Amplitudes and the Quest for Precision

Michael Ruf
Amplitudes 2026 @ QMUL, July ’29

Based on:

[2511.13845, 26XX.XXXXX++] Guan, Mistlberger
[2605.05009] De Laurentis, Ita, Kuschke, Sotnikov
[26XX.XXXXX] Bhanik, Guan, Mistlberger, Suresh

Colliders

GW observatories

[Talks by Gehrmann, Zoia, Chicherin, Vanhove]

[Talks by Kosower, Jacobsen, Bohnenblust, Isabella, Bern]

Precise measurements

$300\,\mathrm{fb}^{-1}$(Run 3)

~ 300 mergers (aLIGO)

Today

$3{,}000\,\mathrm{fb}^{-1}$ (HL-LHC) $\sim 10^5$ Higgs

$\sim 10^5 - 10^6$ mergers (ET, CE)

2030s Electron-Ion Collider

$10^{12} Z$ (FCC-ee)

2040s +

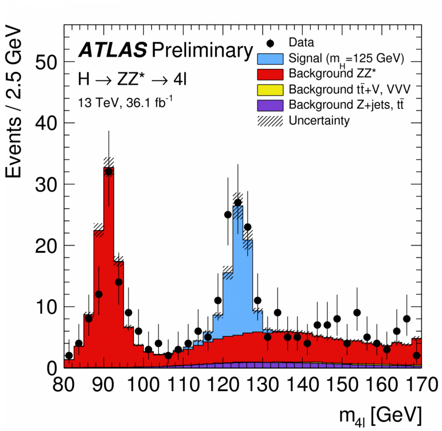
Colliders

GW

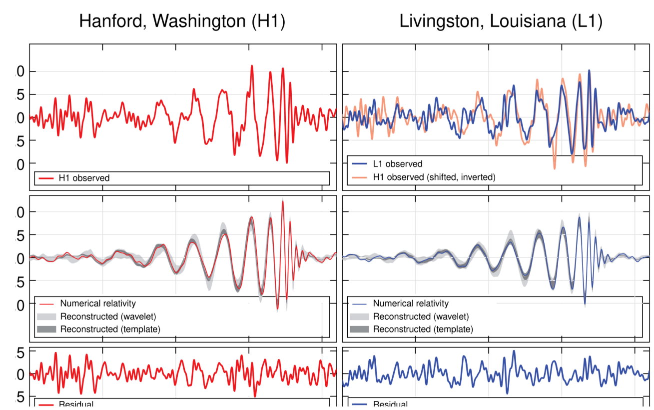
Precise measurements

+

Precise theory predictions



- Detect (Higgs/GW)
- Map (couplings, NS equation of states)
- Constrain (beyond SM/GR)
- Discover? (Exotic compact objects, dark matter)

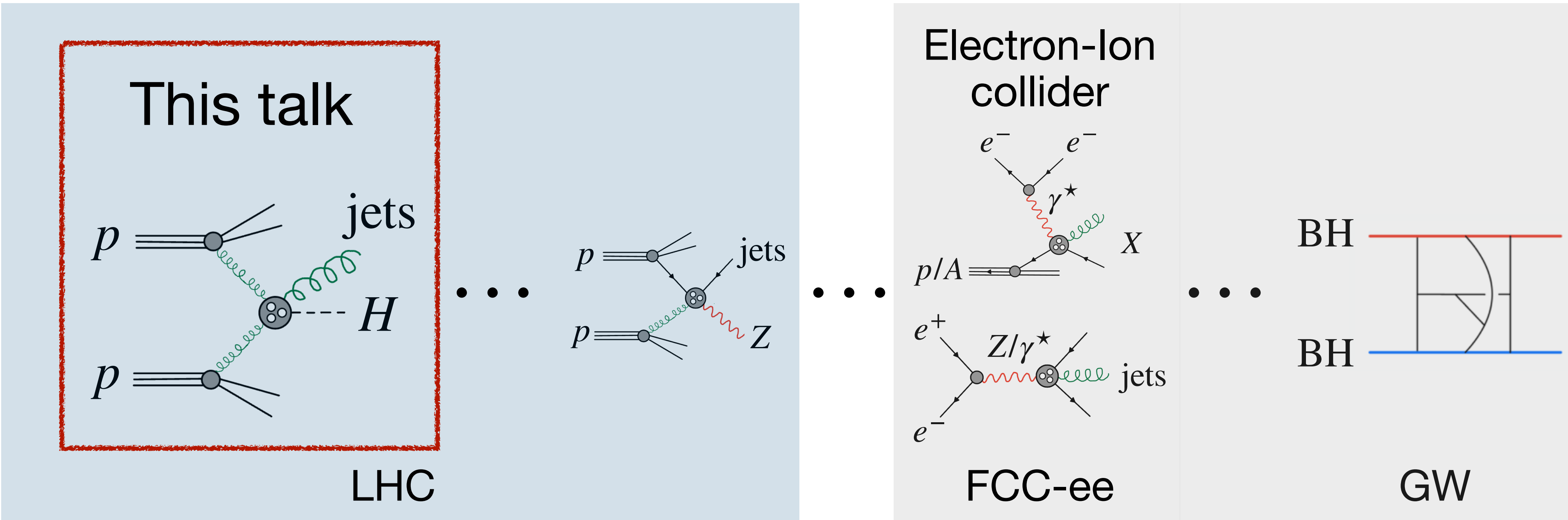


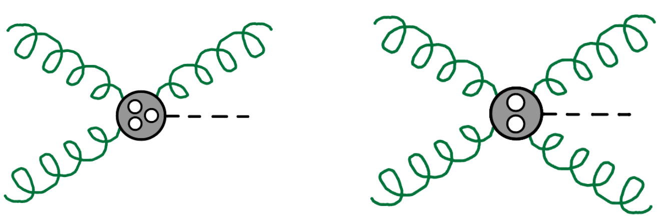
Amplitudes

Precision for collider observables

Common language
Tools & ideas

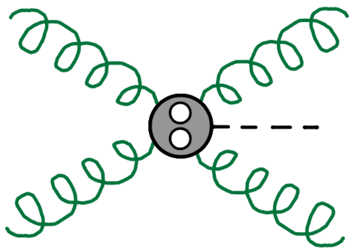
Precision for GW observables





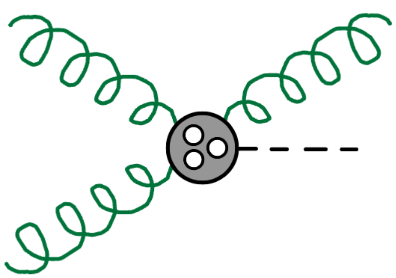
- General methods
- Need robust & flexible pipelines
- New tools, ideas have mayor impact

Amplitudes for Higgs Precision



Part I: $H + jj$ Amplitudes — algebraic complexity/rational functions

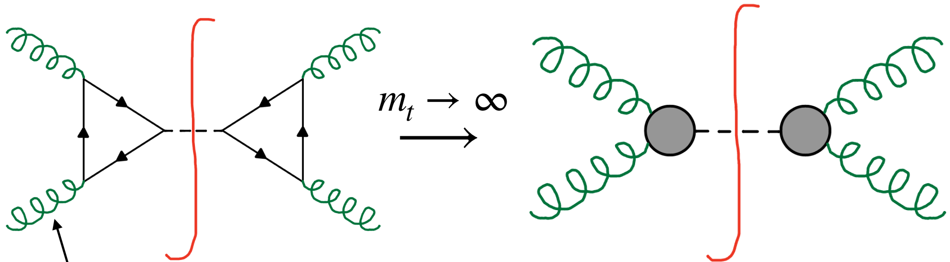
Part II: $H + j$ Amplitudes — integrals



Higgs Production Cross-section

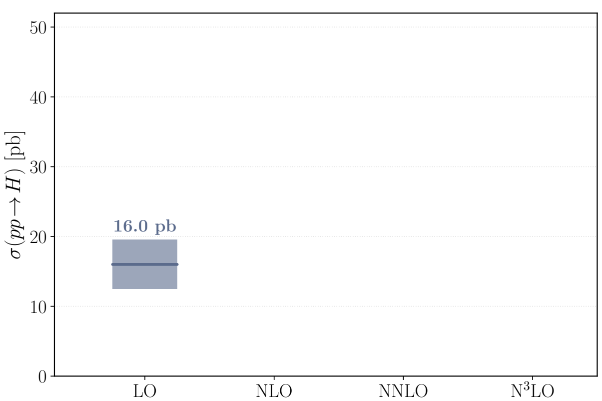
LO

$$\sigma_{gg\to H} = \int \mathrm{d}\Phi(p_H) M^{\star}_{\mathrm{LO}} M_{\mathrm{LO}} =$$



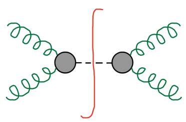
[Georgi, Glashow, Machacek, Nanopoulos, ’78]

gg dominates ~ 90%



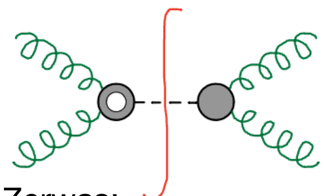
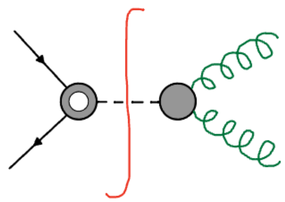
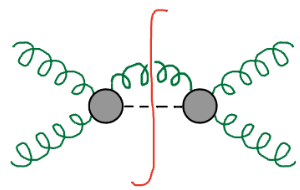
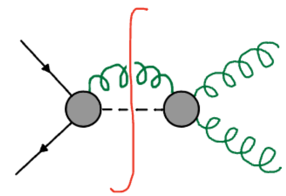
Higgs Production Cross-section

LO



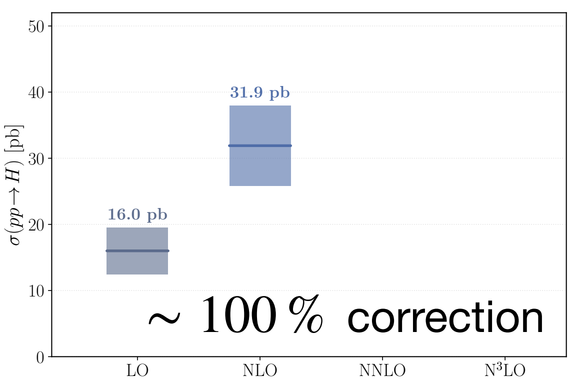
1978

N¹LO

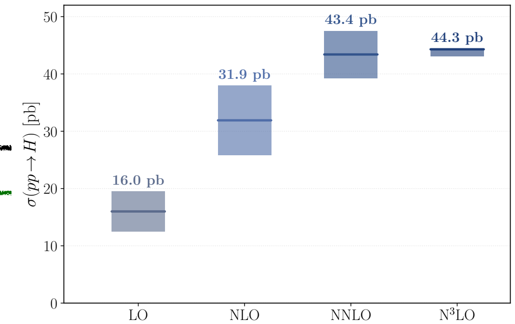
$2\,\mathrm{Re}($  $+$  $) +$  $+$ 

[Djouadi, Spira, Zerwas;
Dawson '91]

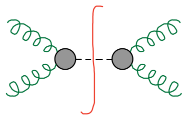
Final state singularities cancel!



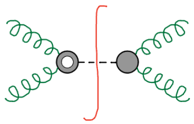
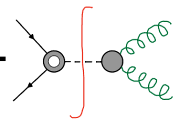
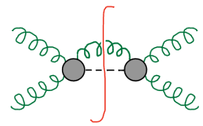
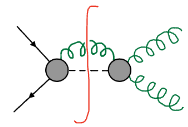
Higgs Production Cross-section



LO
1978

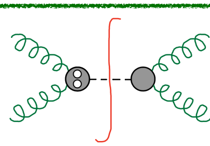
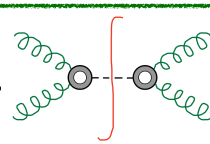
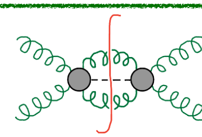


N¹LO
1991

2 Re( + )+  + 

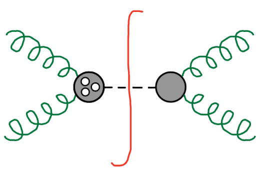
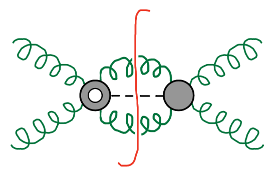
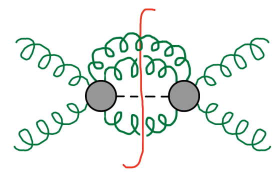
$\sim 100\,\%$

N²LO
2002

2 Re() +  +  + …

$\sim 36\,\%$

N³LO

2 Re() + … + 2 Re() + 

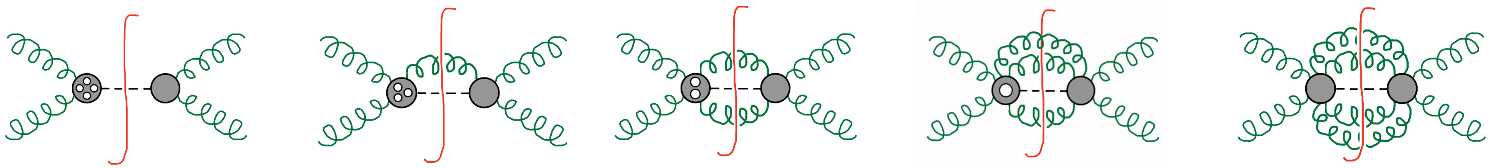
$\sim 2\,\%$

[Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Mistlberger,'14, Anastasiou, Duhr, Dulat, Herzog, Mistlberger,'15, Mistlberger '18]

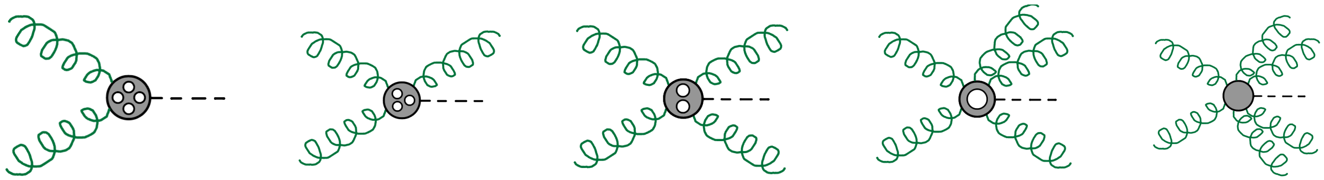
Elliptic functions!

Amplitudes for Higgs Precision

N^{4}LO



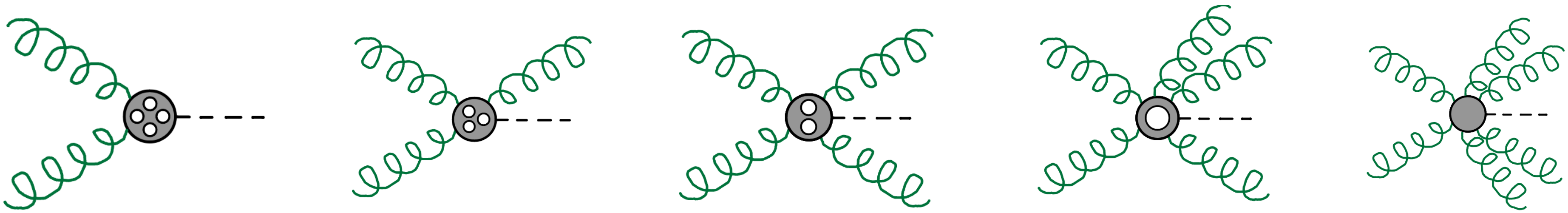
Amplitude
data



$$\mu = \frac{\sigma_{\text{obs}}(pp \to H)}{\sigma_{\text{SM}}(pp \to H)} = 1.023 \pm 0.028\,(\text{stat.})\,^{+0.026}_{-0.025}\,(\text{exp.})\,^{+0.039}_{-0.036}\,(\text{sig. th.}) \pm 0.012\,(\text{bkg. th.})$$

Theory error dominates!

Amplitudes for Higgs Precision



Loops ⟷ # Legs

Many loop integrals

One scale $s = m_H^2$

Dependence on kinematics trivial

PS integration trivial

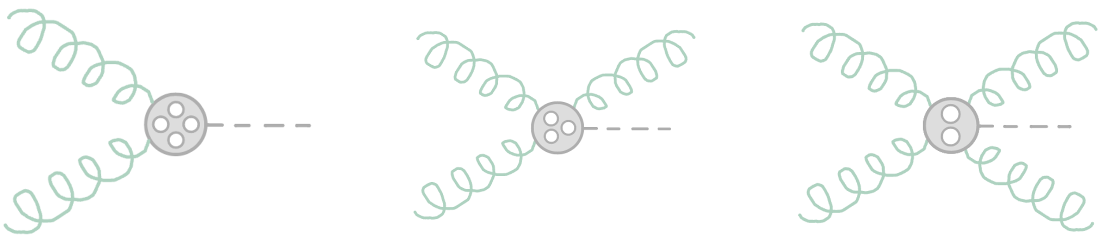
No loop integrals

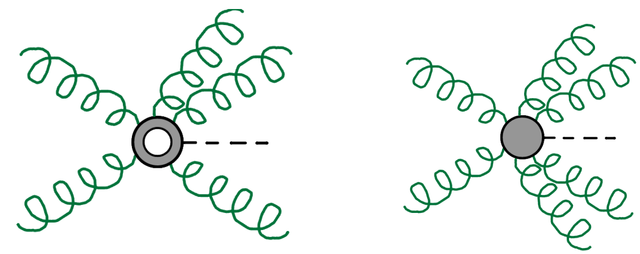
Many scales $s, s_{Hj_1}, s_{Hj_2}, \dots$

Complicated roots and special functions

Need efficient evaluation over PS

Amplitudes for Higgs Precision



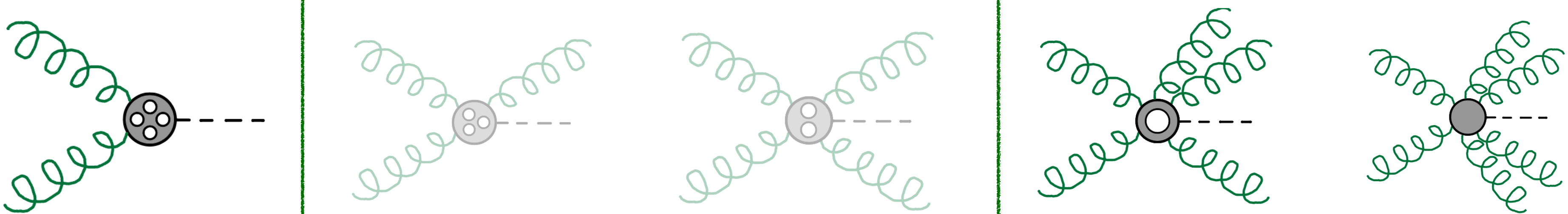


automatized*

[Openloops 2: Buccioni, Lang, Lindert, Maierhöfer, Pozzorini, Zhang, Zoller,…]

*PS hard for N^{4}LO inclusive

Amplitudes for Higgs Precision



4 loop formfactors

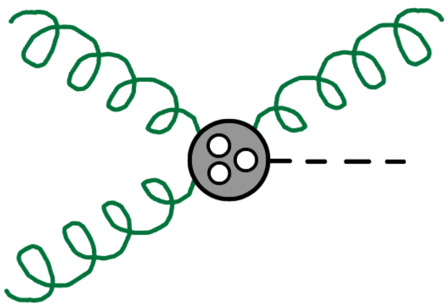
[Lee, von Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser, ’21,’22]

$$
\begin{aligned}
F_g^{(4)}\Big|_{\epsilon^0} = C_A^4 \Big(& -\frac{2591}{90}\zeta_{5,3} + \frac{1018949}{90}\zeta_5\zeta_3 - \frac{35689}{27}\zeta_3^2\zeta_2 + \frac{18282694}{7875}\zeta_2^4 - \frac{27705161}{504}\zeta_7 + \frac{1160731}{270}\zeta_5\zeta_2 - \frac{1928564}{405}\zeta_3\zeta_2^2 \\
& - \frac{1296845}{1458}\zeta_3^2 - \frac{727183}{1134}\zeta_2^3 + \frac{6161623}{243}\zeta_5 - \frac{3233651}{729}\zeta_3\zeta_2 + \frac{54443689}{14580}\zeta_2^2 + \frac{839716507}{104976}\zeta_2 - \frac{84995881}{52488}\zeta_3 + \frac{96887974603}{3779136}\Big) \\
& + \frac{d_A^{abcd}d_A^{abcd}}{N_A}\Big(260\,\zeta_{5,3} - 5092\,\zeta_5\zeta_3 - 16\,\zeta_3^2\zeta_2 - \frac{496766}{525}\zeta_2^4 - \frac{6776}{3}\zeta_7 - 5016\,\zeta_5\zeta_2 + \frac{2992}{3}\zeta_3\zeta_2^2 + \frac{31588}{3}\zeta_3^2 \\
& + \frac{1073972}{945}\zeta_2^3 - 6460\,\zeta_5 + \frac{6752}{9}\zeta_3\zeta_2 + \frac{24616}{45}\zeta_2^2 - \frac{4682}{27}\zeta_2 - \frac{1310}{9} + \frac{68410}{9}\zeta_3\Big) \\
& + \text{contributions with closed fermion loop from Ref. [35].}
\end{aligned} \tag{9}
$$

The Missing Pieces

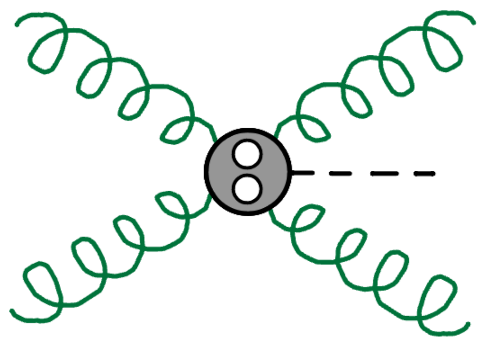
Integral reduction

Master integrals



$H + j$ @ 3 loops

[Chen, Guan, Mistlberger ’25]
[Guan, Mistlberger, **MSR** wip]



Integral reduction

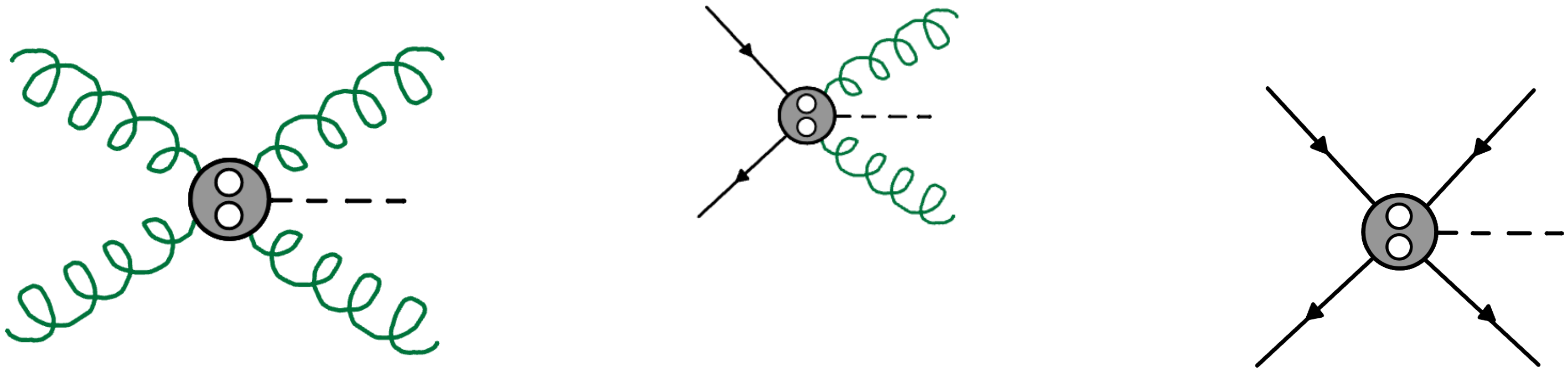
Algebraic coefficients

Special functions

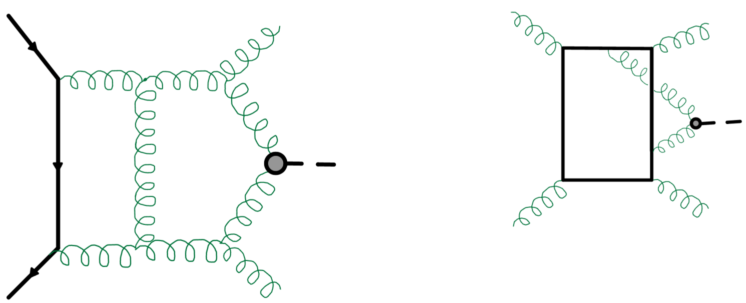
$H + jj$ @ 2 loops

[Hartanto, Poncelet ’26] (quark channels)
[De Laurentis, Ita, Kuschke, **MSR**, Sotnikov ’26]

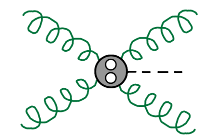
Each with its own challenges !



Higgs + 2 Jets @ 2 Loops

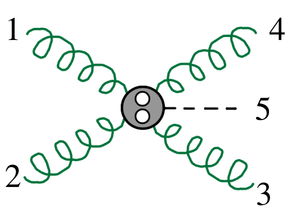


$H + jj$ Amplitudes



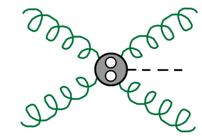
- $H + jj$ amplitudes are quite complicated: 6 scales, non-planar integrals
- Analytic computation using Feynman rules/IBP hard,
  many variables intermediate expression swell
- Solution [van Manteuffel, Schabinger '15; Peraro '16,…]
  - Sample numerical kinematic points in finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$
  - Lift once sufficient samples are obtained
- Efficient numerical evaluations using Caravel [Abreu et al. '20]

Caravel

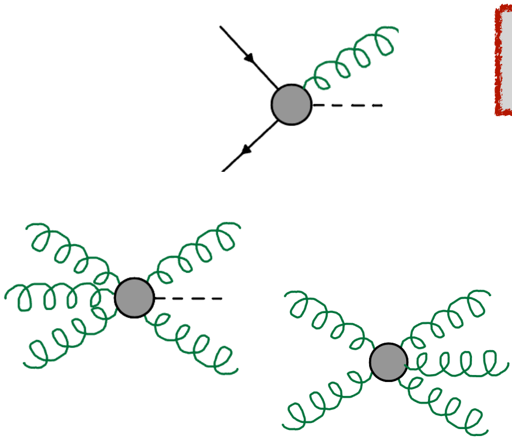


CARAVEL: A C++ Framework for the Computation of Multi-Loop Amplitudes with Numerical Unitarity

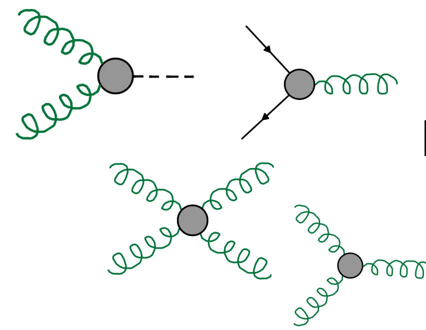
S. Abreu[a], J. Dormans[b], F. Febres Cordero[c,*], H. Ita[b], M. Kraus[c], B. Page[d,*], E. Pascual[b], M. S. Ruf[b], V. Sotnikov[e,*]



$H + jj$ Amplitudes



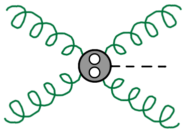
Tree amplitudes ($\mathbb{F}_p$) ⟵ Berends-Giele



Caravel

[Abreu et al. ‘20]

$H + jj$ Amplitudes



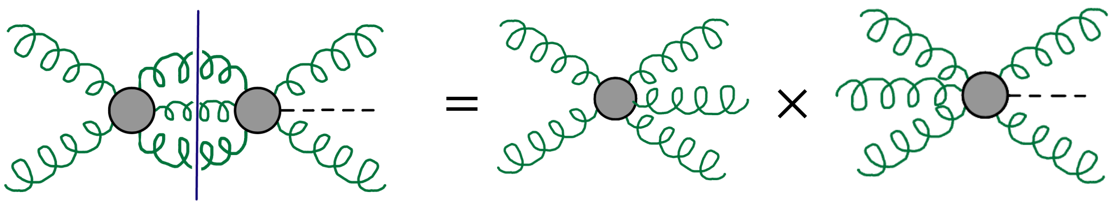
Tree amplitudes ($\mathbb{F}_p$)

Sew

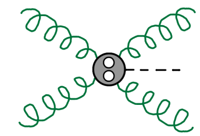
Unitarity cuts ($\mathbb{F}_p$)

Caravel

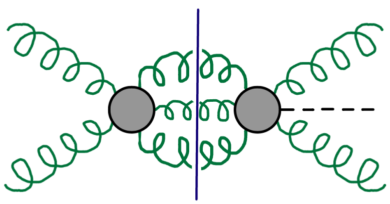
[Abreu et al. '20]



$H + jj$ Amplitudes



Tree amplitudes ($\mathbb{F}_p$)



Unitarity cuts ($\mathbb{F}_p$)

Match

Master-Surface integrand ($\mathbb{F}_p$)

Caravel

[Abreu et al. '20]

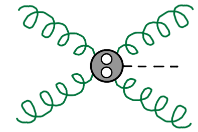
$$A = \int \mathrm{d}^D \ell_1 \mathrm{d}^D \ell_2 \left[\sum_{i=1}^{\mathcal{O}(100)} c_i M_i(\ell) + \sum_j \tilde{c}_j S_j(\ell) \right]$$

[Ita, '15]
[Abreu, Febres Cordero, Ita, Jaquier, Page, Zeng '17]
[Abreu, Febres Cordero, Ita, Jaquier, Page '17]

Master integrals
[Abreu, Ita, Page, Tschernow, '21;
+ Moriello, Zeng '22; + Chicherin, Sotnikov, '23]

Total derivative

$H + jj$ Amplitudes

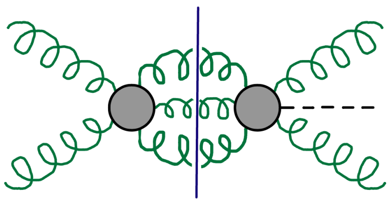


Tree amplitudes ($\mathbb{F}_p$)

Caravel

[Abreu et al. '20]

Unitarity cuts ($\mathbb{F}_p$)



Master-Surface integrand ($\mathbb{F}_p$)

Integrate, IR subtract [Catani '98], renormalize

Remainder function ($\mathbb{F}_p$)

$$f_{17}^{(2)} = \mathrm{Li}_2\left(1 - \frac{s_{12}s_{15}}{p_1^2 s_{34}}\right)$$
$$f_{19}^{(2)} = \mathrm{Li}_2\left(1 - \frac{s_{13}s_{15}}{p_1^2 s_{24}}\right)$$
$$f_{21}^{(2)} = \mathrm{Li}_2\left(1 - \frac{s_{12}s_{13}}{p_1^2 s_{45}}\right)$$

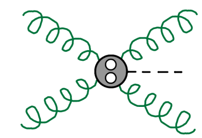
$$R = \sum_i r_i f_i$$

coefficients evaluated in $\mathbb{F}_p$

Pentagon functions

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, '23]

$H + jj$ Amplitudes — Reconstruction

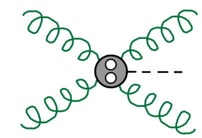


Numeric ($\mathbb{F}_p$) phase-space point → Caravel → Numeric ($\mathbb{F}_p$) coefficient r_i → ? → Analytic coefficient r_i

- Analytic r_i fixed given enough probes
- Sampling along univariate slice → numerator/denominator degree
- For $H + jj$ degree 168: dense ansatz $\sim 8 \times 10^7$ probes!

Can we prune the ansatz?

$H + jj$ Amplitudes — Reconstruction



- Not everything goes:
  - Denominator reconstruction on univariate slice [Abreu, Dormans, Febres Cordero, Ita, Page, ’18] [Abreu, Febres Cordero, Ita, Page, Sotnikov, ’18]
  - 5-pt alphabet: $D_i \in \{W_1 = p_1^2, W_2 = s_{34}, \dots\}$ [Abreu, Ita, Page, Tschernow, ‘21] [Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov ’23]
  - New: denominator pairings from bivariate slices [De Laurentis, Ita, Kuschke, **MSR**, Sotnikov ’26] [De Laurentis, Maitre ’19, De Laurentis, Page ’22]

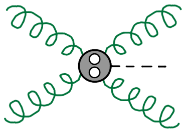
$$r = \frac{w + x^2 y}{x^2 y w} = \frac{1}{x^2 y} + \frac{1}{w} = \frac{1}{w} + \frac{1}{yw} + \frac{z}{x^2 w} \longrightarrow N \in \left(x^2, y\right) \cap \left(x^2, w\right) \cap \left(y, w\right)$$

$w = x^2 + zy$ no $x^2 w, yw$ no $x^2 y$

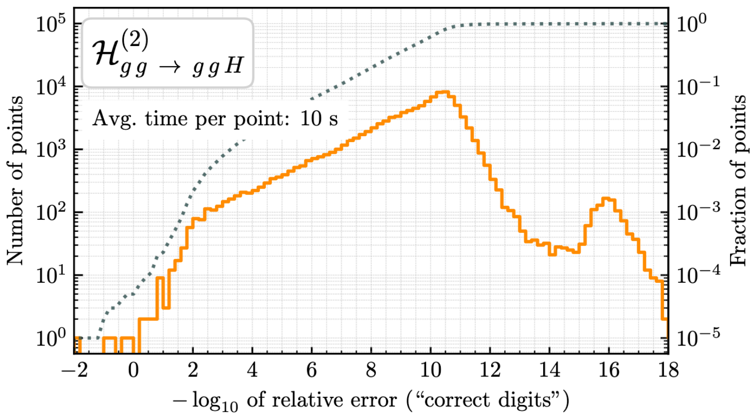
Reconstruction in Antares
[De Laurentis]

$$\boxed{8 \times 10^7 \rightarrow 5 \times 10^3 \text{ probes, } H + jj \text{ is doable!}}$$

$H + jj$ Amplitudes — Reconstruction



- Compact (< 5MB) expressions for amplitudes for $H + jj$
  - Evaluation times: gluons <10s, quarks <3s
- Checked thoroughly:
  - Quark channels reproduce [Hartanto, Poncelet '26]
  - IR-Stucture matches prediction
  - Collinear factorization
- Numerics from PentagonFunctions++
  [Chicherin, Sotnikov, Zoia '21;
  Abreu Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia '23]



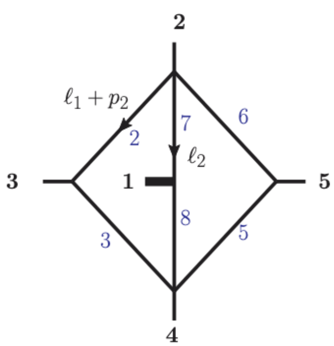
Sampled on ~100k phase-space points

$\boxed{H + jj \text{ amplitudes ready for pheno!}}$

A Spurious Threshold?

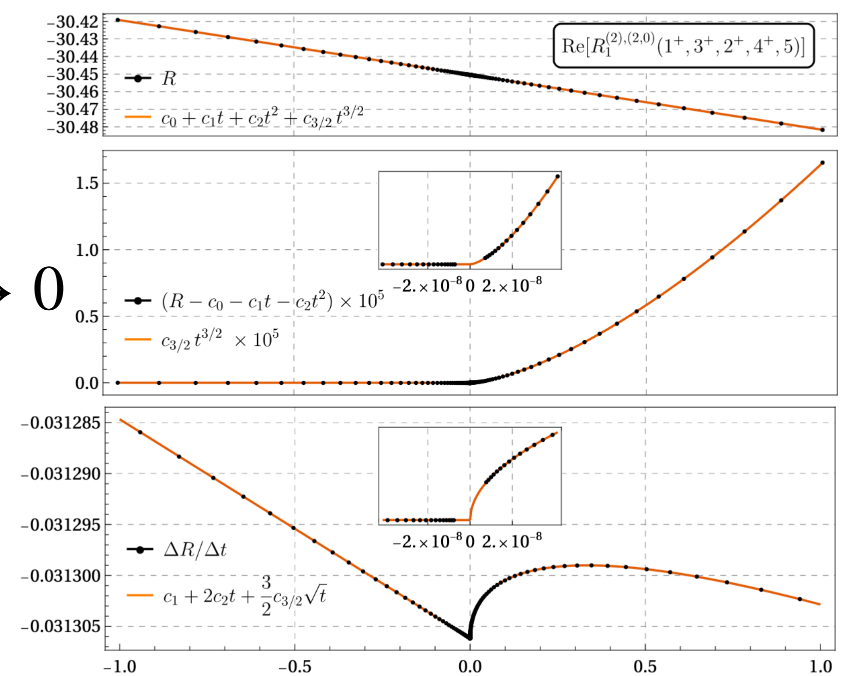
[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, '23]

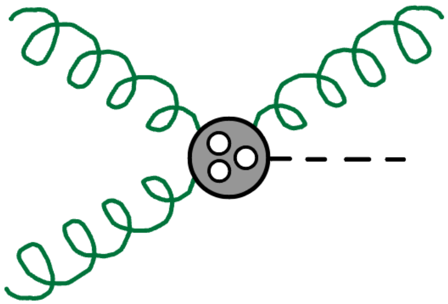
- New algebraic letter $\Sigma_5 = (s_{12}s_{15} - s_{12}s_{23} - s_{15}s_{45} + s_{34}s_{45} + s_{23}s_{34})^2 - 4s_{23}s_{34}s_{45}(s_{34} - s_{12} - s_{15})\,.$
- Genuine 2 loop leading singularity, no simple gram of external vectors
- $\Sigma_5 = 0$ reached in scattering kinematics!
- Singularity of the individual integrals
- Amplitude continuous but not smooth as $\Sigma_5 \to 0$



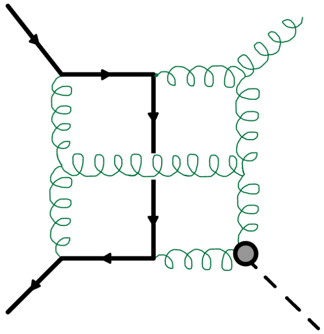
$= \dfrac{1}{\sqrt{\Sigma_5}}$

$\boxed{\text{Will } \Sigma_5 \text{ show up in observables?}}$

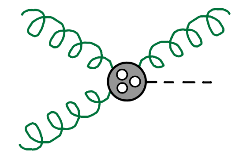


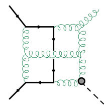


Higgs + jet @ 3 loops



$H + j$ Amplitudes



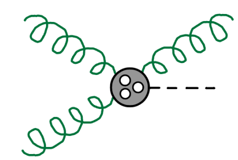
Standard toolchain: 

	Feynman diagrams	CalcLoop [Ma et al.]
		Lorentz algebra
		Tensor projection
$A = \sum_{i=1} T_i A_i$	Integrand	
		IBP w/ Blade [Guan, Liu, Ma '20]
$A_i = \sum_{k=1} c_k M_i$	Master integrals	

- Some steps very hard (IBP: 3 loops, 3 scales, EFT power-counting)
- Heavy use of $\mathbb{F}_p$-sampling
- Missing piece: complete set of 3-loop master integrals

How to compute 3 loop MIs?

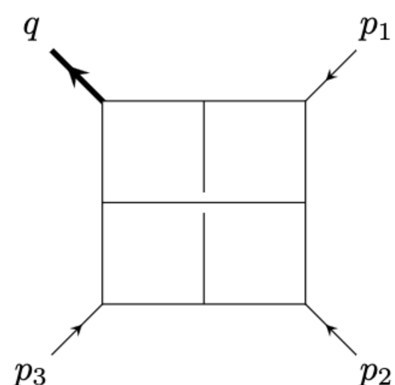
$H + j$ Master Integrals



- Amplitudes need master integrals
- Modest number of scales (s, t, p_1^2) but complicated differential equations
  - $\mathcal{O}(1000)$ MI, 20 families (3 planar + 5 leading color + 12 subleading color)
  - Differential equations GB-sized
- Integrals can be reused in other processes ($e^+e^- \to jjj, Z + j$, DIS,…, SYM form-factors)
- Canonical form [Henn'13] exists, how to find?

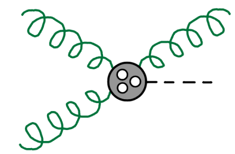
$$\mathrm{d}\vec{I} = C\vec{I} = \epsilon \sum_k \mathrm{d}\log(W_k)\vec{I}$$

the most complicated (by far)

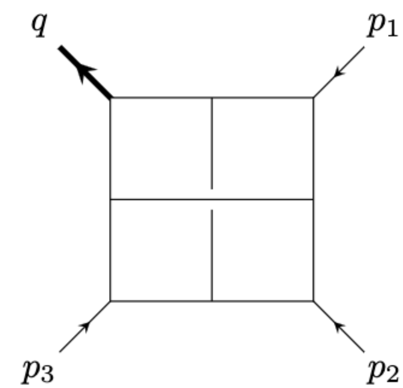


371 integrals,
19 on maximal cut

$H + j$ Master Integrals



- Lots of algorithms for canonical basis:
  - INITIAL [Dlapa, Henn, Wagner ’22,’23]
  - Leading-Singularities/dLogbasis [Henn, Mistlberger, Smirnov, Wasser ’17]
  - CANONICA [Meyer ‘17]
  - Lee [Lee ’14,’18,’20; Prausa ’17; Gituliar, Magerya ’17]
  - ...



$\boxed{\text{The Window graph resists!}}$

Canonical DEs

CANONICA [Meyer '17]: Canonical DEs from ansatz

$$\epsilon C = T^{-1}AT - T^{-1}\mathrm{d}T \qquad \epsilon TC = AT - \mathrm{d}T$$

- Number of parameters grows quickly (millions+)
- Clean results (polynomial in ϵ,…)

Lee [Lee '14,'18,'20; Prausa '17; Gituliar, Magerya '17]: Algorithmic by balancing moves

- Multivariate symbolic algebra (Jordan decomposition,…)
- Messy results (100 digit rational numbers,…)
- Scales to bigger problems

Can we combine the strengths of both?

Canonical DEs

Canonical DE in 3 steps:

Step 1: Lee on univariate $\mathbb{F}_p$ slice, $\epsilon = 10^{-3}$

$\rightarrow$ determines C and numerical $T(\epsilon_0, s, t_0, m_0 \ldots)$

$\mathcal{O}(s - \mathrm{min/probe})$

Step 2: Guess denominators + numerator degrees of T

Step 3: Solve linear system $\epsilon TC = AT - \mathrm{d}T$

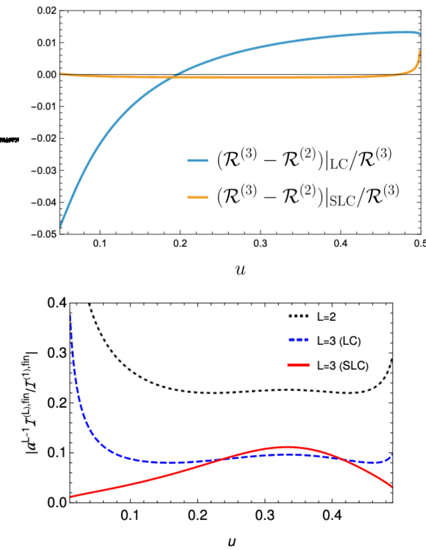
Hybrid approach combines strengths!

✓ No symbolic algebra in Lee step

✓ No big ansatze in CANONICA step, linear system, C is known

✓ Trade ansatz size ↔ probes (extreme case functional reconstruction).

$H + j$ Master Integrals

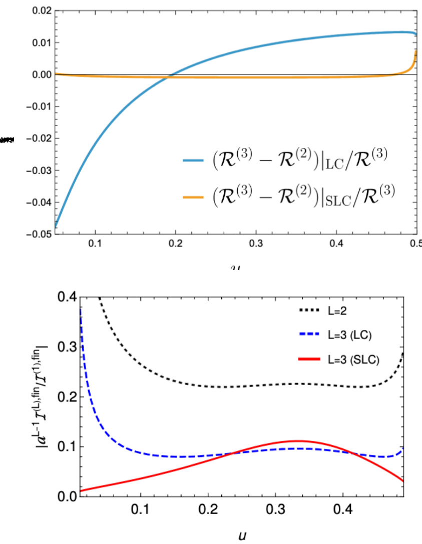


[Guan, Mistlberger, **MSR** ’25]

- A finite-field/sampling realization of canonicalization (Scorpio)
  - Parallelizable, small memory footprint
  - Sequential ~10×–10^4× vs. existing tools their own benchmarks
  - Handles 4-loop gravity integrals and other tests → more problems to test?
- Computed all 20 one-mass integral families in terms of GPLs
- Reproduce 8 known families [Vita, Mastrolia, Schubert Yundin ’14; Canko, Syrrakos ’22; Gehrmann et al ’24] + AMFlow numerics [Liu Ma ’20]
- Directly usable for other processes ($e^+e^- \to 3j$, DIS, $H \to b\bar{b} + j$, $pp \to Z + j$,...)
  [Guan, Mistlberger, **MSR** wip]

$H + j$ Master integrals

- With the master integrals all pieces are in place
- $H + j$ amplitudes in terms of GPLs
- Square-root $r = \sqrt{stuq^2}$ remains in the final result (cancels in $\mathcal{N} = 4$ [Guan, Mistlberger, **MSR '25**])
- Evaluation times < 1s with Multiple-Polylogarithm [Li]

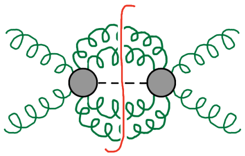


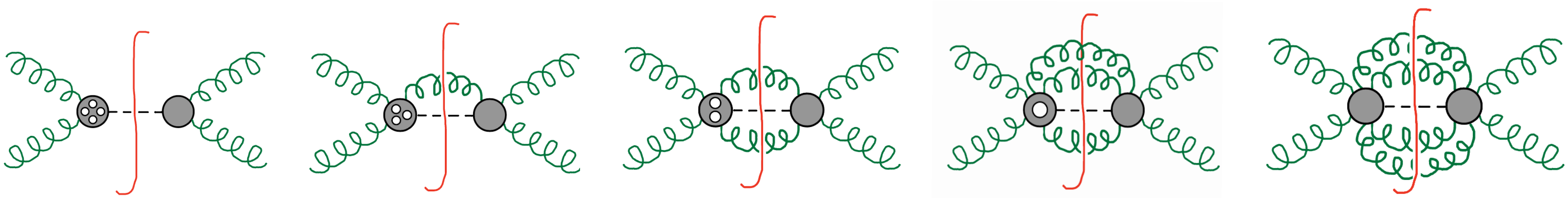
[Guan, Mistlberger, **MSR** '25]

Master integrals → Generalized Polylogs → Numbers

$H + j$amplitudes ready for pheno!

Phase-space integration



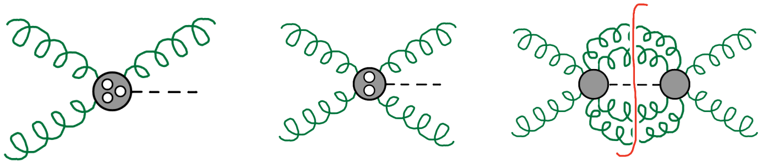


- Squared amplitudes need to integrate over phase space
- Singular limits, subtractions, …

→ As challenging as the loops if not harder

- Different ways to evaluate PS integrals:
  - Numerical: generic observables, cuts → fast evaluators for 2/3loop amplitudes <10s/pep pep, real PSP integrals complicated
  - Semi-analytical as at N^{3}LO [Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Mistlberger,'14, Anastasiou, Duhr, Dulat, Herzog, Mistlberger,'15, Mistlberger '18]
  - First steps taken [Mistberger, Suresh], progress and new ideas for other contributions

Conclusions

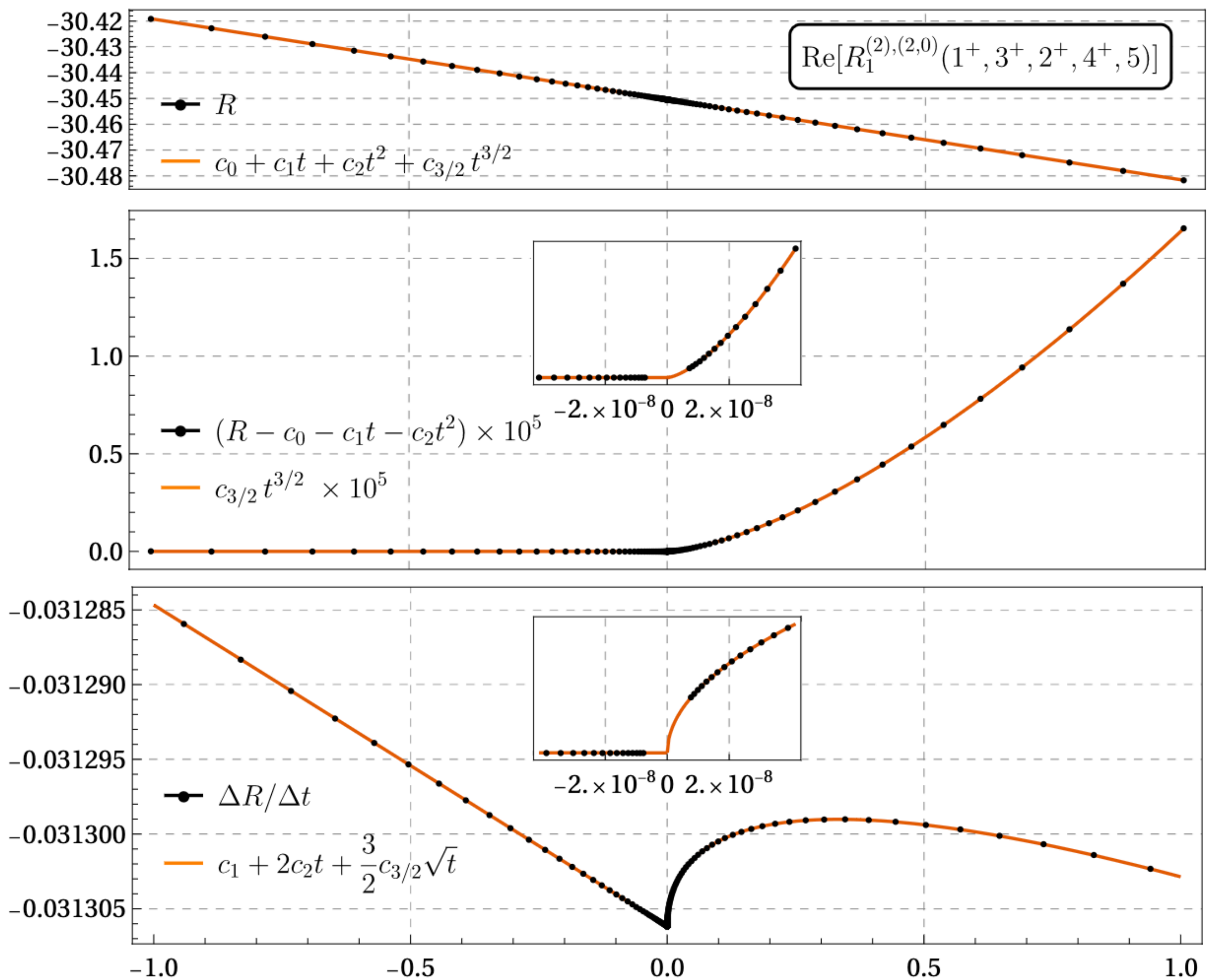


- Mature pipelines produce challenging amplitudes for colliders and beyond

  → Tools for full amplitudes (Caravel) and master integrals (Scorpio)

- Complete set of H+jet amplitudes for N^{4}LO pheno (missing full color $H + jj$)
- Interesting analytic features, whats the purpose of Σ_5?
- Other processes to be explored with same toolchain:
  $e^+e^- \to 3j$, DIS, $H \to b\bar{b} + j$, $pp \to Z + j$,…
- Phase space integrals challenging, exploring new ideas for semi-analytic evaluation
- Progress will have impact everywhere amplitudes are used: GW, cosmology, …

Lots to do but expect progress soon!



$\mathrm{Re}[R_1^{(2),(2,0)}(1^+,3^+,2^+,4^+,5)]$
R
$c_0 + c_1 t + c_2 t^2 + c_{3/2}\, t^{3/2}$
$(R - c_0 - c_1 t - c_2 t^2) \times 10^5$
$c_{3/2}\, t^{3/2} \times 10^5$
$\Delta R/\Delta t$
$c_1 + 2c_2 t + \frac{3}{2} c_{3/2} \sqrt{t}$
$-2.\times 10^{-8}$ 0 $2.\times 10^{-8}$
-30.42
-30.43
-30.44
-30.45
-30.46
-30.47
-30.48
0.0
0.5
1.0
1.5
-0.031285
-0.031290
-0.031295
-0.031300
-0.031305
-1.0
-0.5
0.0
0.5
1.0