Lecture on Color Glass Condensate

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Motivations

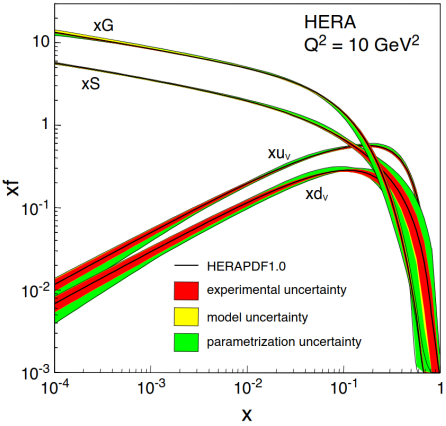
Rise of the gluon distribution

- Gluon distribution $xg(x, Q^2)$ increases at small x.
- Occupation number : number of gluons per unit of phase space volume

$$n(x, Q^2) \sim \frac{xg(x, Q^2)}{\pi R^2 Q^2}$$

- High occupancy $n(x, Q^2) \sim 1/\alpha_s$: breakdown of perturbation theory, non-linear interactions, etc
- Occurs at the **emergent** transverse scale Q_s^2 parametrically defined as

$$Q_s^2 \sim \alpha_s \frac{xg(x, Q_s^2)}{\pi R^2}$$



H1, ZEUS Collaboration, arXiv:0911.0884

High-energy quantum scattering off a hard sphere

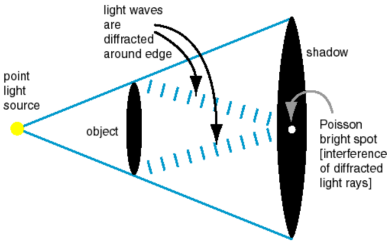
- Total cross-section is twice the geometric cross-section

$$\sigma_{\rm tot} = 2\pi R^2$$

- Related to Arago-Poisson spot in optics.
- Consequence of unitarity through optical theorem.

$$\sigma_{\rm tot} = \sigma_{\rm el} + \sigma_{\rm inel}\,, \quad \sigma_{\rm el} = \sigma_{\rm inel}$$

- Does a nucleon behaves as a black disk in the high-energy limit?



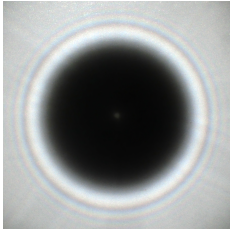


Fig. from Wikipedia

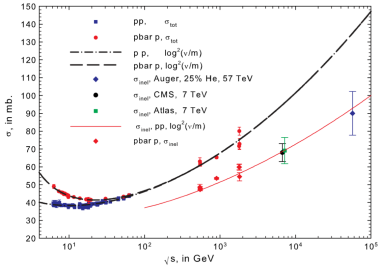
Froissart bound and experimental results

- Interaction probability between a high-energy probe and a hadron

$$T(s, b_\perp) \sim (s/s_0)^\lambda e^{-2m_\pi b_\perp}$$

- $T \leq 1$ implies

$$\sigma_{\rm tot} \leq \frac{\pi}{2}\left(\frac{\lambda}{m_\pi}\right)^2 \ln^2(s/s_0)$$



$$\lim_{s\to\infty} \frac{\sigma_{\rm inel}(s)}{\sigma_{\rm tot}(s)} = 0.509 \pm 0.011$$

Block, Halzen, PRL 107 (2011)
"Experimental Confirmation that the Proton is Asymptotically a Black Disk"

Goals and outline of this lecture

Questions

1. How to describe unitarization of high-energy cross-sections within QCD?
2. How to experimentally test this mechanism?

- The Color Glass Condensate EFT in a nutshell
- CGC predictions at tree-level
- Quantum corrections
- 3D parton tomography at small-x

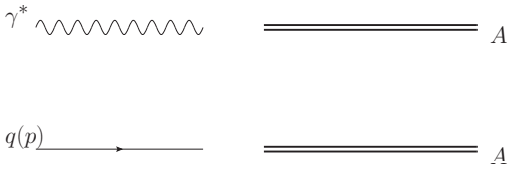
Basics of the Color Glass Condensate

Dilute-dense collisions

- Focus on dilute-dense collisions:
  - DIS,
  - $\gamma - A$ UPC
  - forward pA collisions
- Fast probe (photon, quark) with large q^+.
- scatters off a fast nucleus with large P^-.
- Light-cone coordinates

$$a^{\pm} = \frac{a^0 \pm a^3}{\sqrt{2}}$$

- Note: CGC also for dense-dense collisions, applications to initial state of heavy-ion collisions.

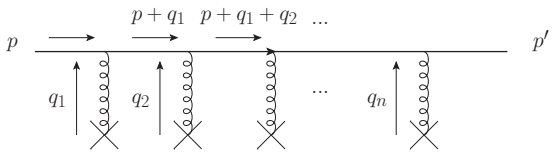


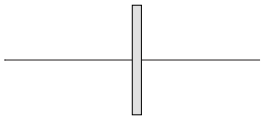
The dilute probe

- Eikonal scattering of a fast particle, $p^+ \gg$ any other scale

- Eikonal propagator for a scalar particle

$$\mathcal{T}^s_{ji}(p', p) = (2\pi)\delta(p'^+ - p^+)(2p^+) \times \int \mathrm{d}^2\boldsymbol{z}_\perp e^{-i\boldsymbol{q}_\perp\cdot(\boldsymbol{p}'_\perp - \boldsymbol{p}_\perp)} V_{ji}(\boldsymbol{z}_\perp)$$





- Infinite light-like Wilson line

$$V(\boldsymbol{x}_\perp) = T \exp\left(ig \int_{-\infty}^{\infty} \mathrm{d}z^+ \; A^-(z^+, \boldsymbol{x}_\perp) \right)$$

The dense target

- High-gluon occupancy $\Rightarrow$ classical field.
- EFT formulated by separating fast $k^- \geq k_F^-$ and slow modes $k^- \leq k_F^-$ within the target wave-function.
- Sources are static
  (much longer lifetime than small-x partons)

$$[D_\nu, F^{\nu\mu}] = j^\mu = \rho(x^+, \boldsymbol{x}_\perp)\delta^{\mu-}$$

- Solution in Lorenz $\partial_\mu A^\mu = 0$ or projectile light-cone gauge $A^+ = 0$

$$A^\mu(x) = \delta^{\mu-}\alpha(x^+, \boldsymbol{x}_\perp), \quad -\boldsymbol{\partial}_\perp^2 \alpha(x^+, \boldsymbol{x}_\perp) = \rho(x^+, \boldsymbol{x}_\perp)$$

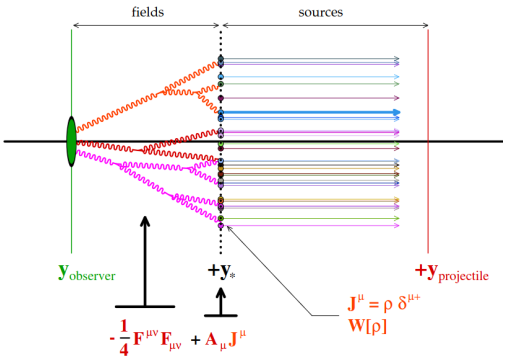


Fig. from François Gelis's review
arXiv:2102.07604

Stochastic colour sources and the McLerran-Venugopalan model

- Stochastic distribution of colour sources
  $\Rightarrow$ need to average over field configurations

$$\langle \mathcal{O} \rangle_{k_F^-} = \int \mathcal{D}\rho \; W_{k_F^-}[\rho] \mathcal{O}[\rho]$$

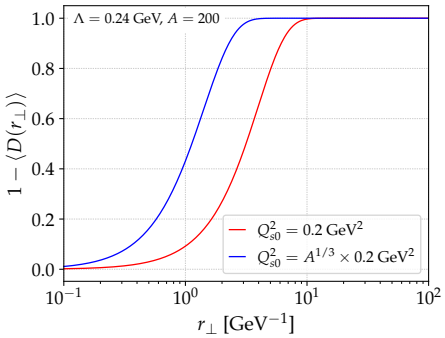
- In the MV model, Gaussian statistics (central limit theorem for a large nucleus)

$$\left\langle \rho_a(x^+, \boldsymbol{x}_1) \rho_b(y^+, \boldsymbol{x}_2) \right\rangle = \delta(x^+ - y^+) \delta(\boldsymbol{x}_{12}) \delta_{ab} \lambda(x^+)$$

with $\int \mathrm{d}x^+ \lambda(x^+) \sim g^2 A / R^2 \sim g^2 A^{1/3}$.

- Example: dipole correlator in the MV model.

$$S_{12} = \frac{1}{N_c} \langle V(\boldsymbol{x}_1) V^\dagger(\boldsymbol{x}_2) \rangle = e^{-\frac{1}{4} Q_0^2 \boldsymbol{x}_{12}^2 \ln\left(\frac{1}{|\boldsymbol{x}_{12}|\Lambda} + e\right)}$$

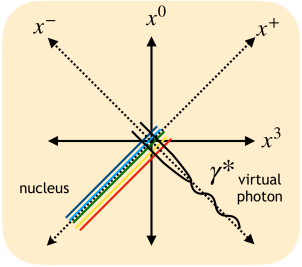


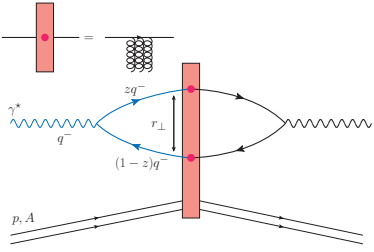
- Saturation scale in the MV model
  $Q_0^2 = \frac{1}{2\pi} g^2 C_F \int_{-\infty}^{\infty} \mathrm{d}z^+ \lambda(z^+)$
- Transition scale between weak and strong scattering regime.

Standard processes at tree-level

The dipole picture of DIS

- The photon splits into a quark-antiquark pair,
- which subsequently probes the dense gluon matter of the nucleus.
- Motivated by time-scale argument: the coherence length of the virtual photon is $\sim 1/(x_{\mathrm{Bj}}P^-) \gg 1/P^-$ longitudinal size of the proton/nucleus.
- Saturation encoded in multiple scattering of the $q\bar{q}$ dipole off small-x gluons.





Geometric scaling at HERA

- Total DIS cross-section at small-x:

$$\sigma^{\gamma^* A}(x, Q^2) \propto 2 \int d^2 \boldsymbol{x}_{12} \int_0^1 dz |\psi^{\gamma^* \to q\bar{q}}(r_\perp, Q^2, z)|^2 (1 - \langle S_{12}(\boldsymbol{x}_{12}) \rangle)$$

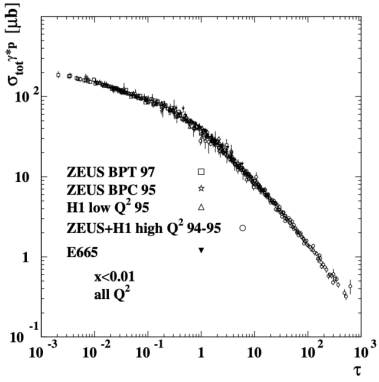
- Predicts geometric scaling of the DIS cross-section:

$$\sigma^{\gamma^* p}(x_{\mathrm{Bj}}, Q^2) = f\left(\tau = \frac{Q^2}{Q_s^2(x_{\mathrm{Bj}})}\right)$$

Stasto, Golec-Biernat, Kwiecinski, PRL 86, 2001

- Universal property of the saturated gluonic matter at small-x.

- But: not the end of the story, because linear DGLAP evolution also leads to geometric scaling...
Kwiecinski, Stasto, PRD 66 2002, Caola, Forte, PRL 101 2008

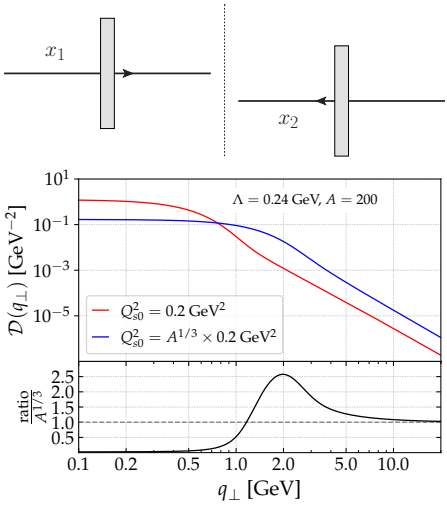


Forward particle production in pA

- Forward $qA \to q + X$ scattering.
- Measures the FT of the dipole correlator.

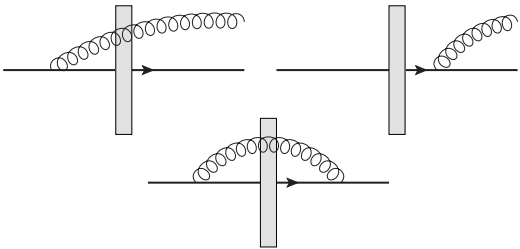
$$\frac{\mathrm{d}\sigma^{qA\to q+X}}{\mathrm{d}^2\boldsymbol{p}_\perp} = \int \frac{\mathrm{d}^2\boldsymbol{x}_1 \mathrm{d}^2\boldsymbol{x}_2}{(2\pi)^2} e^{-i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_{12}} \langle S_{12}\rangle$$

- Typical Cronin peak in the nuclear modification factor.



CGC at NLO

NLO corrections to particle production in pA collisions



$$\frac{\mathrm{d}\sigma^{qA\to q+X}}{\mathrm{d}^2\boldsymbol{p}_\perp} = \int \frac{\mathrm{d}^2\boldsymbol{x}_1\mathrm{d}^2\boldsymbol{x}_2}{(2\pi)^2} e^{-i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_{12}} \left\{ \left\langle S_{12} \right\rangle + \frac{\alpha_s N_c}{2\pi^2} \int \frac{\mathrm{d}k_g^+}{k_g^+} \int \mathrm{d}^2\boldsymbol{x}_3 \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2\boldsymbol{x}_{32}^2} \left\langle S_{13}S_{32} - S_{12} \right\rangle + O(\alpha_s) \right\}$$

Non-linear evolution of the dipole amplitude

- Absorb the divergence as $k_g^+ \to 0$ in the renormalization of the dipole correlator (or equivalently, the CGC weight-functional). In cut-off regularization

$$\int^{q^+} \frac{\mathrm{d}k_g^+}{k_g^+} = \underbrace{\int_{\Lambda^+}^{k_F^+} \frac{\mathrm{d}k_g^+}{k_g^+}}_{\text{evolution}} + \underbrace{\int_{k_F^+}^{q^+} \frac{\mathrm{d}k_g^+}{k_g^+}}_{\text{NLO coef. func.}}$$

- k_F^+ invariance of the cross-section $\Rightarrow$ Balitsky-Kovchegov equation:

$$\frac{\partial \langle S_{12} \rangle}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int \mathrm{d}^2 \boldsymbol{x}_3 \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2 \boldsymbol{x}_{32}^2} \left[\langle S_{13} S_{32} \rangle - \langle S_{12} \rangle \right]$$

  with $Y = \ln(q^+/k_F^+)$.

- For $k_F^+ \sim Q^2/(2P^-)$, $Y \sim \ln(1/x)$. (More on this after.)
- For other operators or beyond the large N_c limit, Jalilian-Marian–Iancu–McLerran–Weigert–Leonidov–Kovner (JIMLWLK) for the weight function $W_{k_F^-}$.

Consequences of non-linear evolution

- Traveling waves, propagation into the unstable state $N_{12} = 1 - S_{12} = 0$.
- Formal connection with reaction-diffusion processes, mapping to FKPP equation.

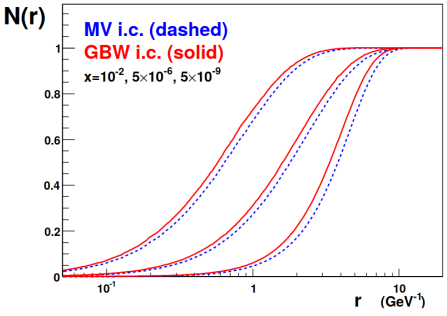
$$\partial_t u = \partial_x^2 u + u - u^2$$

Munier, Peschanski, PRL 91 ,2003

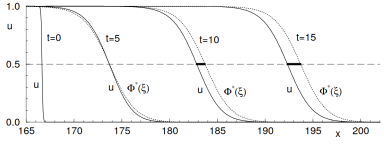
- Growth of Q_s with $1/x$ controled by the BFKL linear regime.

$$Q_s^2(x) \simeq Q_0^2 \left(\frac{1}{x}\right)^{\bar{\alpha}_s \frac{\chi(\gamma_c)}{\gamma_c}} \quad , \quad \frac{\chi(\gamma_c)}{\gamma_c} \simeq 4.88$$

- Unitarity limit $\sigma_{\rm tot} = 2\int \mathrm{d}^2\boldsymbol{b}_\perp N_{12} \to 2\pi R^2$



Albacete, Armesto, Milhano, Salgado, 0902.1112



van Saarloos, *Front propagation into unstable states*, cond-mat/0308540

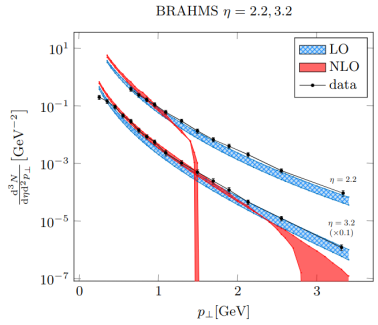
State of the art in multi-loop calculations in the CGC

- NLL high-energy evolution equation
  - BK equation Balitsky, Chirilli, 0710.4330
  - JIMWLK generalization Kovner, Lublinsky, Mulian, 1310.0378
  - with massivie quarks Dai, Lublinsky,2203.13695
- N^2LL high-energy evolution equation
  - Conformal kernel Caron-Huot, Herranen, 1806.01361
  - Full kernel in the planar limit Brunello, Caron-Huot, Crisanti, Giroux, Smith, 2508.03794
- LL evolution of helicity TMDs Kovchegov Sievert, 1808.09010, Cougoulic Kovchegov 1910.04268, Cougoulic, Kovchegov, Tarasov, Tawabutr, 2204.11898, Kovchego Santiago 2108.03667

- Forward pA processes
  - Single inclusive hadron/jet Chirilli, Xiao, Yuan 1203.6139, Liu, Xie, Kang, Liu, 2204.03026
  - Drell-Yan+jet Taels, 2308.02449
- DIS processes
  - structure function (light quarks) Balitsky, Chirilli, 1207.3844, Beuf 1708.06557, Hänninen, Lappi, Paatelainen, 1711.08207
  - massive quarks Beuf, Lappi, Paatelainen, arXiv:2103.14549, 2204.02486
  - diffractive Beuf, Lappi, Mäntysaari, Paatelainen, Penttala, 2401.17251
  - Single inclusive/exclusive hadron/jet Bergabo, Jalilian-Marian, arXiv:2210.03208 Fucilla, Grabovsky, Li, Szymanowski, Wallon, arXiv:2310.11066 PC, Ferrand, Salazar, 2401.01934
  - Inclusive/exclusive dijet Boussarie, Grabovsky, Szymanowski, Wallon, 1606.00419 PC, Salazar, Venugopalan, 2108.06347 Taels, Altinoluk, Beuf, Marquet, 2204.11650 Fucilla, Grabovsky, Li, Szymanowski, Wallon,2211.05774
  - Inclusive jet+photon Roy, Venugopalan, 1911.04530
  - Exclusive vector meson Mäntysaari, Penttala, 2204.14031

Two "historical" issues in small-x NLO calculation

- NLO coefficient function large and give negative cross-section.

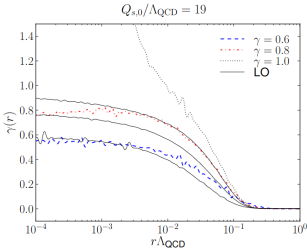


Staśto, Xiao, Zaslavsky, 1307.4057

- Instable NLL BK/JIMWLK equation. Lappi, Mäntysaari, 1502.02400
- Large negative double collinear logarithm in the NLL kernel.

$$\frac{\partial S_{12}}{\partial Y} = \bar{\alpha}_s \int \frac{d^2\boldsymbol{x}_3}{2\pi} \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2 \boldsymbol{x}_{32}^2} \left\{ 1 - \frac{\bar{\alpha}_s}{2} \ln \frac{\boldsymbol{x}_{13}^2}{\boldsymbol{x}_{12}^2} \ln \frac{\boldsymbol{x}_{32}^2}{\boldsymbol{x}_{12}^2} + \ldots \right\} [S_{13} S_{32} - S_{12}]$$

- Similar problem in the dilute BFKL limit. Salam hep-ph/9806482, Ciafaloni, Colferai hep-ph/9812366, Sabio Vera hep-ph/0505128, ...



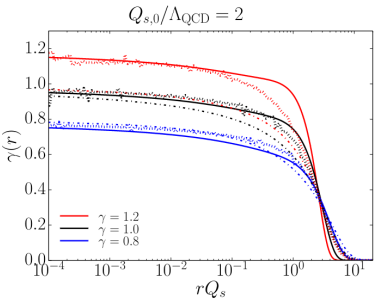
(b) $y = 5$

Resummation of double collinear logarithms

- Double log comes from misidentification of ordering variable (k^+ vs k^-) $\Rightarrow$ infer the resummation of these double logs.
- Implemented via a modified kernel or via a kinematic constraint which enforces life-time (k^-) ordering. Beuf, 1401.0313, Ducloué, Iancu, Mueller, Soyez, Triantafyllopoulos, 1902.06637

$$\frac{\partial S_{12}}{\partial Y} = \bar{\alpha}_s \int \frac{\mathrm{d}^2 \boldsymbol{x}_3}{2\pi} \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2 \boldsymbol{x}_{32}^2} \frac{\mathrm{J}_1\left(2\sqrt{\bar{\alpha}_s \rho_{123}^2}\right)}{\sqrt{\bar{\alpha}_s \rho_{123}^2}} \left[S_{13} S_{32} - S_{12}\right]$$

$$\rho_{123}^2 = \ln \frac{\boldsymbol{x}_{13}^2}{\boldsymbol{x}_{12}^2} \ln \frac{\boldsymbol{x}_{32}^2}{\boldsymbol{x}_{13}^2}$$



$Y = 5$

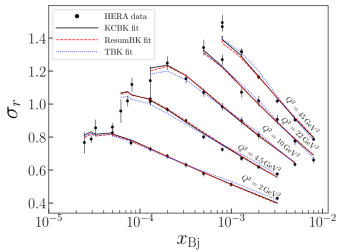
Lappi, Mäntysaari, 1601.06598

NLO phenomenology (selected examples)

Nota bene

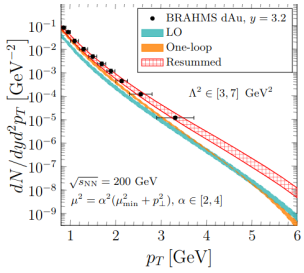
In general, not full NLO, as partial NLL accuracy is reached.

- Inclusive DIS cross-section.
- Various implementation of "collinearly improved" BK equation studied, including k_g^- ordered evolution.



G. Beuf, H. Hänninen, T. Lappi, H. Mäntysaari, 2007.01645

- Single inclusive hadron production in pA.
- Kinematic constraint effectively implemented in the NLO coefficient function.



Yu Shi, Lei Wang, Shu-Yi Wei, Bo-Wen Xiao, 2112.06975

Recent developments: scheme-transformation in the non-linear regime

- NLO calculation are most conveniently done using k_g^+ factorization.
- Yet, the physical choice of ordering variable should be k_g^-.
- Scheme transformation from projectile to target factorization is mediated by an operator in the space of CGC correlators, $|\bar{S}) = \exp(-\alpha_s L(\mu^2))|S)$ Boussarie, PC, Korcyl, Mehtar-Tani, 2509.20236 2511.07509

$$\bar{S}_{12} = S_{12} - \frac{\alpha_s N_c}{\pi} \int \mathrm{d}^2 \boldsymbol{x}_3 \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2 \boldsymbol{x}_{32}^2} \ln(\mu^2 \boldsymbol{r}_{123}^2) \left[S_{12} S_{32} - S_{12}\right]$$

- μ : collinear scale $\sim Q^2$, fixed by minimizing high-energy logarithms in the NLO coefficient function.
- A cross-section $\sigma = (H|S) = (H|e^L e^{-L}|S) = (\bar{H}|\bar{S})$ is scheme-independent

$$\bar{H}_{\mathrm{NLO}} = H_{\mathrm{NLO}} + \frac{\alpha_s N_c}{\pi} H_{\mathrm{LO}} \otimes \int \mathrm{d}^2 \boldsymbol{x}_3 \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2 \boldsymbol{x}_{32}^2} \ln(\mu^2 \boldsymbol{r}_{123}^2) \left[S_{12} S_{32} - S_{12}\right]$$

- Same counterterm arises when computing directly with a k^- regulator. Altinoluk, Beuf, Penttala, 2510.08550

Concrete example: DIS at NLO at small-x in the collinear limit

- In k_g^+ factorization

$$F_L^{\mathrm{NLO}} = F_L^{\mathrm{LO}} \times \underbrace{\frac{\alpha_s N_c}{\pi^2} \int_{1/Q^2}^{1/Q_s^2} \frac{\mathrm{d}^2 \boldsymbol{x}_{13}}{\boldsymbol{x}_{13}^2} \ln\left(\frac{q^+}{k_F^+ Q^2 \boldsymbol{x}_{13}^2}\right)}_{=-2\bar{\alpha}_s \ln^2(Q/Q_s)+\ldots} + \ldots$$

- Add counterterm from the scheme transformation

$$\Delta R = F_L^{\mathrm{LO}} \times \frac{\alpha_s N_c}{\pi^2} \int_{1/Q^2}^{1/Q_s^2} \frac{\mathrm{d}\boldsymbol{x}_{13}^2}{\boldsymbol{x}_{13}^2} \ln(\boldsymbol{x}_{13}^2 \mu^2)$$

- Get:

$$F_L^{\mathrm{NLO}} + \Delta R = F_L^{\mathrm{LO}} \times \underbrace{\frac{\alpha_s N_c}{\pi^2} \int_{1/Q^2}^{1/Q_s^2} \frac{\mathrm{d}^2 \boldsymbol{x}_{13}}{\boldsymbol{x}_{13}^2} \ln\left(\frac{q^+ \mu^2}{k_F^+ Q^2}\right)}_{=\bar{\alpha}_s \ln(Q/Q_s) \ln(x_B/x_F)} + \ldots$$

- For $\mu^2 = Q^2$, NLO coefficient function is $\ll 1$ for the natural choice $k_F^+ \sim q^+$.

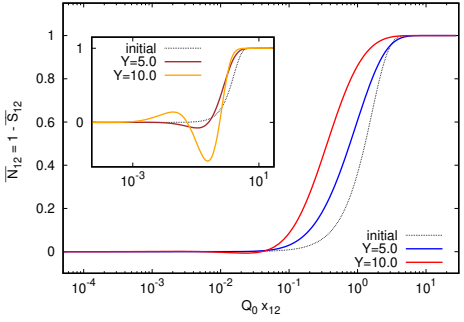
Stabilizing the NLL BK equation

- New NLL BK equation after rotation, in k^--scheme $\eta = \ln(k_F^+ P^-/Q^2) = \ln(1/x_F)$

$$\frac{\partial \bar{S}_{12}}{\partial \eta} = \alpha_s K_{\rm LL} \otimes \bar{S}_{12} + \alpha_s^2 K_{\rm NLL} \otimes \bar{S}_{12} + \alpha_s^2 [K_{\rm LO}, L_{\rm LO}] \otimes \bar{S}_{12}$$

Boussarie, PC, Korcyl, Mehtar-Tani, 2511.07509

- Equation free of double collinear logarithm, μ independent to all orders.



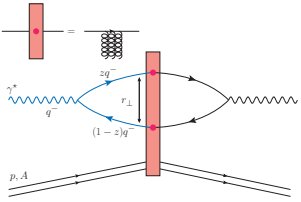
TMD physics from the CGC

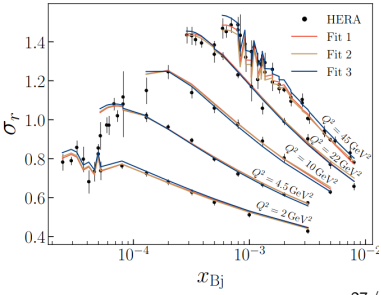
based on Review for the Encyclopedia of Nuclear Physics, PC, F. Salazar, arXiv:2610.26004
(feedback welcome!)

Why going beyond inclusive observables?

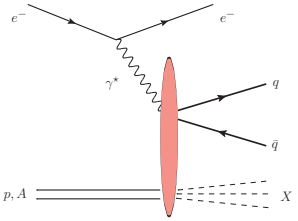
- Total DIS cross-section sensitive to saturation when $Q^2 \sim 1/r_\perp^2 \lesssim Q_s^2 \sim 1$ GeV^2.
- **But** NP contamination for semi-hard Q^2.

  Mäntysaari, Zurita, 1804.05311, Mäntysaari, Schenke, 1806.06783
- In general, no single observable establishes saturation.
- Conceptual interest: how to perform small-x resummation in the non-linear regime within the established TMD framework?

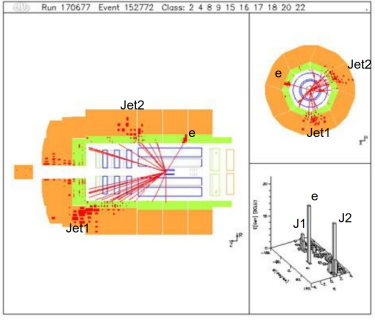




A "golden channel": inclusive back-to-back dijet production in DIS



- Production of two collimated sprays of hadrons, nearly back-to-back in the transverse plane.



- Measure azimuthal correlation between the two jets.

Interest of semi-inclusive processes: back-to-back dijets

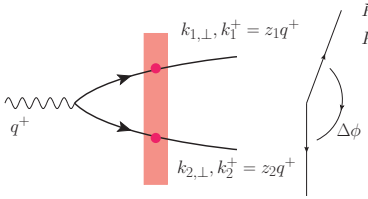
- More than one transverse scale! For back-to-back dijet: $P_\perp \gg K_\perp$
- $P_\perp$ hard, $K_\perp \sim Q_s$ semi-hard.
- Imprint of saturation on final state correlations.

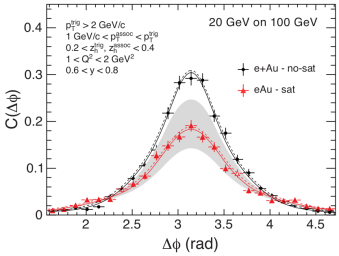
  Dominguez, Marquet, Xiao, Yuan, 1101.0715, Dumitru, Lappi, Skokov, 1508.04438, Dumitru, Skokov, Ullrich, 1809.02615
- **But:** soft gluon radiation effects spoil this nice picture.

  Mueller, Xiao, Yuan, 1308.2993

Ideal probe

Semi-inclusive observables with a hard scale, sensitive to Q_s through final state correlations.





Zheng, Aschenauer, Lee, Xiao, 1403.2413

LO cross-section in the CGC

- Differential cross-section at leading order:

$$\left.\frac{\mathrm{d}\sigma^{\gamma^*_\lambda + A \to q\bar{q}+X}}{\mathrm{d}^2\boldsymbol{k}_1 \mathrm{d}^2\boldsymbol{k}_2 \mathrm{d}\eta_1 \mathrm{d}\eta_2}\right|_{\mathrm{LO}} = \frac{\alpha_{\mathrm{em}} e_f^2 N_c}{(2\pi)^6} \int_{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_{1'},\boldsymbol{x}_{2'}} e^{-i\boldsymbol{k}_1\cdot\boldsymbol{x}_{11'}} e^{-i\boldsymbol{k}_2\cdot\boldsymbol{x}_{22'}} \Xi_{\mathrm{LO}}(\boldsymbol{x}_1,\boldsymbol{x}_2;\boldsymbol{x}_{1'},\boldsymbol{x}_{2'}) \mathcal{R}^{\lambda}_{\mathrm{LO}}(\boldsymbol{x}_{12},\boldsymbol{x}_{1'2'})$$

- Convolution between perturbative factor describing the $\gamma^\star \to q\bar{q}$ splitting...

$$\mathcal{R}^{\mathrm{L}}_{\mathrm{LO}}(\boldsymbol{r}_{xy},\boldsymbol{r}'_{xy}) = 8 z_1^3 z_2^3 Q^2 K_0(\bar{Q}|\boldsymbol{x}_{12}) K_0(\bar{Q}\boldsymbol{x}_{1'2'})$$

- ... and a color structure describing the interaction of $q\bar{q}$ with the dense target

$$\Xi_{\mathrm{LO}}(\boldsymbol{x}_1,\boldsymbol{x}_2;\boldsymbol{x}_{2'},\boldsymbol{x}_{1'}) = \left\langle \underbrace{Q(\boldsymbol{x}_1,\boldsymbol{x}_2;\boldsymbol{x}_{2'},\boldsymbol{x}_{1'})}_{\textit{quadrupole}} - S_{12} - \underbrace{S_{2'1'}}_{\textit{dipole}} + 1 \right\rangle_Y$$

$$\text{Quadrupole:} \qquad Q(\boldsymbol{x}_1,\boldsymbol{x}_2;\boldsymbol{x}_{2'},\boldsymbol{x}_{1'}) = \frac{1}{N_c}\mathrm{Tr}(V(\boldsymbol{x}_1)V^\dagger(\boldsymbol{x}_2)V(\boldsymbol{x}_{2'})V(\boldsymbol{x}_{1'})^\dagger)$$

- Leading power in x, **but all orders in** $K_\perp/P_\perp$.

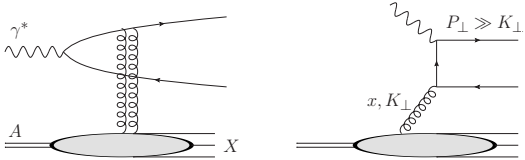
The back-to-back limit at LO

- Power expansion of LO cross-section: keep the LP term in $K_\perp/P_\perp$ and $Q_s/P_\perp$.
- $\Rightarrow$ TMD factorization Dominguez, Marquet, Xiao, Yuan, 1101.0715

$$\left.\frac{\mathrm{d}\sigma^{\gamma^*\to q\bar{q}+X}}{\mathrm{d}^2\boldsymbol{P}_\perp \mathrm{d}^2\boldsymbol{K}_\perp \mathrm{d}\eta_1 \mathrm{d}\eta_2}\right|_{\mathrm{LO}} = \alpha_s \mathcal{H}^{ij}_{\mathrm{LO}}(\boldsymbol{P}_\perp) \int \frac{\mathrm{d}^2\boldsymbol{b}_\perp \mathrm{d}^2\boldsymbol{b}'_\perp}{(2\pi)^4} e^{-i\boldsymbol{K}_\perp\cdot(\boldsymbol{b}_\perp-\boldsymbol{b}'_\perp)} \mathcal{W}^{ij}(\boldsymbol{b}_\perp,\boldsymbol{b}'_\perp) + \mathcal{O}\left(\frac{q_\perp}{P_\perp}\right) + \mathcal{O}\left(\frac{Q_s}{P_\perp}\right)$$

- LO hard factor **identical** to moderate x case. $\boldsymbol{K}_\perp$-dependence in the WW correlator.

$$\mathcal{W}^{ij}(\boldsymbol{b}_\perp,\boldsymbol{b}'_\perp) \equiv -\frac{2}{\alpha_s}\left\langle \frac{1}{N_c}\mathrm{Tr}\left[\partial^i V^\dagger(\boldsymbol{b}_\perp) V(\boldsymbol{b}'_\perp) \partial^j V^\dagger(\boldsymbol{b}'_\perp) V(\boldsymbol{b}_\perp)\right]\right\rangle_x$$



Gluon TMDs at small x

WW gluon TMD

$$xG_{WW}(x,\boldsymbol{K}_\perp) = \int \frac{\mathrm{d}\xi^+\mathrm{d}^2\boldsymbol{\xi}}{4\pi^3 P^-} e^{-ixP^-\xi^+ + i\boldsymbol{K}_\perp\cdot\boldsymbol{\xi}} \langle P|\mathrm{Tr}F^{-i}(\xi^+,\boldsymbol{\xi}) W_{\sqsupset}(\xi^\mu,0) F^{-i}(0,0) W^\dagger_{\sqsupset}(\xi^\mu,0)|P\rangle$$

$\Rightarrow$ Limit $x \to 0$, projectile light-cone gauge

$$xG_{WW}(x,\boldsymbol{K}_\perp) = -\frac{2}{\alpha_s}\int \frac{\mathrm{d}^2\boldsymbol{b}\mathrm{d}^2\boldsymbol{b}'}{(2\pi)^4} e^{-i\boldsymbol{K}_\perp\cdot(\boldsymbol{b}-\boldsymbol{b}')} \frac{1}{N_c} \left\langle \mathrm{Tr}\left[\partial^i V^\dagger(\boldsymbol{b}) V(\boldsymbol{b}') \partial^i V^\dagger(\boldsymbol{b}') V(\boldsymbol{b})\right]\right\rangle_x$$

Dipole gluon TMD

$$xG_{D}(x,\boldsymbol{K}_\perp) = \int \frac{\mathrm{d}\xi^+\mathrm{d}^2\boldsymbol{\xi}}{4\pi^3 P^-} e^{-ixP^-\xi^+ + i\boldsymbol{K}_\perp\cdot\boldsymbol{\xi}} \langle P|\mathrm{Tr}F^{-i}(\xi^+,\boldsymbol{\xi}) W_{\sqsubset}(\xi^\mu,0) F^{-i}(0,0) W^\dagger_{\sqsupset}(\xi^\mu,0)|P\rangle$$

$\Rightarrow$ Limit $x \to 0$, projectile light-cone gauge

$$xG_{D}(x,\boldsymbol{K}_\perp) = \frac{2}{\alpha_s}\int \frac{\mathrm{d}^2\boldsymbol{b}\mathrm{d}^2\boldsymbol{b}'}{(2\pi)^4} e^{-i\boldsymbol{K}_\perp\cdot(\boldsymbol{b}-\boldsymbol{b}')} \frac{1}{N_c} \left\langle \mathrm{Tr}\left[\partial^i V(\boldsymbol{b}) \partial^i V^\dagger(\boldsymbol{b}')\right]\right\rangle_x$$

WW vs dipole gluon TMD in the saturation regime

- LO in photon-gluon fusion channel: TMD factorization Dominguez, Marquet, Xiao, Yuan, 1101.0715

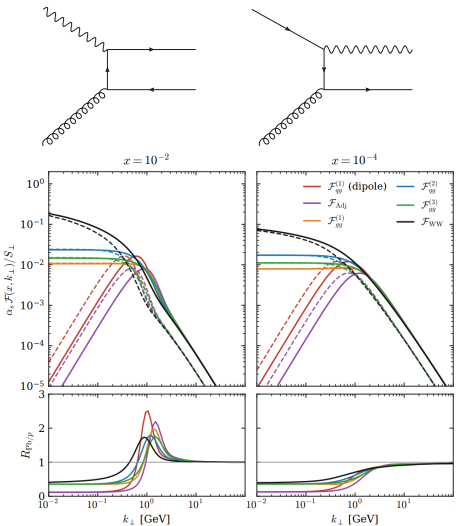
$$\left.\frac{\mathrm{d}\sigma^{\gamma^\star \to q\bar{q}+X}}{\mathrm{d}^2P_\perp \mathrm{d}^2K_\perp}\right|_{\mathrm{LO}} \propto \mathcal{H}(P_\perp) x G_{WW}(x, K_\perp)$$

See also del Castillo, Echevarria, Makris, Scimemi, 2008.07531

- Forward γ-jet production in pA

$$\left.\frac{\mathrm{d}\sigma^{qA \to q\gamma+X}}{\mathrm{d}^2P_\perp \mathrm{d}^2K_\perp}\right|_{\mathrm{LO}} \propto \mathcal{H}(P_\perp) x G_D(x, K_\perp)$$

- Weizsäcker-Williams: particle number interpretation
- Behaves like $Q_s^2/k_\perp^2$ at large $k_\perp$ and $\ln(Q_s^2/k_\perp^2)$ for $k_\perp < Q_s$ in the MV model.

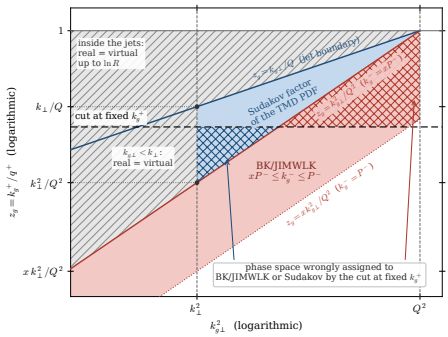


Sudakov logarithms vs high-energy logarithms

- NLO coefficient function features large double and single Sudakov logarithms. Mueller, Xiao, Yuan, 1210.5792

$$\mathrm{d}\sigma^{\gamma^*\to q\bar{q}+X} = \mathcal{H}_{\mathrm{LO}}(P_\perp)\int \mathrm{d}^2\boldsymbol{b}_\perp e^{-i\boldsymbol{K}_\perp\cdot\boldsymbol{b}_\perp} x_F G_{WW}(x_F,\boldsymbol{b}_\perp)$$
$$\times\left[1-\frac{\alpha_s N_c}{4\pi}\ln^2\left(\frac{\boldsymbol{P}_\perp^2\boldsymbol{b}_\perp^2}{c_0^2}\right)-\frac{\alpha_s s_L}{\pi}\ln\left(\frac{\boldsymbol{P}_\perp^2\boldsymbol{b}_\perp^2}{c_0^2}\right)+\ldots\right]$$
$$+\mathcal{H}_{\mathrm{LO}}(P_\perp)\ln\left(\frac{x_F}{x_g}\right)\int \mathrm{d}^2\boldsymbol{b}_\perp e^{-i\boldsymbol{K}_\perp\cdot\boldsymbol{b}_\perp} H_{\mathrm{LL}}\otimes x_F G_{WW}(x_F,\boldsymbol{b}_\perp)$$

- In k^- scheme, no overlap in phase-space, at single log accuracy. PC, Salazar, Schenke, Venugopalan 2208.13872
  $s_L = -\beta_0 - C_F\ln(z_1 z_2 R^2) + N_c\ln(1+Q^2/M_{q\bar{q}}^2)$
- "Naive" k^+ factorization leads to overlapping contribution. Taels, Altinoluk, Beuf, Marquet, 2204.11650
- Hard scale dependence of the WW TMD agrees with the CSS solution with $\mu^2=\zeta=Q^2$.



NLO coefficient function for inclusive dijet

- NLO coefficient function involves new operators beyond WW correlator.
  - (adjoint) dipole gluon TMD
  - 3-point correlator

$$xG_{D,A}(x,\ell) \equiv \frac{C_F \ell^2}{2\pi^2 \alpha_s} \int \frac{\mathrm{d}^2 \boldsymbol{r}_\perp}{(2\pi)^2} \frac{e^{i\boldsymbol{q}_\perp \cdot \boldsymbol{r}_\perp}}{N_c^2 - 1} \left\langle \mathrm{Tr}\left[U_{\boldsymbol{b}_\perp + \frac{\boldsymbol{r}_\perp}{2}} U^\dagger_{\boldsymbol{b}_\perp - \frac{\boldsymbol{r}_\perp}{2}} \right] \right\rangle_x$$

$$\mathcal{Z}^n(x, \boldsymbol{q}_1, \boldsymbol{q}_2) \equiv \frac{iC_F \boldsymbol{q}_2^2}{2\pi^2 \alpha_s} \int \frac{\mathrm{d}^2 \boldsymbol{r}_1 \mathrm{d}^2 \boldsymbol{r}_2}{(2\pi)^4} \frac{e^{i\boldsymbol{q}_1 \cdot \boldsymbol{r}_1 + i\boldsymbol{q}_2 \cdot \boldsymbol{r}_2}}{N_c^2 - 1} \left\langle \mathrm{Tr}\left[U_{\boldsymbol{b}_\perp + \boldsymbol{r}_2/2} U^\dagger_{\boldsymbol{b}_\perp - \boldsymbol{r}_2/2} \partial^n U_{\boldsymbol{b}_\perp + \boldsymbol{r}_2/2 + \boldsymbol{r}_1} U^\dagger_{\boldsymbol{b}_\perp + \boldsymbol{r}_2/2 + \boldsymbol{r}_1} \right] \right\rangle_x$$

PC, Iancu, Salazar, Yuan 2510.08454

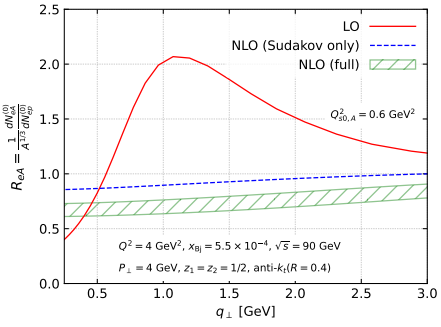
- Leading power in x and in $K_\perp/P_\perp$.
- Expected since high-energy evolution does not generally give closed equations.

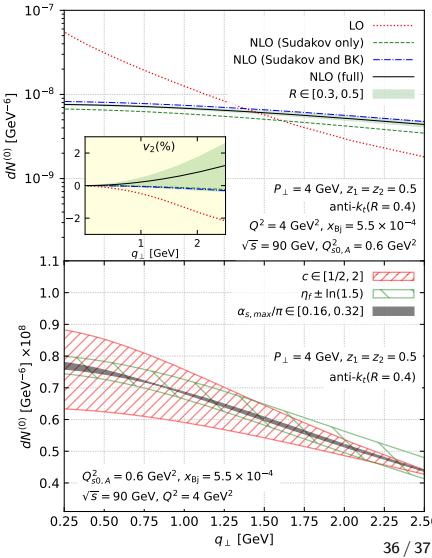
Phenomenology for the EIC

- Factorization of the back-to-back dijet cross-section in terms of the WW gluon TMD persists at NLO.

  PC, Salazar, Schenke, Stebel, Venugopalan, 2308.00022

$$\frac{\mathrm{d}\sigma^{\gamma^\star \to q\bar{q}+X}}{\mathrm{d}^2\boldsymbol{P}_\perp \mathrm{d}^2\boldsymbol{K}_\perp} = \mathcal{H}(P_\perp) \int \mathrm{d}^2\boldsymbol{b}_\perp e^{-i\boldsymbol{K}_\perp\cdot\boldsymbol{b}_\perp} x_g G_{WW}(x_g, \boldsymbol{b}_\perp) S(\boldsymbol{P}_\perp, \boldsymbol{b}_\perp)$$





Concluding remarks

- Many recent developments I didn't cover, in particular subeikonal corrections
  1. bridge towards the moderate x regime, especially relevant for EIC kinematics
  2. access some spin-dependent TMD distributions at small x (helicity TMD g_{1L} notably)
- Rich CGC phenomenology in pA and UPC.

- Progress towards achieving full NLO accuracy. Crucial to go beyond back-of-the envelop calculations.
- TMD factorization at small x: theoretically more challenging, with precision studies still in their infancy.
- High-energy QCD is a rich field with many connections yet to be explored further. *Statistical physics, black hole physics, quantum information science, ...*

Backup

- CGC-TMD correspondence manifest using

$$\partial_i V(\boldsymbol{b}_\perp) = ig \int_{-\infty}^{\infty} \mathrm{d}z^+ V_{[-\infty,z^+]}(\boldsymbol{b}_\perp) \partial_i A^-(z^+, \boldsymbol{b}_\perp) V_{[z^+,\infty]}(\boldsymbol{b}_\perp)$$