

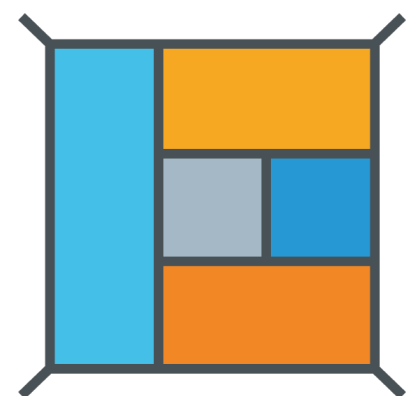
Advances in High-Precision Gravitational Scattering from Worldline QFT

Gustav Uhre Jakobsen

29 June

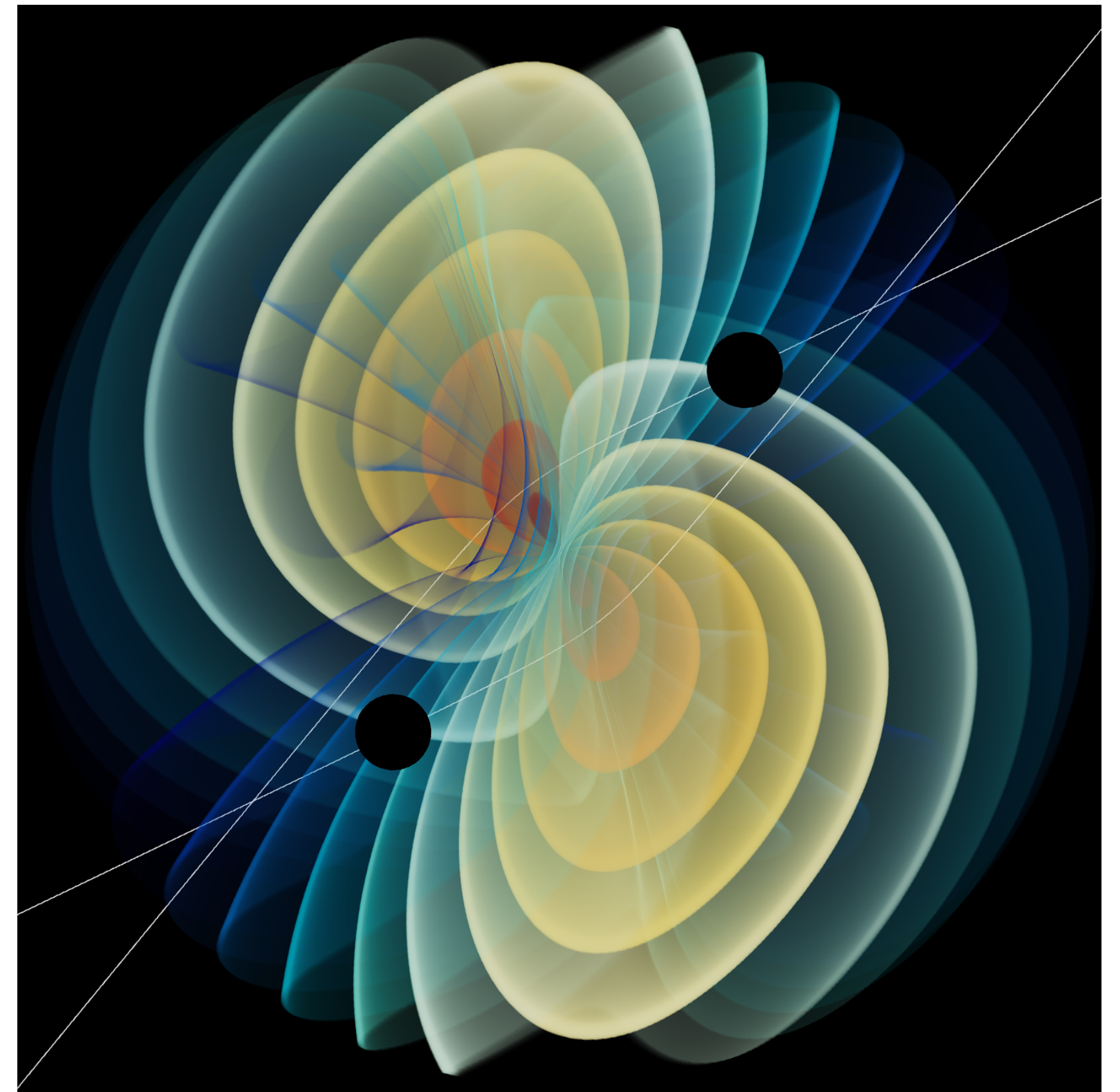
Amplitudes 2026

Queen Mary University London



RTG 2575:

**Rethinking
Quantum Field Theory**

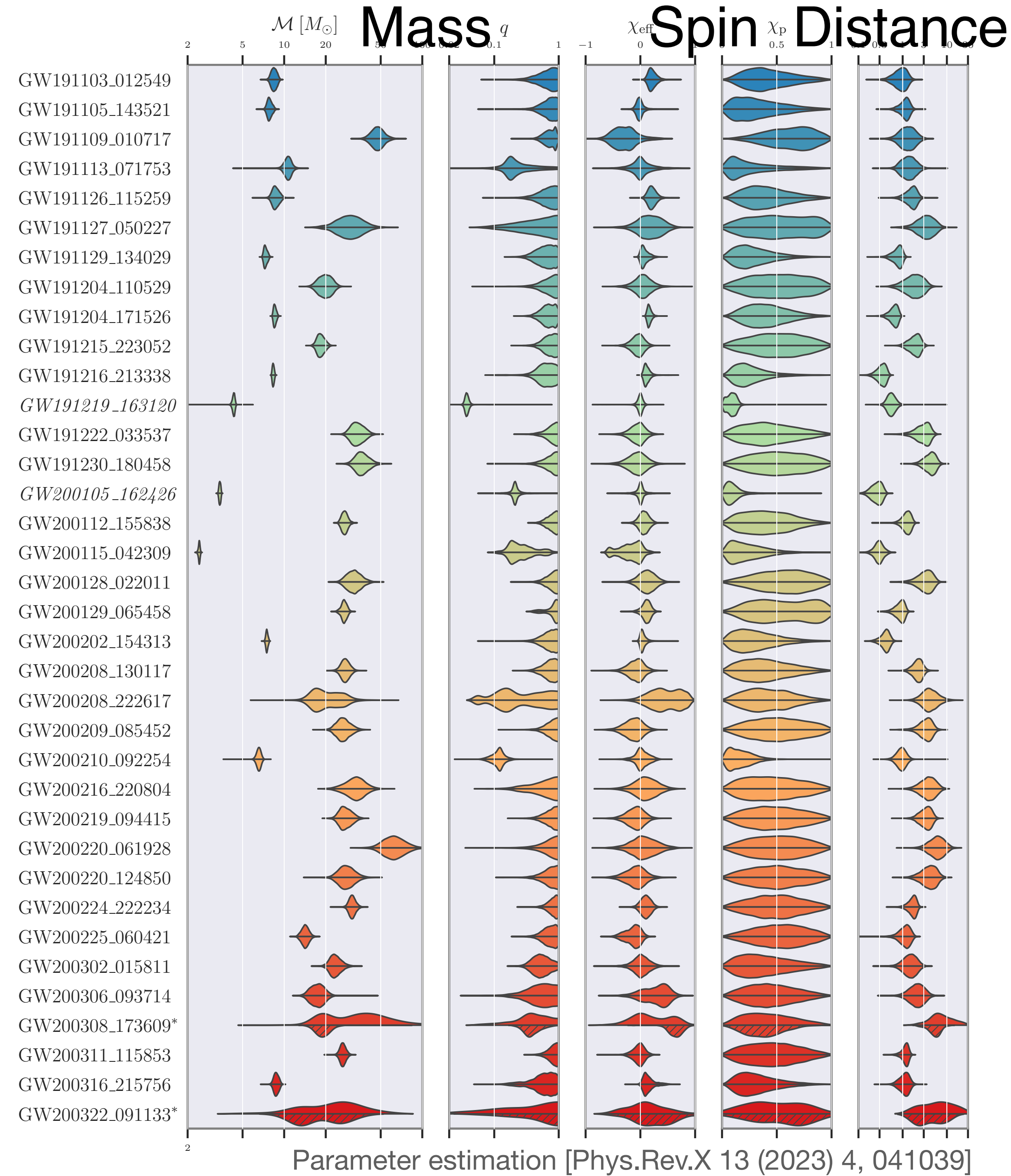


Credit: Mathias Driesse

- Improve analytic gravitational wave (GW) templates
- QFT methods for precision GW physics
- Third generation GW observatories with 10^2 sensitivity increase

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Virgo, Cascina, Italy



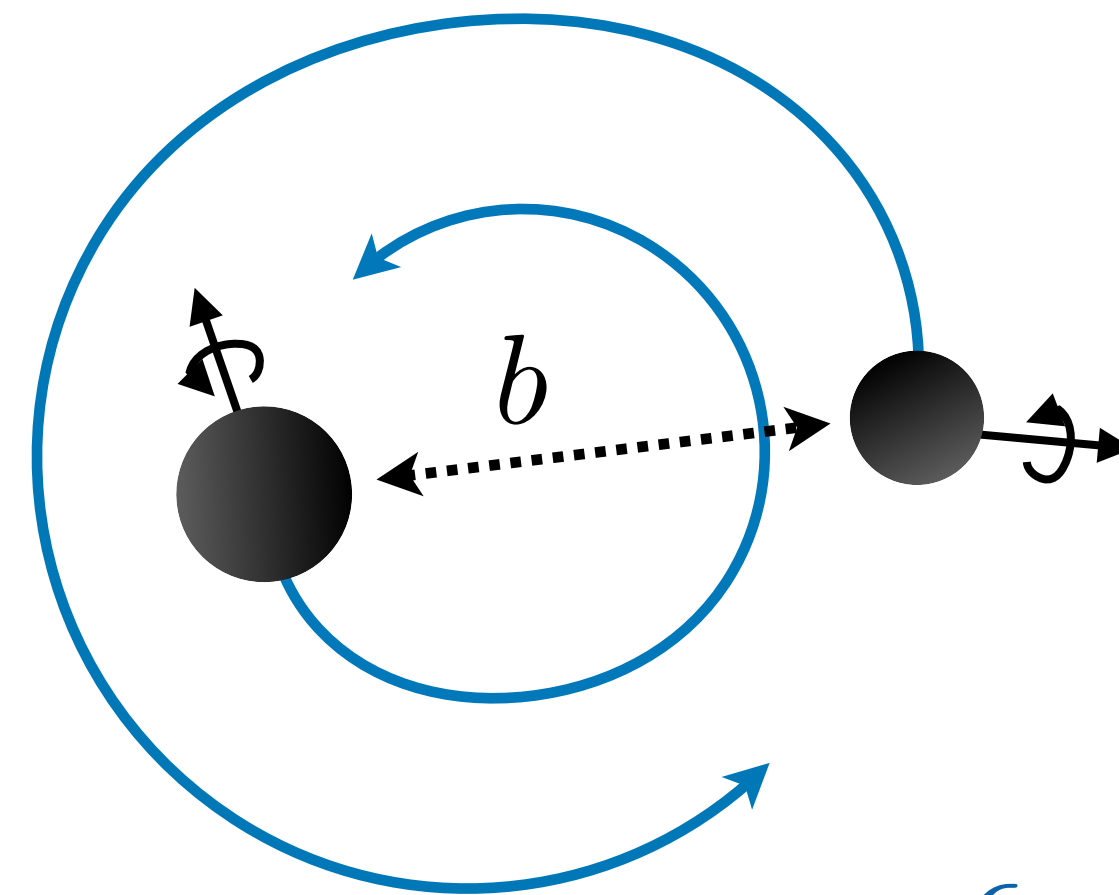
General Relativistic Two-Body Problem

Gravitational Self-Force (GSF)

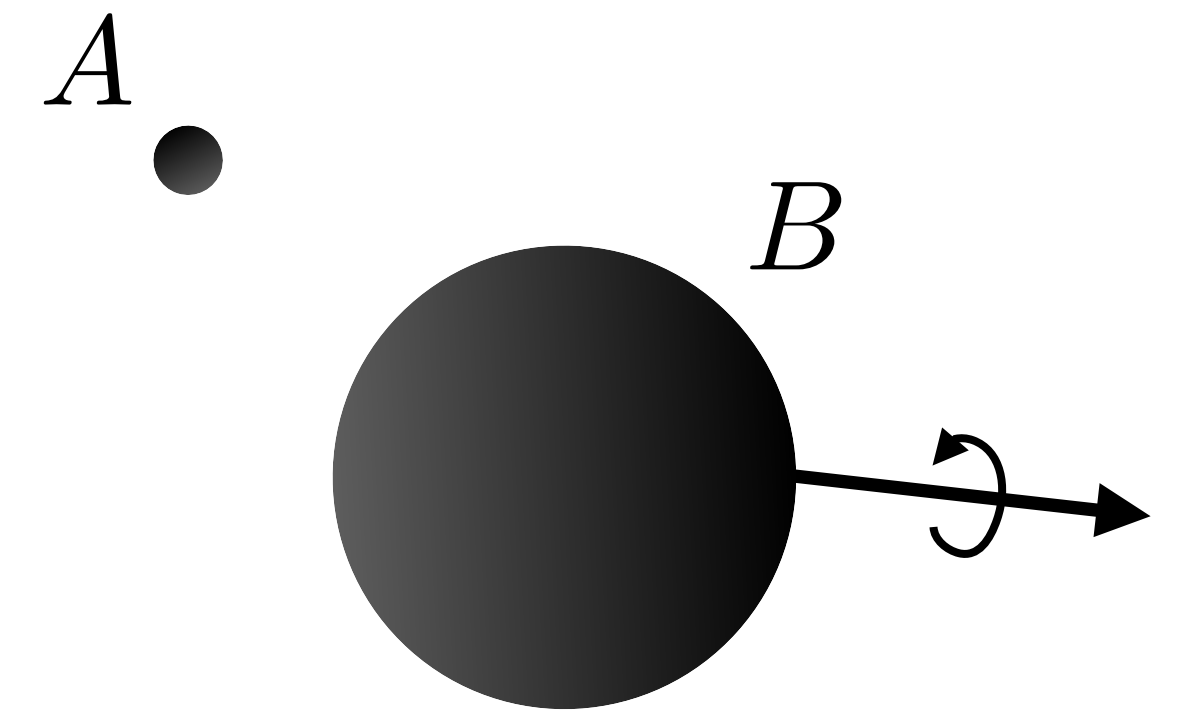
Virial theorem

$$\frac{v^2}{c^2} \sim \frac{r_{\text{Sch}}}{b}$$

Post-Newtonian (PN)

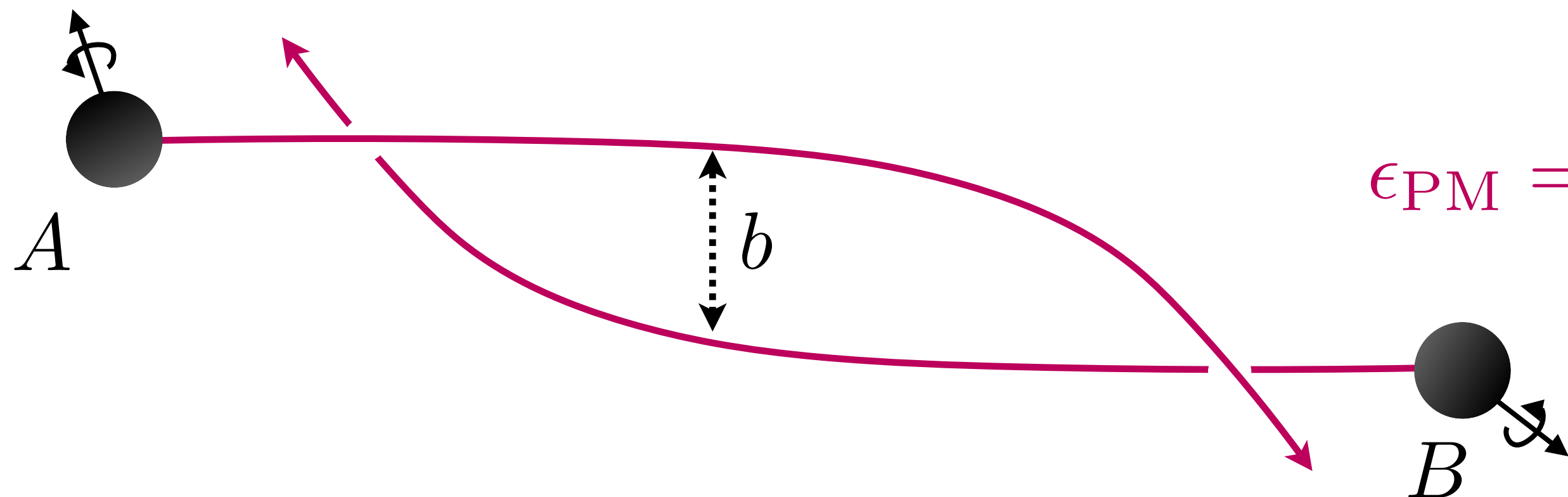


$$\epsilon_{\text{PN}} = \frac{v^2}{c^2}$$



$$\epsilon_{\text{SF}} = \frac{m_A}{m_B}$$

Post-Minkowskian (PM)



$$\epsilon_{\text{PM}} = \frac{2GM}{bv^2} = \frac{c^2 r_{\text{Sch}}}{v^2 b}$$

Phase space:

$$\theta = \theta(\epsilon_{\text{PM}}, \epsilon_{\text{PN}}, \epsilon_{\text{SF}})$$

(+ spin)

PM resummations of PN terms

	0PN	1PN	2PN	3PN	4PN		PM-resummed
$V(\mathbf{r}, \mathbf{p})$	$= -\frac{m_A m_B G}{ \mathbf{r} }$	$\left[1 + c_{01} \mathbf{p}^2 + c_{02} \mathbf{p}^4 + c_{03} \mathbf{p}^6 + c_{04} \mathbf{p}^8 + \dots \right]$				$\rightarrow$	$-\frac{m_A m_B G}{ \mathbf{r} } f_1(\mathbf{p}^2)$
	$+\frac{m_A m_B M G^2}{ \mathbf{r} ^2 c^2}$	$\left[c_{10} + c_{11} \mathbf{p}^2 + c_{12} \mathbf{p}^4 + c_{13} \mathbf{p}^6 + \dots \right]$				$\rightarrow$	$\frac{m_A m_B M G^2}{ \mathbf{r} ^2 c^2} f_2(\mathbf{p}^2)$
	$+\frac{m_A m_B M^2 G^3}{ \mathbf{r} ^3 c^4}$	$\left[c_{20} + c_{21} \mathbf{p}^2 + c_{22} \mathbf{p}^4 + \dots \right]$				$\rightarrow$	$\frac{m_A m_B M^2 G^3}{ \mathbf{r} ^3 c^4} f_3(\mathbf{p}^2)$
	$+\frac{m_A m_B M^3 G^4}{ \mathbf{r} ^4 c^6}$	$\left[c_{30} + c_{31} \mathbf{p}^2 + \dots \right]$				$\rightarrow$	$\frac{m_A m_B M^3 G^4}{ \mathbf{r} ^4 c^6} f_4(\mathbf{p}^2)$
		$+ \dots$					

Outline

I) Worldline Quantum Field Theory

II) Gravitational Wave Scattering at three loops

[Y.F. Bautista, M. Driesse, K. Haddad, GUJ: 2602.06125, 2606.TODAY]

III) Impulse at four loops and second self-force order

[M. Driesse, GUJ, G. Mogull, C. Nega, J. Plefka, B. Sauer, J. Usovitsch: 2601.16256]

Worldline Effective Field Theory in classical general relativity

- Black holes (BH) and neutrons stars (NS) as point-like particles
- Finite size effects as Wilsonian effective couplings
- Established approach in post-Newtonian and post-Minkowskian regimes

[W. Goldberger, I. Rothstein: 0409156], [M. Levi, J. Steinhoff: 1506.05056], [R. Porto: 1601.04914], [M. Saketh, J. Steinhoff, J. Vines, A. Buonanno: 2212.13095] ...
“NRGR”

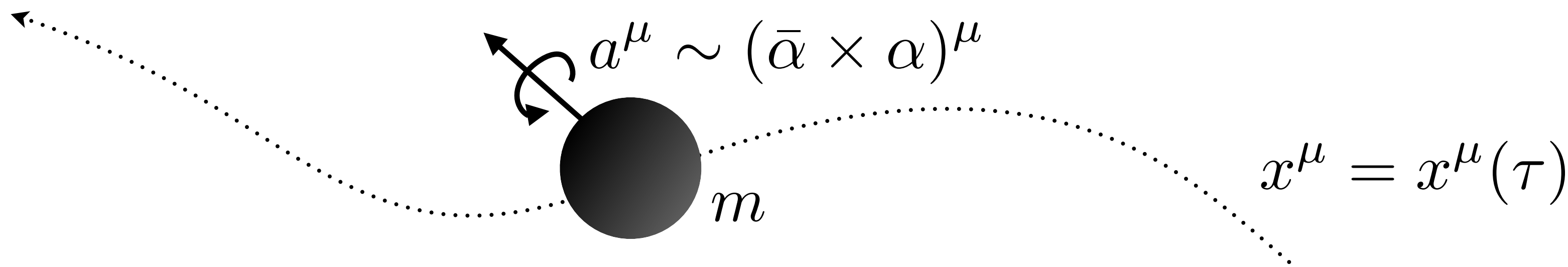
“Minimal”

“Non-minimal”

$$S_{\text{BH/NS}} = -m \int d\tau \left(\frac{1}{2} \dot{x}^\mu \dot{x}_\mu + i \bar{\alpha}_\mu \frac{D\alpha^\mu}{d\tau} + \lambda_1 a^\mu a^\alpha \dot{x}^\nu \dot{x}^\beta R_{\mu\nu\alpha\beta} + \lambda_2 (mG)^4 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \dots \right)$$

Spin-induced quadrupole moment

Tidal deformability



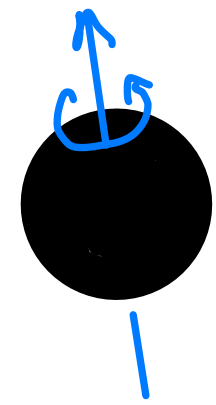
[K. Haddad, GUJ, G. Mogull, J. Plefka: 2411.08176]

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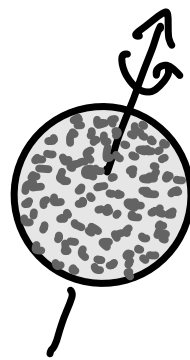
Black Hole



mass, spin

$\lambda_1 = 1, \lambda_2 = 0$

Neutron Star



mass, spin, radius,
tidal deformability

$\lambda_1, \lambda_2 = ??$

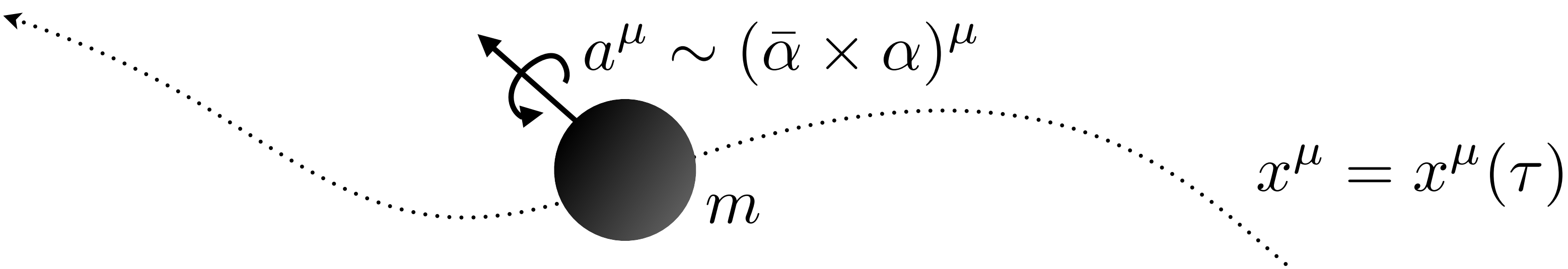
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Spin-induced quadrupole moment

Tidal deformability



[K. Haddad, GUJ, G. Mogull, J. Plefka: 2411.08176]

Worldline Quantum Field Theory (WQFT)

[G. Mogull, J. Plefka, J. Steinhoff: 2010.02865]
[GUJ, G. Mogull, J. Plefka, B. Sauer: 2207.00569]

Ignore spins and finite size effects:

$$S = - \sum_i \frac{1}{2} m_i \int d\tau \dot{x}_i^\mu(\tau) \dot{x}_i^\nu(\tau) g_{\mu\nu}(x_i(\tau)) - \frac{2}{\kappa^2} \int d^{4-2\epsilon} x \sqrt{-g} R$$

$$\kappa^2 = 32\pi G$$

Post-Minkowskian scattering regime:

$$x_i^\mu(\tau) = b_i^\mu + \tau v_i^\mu + z_i^\mu(\tau)$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)$$

Solve classical EOMs:

$$\frac{\delta S}{\delta g_{\mu\nu}} = \frac{\delta S}{\delta x_i^\mu} = 0$$

Set up perturbative solution using known QFT techniques!

Bonus: only tree-level diagrams required!

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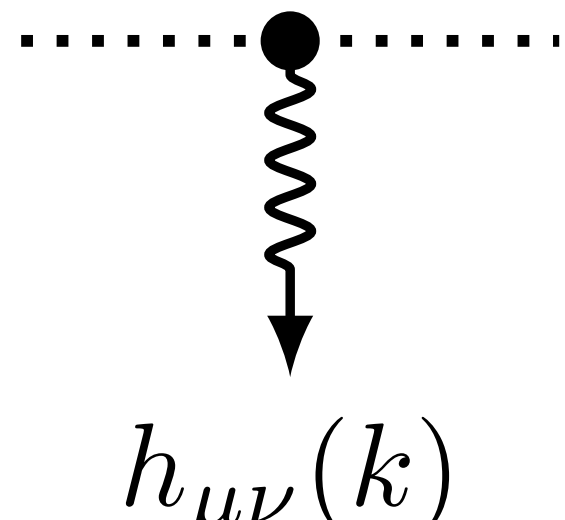
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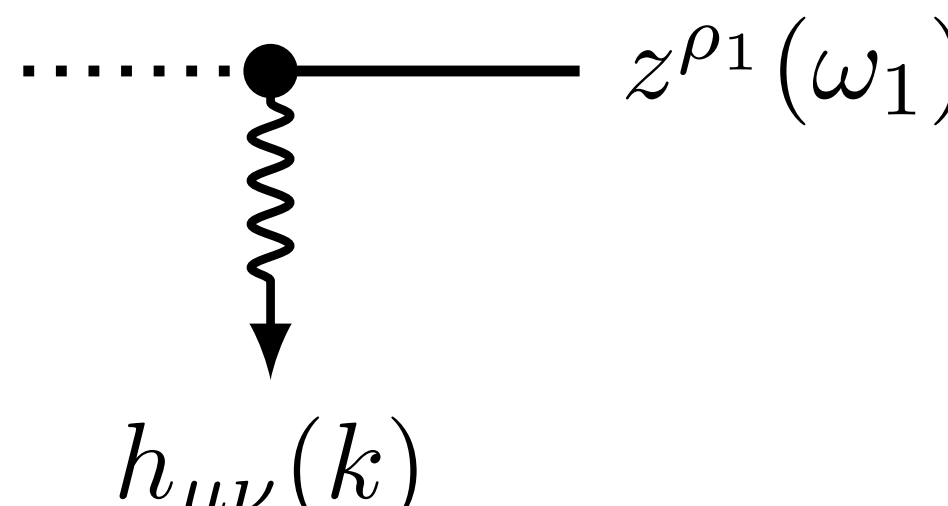
Bonus: only tree-level diagrams required!

Worldline QFT

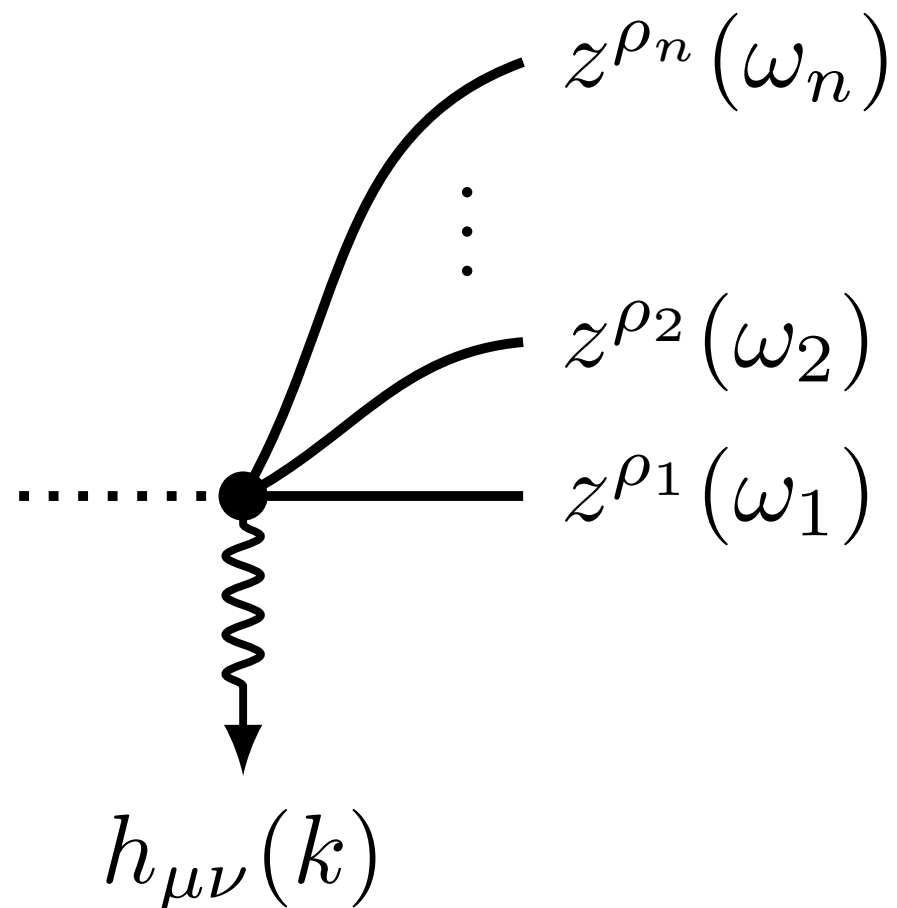
- Worldlines interact with gravitons
- Background parameters:
 $\{v^\mu, b^\mu\}$
- Impact parameters break translational invariance



$$= -i \frac{m\kappa}{2} e^{ik \cdot b} \delta(k \cdot v) v^\mu v^\nu$$

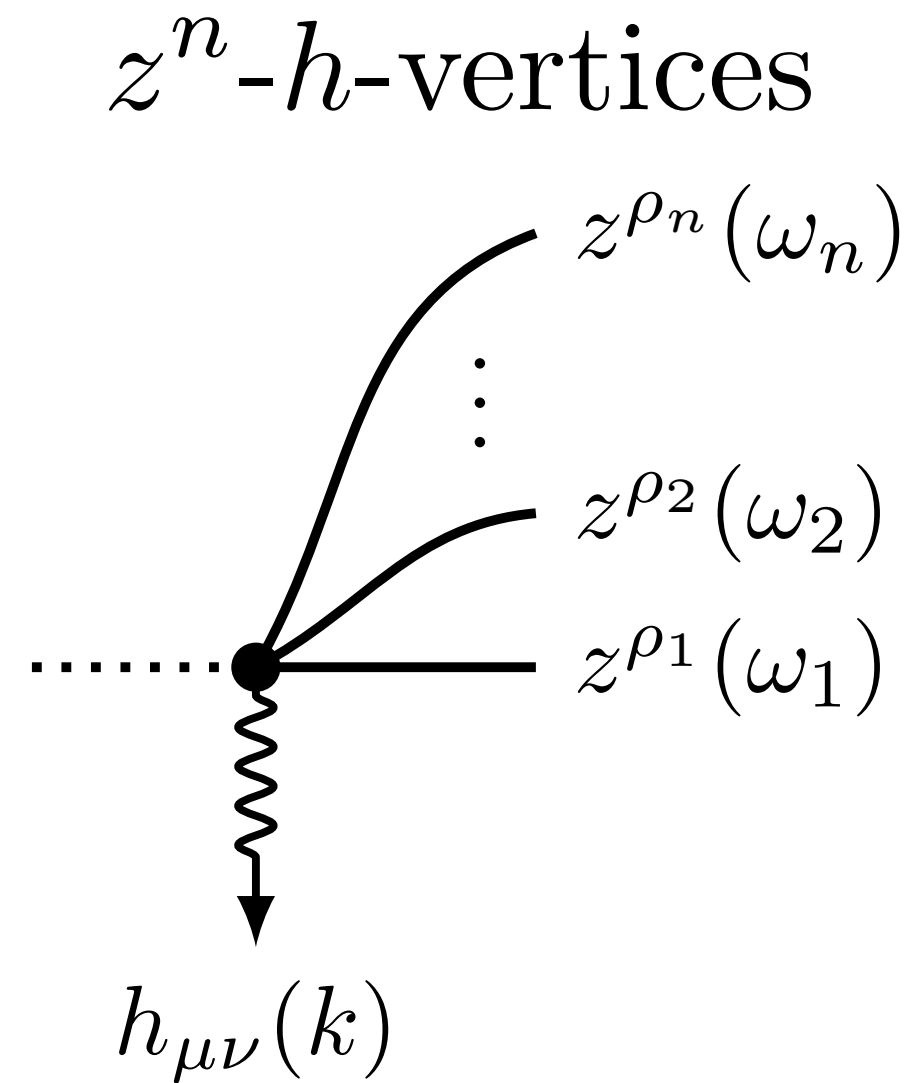


$$= \frac{m\kappa}{2} e^{ik \cdot b} \delta(k \cdot v + \omega) \left(2\omega v^{(\mu} \delta_{\rho}^{\nu)} + v^\mu v^\nu k_\rho \right)$$

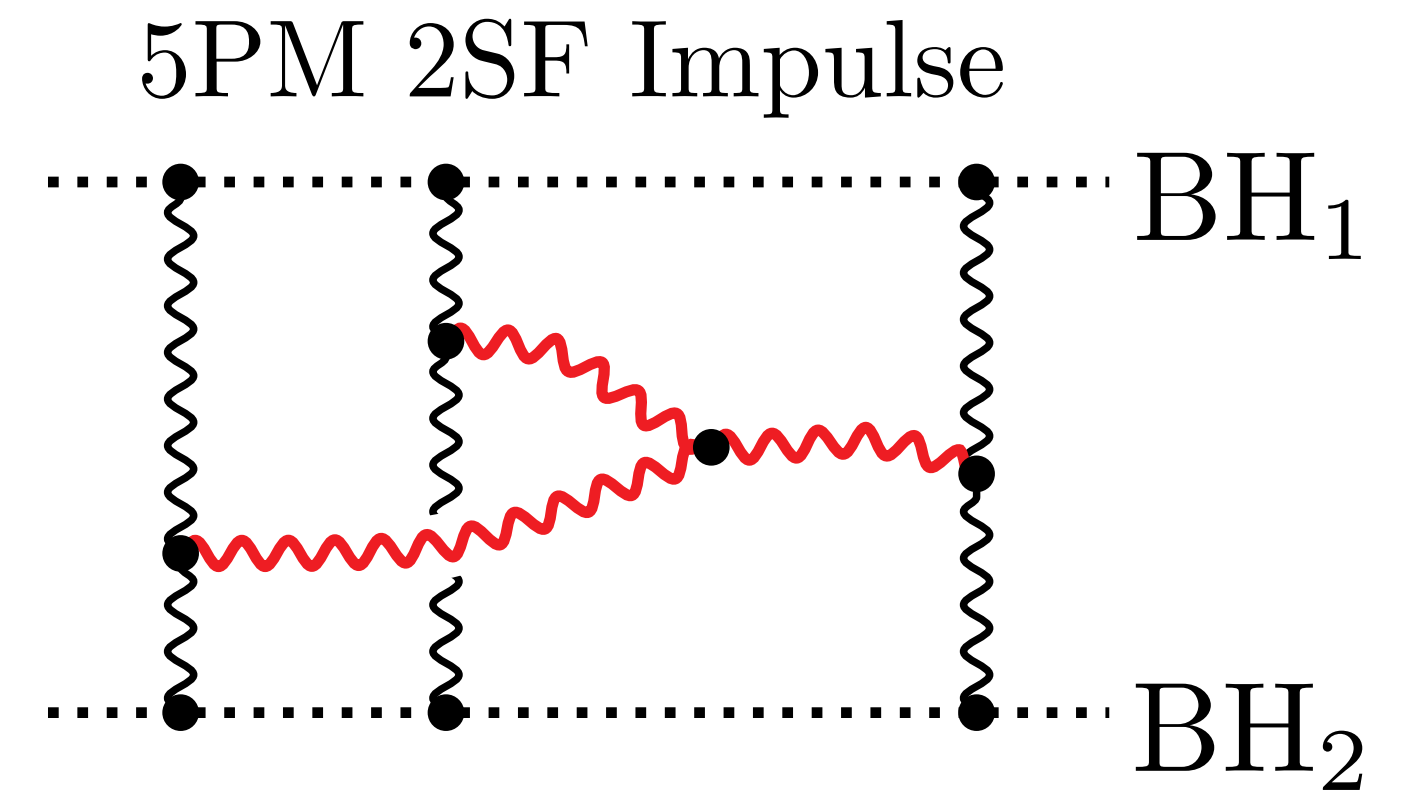
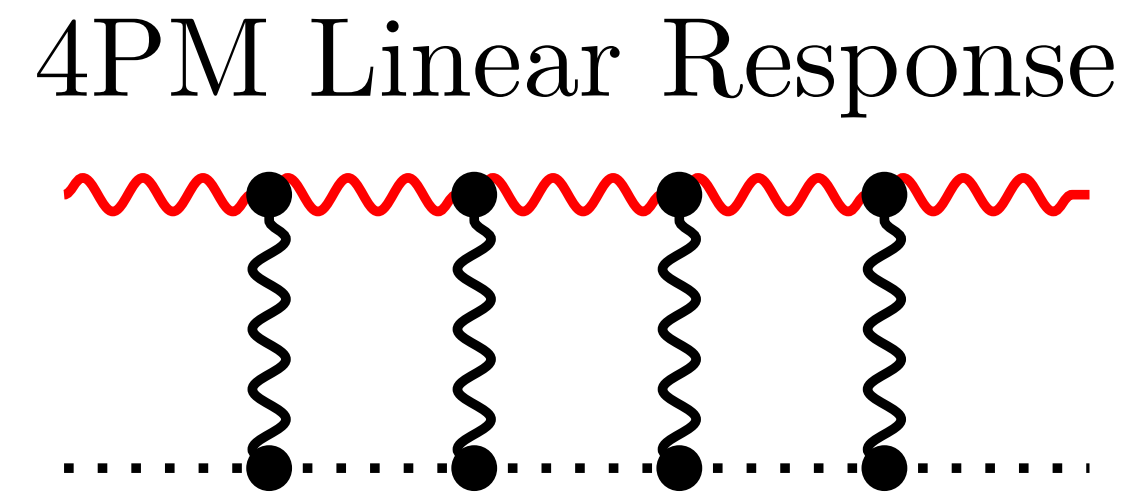


$$= m\kappa e^{-ik \cdot b} \delta(\omega + v \cdot \sum_{i=1}^n k_i) \left[\frac{1}{2} v^\mu v^\nu k_{\sigma_i} k_{\sigma_j} + \sum_{i=1}^n \omega_i \delta_{\sigma_i}^{(\mu} v^{\nu)} k_{\sigma_j} + \sum_{i < j}^n \omega_i \omega_j \delta_{\sigma_i}^{(\mu} \delta_{\sigma_j}^{\nu)} \right] \prod_{l \neq i, j}^n k_{\sigma_l}$$

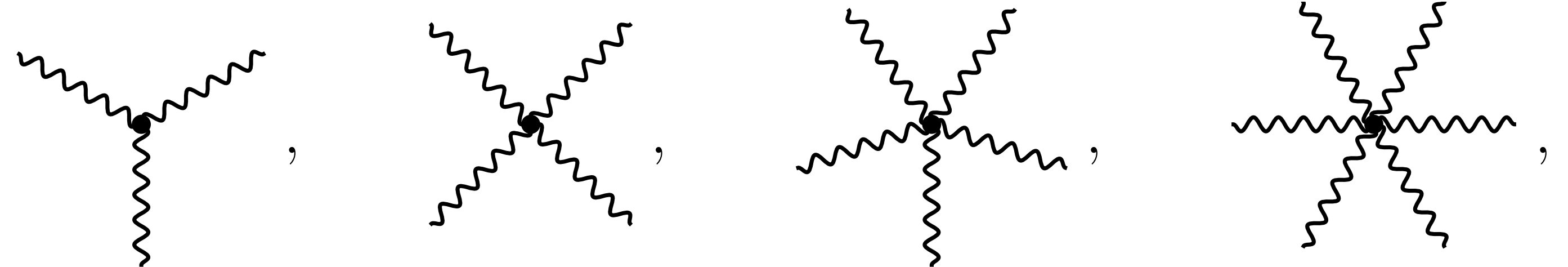
Worldline QFT



$(33)^4$ terms = 1.185.921 terms



h^n -vertices



Number of terms: 33 366 4630 51360

In an optimized generalized gauge:

$$\begin{aligned}
 G^\mu = & \partial_{\nu_1} h^{\mu\nu_1} - \frac{1}{2} \partial^\mu h + \frac{1}{2} \partial_{\nu_1} h h^{\mu\nu_1} + \frac{1}{2} \partial^\mu h_{\nu_1\nu_2} h^{\nu_1\nu_2} - \partial^{\nu_1} h^{\mu\nu_2} h_{\nu_1\nu_2} - \frac{1}{4} \partial^\mu h h \\
 & + \partial_{\nu_1} h^{\nu_3\mu} h^{\nu_1\nu_2} h_{\nu_2\nu_3} + \frac{1}{2} \partial_{\nu_1} h^{\nu_1\nu_2} h_{\nu_2\nu_3} h^{\nu_3\mu} - \frac{1}{4} \partial_{\nu_2} h h_{\nu_2\nu_3} h^{\nu_3\mu} + \partial_{\nu_1} h_{\nu_2\nu_3} h^{\nu_1\nu_2} h^{\nu_3\mu} \\
 & - \frac{1}{2} \partial^{\nu_2} h_{\nu_1\nu_2} h^{\mu\nu_1} h + \frac{1}{4} \partial^\mu h_{\nu_1\nu_2} h^{\nu_1\nu_2} h - \frac{1}{2} \partial^{\nu_1} h^{\mu\nu_2} h_{\nu_1\nu_2} h + \frac{1}{8} \partial_{\nu_1} h^{\mu\nu_1} h^2 .
 \end{aligned}$$

Worldline QFT

Classical observables from **on-shell, one-point, tree-topology** functions

Gravitational Waveform: $k^2 \epsilon^\mu \epsilon^\nu \langle h_{\mu\nu}(k) \rangle \Big|_{k^2 \rightarrow 0}$

Black Hole or Neutron Star Impulse: $\omega^2 \langle z^\mu(\omega) \rangle \Big|_{\omega \rightarrow 0}$

Angular Momentum Loss (indirectly): $\frac{d}{d\omega} \left(\omega^2 \langle z^\mu(\omega) \rangle \right) \Big|_{\omega \rightarrow 0}$

Dissipative effects with **retarded** propagators oriented to the single outgoing line

$$\mu\nu \xrightarrow[k]{\text{wavy}} \alpha\beta = \frac{\mathcal{P}_{\mu\nu\alpha\beta}}{(k_0 + i\epsilon)^2 - \mathbf{k}^2}$$

$$\sigma \xrightarrow[\text{solid}]{\omega} \rho = \frac{\eta_{\sigma\rho}}{(\omega + i\epsilon)^2}$$

“in-in field theory”

High precision WQFT

- Draw all diagrams at given order
 - Insert Feynman rules
 - Reduce to sum of scalar integrals with tensor reduction
 - Reduce to sum of master integrals with **integration-by-parts (IBP)**
 - Bootstrap master integrals with **differential equations** (put in **canonical form**)
 - Determine boundary conditions with **method of regions**
- Construct and simplify integrand
- Compute integrals

Gravitational wave scattering @ three loops (4PM) **Part II**

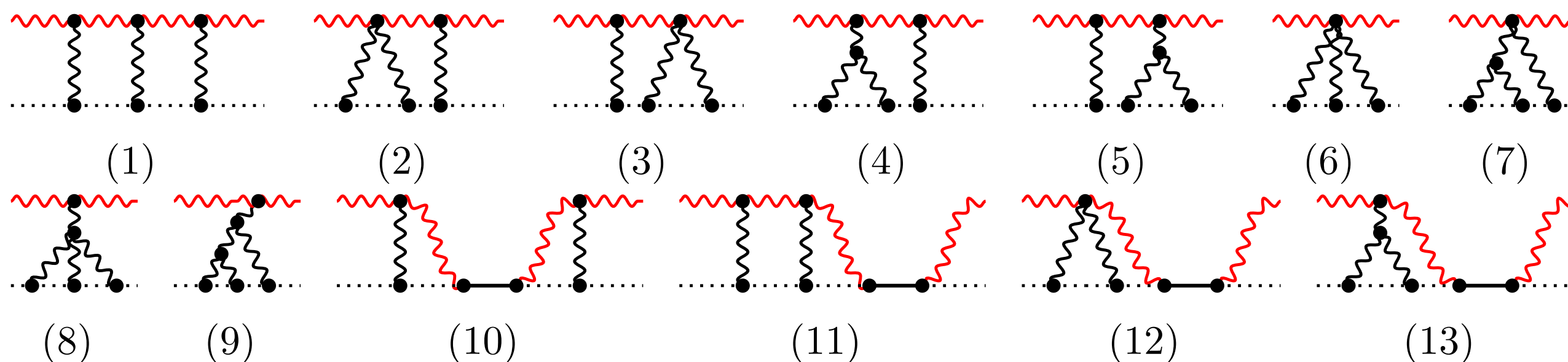
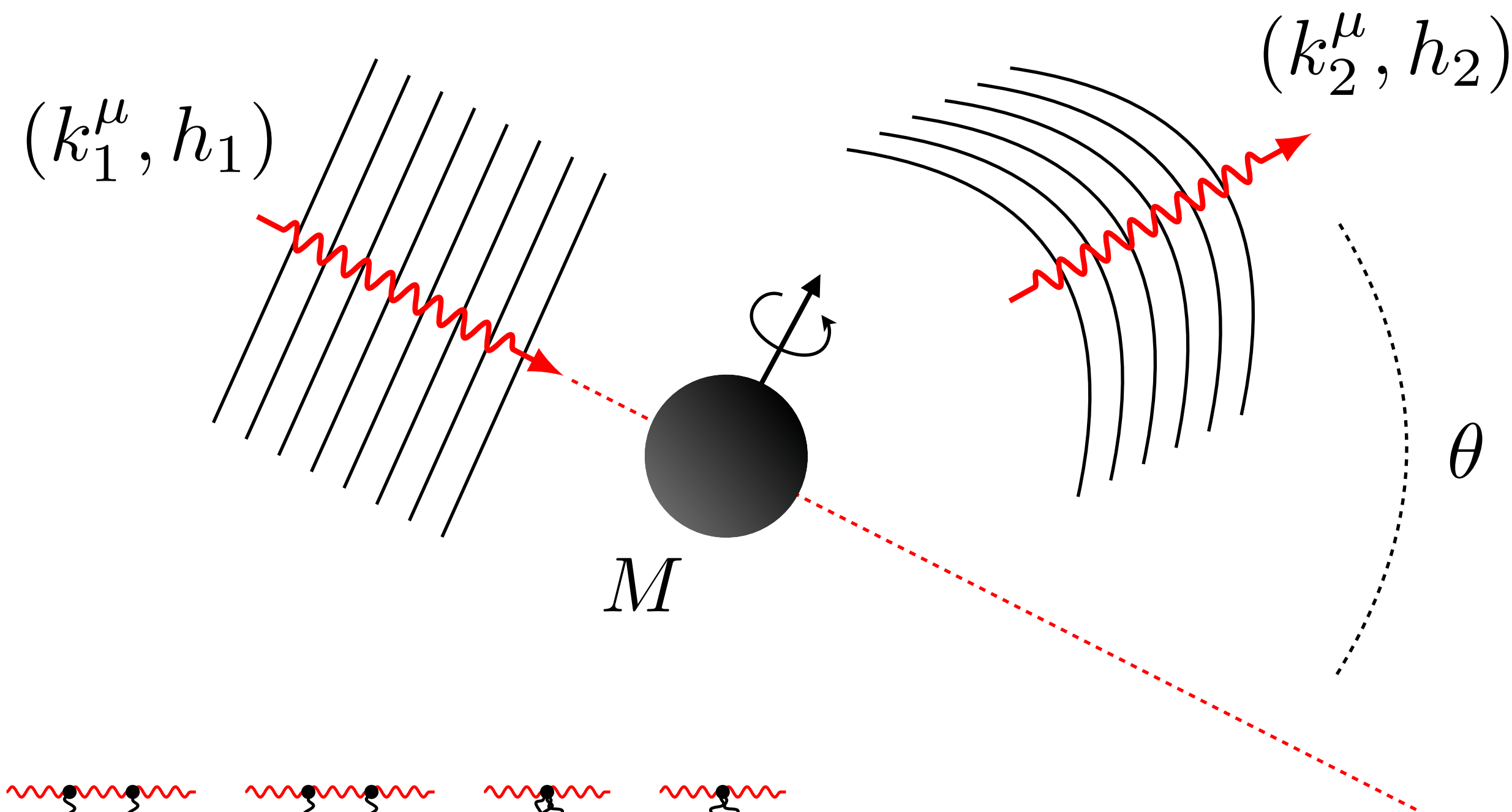
Benchmarks: Impulse @ four loops (5PM, 1SF & conservative 2SF) **Part III**

Angular momentum flux @ two loops (3PM, higher orders to come!)

Part II: Gravitational Wave Scattering

See also Lara's, Giulia's and Gang's talks!

- General Relativity: Black hole and stellar perturbation theory
- EFT: Matching strategy for Wilson couplings



Perturbative regime: $GM\omega \ll 1$

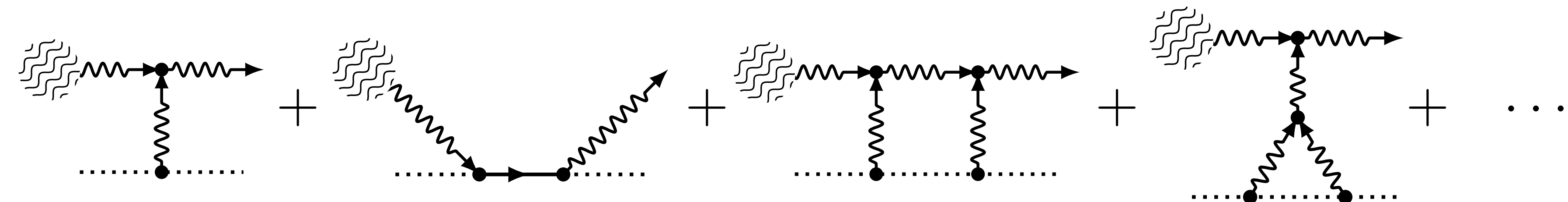
GW Scattering from WQFT

[Y.F. Bautista, M. Driesse, K. Haddad, GUJ: 2602.06125, 2606.TODAY]

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}^{(\text{inc})}(x) + \kappa h_{\mu\nu}(x)$$

$$= \eta_{\mu\nu} + \text{[wavy line with arrow]} + \kappa h_{\mu\nu}(x)$$

Linear response

$$k^2 \epsilon^\mu \epsilon^\nu \langle h_{\mu\nu}(k) \rangle \Big|_{k^2 \rightarrow 0} =$$


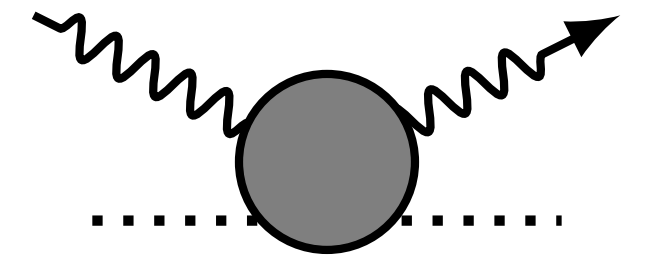
Zero classical loops $\sim G$ One classical loop $\sim G^2$

$$+ \text{[diagram: two wavy lines with arrows pointing to a single vertex, which then connects via a wavy line with an arrow to a horizontal dotted line]} + \dots$$

$$+ \dots$$

Non-linear response

Computation of the GW scattering amplitude

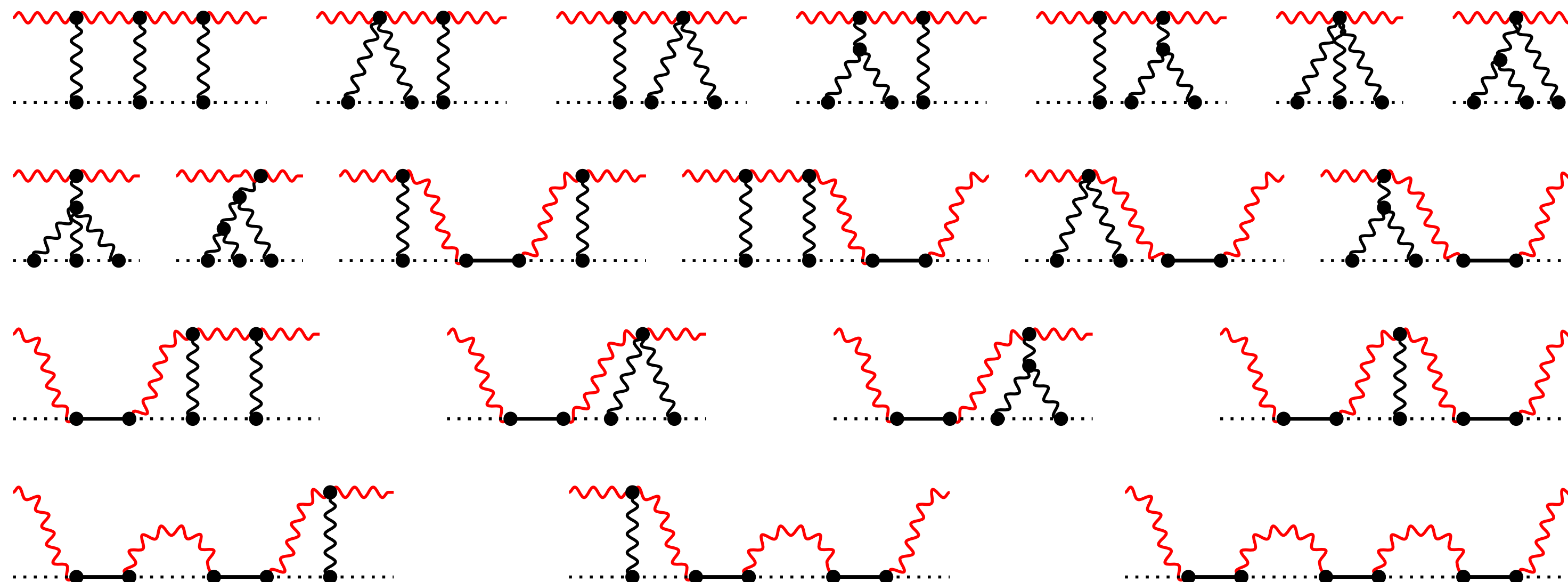


[N.E.J. Bjerrum-Bohr, G. Chen, C.J. Eriksen, N. Shah: 2506.19705]

With spin: [M.V.S. Saketh, J. Vines: 2208.03170]

With spin: [D. Akpinar: 2511.10280]

(2 diagrams @ tree) + (6 diagrams @ one loop) +



(20 diagrams)

[M.M. Ivanov, Y.-Z. Li, J. Parra-Martinez, Z. Zhou: 2602.06951]

[Y.F. Bautista, M. Driesse, K. Haddad, GUJ: 2602.06125]

[N.E.J. Bjerrum-Bohr, G. Chen, C.J. Eriksen, N. Shah: 2602.06947]

+ (70 diagrams @ three loops) + ...

Alternative approaches:

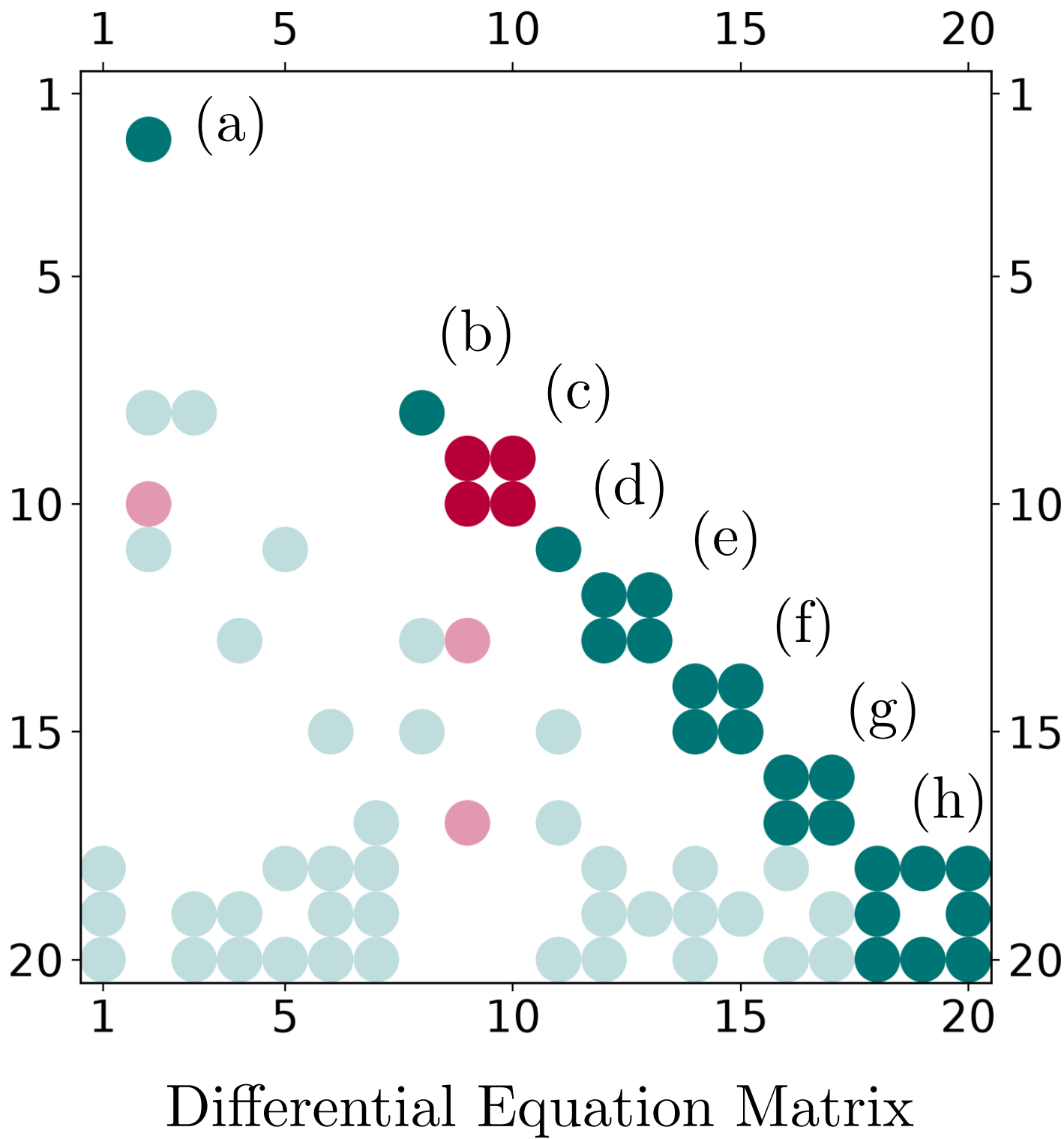
Shell-EFT: [D. Kosmopoulos, D. Perrone, M. Solon: 2512.04002]

Born series: [S. Caron-Huot, M. Correia, G. Isabella, M. Solon: 2503.13593]

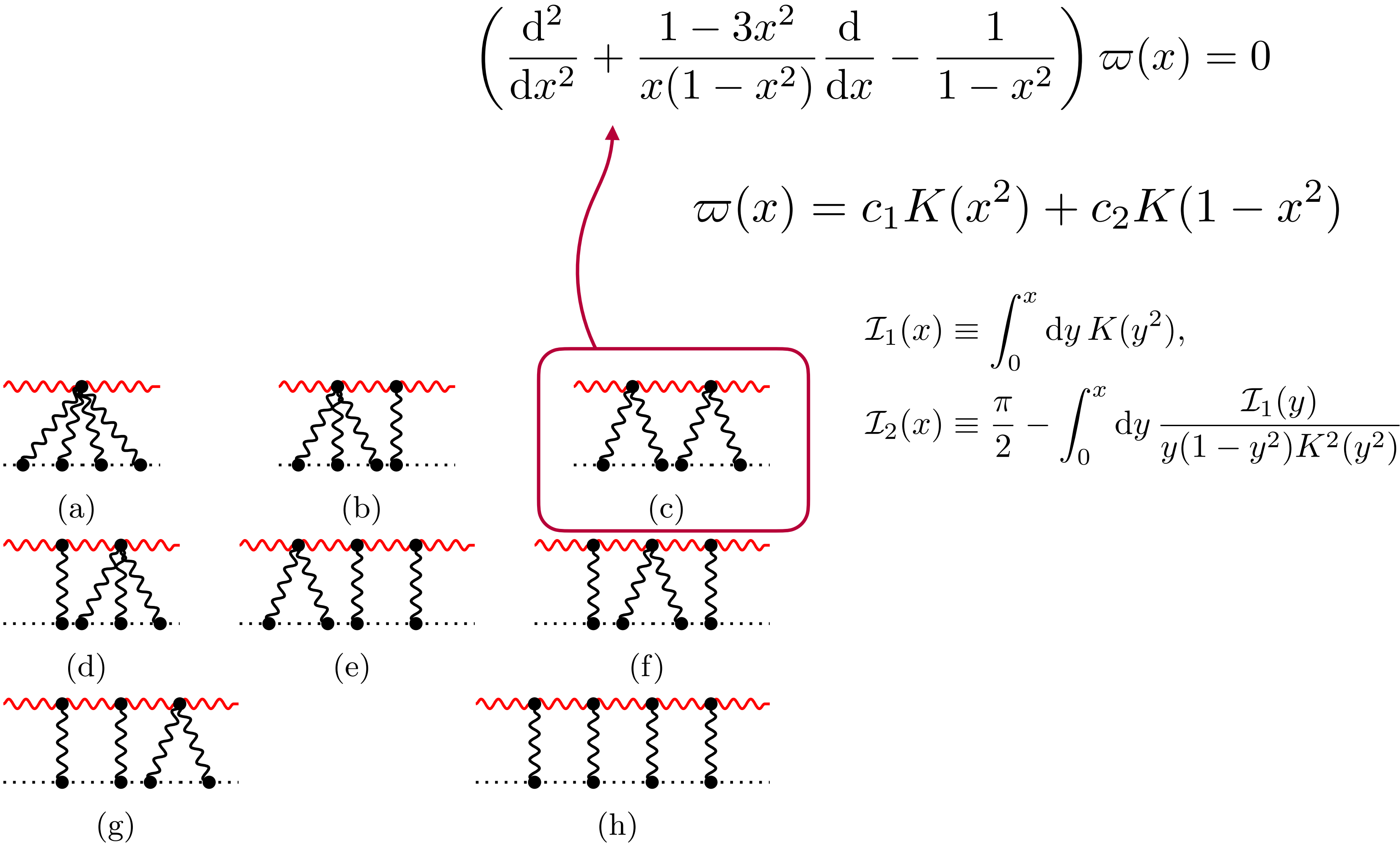
[Y.F. Bautista, M. Driesse, K. Haddad, GUJ: 2606.TODAY]

[M. Meo, S. Smith, G. Brunello: 2606.TODAY]

Complete Elliptic Integrals at $\mathcal{O}(G^4)$



Loops	1	2	3
Master integrals	3	6	20



Magnusian matrix elements

[P.H. Damgaard, L. Plante, P. Vanhove: 2107.12891]

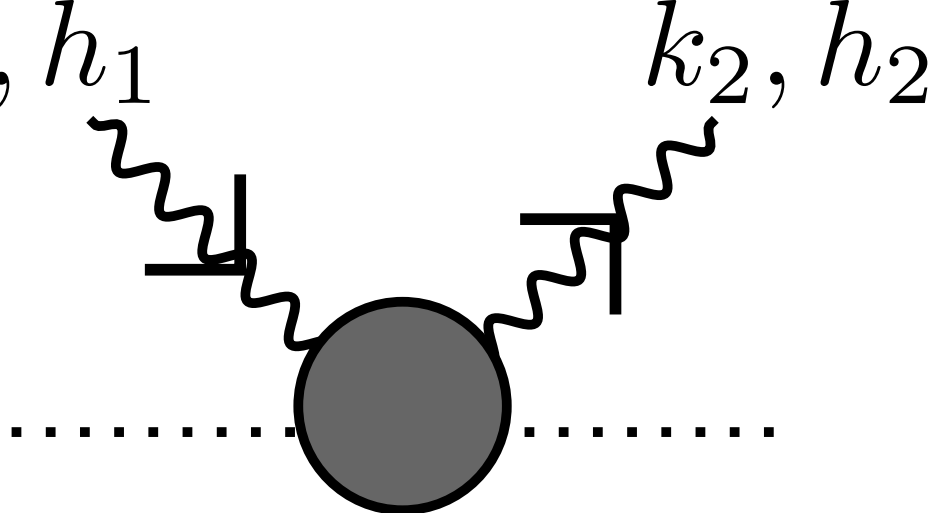
[J.-H. Kim, J.-W. Kim, S. Kim, S. Lee: 2410.22988]

[J.-W. Kim, R. Patil, T. Scheopner, J. Steinhoff: 2511.05649]

[A. Brandhuber, G. R. Brown, P. Pichini, G. Travaglini, P. V. Matasian: 2512.05017]

[R. Gonzo, G. Mogull: 2603.05237]

$$\hat{S} = 1 + i\hat{T} = e^{i\hat{N}}$$

$$\langle k_2, h_2 | \hat{T} | k_1, h_1 \rangle = \text{Diagram} = \delta(\omega_1 - \omega_2) i \mathcal{M}_\sigma(x)$$


“Conventional scattering amplitude”

Complex and infrared divergent

$$\langle k_2, h_2 | \hat{N} | k_1, h_1 \rangle = \delta(\omega_1 - \omega_2) i N_\sigma(x)$$

Magnusian (N-matrix, Magnus operator)

Real and infrared finite* to $\mathcal{O}(G^4)$

$$x = \sin \frac{\theta}{2}$$

$$\sigma = \frac{h_2}{h_1}$$

Magnusian matrix elements

[P.H. Damgaard, L. Plante, P. Vanhove: 2107.12891]

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“Conventional scattering amplitude”

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$$\langle k_2, h_2 | \hat{N} | k_1, h_1 \rangle = \delta(\omega_1 - \omega_2) i N_\sigma(x)$$

Maps directly onto the black hole perturbation theory phase shift!

$$x = \sin \frac{\theta}{2}$$

$$\sigma = \frac{h_2}{h_1}$$

[Y.F. Bautista, M. Driesse, K. Haddad, GUJ: 2602.06125]

See also Graham’s talk!

Magnusian from Murua

[J.-H. Kim, J.-W. Kim, S. Kim, S. Lee: 2410.22988]

[A. Brandhuber, G. R. Brown, P. Pichini, G. Travaglini, P. V. Matas: 2512.05017]

[R. Gonzo, G. Mogull: 2603.05237]

$$\text{red wavy line with } \rangle = \frac{1}{\omega^2 - \mathbf{k}^2 + i0^+ \omega} \quad \text{Retarded (Feynman)}$$

$$\text{red wavy line with } \langle = \frac{1}{\omega^2 - \mathbf{k}^2 - i0^+ \omega} \quad \text{Advanced (Dyson)}$$

T-matrix (response function)

$$T \left(\text{red wavy line} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \right) = \text{red wavy line} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array}$$

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N-matrix (Magnusian)

$$M \left(\text{red wavy line} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \right) = \frac{1}{2} \text{red wavy line} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} + \frac{1}{2} \text{red wavy line} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} + c.c.$$

$$M \left(\text{red wavy line} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \right) = \frac{1}{3} \text{red wavy line} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} + \frac{1}{6} \text{red wavy line} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} + \frac{1}{6} \text{red wavy line} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} + \frac{1}{3} \text{red wavy line} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{wavy} \\ \bullet \end{array} + c.c.$$

May be applied at the level of master integrals!

At n loop order with k retarded propagators: $\left[(n+1) \binom{n}{k} \right]^{-1} = \text{“Murua coefficients”}$

Magnusian from Murua

[J.-H. Kim, J.-W. Kim, S. Kim, S. Lee: 2410.22988]

[A. Brandhuber, G. R. Brown, P. Pichini, G. Travaglini, P. V. Matasari: 2512.05017]

[R. Gonzo, G. Mogull: 2603.05237]

$$\text{wavy line with } \rangle = \frac{1}{\omega^2 - \mathbf{k}^2 + i0^+\omega} \quad \text{Retarded (Feynman)}$$

$$\text{wavy line with } \langle = \frac{1}{\omega^2 - \mathbf{k}^2 - i0^+\omega} \quad \text{Advanced (Dyson)}$$

N-matrix (Magnusian)

$$M \left(\begin{array}{c} \text{wavy line} \\ \vdots \end{array} \right) = \frac{1}{2} \begin{array}{c} \text{wavy line} \rangle \\ \vdots \end{array} + \frac{1}{2} \begin{array}{c} \text{wavy line} \langle \\ \vdots \end{array} + c.c.$$

$$M \left(\begin{array}{c} \text{wavy line} \\ \vdots \end{array} \right) = \frac{1}{3} \begin{array}{c} \text{wavy line} \rangle \rangle \\ \vdots \end{array} + \frac{1}{6} \begin{array}{c} \text{wavy line} \langle \rangle \rangle \\ \vdots \end{array} + \frac{1}{6} \begin{array}{c} \text{wavy line} \rangle \langle \rangle \\ \vdots \end{array} + \frac{1}{3} \begin{array}{c} \text{wavy line} \langle \langle \rangle \\ \vdots \end{array}$$

T-matrix (response function)

$$T \left(\begin{array}{c} \text{wavy line} \\ \vdots \end{array} \right) = \begin{array}{c} \text{wavy line} \rangle \\ \vdots \end{array}$$

$$T \left(\begin{array}{c} \text{wavy line} \\ \vdots \end{array} \right) = \begin{array}{c} \text{wavy line} \rangle \rangle \\ \vdots \end{array}$$

$$T \left(\begin{array}{c} \text{wavy line} \\ \vdots \end{array} \right) = i\mathcal{M}_\sigma(x)$$

$$M \left(\begin{array}{c} \text{wavy line} \\ \vdots \end{array} \right) = iN_\sigma(x)$$

At n loop order with k retarded propagators: $\left[(n+1) \binom{n}{k} \right]^{-1} = \text{“Murua coefficients”}$

GW Scattering Magnusian at $\mathcal{O}(G^4)$

[Y.F. Bautista, M. Driesse, K. Haddad, GUJ: 2606.TODAY]

[M. Meo, S. Smith, G. Brunello: 2606.TODAY]

$$N_{-1}^{(4)}(x) = 0$$

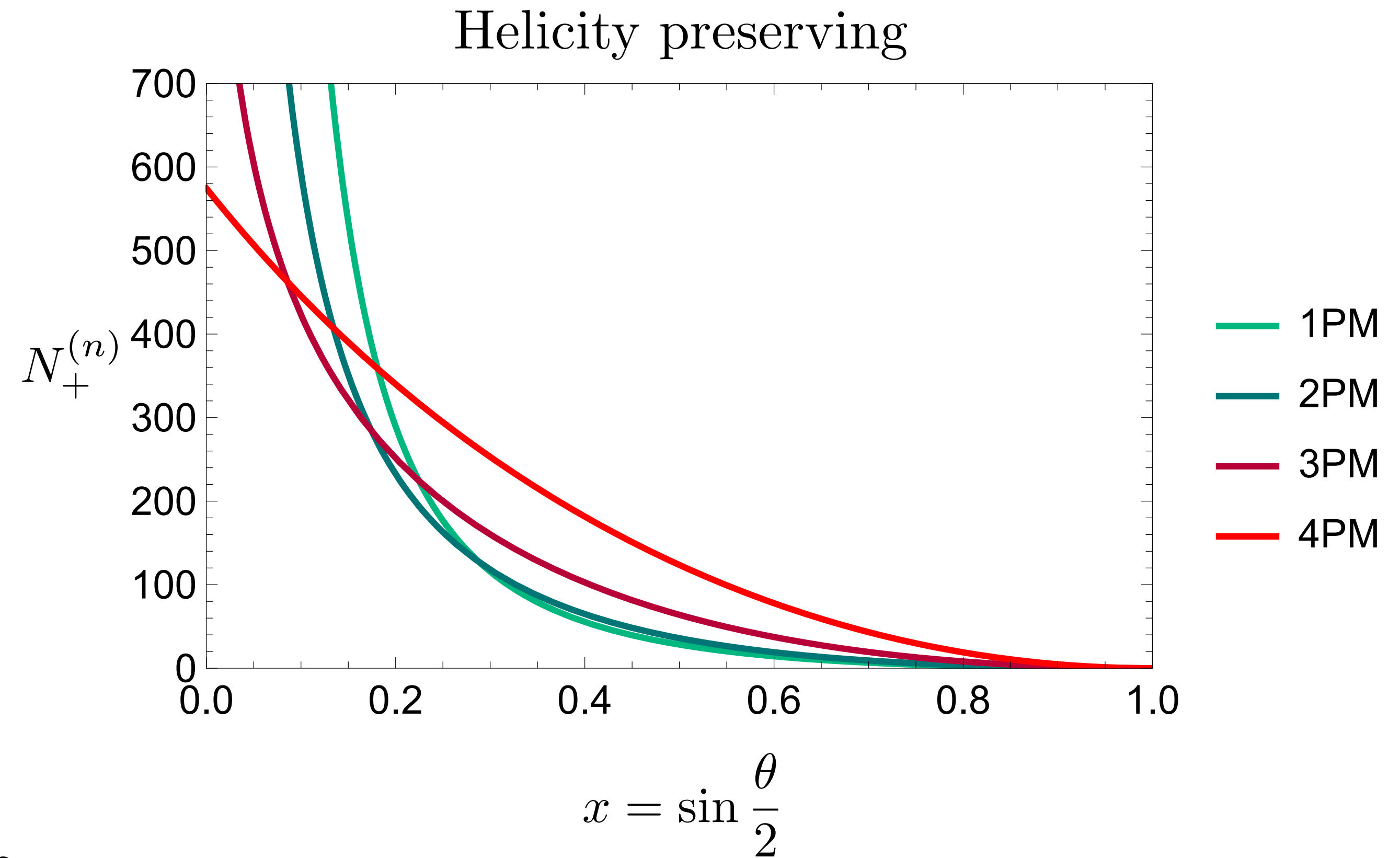
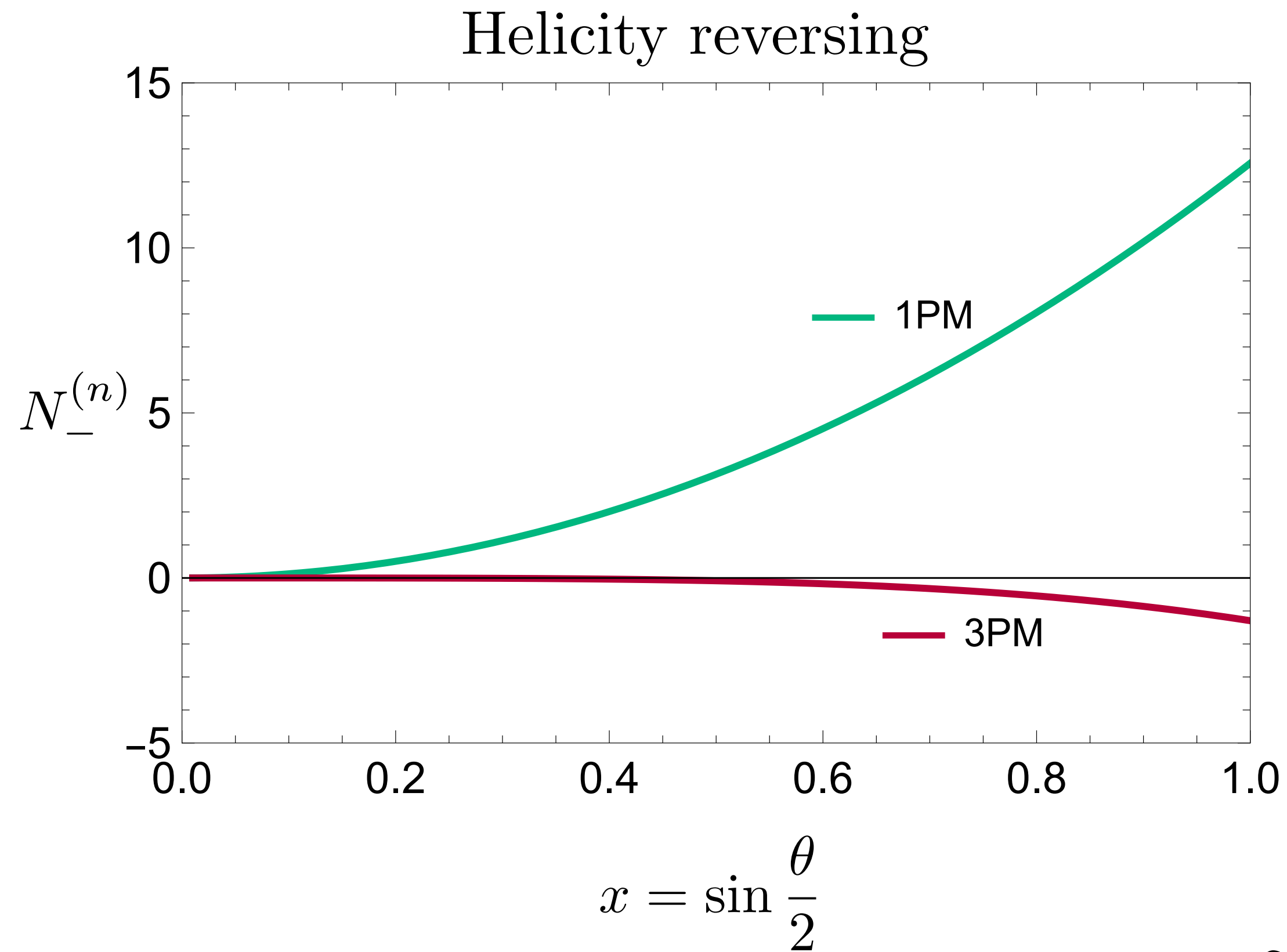
$$\begin{aligned} N_1^{(4)}(x) = & \frac{1575x^6 + 47655x^5 + 827547x^4 + 366747x^3 + 3142262x^2 + 166262x + 722432}{23040\pi^2(x+1)^2(x-1)} \\ & + \frac{4(7+11x^2)}{\pi^2(1-x^2)} G(-1; x) - \left[1 - \frac{12}{\pi^2} G(0, -1; x) \right] \frac{4}{3} \frac{1+6x^2+2x^4}{(x^2-1)^2} \\ & - \frac{17805x^4 + 113738x^2 + 44273}{768\pi^2(x^2-1)^2} \left[\frac{\mathcal{I}_1(x)}{K(x^2)} - E(x^2)\mathcal{I}_2(x) \right] + \frac{10125x^4 + 107121x^2 + 58570}{1536\pi^2(x^2-1)} K(x^2)\mathcal{I}_2(x), \end{aligned}$$

Iterated integrals:

$$\mathcal{I}_1(x) \equiv \int_0^x dy K(y^2),$$

$$\mathcal{I}_2(x) \equiv \frac{\pi}{2} - \int_0^x dy \frac{\mathcal{I}_1(y)}{y(1-y^2)K^2(y^2)} = \frac{\pi}{2} + \frac{2}{\pi} \mathcal{I}_1(x) \frac{K(1-x^2)}{K(x^2)} - \frac{2}{\pi} \int_0^x dy K(1-y^2)$$

GW Scattering Magnusian at $\mathcal{O}(G^4)$



$$N_\sigma = \sum_{n=1}^{\infty} (Gm\omega)^n N_\sigma^{(n)}$$

GW Scattering at $\mathcal{O}(G^4)$

(Numerical) Match with black hole perturbation theory in partial wave space!

$$N_{\ell;2,2}^{(4)} = \omega \int d\Omega_{2,2} Y_{\ell,-2}(\theta, \phi) N_{2,2}^{(4)}(\omega; \theta, \phi) \quad \text{for } \ell \in [2, 20] \quad \text{with} \quad \frac{|N - \delta|}{|\delta|} < 10^{-34}$$

$${}_{-2}\delta_{\ell 2}^{P,(4)} = \frac{\pi \epsilon_{\text{PM}}^4}{16} \frac{18480L^8 - 61320L^7 + 2415L^6 - 85775L^5 - 123233L^4 - 51522L^3 + 953424L^2 - 102816L - 51840}{(2\ell + 1)^3(L - 2)(4L - 15)(4L - 3)^3L^3}$$

$$L \equiv \ell(\ell + 1)$$

Nekrasov: [Y.F. Bautista, G. Bonelli, C. Iossa, A. Tanzini, Z. Zhou: 2312.05965]

MST: [S. Mano, H. Suzuki, E. Takasugi: 9605057]

T-matrix amplitude satisfies Weinberg's theorem:

$$i\mathcal{M}_\sigma = \exp(-2iGm\omega/\varepsilon) i\mathcal{M}_{\sigma,\text{fin}}$$

Can also be used for matching to BHPT

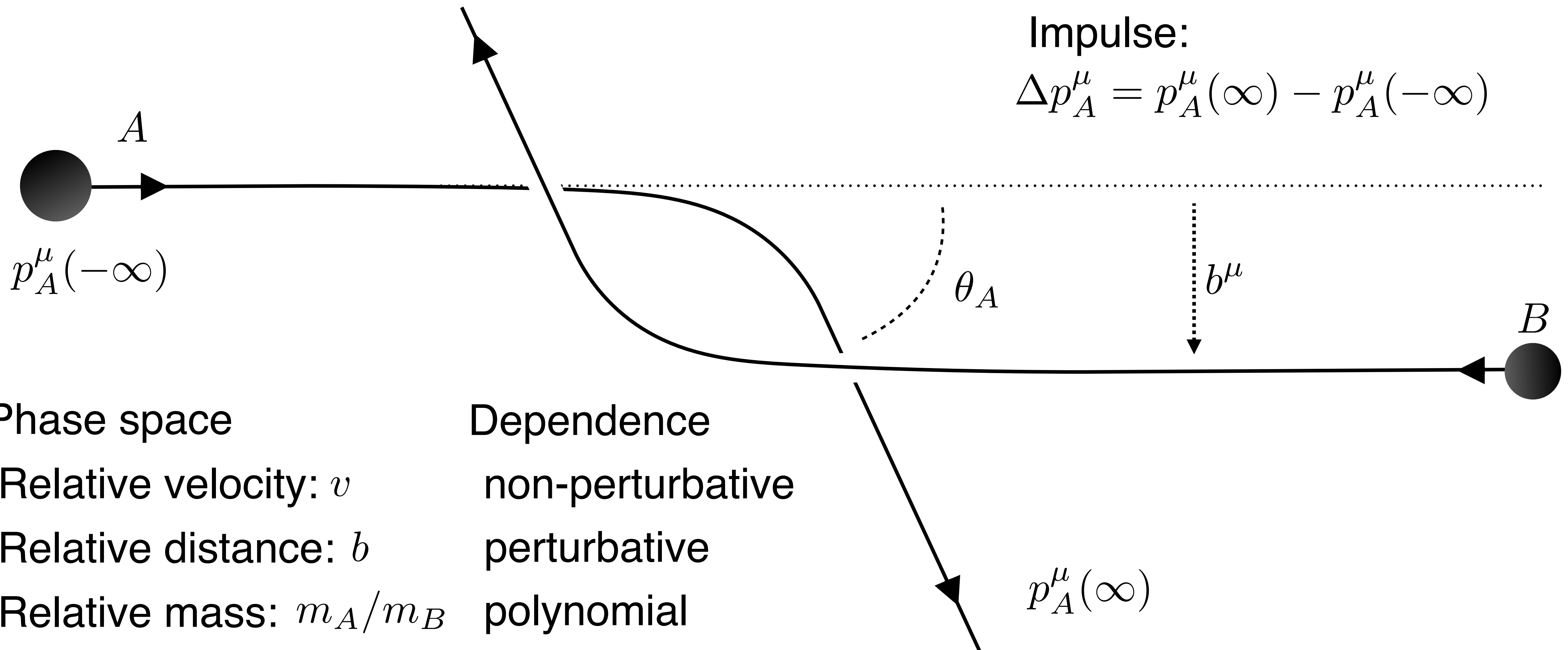
Check requires ε^1 order at one-loop: $\mathcal{M}_\sigma^{(4,-1)} = \frac{1}{2}\mathcal{M}_\sigma^{(2,1)} - i\mathcal{M}_\sigma^{(3,0)}$

Part III: Black Hole Impulses at 5PM

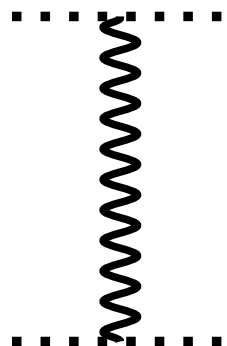
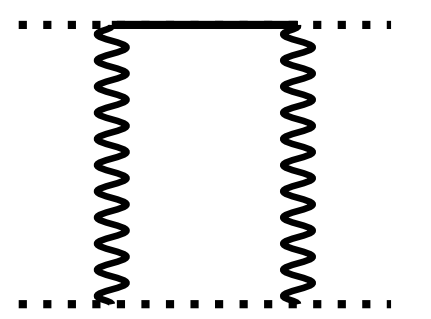
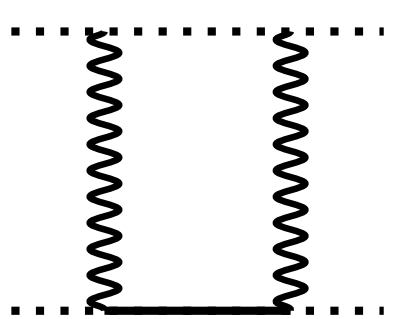
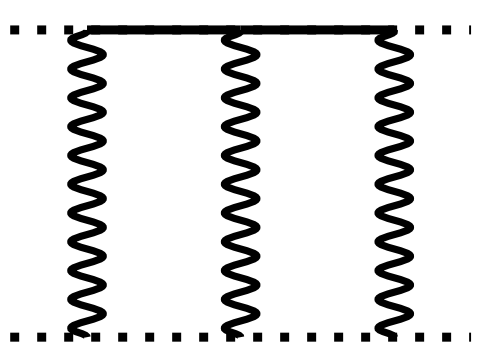
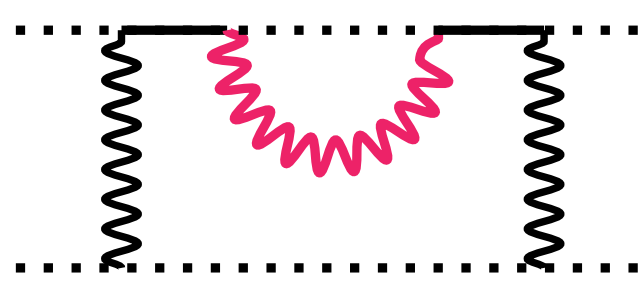
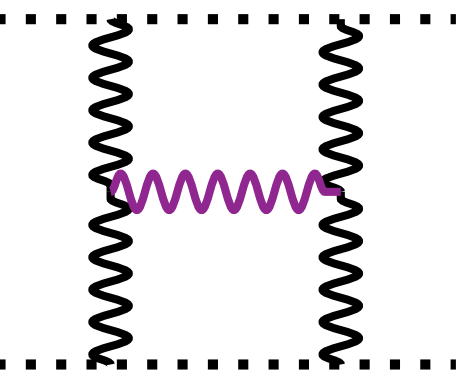
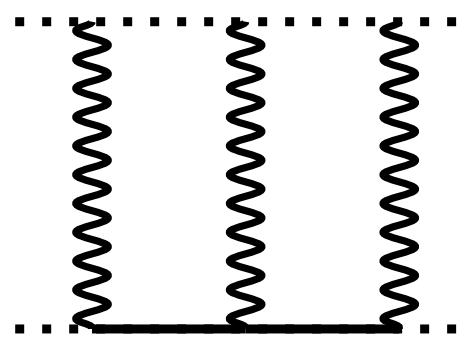
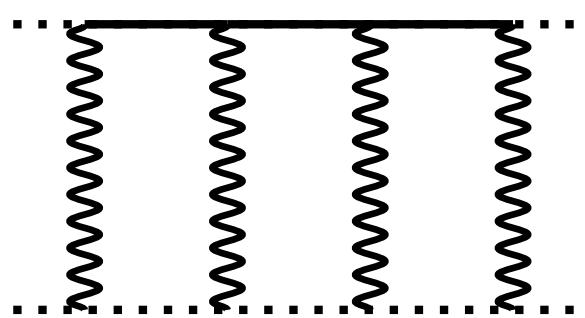
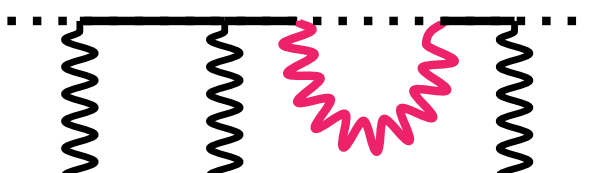
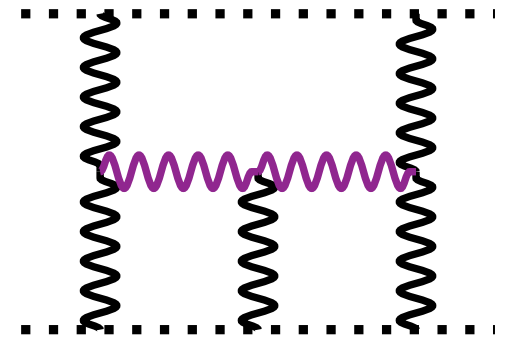
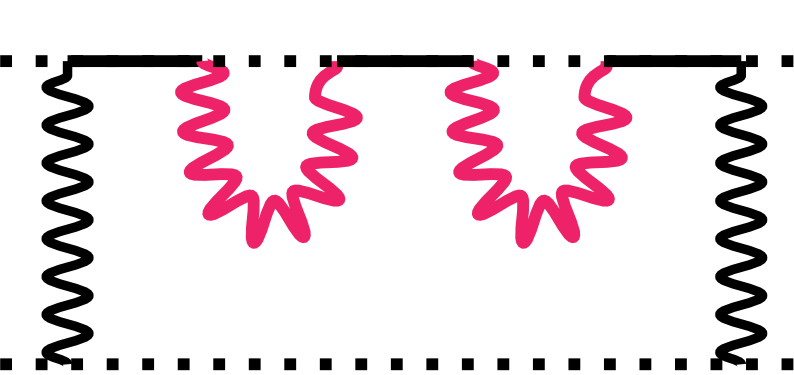
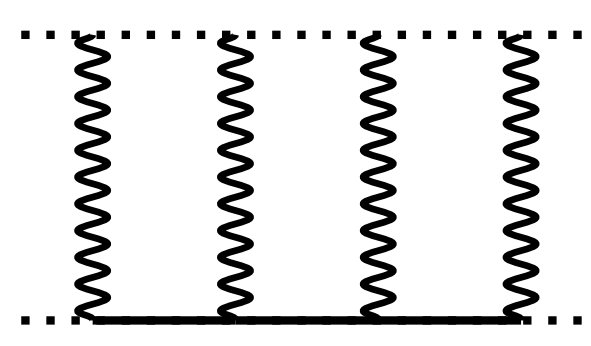
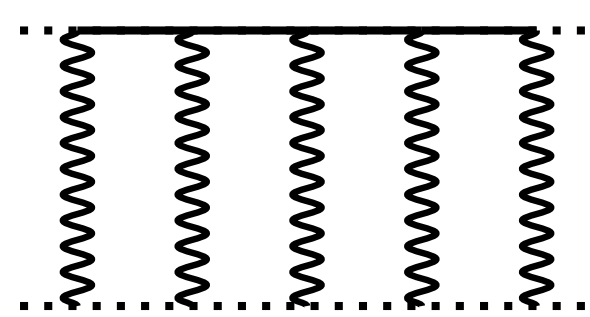
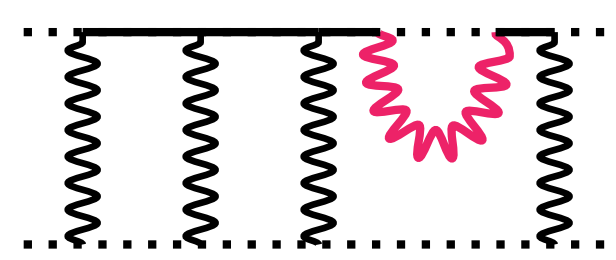
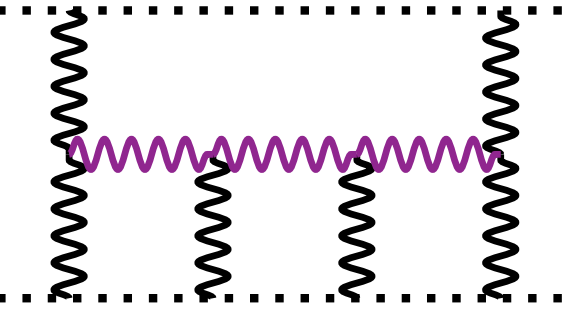
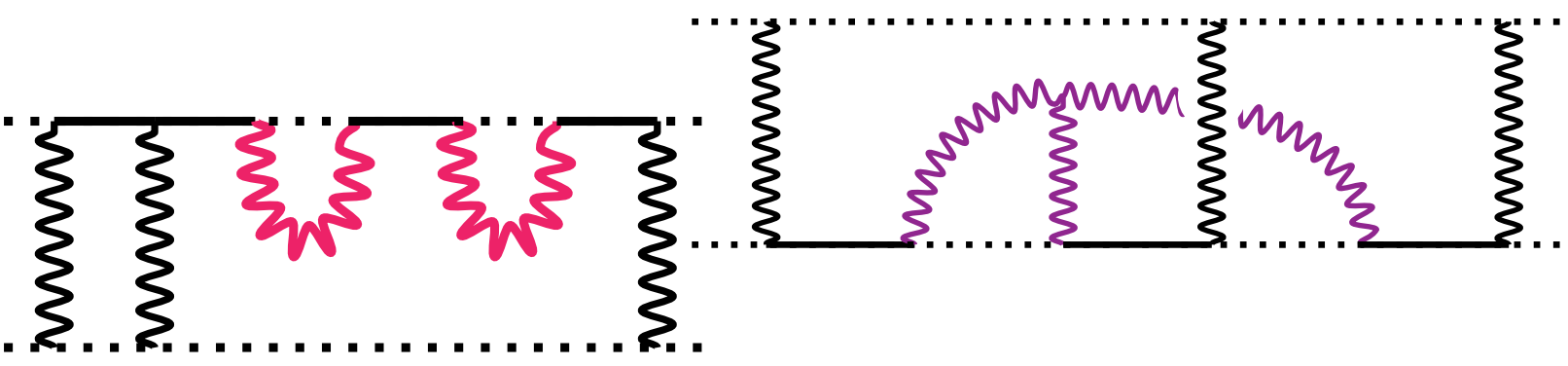
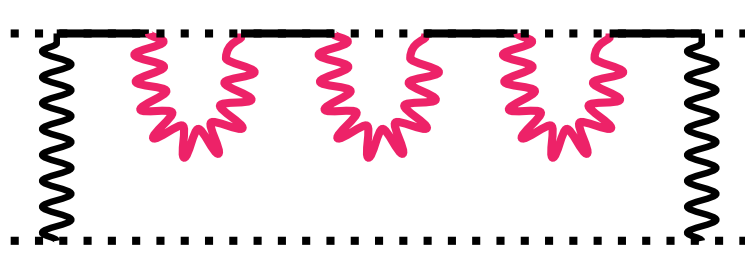
[M. Driesse, GUJ, G. Mogull, J. Plefka, B. Sauer, J. Usovitsch: 2403.07781]

[M. Driesse, GUJ, A. Klemm, G. Mogull, C. Nega, J. Plefka, B. Sauer, J. Usovitsch: 2411.11846]

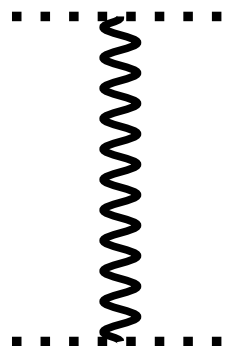
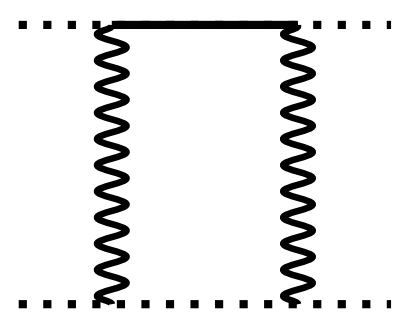
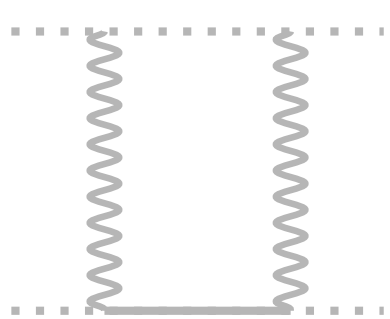
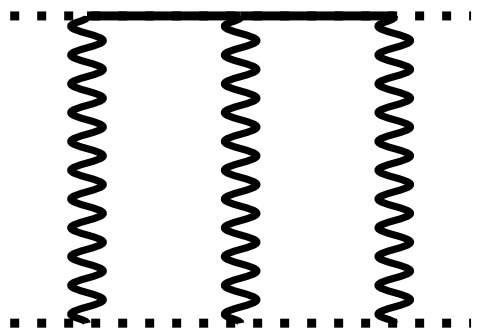
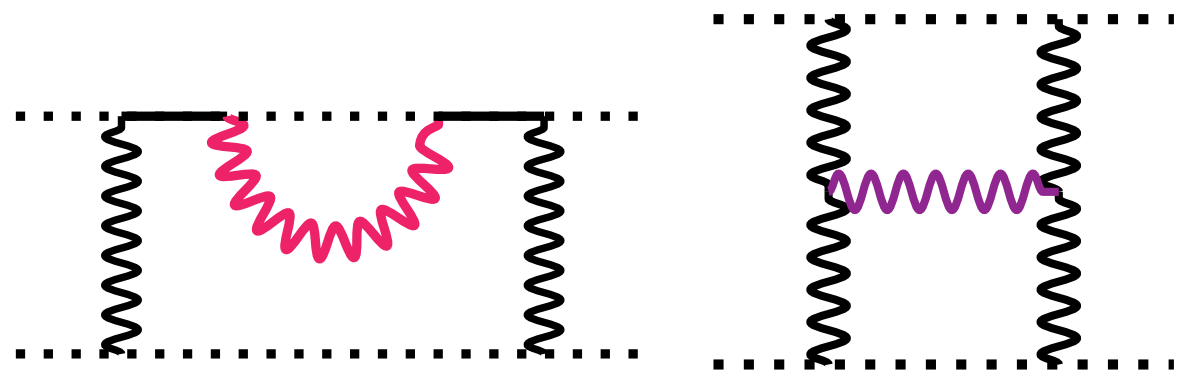
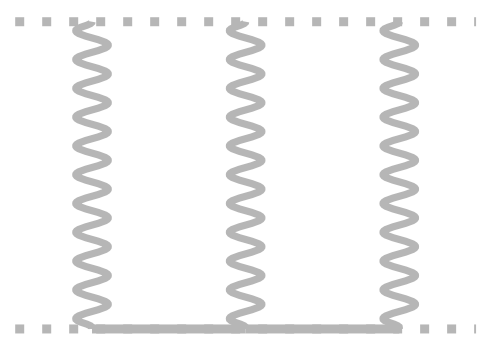
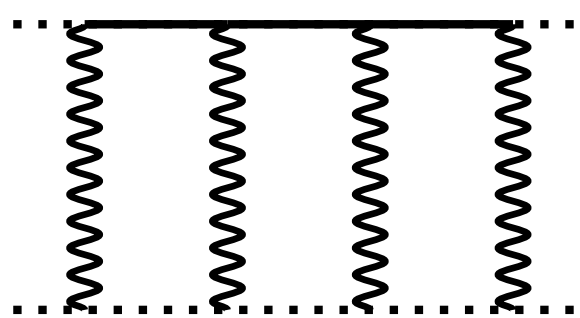
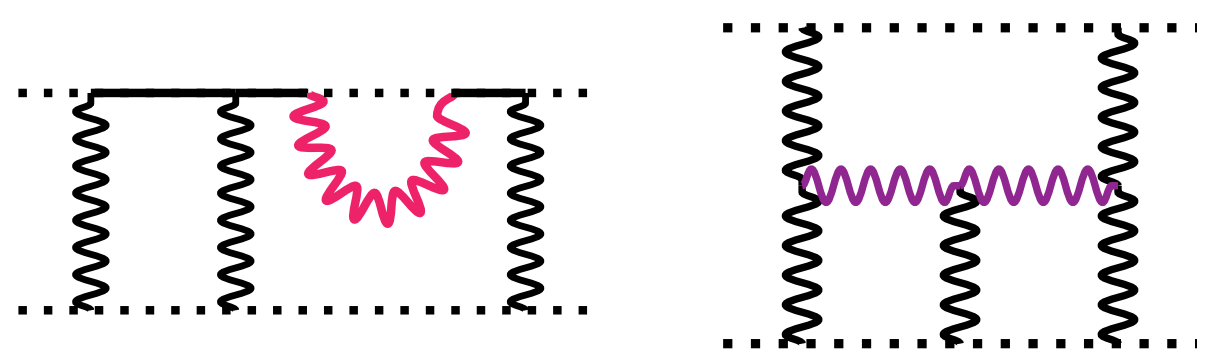
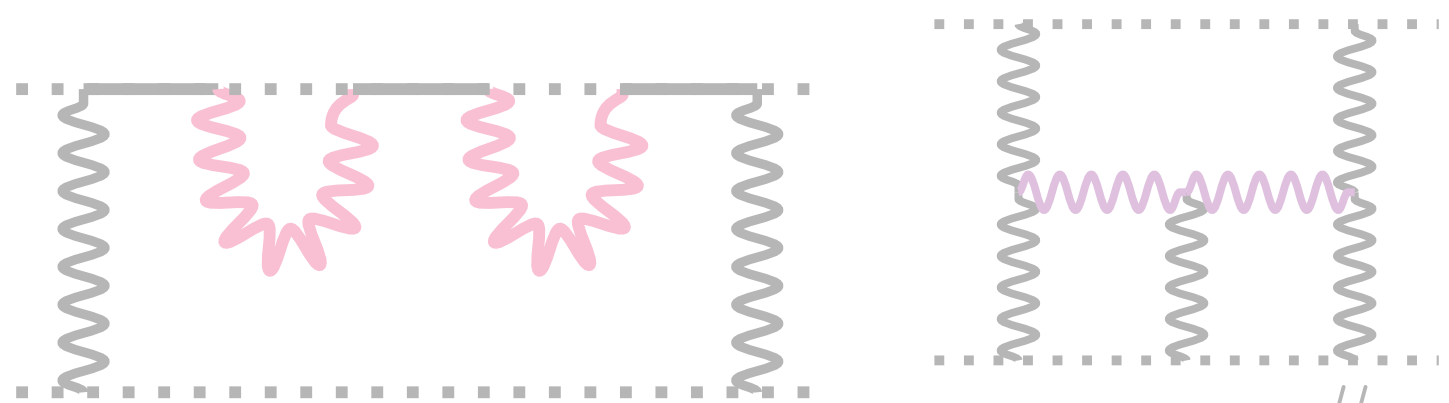
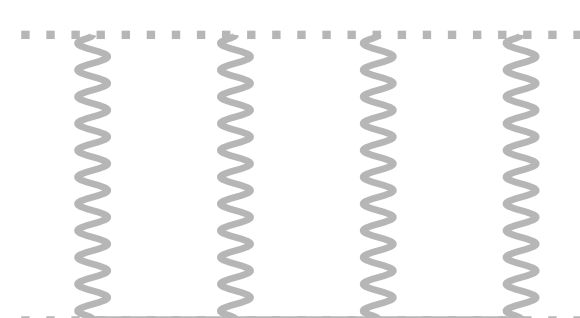
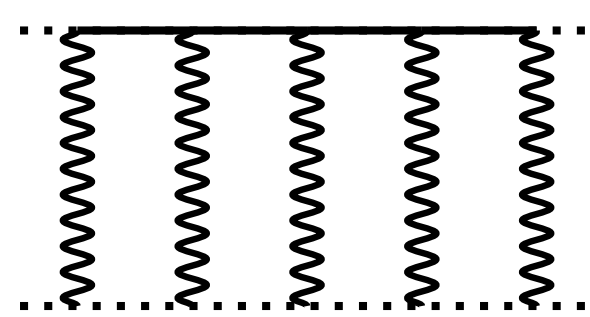
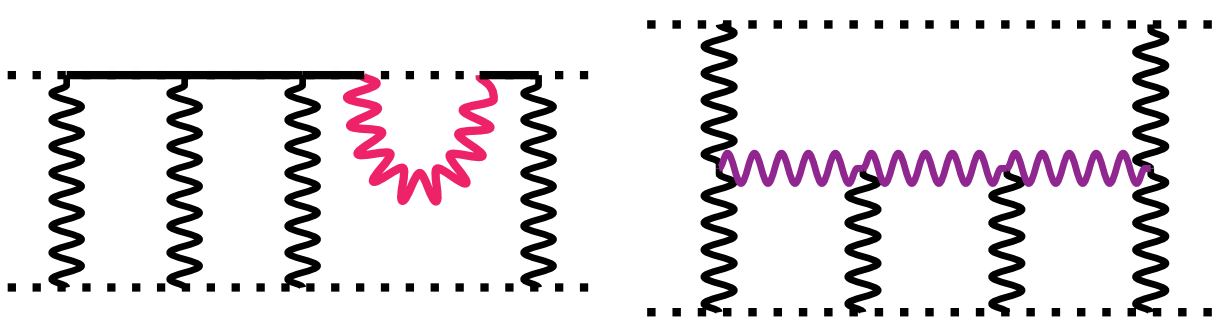
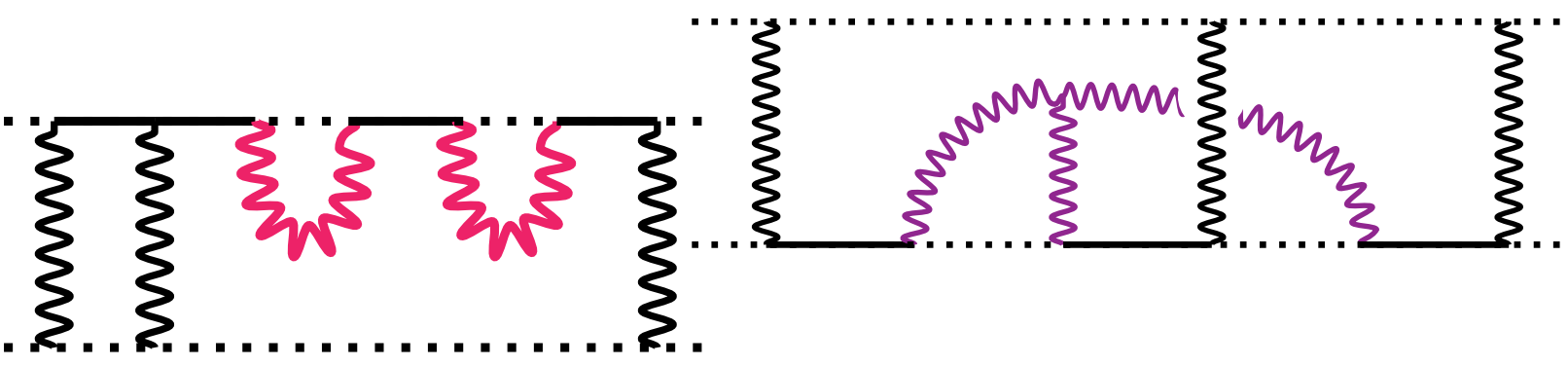
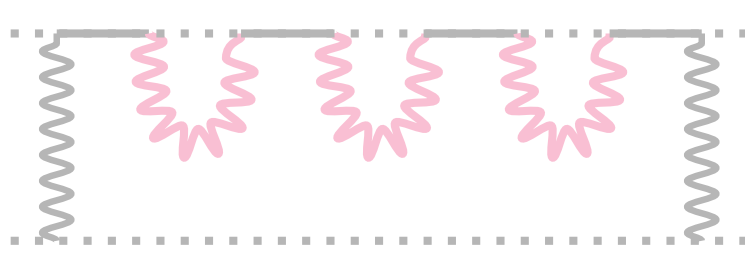
[M. Driesse, GUJ, G. Mogull, C. Nega, J. Plefka, B. Sauer, J. Usovitsch: 2601.16256]



Loops and Self Force Orders

		Loops and Self Force Orders				
Tree ϵ_{PM}^1	ϵ_{SF}^0			potential:  radiative:  active: 	$\epsilon_{\text{PM}} = \frac{2(m_A + m_B)}{bv^2}$ $\epsilon_{\text{SF}} = \frac{m_A}{m_B}$	
		ϵ_{SF}^1				
1 loops ϵ_{PM}^2	 + ...	 + ...		ϵ_{SF}^2		
	2 loops ϵ_{PM}^3	 + ...	 + ...	 + ...	 + ...	ϵ_{SF}^3
3 loops ϵ_{PM}^4		 + ...	 + ...	 + ...	 + ...	 + ...
	4 loops ϵ_{PM}^5	 + ...	 + ...	 + ...	 + ...	 + ...

Loops and Self Force Orders

		Loops and Self Force Orders				
Tree ϵ_{PM}^1 1 loops ϵ_{PM}^2 2 loops ϵ_{PM}^3 3 loops ϵ_{PM}^4 4 loops ϵ_{PM}^5	ϵ_{SF}^0	 # = 1	ϵ_{SF}^1	potential:  radiative:  active: 	$\epsilon_{\text{PM}} = \frac{2(m_A + m_B)}{bv^2}$	$\epsilon_{\text{SF}} = \frac{m_A}{m_B}$
	 # = 2	 # = 2	ϵ_{SF}^2			
	 # = 6	 # = 14	 # = 6	ϵ_{SF}^3		
	 # ~ 20	 # ~ 75	 # ~ 75	 # ~ 20		
	 # ~ 60	 # ~ 350	 # ~ 650	 # ~ 350	...	

Method of regions

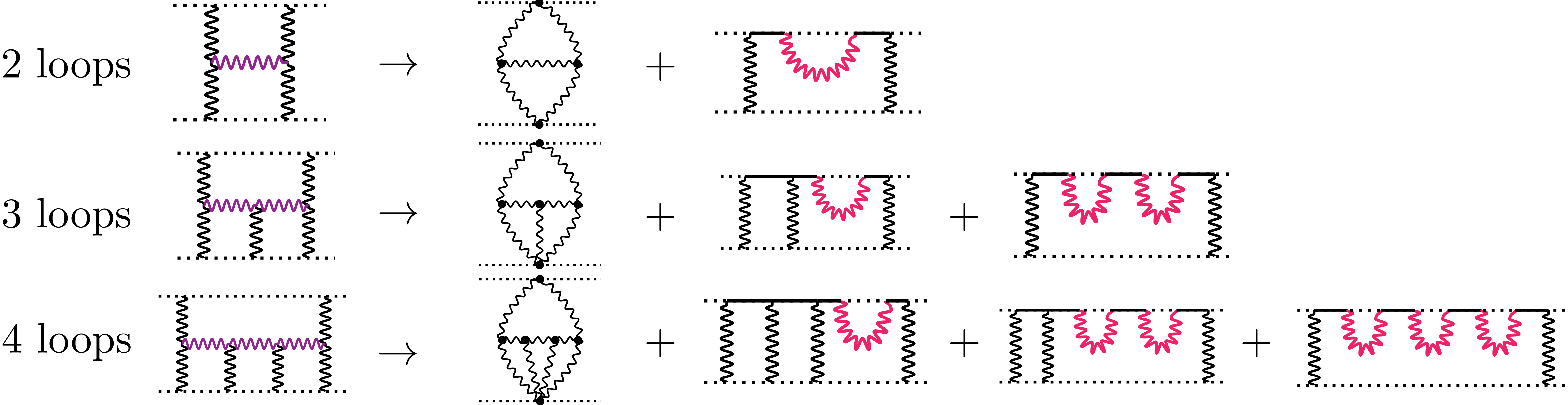
Small-velocity (PN) expansion of integrals

$v \rightarrow 0$

active: 

P = potential : $\ell^\mu_{\text{pot}} = (\ell^0_{\text{pot}}, \boldsymbol{\ell}_{\text{pot}}) \sim (v, 1)$
R = radiative : $\ell^\mu_{\text{rad}} = (\ell^0_{\text{rad}}, \boldsymbol{\ell}_{\text{rad}}) \sim (v, v)$

$R^0 \sim f(v^2)$
 $R^1 \sim v^{1-2\epsilon} f(v^2)$
 $R^2 \sim v^{-4\epsilon} f(v^2)$
 $R^3 \sim v^{1-6\epsilon} f(v^2)$



Black Hole Scattering at 4 loops and 1SF

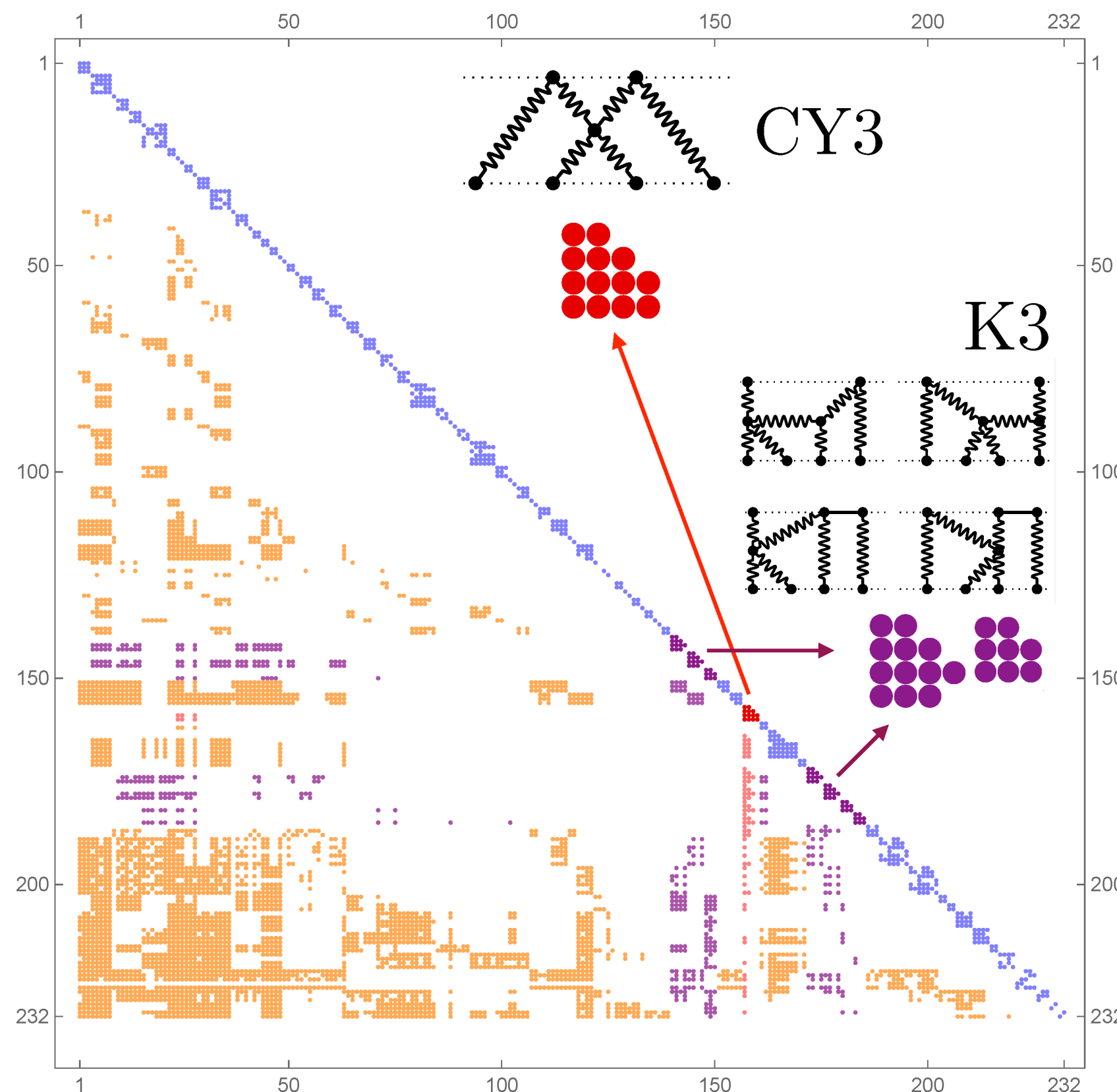
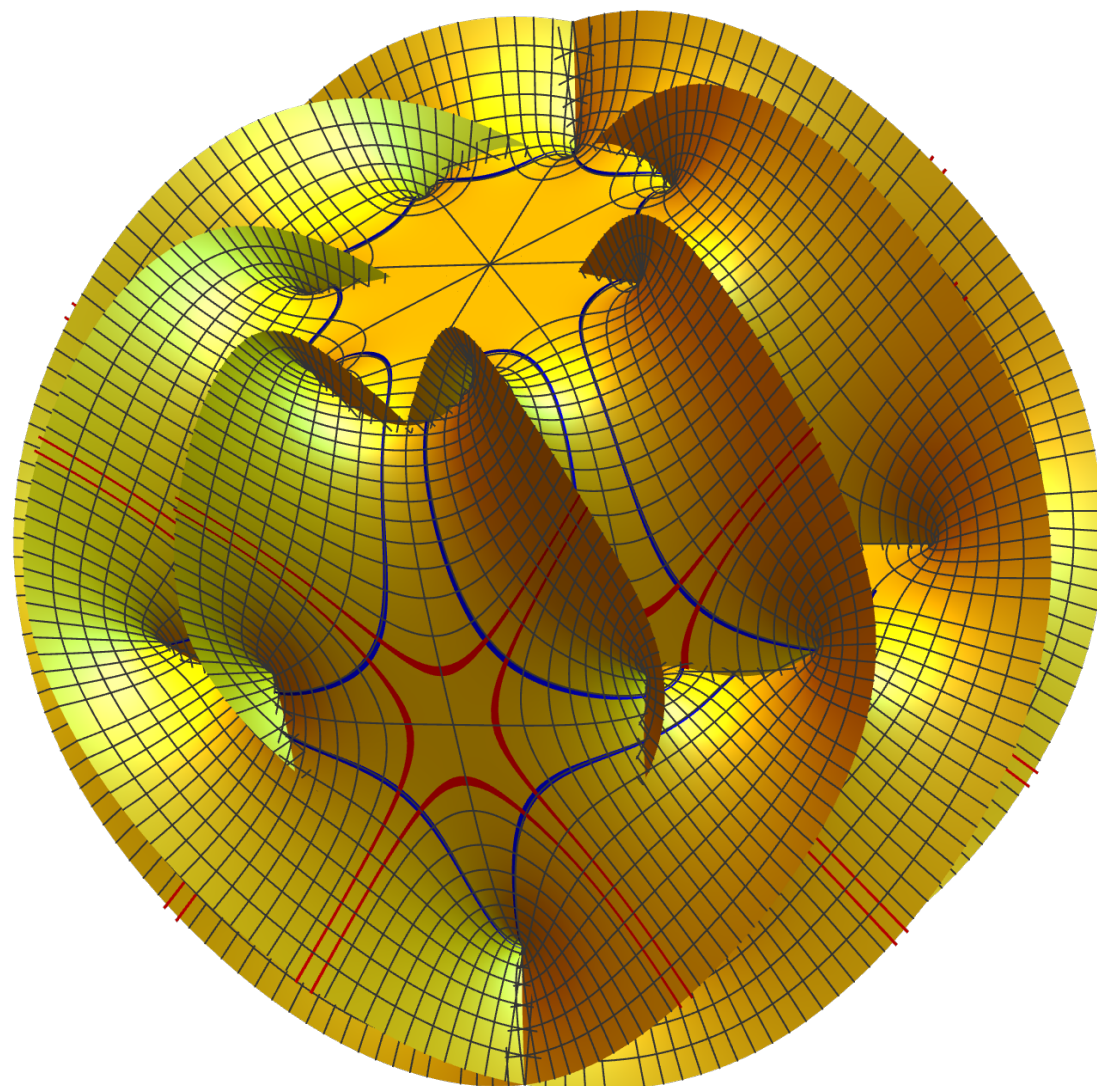
Conservative: [M. Driesse, GUJ, G. Mogull, J. Plefka, B. Sauer, J. Usovitsch: 2403.07781]
Dissipative: [M. Driesse, GUJ, A. Klemm, G. Mogull, C. Nega, J. Plefka, B. Sauer, J. Usovitsch: 2411.11846]

- IBP reduction of $\sim 10^6$ scalar integrals to 468 master integrals

- KIRA essential for $\sim 10^5$ core hours of IBP reduction

[F. Lange, J. Usovitsch, Z. Wu: 2505.20197]

- Calabi-Yau periods appear in dissipative parts

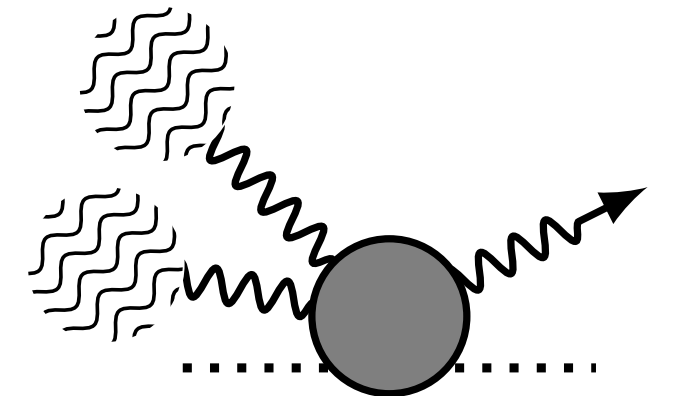


Black Hole Scattering at 4 loops and 2SF

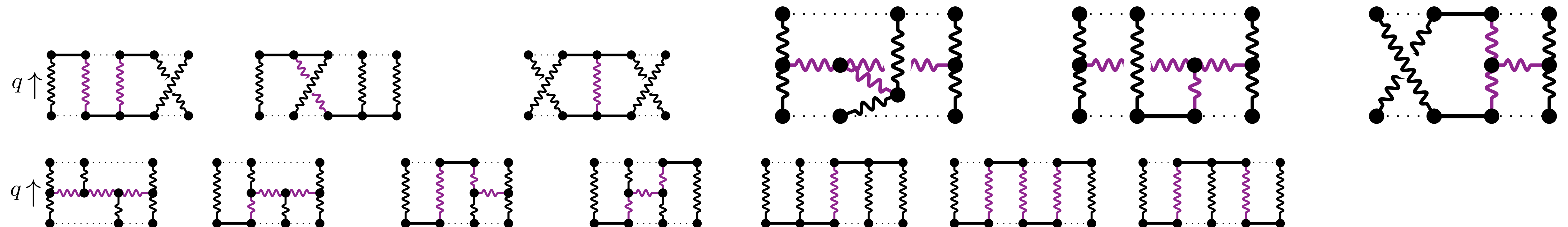
Conservative: [M. Driesse, GUJ, G. Mogull, C. Nega, J. Plefka, B. Sauer, J. Usovitsch: 2601.16256]

Potential: [Z. Bern, E. Herrmann, R. Roiban, M.S. Ruf, A. Smirnov, S. Smith, M. Zeng: 2512.23654]

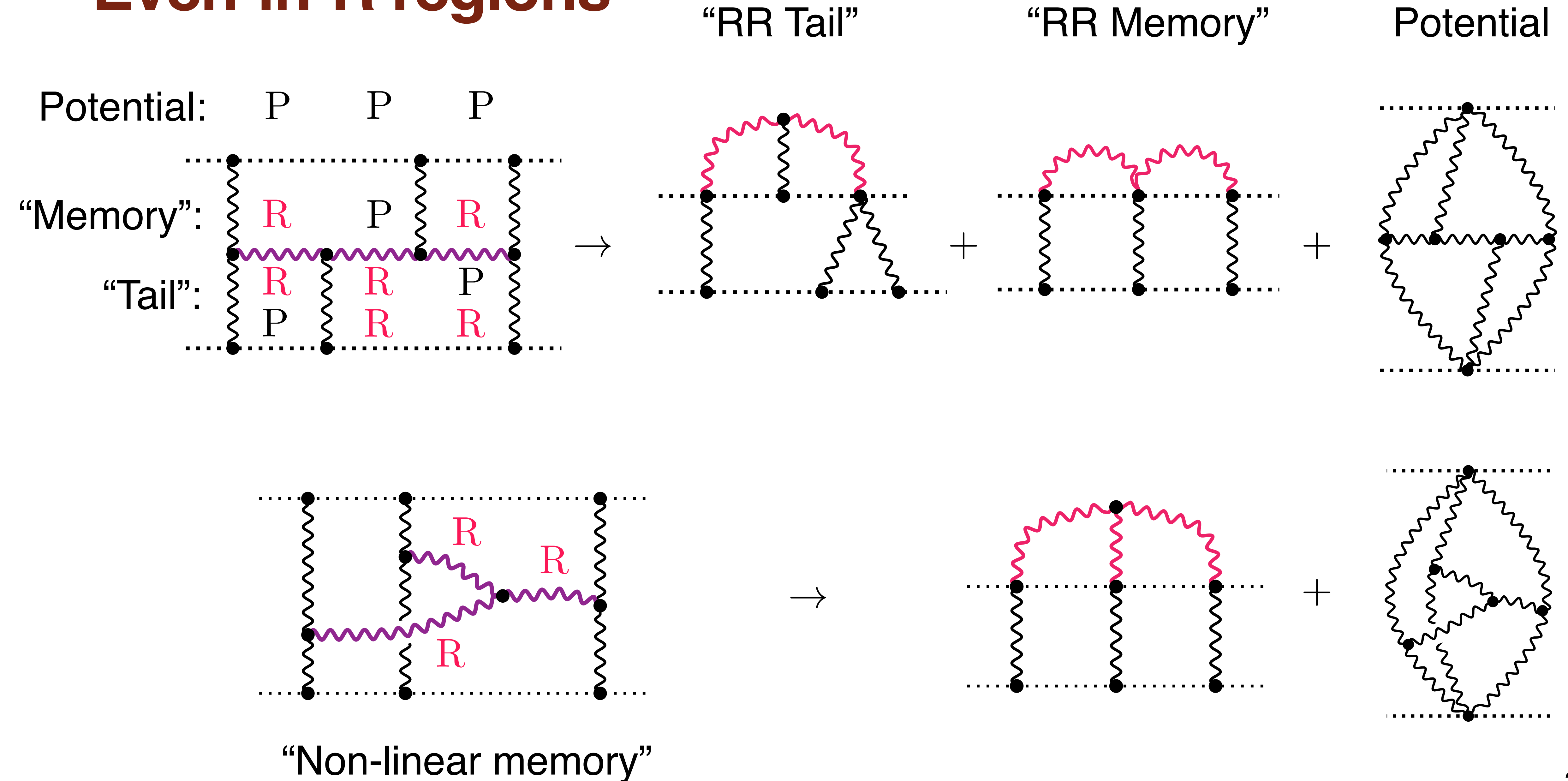
- Non-linear black hole perturbation theory $\rightarrow$ non-planar diagrams
- IBP reduction $\sim 10x$ more core-hours (after crucial optimisations)
- Spurious pole in $v = \sqrt{8}/3 \approx 0.94$ cancel between regions
- “Real part of Feynman” conservative prescription breaks down
- Calabi-Yau period from DEs drops out of final result



[Z. Bern, A. Jackman, G. Mansfield, M.S. Ruf: 2603.15383]

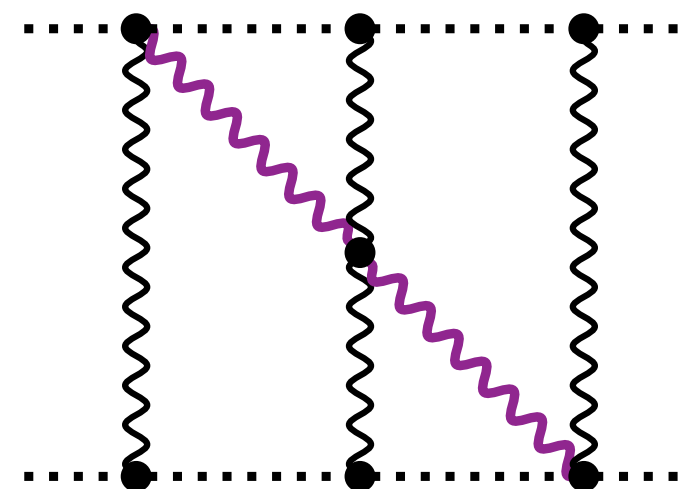


Even-in-R regions



γ -3 divergence and Beukers' K3

Graphs with

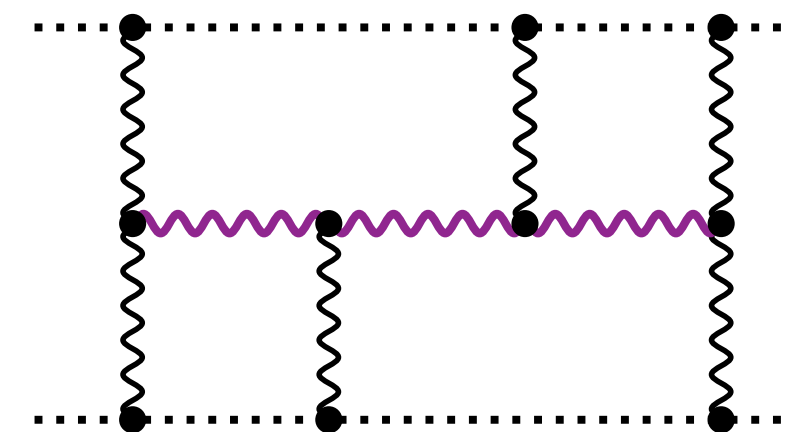


as sub-graph lead to spurious divergences

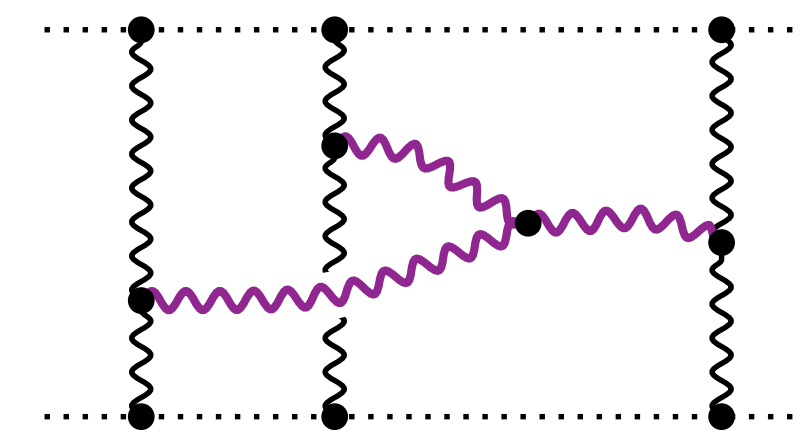
5PM 2SF, K3':

"Beukers' K3" related to Apéry's operator

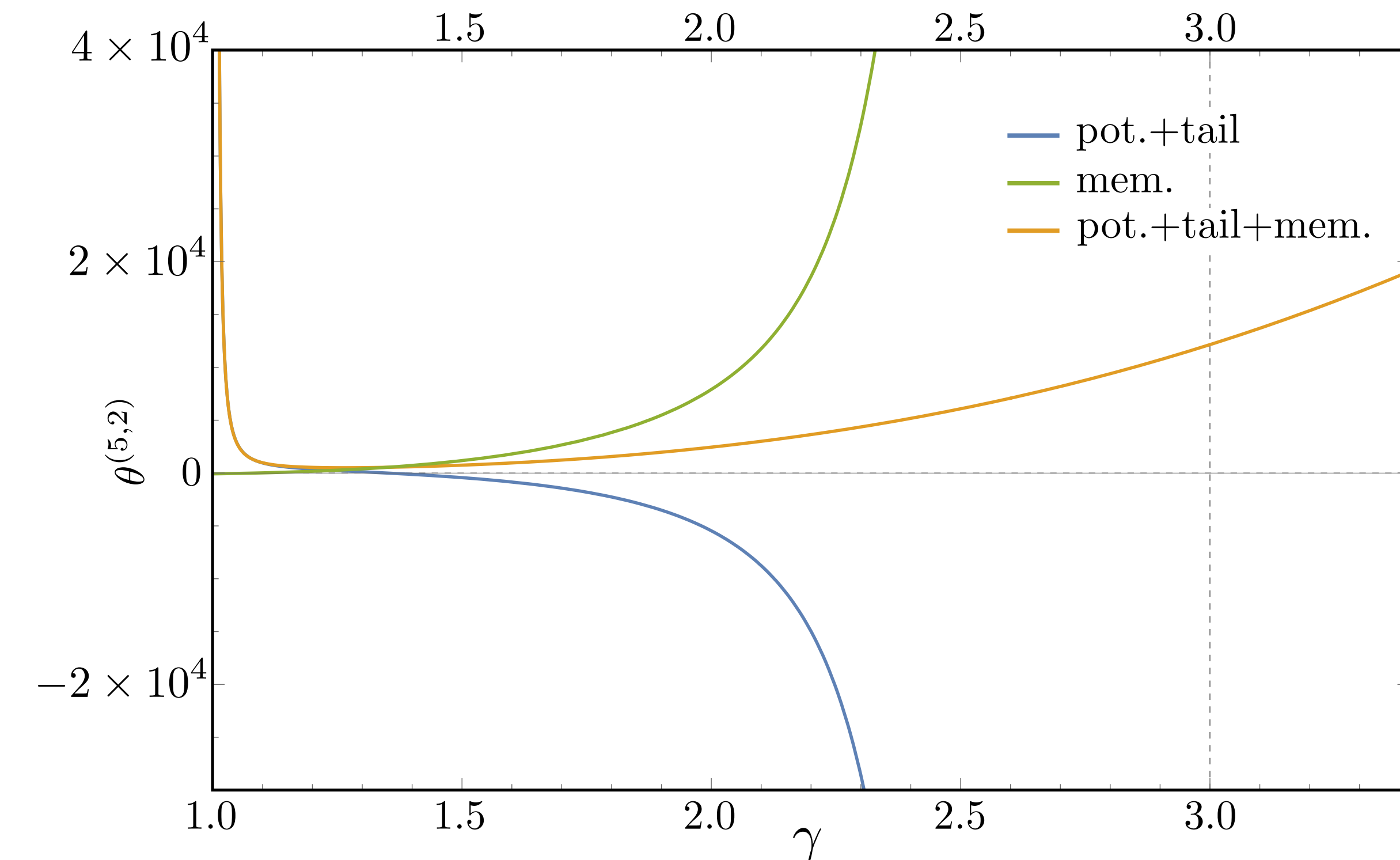
E.g.



and



[Only non-planar graphs?]

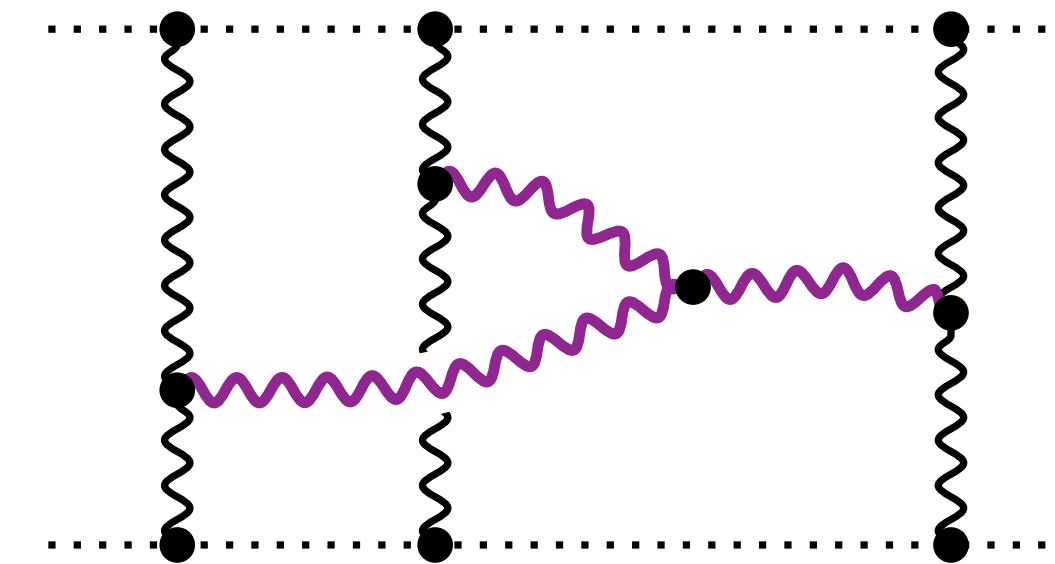


Divergence at $\gamma = 3$ for (potential+tail)
and for memory scaling as $(3 - \gamma)^{-15/2}$

($v = \sqrt{8}/3 \approx 0.94$)

Conservative memory effects

- Separating conservative and dissipative effects is subtle at 2SF A. Pound: [1404.1543]
- Non-linear memory blurs the distinction



- Required boundary integrals:

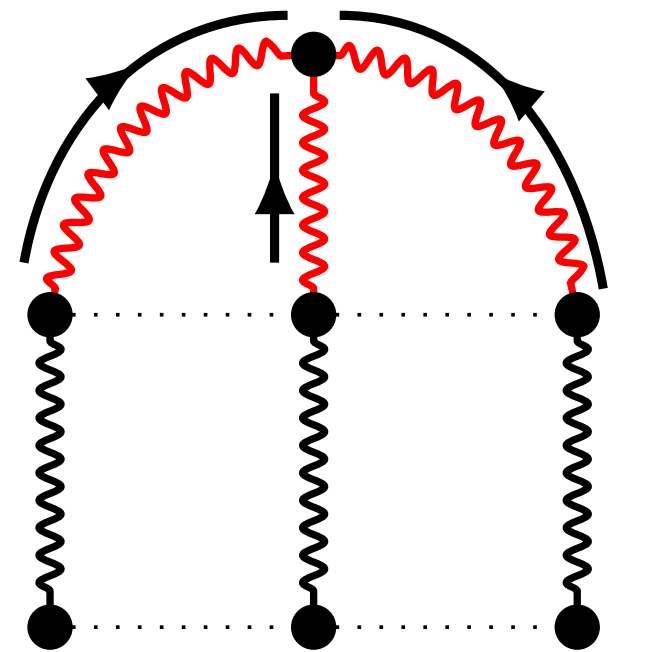
$$I_1^{(M)} = \text{diagram}, \quad I_2^{(M)} = \text{diagram},$$

The diagrams show two boundary integrals, $I_1^{(M)}$ and $I_2^{(M)}$, represented by Feynman diagrams. Both diagrams consist of a square loop with wavy lines. The top horizontal line is red, and the other three lines are black. The vertices are marked with black dots. The diagrams are enclosed in a rectangular frame defined by dotted lines.

- Conventional “real part of Feynman” leads to ϵ pole and does not cancel γ pole

The γ -3 prescription for conservative memories

- Evaluate memory boundaries with retarded propagators pointing towards symmetric point
- Equivalent symmetries as Feynman prescription
- Cancels ϵ and γ poles
- Numerical evaluation of boundary integrals to sub-sub-leading order in ϵ :



$$I_1^{(M)} \propto \int_{-\infty}^{\infty} d\omega_1 d\omega_2 d\omega_3 \delta(\omega_1 + \omega_2 - \omega_3) (0^+ - \omega_1)^{1-2\epsilon} (0^+ - \omega_2)^{1-2\epsilon} \prod_{i=1}^3 |\omega_i|^\epsilon K_\epsilon(|\omega_i|)$$

$$I_3^{(M)} \propto \int_{-\infty}^{\infty} d\omega_1 d\omega_2 d\omega_3 \delta(\omega_1 + \omega_2 - \omega_3) (0^+ - \omega_1)^{-1-2\epsilon} (0^+ - \omega_2)^{-1-2\epsilon} \omega_1^3 \omega_2^3 \prod_{i=1}^3 |\omega_i|^\epsilon K_\epsilon(|\omega_i|)$$

Conservative 2SF Scattering Angle

Iterated integrals up to weight three:

$$\mathcal{I}[\varphi_1, \dots, \varphi_n; x] = \int_1^x dx' \varphi_1(x') \mathcal{I}[\varphi_2, \dots, \varphi_n; x']$$

$$\phi_n \in \left\{ \frac{1}{x}, \frac{1}{1+x^2}, \frac{1+x}{x(x-1)}, \frac{-1+x}{x(x+1)}, \frac{1-x^2}{x+x^2+x^3}, \frac{1-x^2}{x-x^2+x^3}, \frac{-1+x^2}{x+x^3}, \frac{(1+x^2)\varpi_{K3}(x)}{x}, \frac{1}{x r(x) \varpi_{K3}(x)} \right\}$$

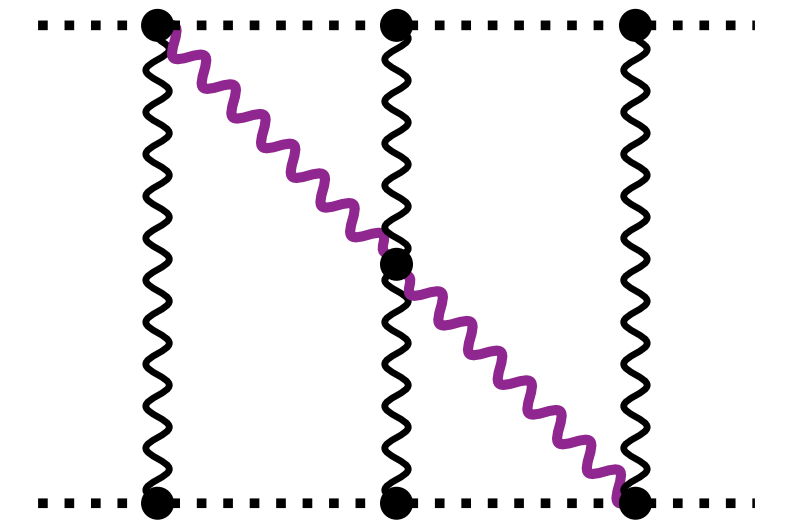
$$x = \gamma - \frac{1}{\sqrt{\gamma^2 - 1}}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2}}$$

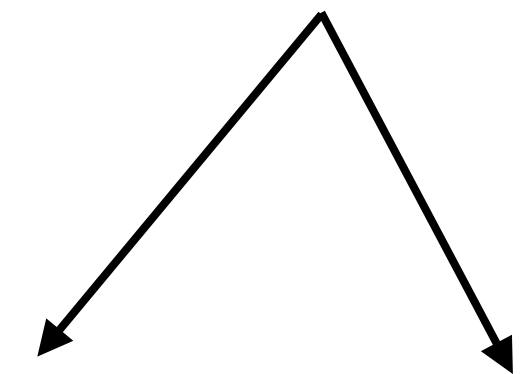
$$r(x) = \sqrt{-1 + 34x^2 - x^4}$$

$$= x\sqrt{(\gamma + 3)(3 - \gamma)}$$

Poles of ϕ_n : $\sqrt[4]{1}$, $\sqrt[6]{1}$



“Beukers’ K3 period”



Outlook

- GW scattering at four loops (5PM): First non-trivial EFT matching for GR
- Dissipative effects at 5PM-2SF
Building on [GUJ, K. Stoldt: 2605.12214]
- High-precision results for angular momentum loss
- How to define conservative dynamics in PM?
- Resummation models and mapping from unbound→bound
- High energy limit of PM observables

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Thanks for listening!

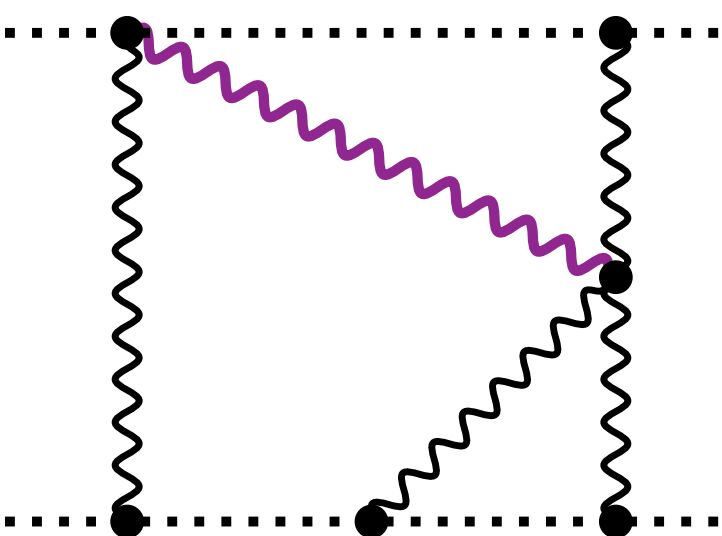
Differential Equations and Special Functions

[A. Klemm, C. Nega, B. Sauer, J. Plefka: 2401.07899]

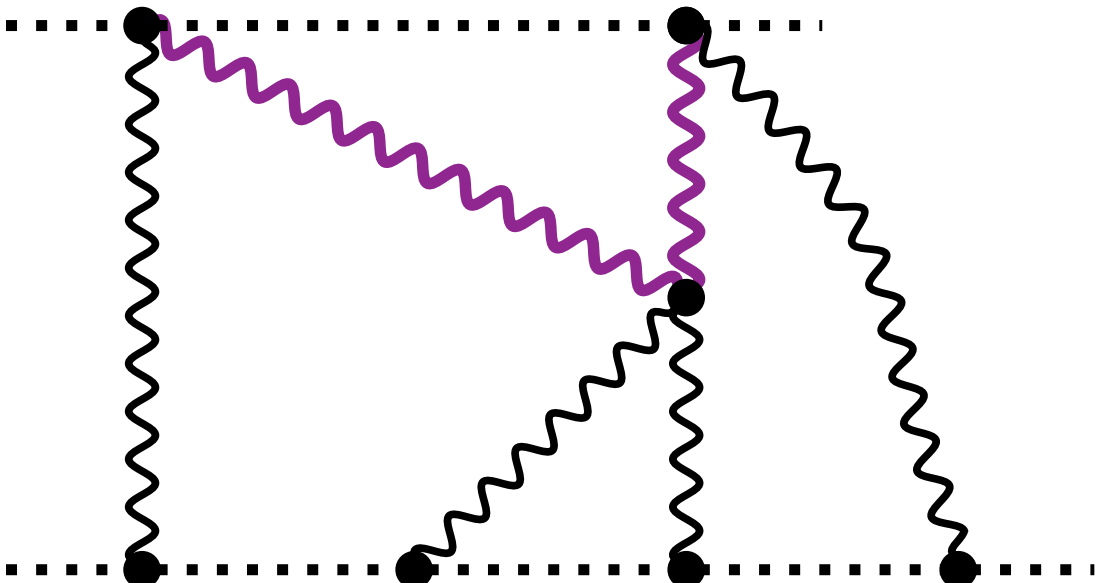
[H. Frellesvig, R. Morales, M. Wilhelm: 2312.11371]

4PM 1SF, K3:

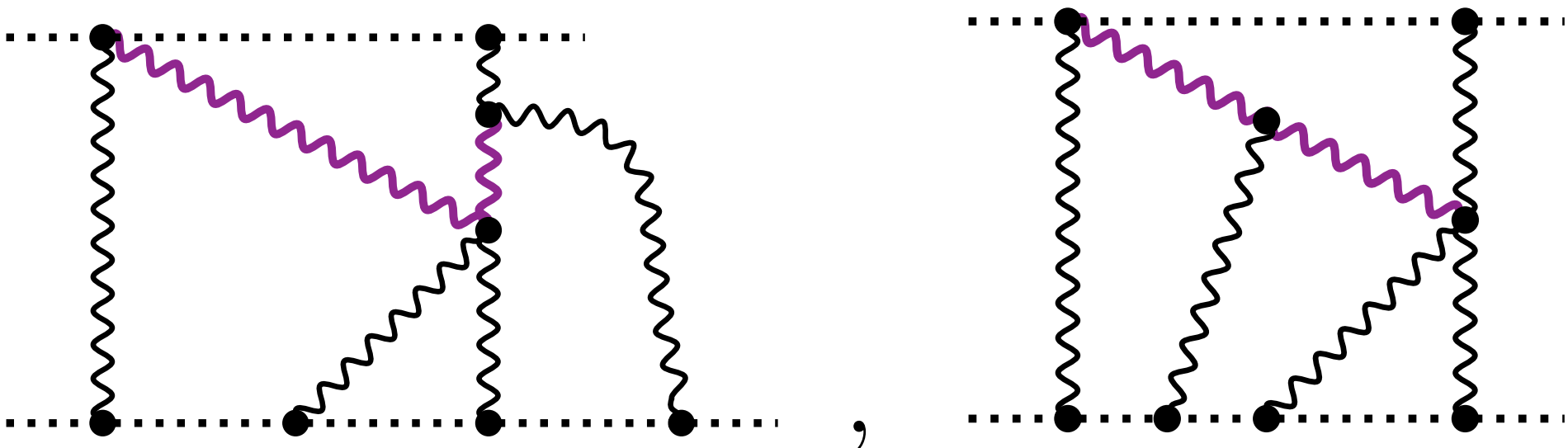
$$E^2(x), K^2(x), E(x)K(x)$$



CY3:



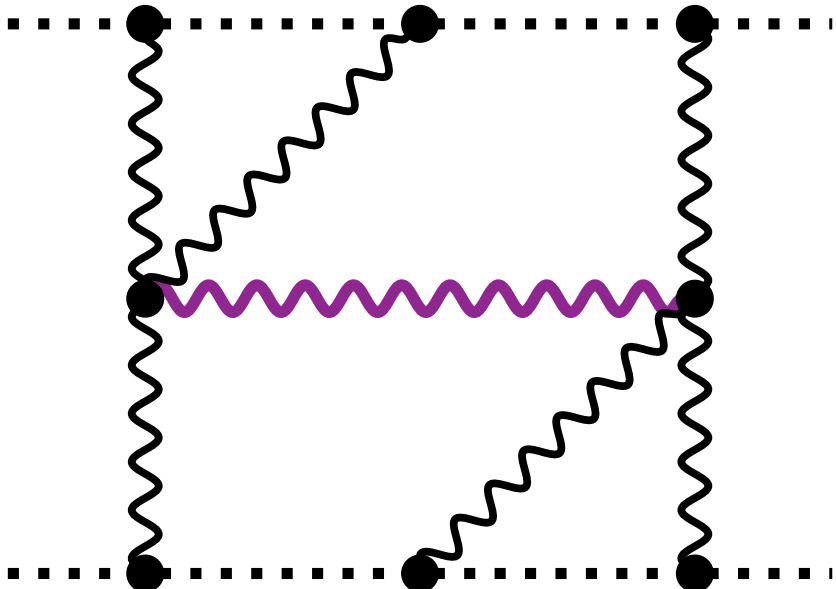
5PM 1SF, K3:



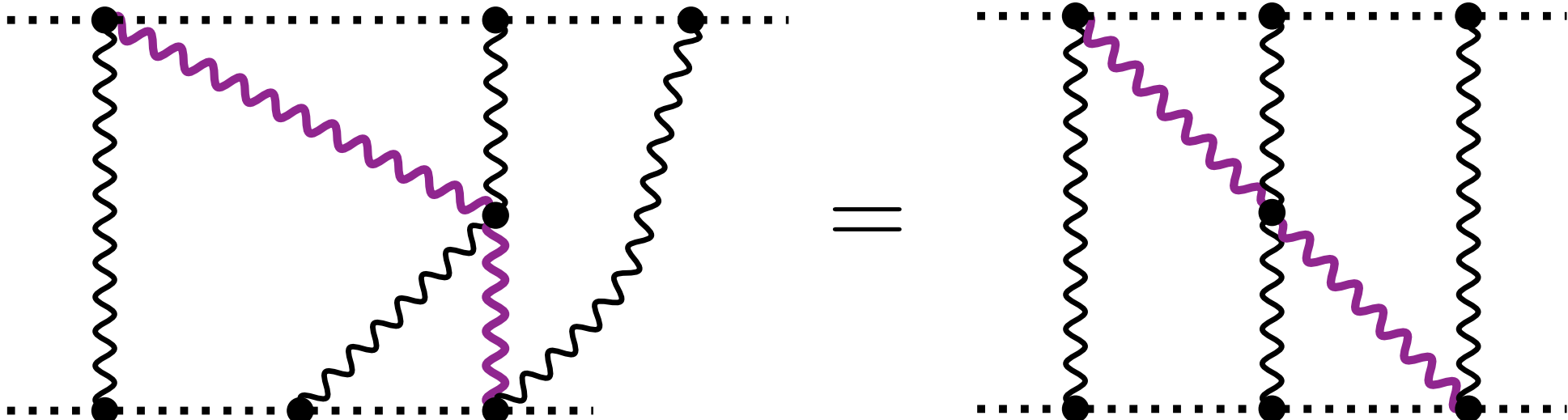
1SF

2SF

CY3':



5PM 2SF, K3':



“Beukers’ K3” related to Apéry’s operator

Conservative 2SF Scattering Angle

Matches known PN results up to 4PN and all terms except (unknown) memory at 5PN!

$$\theta_{\text{cons}}^{2\text{SF}} = \Gamma \frac{M^5 \nu^2}{|b|^5 v^{10}} \left[\frac{2}{5} v^4 + \frac{53}{5} v^6 + \left(\frac{45341}{360} - \frac{41\pi^2}{12} \right) v^8 \right. \\ \left. + \left(\frac{46629199}{15120} + \left[\frac{64 c_M}{5} - \frac{11456}{135} \right] - \frac{221597\pi^2}{720} + \frac{2816 \log[2v]}{45} \right) v^{10} + \mathcal{O}(v^2) \right]$$

“Memory”

Non-local effects
 (“Tail”)

Potential only

Bini, Damour, Geralico: [2007.11239], [2107.08896], [2210.07165]

Blümlein, Maier, Marquard, Schäfer: [2003.01692], [2110.13822]

Porto, Riva: [upcoming work]

