frac{1}{2}ka^2\cos 2\theta + \cdots$
$\theta = kx - \omega t, \quad ka = 0.35$
orbits don't close: Stokes drift

Resonant Hamiltonian

- Expanding the Hamiltonian: $H = H_2 + \epsilon H_3 + \epsilon^2 H_4 + \cdots$

$$H_2 = \int dk \omega_k a_k^* a_k \qquad \omega_k = \sqrt{g|k|}$$

- A generic interaction Hamiltonian contains monomials:

$$a_{k_1}^* \cdots a_{k_m}^* a_{p_1} a_{p_2} \cdots a_{p_r}$$

- In the interaction picture this term generally contains the oscillating piece:

$$e^{i\Delta\omega t} = \exp(i(\omega_{k_1} + \cdots + \omega_{k_m} - \omega_{p_1} - \cdots - \omega_{p_r})t)$$

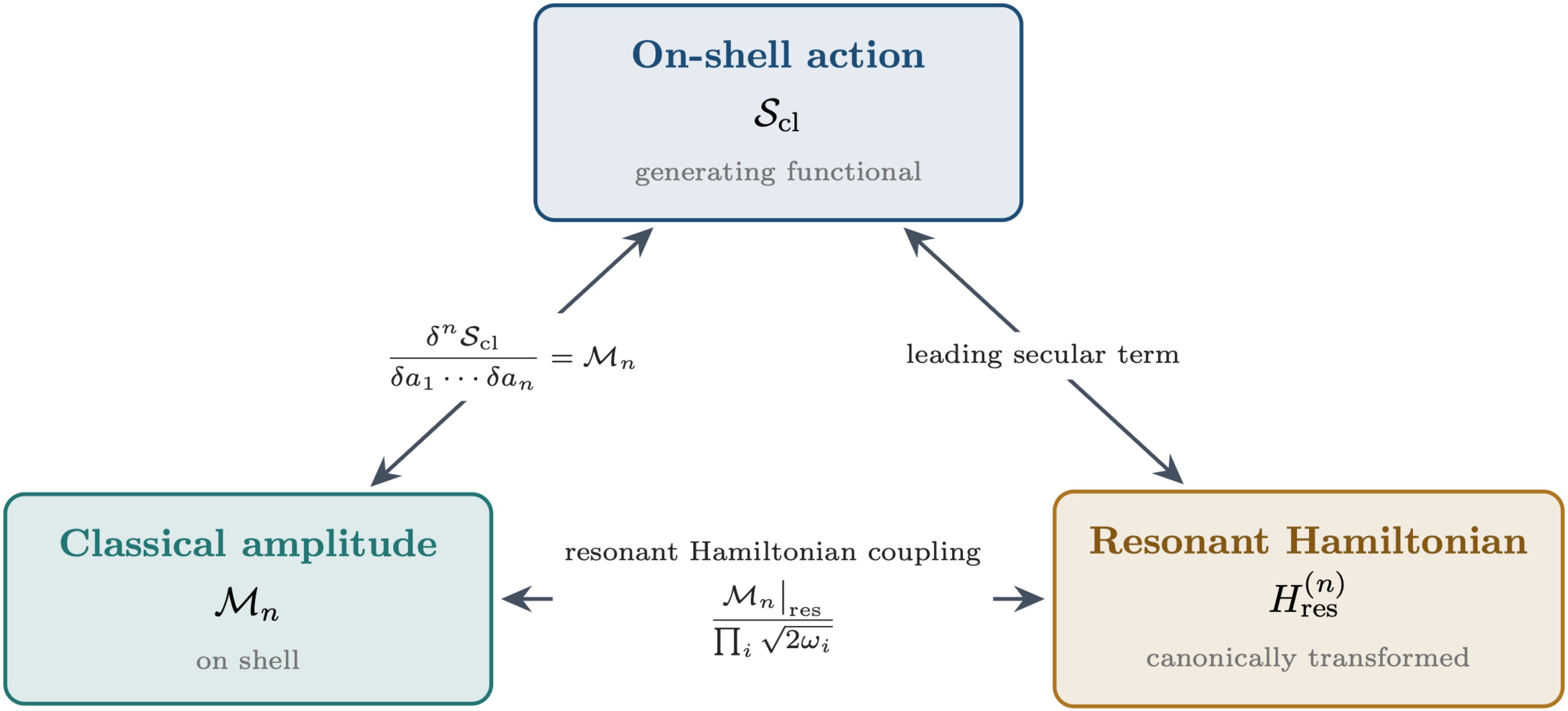
- The fast-oscillating piece can be removed by near-identity canonical transformation:

$$H \to H_{\text{res}} = H_2 + \sum_{\text{resonance}} \int T_{\text{res}} a^* \cdots a^* a \cdots a \delta(\underline{\Delta k}) \qquad \underline{\Delta\omega} = 0$$

[Zakharov (1968); Krasitskii (1990, 1994)]

Lessons from Amplitude for GWs:

Classical scattering amplitude, on-shell action & resonant Hamiltonian



A 30-Year-Old Puzzle

Lvov's 1997 Puzzle

Table 1
Results of the calculations for configurations (i)–(v)

Parametrization	Relation	Matrix element value
(i) all vectors positive		
$\omega_1 = c(a^2 - b^2 + 1 - 2a)$		$T_{pq}^{k_1 k_2 k_3} = (2/g^{1/2}\pi^{3/2})\sqrt{\omega_{k_1}\omega_{k_2}\omega_{k_3}/\omega_p\omega_q}$
$\omega_2 = c(a^2 - b^2 + 1 + 2a)$		$\times k_1 k_2 k_3 pq/\max(k_1, k_2, k_3)$
$\omega_3 = 4c$		$= (2/g^{1/2}\pi^{3/2})\omega_1^{5/2}\omega_2^{5/2}\omega_3^{5/2}\omega_p^{3/2}\omega_q^{3/2}/\max(\omega_1^2, \omega_2^2, \omega_3^2)g^4$
$\omega_p = c(a^2 - b^2 + 3 - 2b)$		
$\omega_q = c(a^2 - b^2 + 3 + 2b)$		
$c > 0,\ 0 < a, b < 1,\ \lvert a \pm b\rvert < 1$		
(ii) positive p and q, and one of k_1, k_2, k_3 negative, choose $k_1 > k_2$		
$\omega_1 = c(a + ab + b),\ k_1 \propto \omega_1^2$	$\omega_1 > \omega_p, \omega_q$	$T_{p,q}^{k_1 k_2 k_3} = (1/g^{9/2}\pi^{3/2})\omega_{k_1}^{3/2}\omega_{k_2}^{11/2}\omega_{k_3}^{1/2}\omega_p^{1/2}\omega_q^{1/2}$
$\omega_2 = c(ab - 1), k_2 \propto \omega_2^2$	$\omega_p, \omega_q > \omega_3 > \omega_2$	
$\omega_3 = c, k_3 \propto -\omega_3^2$		
$\omega_p = c(a + 1)b, p \propto \omega_p^2$	$\omega_1 > \omega_p, \omega_q$	$T_{pq}^{k_1 k_2 k_3} = (1/g^{9/2}\pi^{3/2})\omega_{k_1}^{3/2}\omega_{k_2}^{3/2}\omega_{k_3}^{5/2}\omega_p^{1/2}\omega_q^{1/2}(2\omega_{k_2}^2 - \omega_{k_3}^2)$
$\omega_q = c(b + 1)a, q \propto \omega_q^2$	$\omega_p, \omega_q > \omega_2 > \omega_3$	
$a, b > 0, ab > 1$		
(iii) positive p and q, and two of k_1, k_2, k_3 negative		zero
(iv) p and q have different signs, k_1, k_2, k_3 positive		zero
(v) p and q have different signs, and one of k_1, k_2, k_3 negative, choose $k_1 > k_2$		
$\omega_p = \omega_1 + (\omega_2^2 + \omega_2\omega_3)/(\omega_1 + \omega_2 + \omega_3)$	$\omega_1 > \omega_2 > \omega_3$	$T_{p,q}^{k_1,k_2,k_3} = -(1/\pi^{3/2}g^{9/2})\omega_{k_1}^{1/2}\omega_{k_2}^{1/2}\omega_{k_3}^{11/2}\omega_p^{3/2}\omega_q^{1/2}$
$\omega_q = (\omega_1 + \omega_3)(\omega_2 + \omega_3)/(\omega_1 + \omega_2 + \omega_3)$	$\omega_p > \omega_q$	
$k_1 = \omega_1^2/g$		
$k_2 = \omega_2^2/g$	$\omega_1 > \omega_3 > \omega_2$	$T_{p,q}^{k_1,k_2,k_3} = (1/\pi^{3/2}g^{9/2})\omega_{k_1}^{1/2}\omega_{k_2}^{5/2}\omega_{k_3}^{3/2}\omega_p^{3/2}\omega_q^{1/2}(\omega_{k_2}^2 - 2\omega_{k_3}^2)$
$k_3 = -\omega_3^2/g$	$\omega_p > \omega_q$	
$p = -\omega_p^2/g$		
$q = \omega_q^2/g$	$\omega_3 > \omega_1 > \omega_2$	$T_{p,q}^{k_1,k_2,k_3} = (1/\pi^{3/2}g^{9/2})\omega_{k_1}^{1/2}\omega_{k_2}^{1/2}\omega_{k_3}^{3/2}\omega_p^{3/2}\omega_q^{1/2}$
	$\omega_4 > \omega_5$	$\times(\omega_1^4 + \omega_2^4 - 2\omega_1^2\omega_3^2 - 2\omega_2^2\omega_3^2 + \omega_3^4)$
	$\omega_3 > \omega_1 > \omega_{k_2}$	$T_{p,q}^{k_1,k_2,k_3} = -(1/\pi^{3/2}g^{9/2})2\omega_1^{5/2}\omega_2^{5/2}\omega_3^{3/2}\omega_p^{3/2}\omega_q^{1/2}$
	$\omega_q > \omega_p$	

[Y. V. Lvov (1997)]

Lvov's 1997 Puzzle

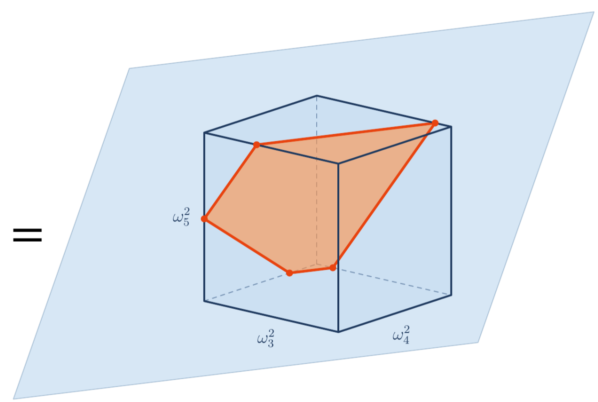
All the matrix elements for the fifth-order interaction in one dimension are calculated in this paper for surface waves on top of an ideal fluid of infinite depth. The expressions obtained are astonishingly simple. For some particular orientation of the wave vectors the interaction matrix element is equal to *zero*. We think that this fact has a deep physical meaning and that there should be simpler ways of getting these results.

This article answers the question *how* waves interact in the case of one-dimensional waves on the surface of an infinitely deep ideal fluid. It is still to be explained *why* they interact in such a way, and *why* some of them do not interact at all.

[Y. V. Lvov (1997)]

$$A_5 = 32\omega_1\omega_2 \mathrm{Vol}(\mathscr{W}_5)$$

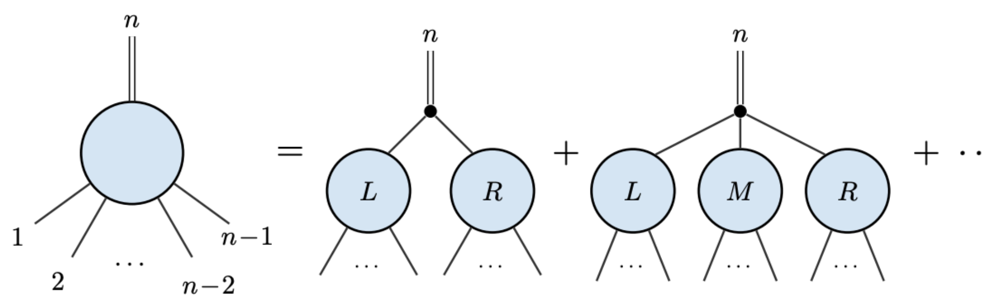
$$\mathrm{Vol}(\mathscr{W}_5) =$$



1D kinematics: $\omega^2 = |k|, g = 1 \;\longrightarrow\; k = \sigma\omega^2, \sigma = \pm 1$

$$\sum_i \omega_i = 0 \qquad \sum_i \sigma_i \omega_i^2 = 0$$

Berends-Giele recursion:



Known results: $A_n^{-+\cdots+} = 0$

Human conjecture: $A^{--+\cdots+} \propto \omega_1\omega_2 \mathrm{min}(\omega_1^2, \omega_2^2)^{(n-3)}$ in some cases

March 13, 2026, 22:03:23

can you test the 1D case numerically with all positive but two negative (to make sure we are on the same convention. I choose the first and second leg to have σ negative) from four point all the way to eight or nine point. I guess the expression would be proportionl to $\omega_1\omega_2 \min(\omega_1^2, \omega_2^2)^{n-3}$, the n the n-point amplitude

AI: Confirms the intuition in some cases and but finds failure in some other cases

March 13, 2026, 22:45:58

try to relax the assumption, $\omega_2 \ldots \omega_{n-1} > 0$, $|\omega_2| > |\omega_1|$ and guess the general formula

AI: At low multiplicity, I find another chamber, but still cannot fully succeed

March 13, 2026, 23:04:14

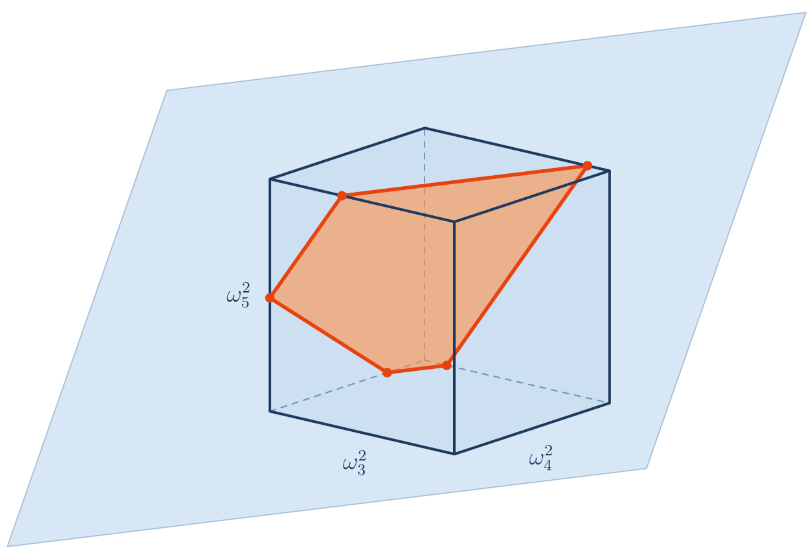
For the failed case, try to guess the formula

AI: $$A_n^{(--+\cdots+)} = \omega_1\omega_2\, 2^{n-1} \sum_{S\subseteq\{3,\ldots,n\}} (-1)^{|S|}\Big[\beta^2 - \sum_{i\in S}\omega_i^2\Big]_+^{n-3}, \quad [x]_+ \equiv \max(x,0)$$

The Hydrotope

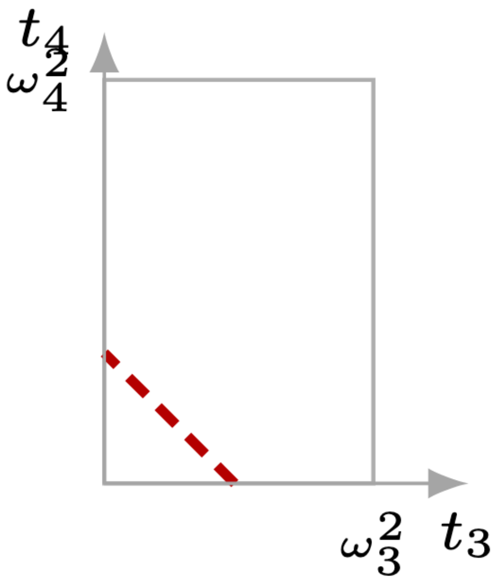
$$A_n^{--+\cdots+} = \omega_1\omega_2 2^{n-1}(n-3)!\,\mathrm{Vol}(\mathscr{W}_n)$$

$$\mathrm{Vol}(\mathscr{W}_n) = \int_0^{\omega_3^2}\cdots\int_0^{\omega_n^2}\Big(\prod_i dt_i\Big)\delta\Big(\sum_i t_i - \beta^2\Big)\ ,\quad \beta^2 = \min(\omega_1^2, \omega_2^2)$$

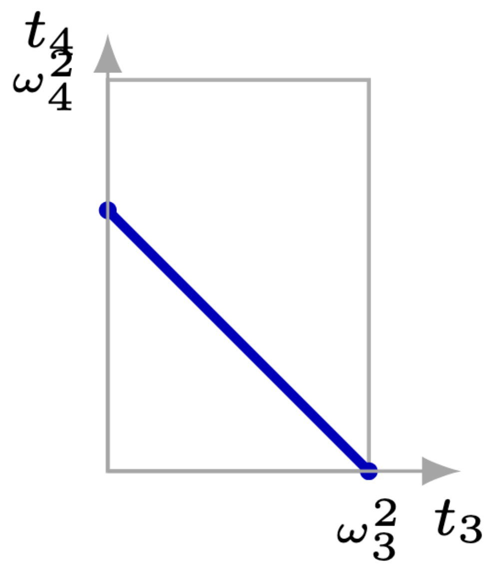


Examples of The Geometry (n=4)

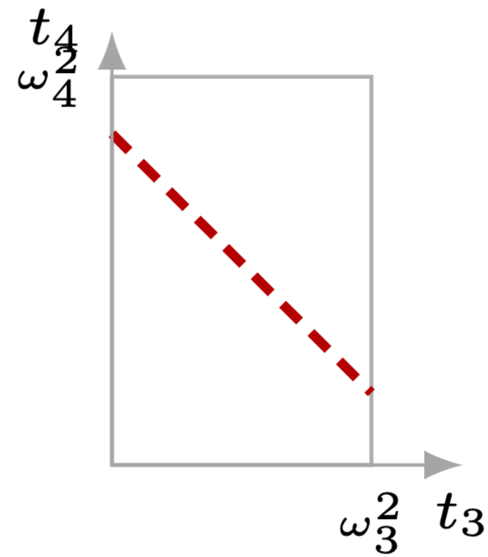
$$A_4^{--++} = 8\omega_1\omega_2 \min(\omega_1^2, \omega_2^2)$$



$\beta^2 < \omega_3^2$

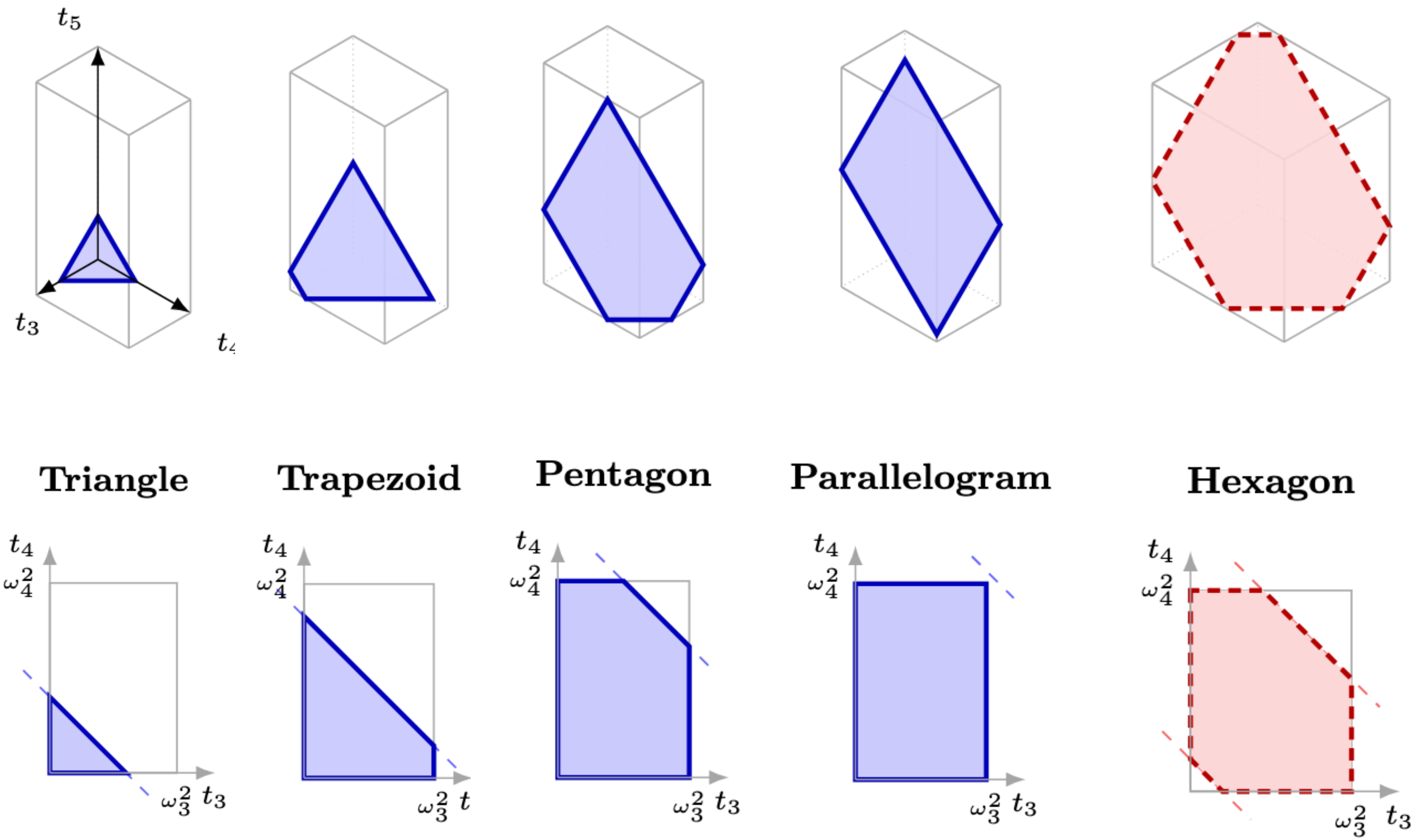


$\beta^2 = \omega_3^2$



$\beta^2 > \omega_3^2$

Examples of The Geometry (n=5)



$\beta^2 < \omega_3^2$

$\omega_3^2 < \beta^2 < \omega_4^2$

$\omega_4^2 < \beta^2 < \omega_3^2+\omega_4^2$

$\omega_3^2+\omega_4^2 < \beta^2 < \omega_5^2$

$\beta^2 > \omega_5^2$

Solution to Lvov's 1997 Puzzle

Branch	Ordering	Lvov matrix element $\pi^{3/2}\,T_{pq}^{k_1k_2k_3}\quad(g=1)$	Regime	Clips
(i)	—	$2\,\omega_{k_1}^{5/2}\omega_{k_2}^{5/2}\omega_{k_3}^{5/2}\omega_p^{3/2}\omega_q^{3/2}/\max(\omega_{k_1}^2,\omega_{k_2}^2,\omega_{k_3}^2)$	parallelogram	2
(ii a)	$\omega_{k_2}<\omega_{k_3}$	$\omega_{k_1}^{3/2}\omega_{k_2}^{11/2}\omega_{k_3}^{1/2}\omega_p^{1/2}\omega_q^{1/2}$	triangle	0
(ii b)	$\omega_{k_2}>\omega_{k_3}$	$\omega_{k_1}^{3/2}\omega_{k_2}^{3/2}\omega_{k_3}^{5/2}\omega_p^{1/2}\omega_q^{1/2}\,(2\omega_{k_2}^2-\omega_{k_3}^2)$	trapezoid	1
(v a)	$\omega_{k_1}>\omega_{k_2}>\omega_{k_3}$	$-\,\omega_{k_1}^{1/2}\omega_{k_2}^{1/2}\omega_{k_3}^{11/2}\omega_p^{3/2}\omega_q^{1/2}$	triangle	0
(v b)	$\omega_{k_1}>\omega_{k_3}>\omega_{k_2}$	$\omega_{k_1}^{1/2}\omega_{k_2}^{5/2}\omega_{k_3}^{3/2}\omega_p^{3/2}\omega_q^{1/2}\,(\omega_{k_2}^2-2\omega_{k_3}^2)$	trapezoid	1
(v c)	$\omega_{k_3}>\omega_{k_1}>\omega_{k_2},\ \omega_p>\omega_q$	$\omega_{k_1}^{1/2}\omega_{k_2}^{1/2}\omega_{k_3}^{3/2}\omega_p^{3/2}\omega_q^{1/2}\,(\omega_{k_1}^4+\omega_{k_2}^4-2\omega_{k_1}^2\omega_{k_3}^2-2\omega_{k_2}^2\omega_{k_3}^2+\omega_{k_3}^4)$	pentagon	2
(v d)	$\omega_{k_3}>\omega_{k_1}>\omega_{k_2},\ \omega_q>\omega_p$	$-\,2\,\omega_{k_1}^{5/2}\omega_{k_2}^{5/2}\omega_{k_3}^{3/2}\omega_p^{3/2}\omega_q^{1/2}$	parallelogram	2

$$T_{pq}^{k_1k_2k_3}=\frac{1}{16\,\pi^{3/2}}\left(\omega_{k_1}\omega_{k_2}\omega_{k_3}\,\omega_p\,\omega_q\right)^{1/2}A_5$$

Hydrotope: The Kinematic Chambers

$$A_n^{(--+\cdots+)} = \omega_1\omega_2\, 2^{n-1} \sum_{S\subseteq\{3,\ldots,n\}} (-1)^{|S|}\Big[\beta^2 - \sum_{i\in S}\omega_i^2\Big]_+^{n-3} = \omega_1\omega_2 2^{n-1}(n-3)!\,\mathrm{Vol}(\mathscr{W}_n)$$

$$\mathrm{Vol}(\mathscr{W}_n) = \int_0^{\omega_3^2}\cdots\int_0^{\omega_n^2}\Big(\prod_i dt_i\Big)\delta\Big(\sum_i t_i - \beta^2\Big)\,, \qquad \beta^2 = \min(\omega_1^2, \omega_2^2)$$

- The kinematic chamber is determined by which cube vertices lie below the plane

Intermediate propagator: $\dfrac{|k_i + \ldots k_j|}{(\omega_i + \cdots\omega_j)^2 - g\,|k_i + \cdots k_j|}$

Discontinuity: $\beta^2 = \omega_S^2 \longrightarrow k_S + k_{1,2} = 0$

- The two-minus amplitude is piecewise-homogeneous polynomial

The no-pole property follows from the vanishing single minus amplitude

- Universal soft limit $A_n(\omega_+ \to 0) = 2(n-3)\omega_+^2 A_{n-1}$

- Spline uniqueness: order + chamber + soft limit

[C. de Boor, K. Hollig (1982)]

The "Scaling" Symmetry

- Soft limit: $A_n(\omega_+ \to 0) = 2(n-3)\omega_+^2 A_{n-1}$

$$\lim_{\omega_+ \to 0} \partial_{\omega_+^2} A_n = 2(n-3) A_{n-1}$$

$$\lim_{\omega_+ \to 0} \partial_{\omega_+^2} A_n = \sum_{i=1}^{n-1} \omega_i \partial_{\omega_i} A_{n-1}$$

$$C = \quad t^2\partial_y \quad + \quad t\partial_t \quad + \quad (\cdots)\partial_\phi$$

- Special "scaling" symmetry for an incompressible/irrotational fluid:

$$C = gt^2\partial_y - t\partial_t + \left(\phi + 2gty - \frac{1}{3}g^2t^3\right)\partial_\phi$$

is a symmetry of the Bernoulli equation on constant-pressure surface with gravity

[Benjamin, Olver (1982)]

Final Comments on AI

Rewritten Chain of Thought for the
Two-Minus Water-Wave Amplitude

Abstract

This document is a first-person reconstruction of the chain of thought for one blind-benchmark attempt at the closed-form tree amplitude A_n in the two-minus sector of one-dimensional deep-water surface waves. It is rebuilt from the visible run trajectory — the raw private reasoning was encrypted, so only the narrated messages, commands, and outputs survive — and it keeps the false starts and hedges rather than tidying them away. It is honest about where the run landed: I found and exactly verified the leading-chamber monomial $A_n = i\,2^{n-1}g^{3-n}\,\omega_1\omega_2\,[\min(\omega_1^2,\omega_2^2)]^{n-3}$, I explicitly mapped the other kinematic chambers and even wrote down their individual rational forms, but I then concluded that no single globally-analytic expression exists and presented the monomial as "the" closed form with a scope caveat. I did *not* assemble the chamber pieces into the inclusion–exclusion / truncated-power formula they were terms of. So this is a partial result, stopped at the leading chamber.

Rewritten Chain of Thought

The ground rules come first, and they are strict. This is a blind benchmark, so I may read *only* `prompt.md` and `OnShellBG.m` — not `KEY.md`, not the sibling cases, nothing else — and I may not use any web or literature lookup. Everything has to come from the Berends–Giele code and data I generate myself. I check the environment: `wolframscript` and a `math` kernel are present, and `sympy/numpy` are available under Anaconda. The one nuisance in `OnShellBG.m` is that its section VI tests auto-run, and they are for the *one-minus* sector $\sigma = (-1,+1,\ldots,+1)$, which vanishes identically — not my sector. So I `head -145 OnShellBG.m > bg_core.m` to keep only the function definitions (sections I–V, ending at `MakeKinematics`) and drive the recursion myself.

The first probes set the tone. At $n=4$ the recursion returns `Indeterminate`: with $\omega = \{-2, 3/2, 2, -3/2\}$ legs 2 and 4 satisfy $\omega_2+\omega_4=0$ and $k_2+k_4=0$, a null sub-channel that puts an internal line exactly on-shell and blows up a propagator as a removable $0/0$. I set $n=4$ aside to recover later as a limit. The higher points are clean and purely imaginary: $A_5 = -\frac{891}{2}i$, $A_6 = -\frac{11907}{4}i$. So I write $A_n = i\,P_n$ and chase P_n.

Before fitting anything I pin the gross structure. Since $|k_i| = \omega_i^2/g$ regardless of the sign σ_i, power-counting the vertices and propagators predicts A_n homogeneous of degree $2n-4$ in ω and $\propto g^{3-n}$. I check both numerically: scaling $\omega \to 2\omega$ at $n=5$ multiplies A_5 by $64 = 2^6$, and $\omega \to 3\omega$ by $729 = 3^6$, so degree $2n-4$ holds; and $A_5(g{=}1) = -3328i$, $A_5(g{=}2) = -832i = -3328i/4$, $A_5(g{=}3) = -3328i/9$, so $A_5 \propto g^{-2} = g^{3-n}$. A clean reference point falls out: $\omega = \{-13/2, 2, 3, 5, -7/2\} \Rightarrow A_5 = -3328i = -2^8\cdot 13\,i$. I also confirm a permutation symmetry: swapping any two plus legs, and (separately) swapping the two minus legs, leaves A_5 fixed, so A_n carries an $S_2 \times S_{n-2}$ symmetry. I will hold onto that — it turns out to be the thing that flags the piecewise structure.

Now I try to fit. The natural ansatz, given the symmetry, is that $P_n = A_n/i$ is a symmetric polynomial in the group power sums — minus group $m_1 = \omega_1+\omega_2$, $m_2 = \omega_1^2+\omega_2^2$, plus group $P_3, \ldots$ — with the two conservation laws eliminating $P_1 = -m_1$ and $P_2 = m_2$. I generate a big exact dataset, 64 points (30 at $n=5$, 20 at $n=6$, 14 at $n=7$, e.g. `5 | {-13/2,2,3,5,-7/2} | -3328*I`), and solve the exact linear system for the polynomial coefficients. It is inconsistent at $n=5$ and again at $n=6$: **P_n is not a polynomial**. That is a clean negative result — it means either a genuine denominator from the internal propagators or piecewise dependence from the absolute values in the kernels.

So I stop fitting the final number and resolve the non-analyticity by hand. The only non-analytic object anywhere is $\mathrm{mag}[k] = |k|$. I redefine `mag`$[z\_] := z\,\mathrm{Sign}[z/.\mathrm{ref}]$ — replace every `Abs` by its sign at a numeric

Why?

Rewritten Chain of Thought for the Solution to the
Two-Minus Water-Wave Amplitude

Abstract

This document is a first-person reconstruction of the chain of thought for the original AI derivation of the closed-form tree amplitude A_n in the two-minus sector of one-dimensional deep-water surface waves. It is rebuilt from the visible run trajectory and keeps the original wording, the false starts, and the hedges — including a wrong first guess and a stretch where the machine ran out of memory — rather than tidying them away.

Rewritten Chain of Thought

I inspect the case directory first so I can infer the deliverables rather than guess from the folder name. The prompt's restrictions are strict, so from here I use only `prompt.md`, `OnShellBG.m`, and files I create in the case directory, and I generate fresh amplitude data rather than relying on any prior attempt. I want a small Wolfram harness that imports only the *definition* portion of `OnShellBG.m`, so the file's built-in demo block does not dominate the runs; the first target is exact low-point data, to see the algebraic pattern.

The first exact probe shows that $n = 5, 6, 7$ are finite and purely imaginary for a generic positive-frequency choice, so I write $A_n = i\,P_n$. The point $n = 4$ is kinematically degenerate in this sector — the recursion returns `Indeterminate` — so I decide to handle it separately, as a limiting/regularized value, rather than treat the raw indeterminate as data.

The low-point values are not just a constant times a simple all-frequency product, so I start by fitting against symmetric combinations allowed by the two-minus constraints. Two facts narrow this. First, the amplitude is homogeneous: scaling all frequencies by λ scales A_n by λ^{2n-4}. Second, on the resonant manifold the two negative-sector frequencies are determined by the elementary symmetric sums of the positive ones, so I should reduce the data to those invariants. For $n = 5$ that makes a seven-term symmetric degree-six polynomial in the three positive-sector frequencies the first thing to rule in or out. I also remind myself that the absolute values in the interaction kernels mean the closed form may be *piecewise* unless it collapses on the resonant manifold, so I check that directly at five points before committing to any ansatz.

A pure symmetric polynomial fails on held-out five-point data. That is a clean negative result: it points to either denominator structure from the internal propagators or a piecewise dependence coming from the absolute-value kernels. So I shift from fitting the final amplitude to simplifying the building blocks. The cubic kernel is already sign-selective, which makes me think a closed form for the higher kernels could collapse the Berends–Giele result into a manageable subset-sum expression.

A strong branch then emerges from the data: in the chambers where the second negative-momentum leg is the smallest scale, the amplitude obeys

$$A_n \;=\; i\,2^{n-1}\,\omega_1\,\omega_2^{2n-5}$$

for $n = 5, 6, 7$. This is encouraging, but it is only one chamber, and I say so explicitly to myself: I am now *mapping the other chambers to see the invariant way to state the all-kinematics formula*. Inside each absolute-value chamber the five-point expression factors cleanly, but there are several chambers, so I ask Wolfram for a compact absolute-value expression after imposing the two-minus conservation laws — which is the right way to get a chamber-independent statement if it simplifies.

It does not simplify quickly. The symbolic run drags, and then the environment turns hostile: the shell cannot fork because the machine reports out of memory, and for a while it refuses to create any new process

How?

Rewritten Chain of Thought for the Solution to the
Two-Minus Water-Wave Amplitude

Abstract

This document is a first-person reconstruction of the chain of thought for the original AI derivation of the closed-form tree amplitude A_n in the two-minus sector of one-dimensional deep-water surface waves. It is rebuilt from the visible run trajectory and tries to keep the original wording, the false starts, and the hedges rather than tidy them away.

Rewritten Chain of Thought

I can only use the prompt and `OnShellBG.m`, and I have to be careful not to read any other pre-existing file. I need a closed-form A_n for $\sigma = (-1,-1,+1,\ldots,+1)$, and I should understand how the kernels work for a faster implementation. First question: is `wolframscript` even available? Let's run `wolframscript`. It is. I think about how to use the code: maybe I copy some definitions, maybe I append to `OnShellBG`, maybe I write a separate loader that does `Get`. I write a wrapper that builds the two-minus sign vector, calls `MakeKinematics`, and feeds `BGAmplitude`, then I just `Get` the file — and that was a mistake: `Get` runs the file's own demonstration tests, which are for a *different* sector ($\sigma = (-1,+1,\ldots,+1)$, where the amplitude is identically zero), and it prints a wall of "$A_5 = 0$, $A_6 = 0$, …" while the $n = 8$ case grinds. I let it run for a bit, decide it is wasting time, and interrupt it. Better idea: import the file as text, split it at the `VI. TESTS` marker, and `ToExpression` only the part before that, so I load the definitions without the demo. (A `Do[...,{{n,fw},...}]` trips a "Tag List is Protected" error too, so I switch the loop to `Scan`; small things.)

Now the oracle behaves. The first real surprise is $n = 4$: the recursion returns `Indeterminate` — "1/0", "0·ComplexInfinity". The on-shell point forces an internal two-leg current to zero momentum, so a propagator blows up and a real cancellation hides behind a removable $0/0$. I set $n = 4$ aside, to come back as a limit, and look at $n = 5, 6, 7$, which give clean, purely imaginary rationals like $-2304\,i$. So I write $A_n = i\,P_n$ and chase P_n.

I keep thinking about kernels, propagators, and the overall dimensions. Scaling $\omega \to \lambda\omega$ multiplies A_n by λ^{2n-4}; changing g shows $A_n \propto g^{-(n-3)}$; and a family where the second negative leg goes soft, $\omega_2 = a \to 0$, shows the amplitude going like a^5 when ω_2 is the smallest scale around. Putting the degree, the g-power, and that soft behaviour together, the simplest thing that fits is the monomial

$$A_n \overset{?}{=} i\,\frac{2^{n-1}}{g^{n-3}}\,\omega_1\,\omega_2^{2n-5}.$$

It is suspiciously clean, and it does reproduce the soft data. But it breaks the instant ω_2 is not the smallest same-sign scale. I notice something intriguing while poking at this: flipping ω_1, ω_2 both negative or both positive gives the same $B \equiv A/(-i)$, so whatever P_n is, it looks symmetric in the two negative legs. That tempts me to fit a symmetric polynomial $f(\omega_1,\omega_2)$ of degree six — $a(u^6{+}v^6) + b(u^5v{+}uv^5) + c(u^4v^2{+}u^2v^4) + d\,u^3v^3$. I solve for the coefficients on four points; the solver returns monstrous fractions, and they miss every other point by huge amounts. So A_n is simply not a polynomial in the two negative frequencies. I toy with a divided-difference / Lagrange form, $\sum_i \sigma_i\omega_i^{2n-4}\prod_{j\neq i}\omega_j/(\omega_i-\omega_j)$, and with B-splines on the sorted (negative) phase velocities, "normal-form coefficients based on the max of the frequencies" — these feel close to the right shape but I have no derivation for them yet.

The derivation is hiding in the absolute values. The only non-analytic thing in the whole construction is $|k_S| = \left|\sum_{i\in S}\sigma_i\omega_i^2\right|/g$ inside each propagator. An absolute value has a corner where its argument changes

What Next?

[Zhou, Wadekar, Zaldarriaga (ongoing)]

Final Comments about AI

- For mathematicians:

  Michael Atiyah: "Algebra is the offer made by the devil to the mathematician"

  Michael Atiyah: "We would probably cheat on the devil: pretend we are selling our soul, and not give it away."

- For physicists:

  Surya Ganguli: "AI is the offer made by the devil to the physicist"

  Surya Ganguli: "Let's dance with the devil without losing our soul"

  [Ganguli (talk at PAI26)]

Thank You !