# RAPIDITY REGULATORS IN TMD FACTORIZATION AND COMPUTATIONS: THE IMPORTANCE OF THE TMD PORTAL@HADRON2030

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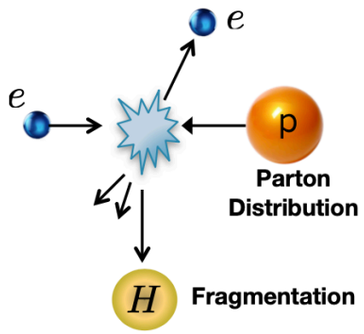
# OUTLOOK

- FACTORIZATION THEOREMS PROOF (COLLINS, SCET, BACKGROUND FIELD METHOD(S))
- TMD DEFINITION AND OVERLAPS
- TMD CALCULATIONS USING DIFFERENT REGULATORS
- STATUS OF PERTURBATIVE CALCULATIONS WITH DIFFERENT REGULATORS
- GOING TO NLP
- GLAUBER GLUONS

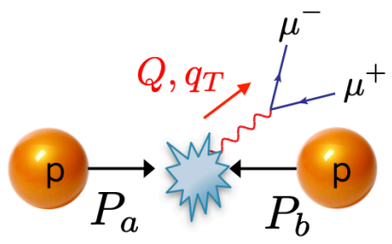
*DISCLAIMER: I will not discuss small-x approaches to TMD factorization, see B. Guiot dedicated talk*

# FACTORIZATION OF THE CROSS SECTION

**Semi-Inclusive DIS**

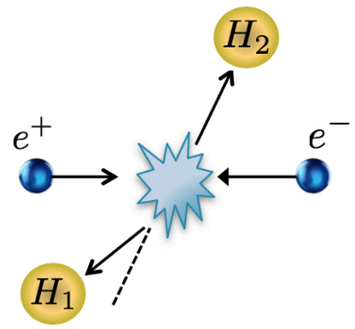



**Drell-Yan**




$q_T \ll Q$

**Dihadron in e⁺e⁻**




$$Q^2 = q^2 = 2q^+q^- - \mathbf{q_T}^2$$

$$W^{\mu\nu}_{\rm DY} = \int \frac{d^4y}{(2\pi)^4} e^{-i(yq)} \sum_X \langle p_1, p_2 | J^{\mu\dagger}(y) | X \rangle \langle X | J^{\nu}(0) | p_1, p_2 \rangle$$

$$W^{\mu\nu}_{\rm SIDIS} = \int \frac{d^4y}{(2\pi)^4} e^{-i(yq)} \sum_X \langle p_1 | J^{\mu\dagger}(y) | p_2, X \rangle \langle p_2, X | J^{\nu}(0) | p_1 \rangle$$

*Factorization is an expansion of the tensor in powers of*

$$\frac{\Lambda}{Q} \lesssim \frac{|q_T| = \text{fixed}}{Q} \ll 1$$

*As a result one obtains universal distributions that reveal quark/gluon motion and interference inside a hadron*

# FACTORIZATION AND REGULATORS

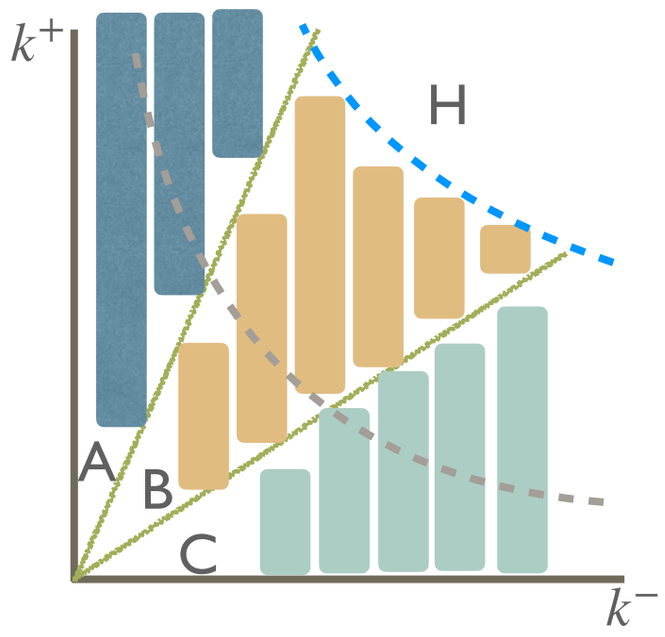
The demonstrations of TMD factorization have been strongly affected by the used/usage of regularization methods.

Dimensional regularization is the standard for UV divergences, and it is common also for IR divergences.

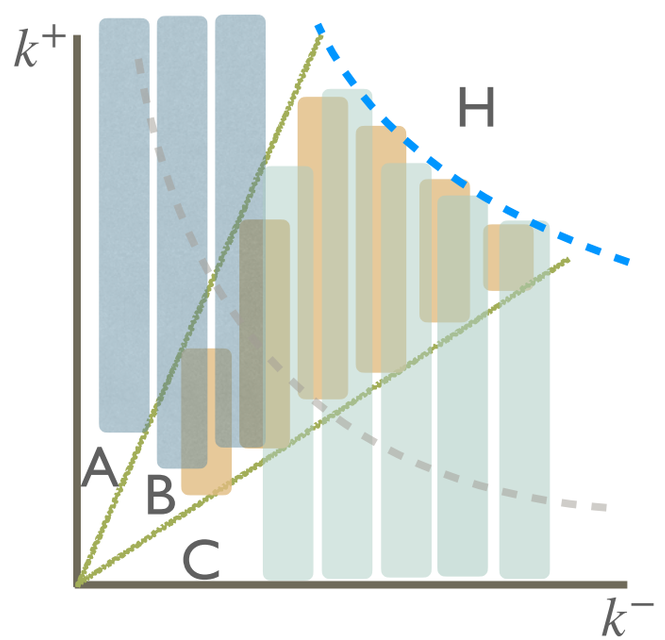
The appearance of rapidity divergences jeopardizes the picture

# FACTORIZATION AND REGULATORS

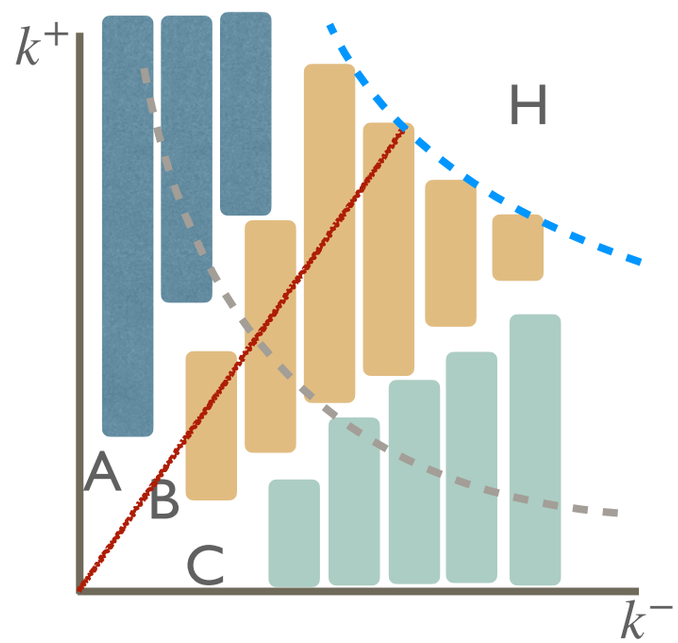
Ideal starting picture




Realistic picture




Final picture




Green lines (rapidity cuts) and overlapping regions are regulator dependent, final factorization is not.

*We have some request for a proper TMD definition:*

- *It must be independent of the rapidity regulator used for its perturbative expansion*
- *In the product of two TMD of the hadronic tensor the regulators and renormalization of each sector are independent*

# BASICS OF TMD FACTORIZATION: MODES AND EXPANSIONS

$\lambda \sim q_T/Q$ is the expansion parameter

We have several modes, fixed by momenta

$$n\text{-collinear} \sim (\lambda^2,1,\lambda)\,, \quad \bar{n}\text{-collinear} \sim (1,\lambda^2,\lambda)\,, \quad \text{soft} \sim (\lambda,\lambda,\lambda)$$

$$\partial^\mu \phi \sim (\lambda^2,1,\lambda)\phi\,, \quad \partial^\mu \phi \sim (1,\lambda^2,\lambda)\phi\,, \quad \partial^\mu \phi \sim (\lambda,\lambda,\lambda)\phi$$

$$A^\nu \sim (\lambda^2,1,\lambda)\,, \quad A^\nu \sim (1,\lambda^2,\lambda)\,, \quad A^\nu \sim (\lambda,\lambda,\lambda)$$

*The explicit Lagrangian agrees at LP, but differs slightly among authors in the literature at NLP:*

- *Collins,* ***LP,*** *2011*
- *SCET,* ***LP:*** *Becher, Neubert; Echevarria, I.S., Idilbi; Chiu et al. 2010-2012,* ***NLP:*** *Ebert et al., 2022*
- *Background field method without soft modes,* ***NLP:*** *Vladimirov, Moos, I.S., 2022*
- *Background field method with soft modes* ***NLP:*** *Del Rio, Jaarsma, I.S. ,Waalewijn, 2025*

# REVIEW OF BACKGROUND FIELD METHOD WITH SOFT MODES (BFM+S)

All fields are re-written in its components

$$\psi = \psi + \psi + \psi + \varphi\,, \qquad A^\mu = A^\mu + A^\nu + A^\mu + B^\mu\,.$$

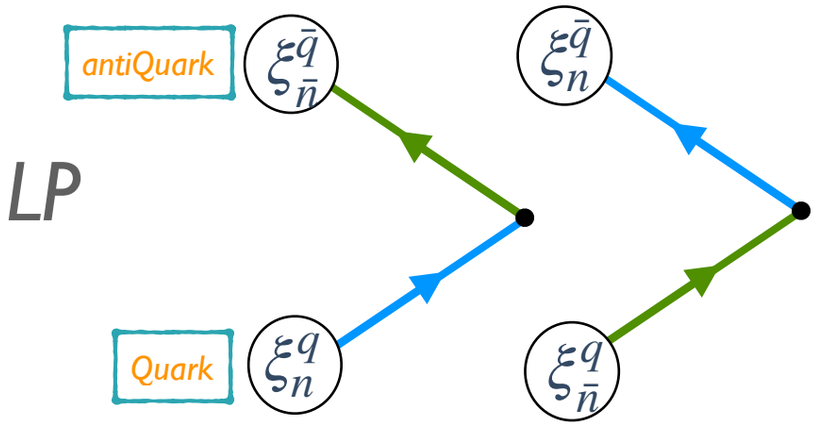
This produces a sum of terms for the QCD action with replicas for each mode

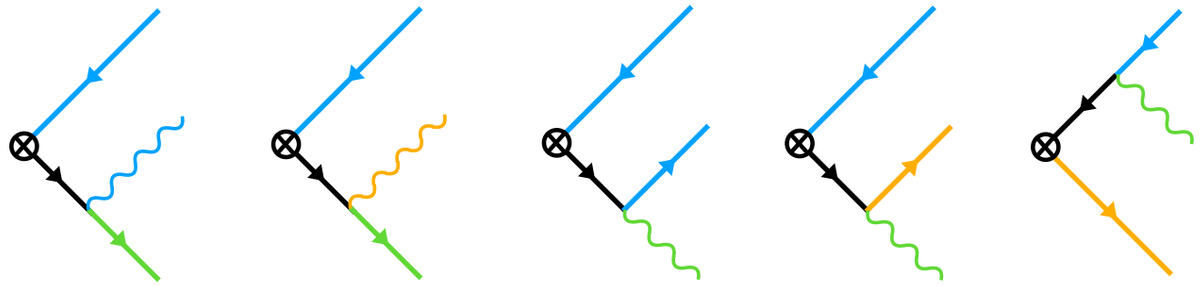
$$S_{\rm QCD}[\bar\psi,\psi,A] = S_{\rm QCD}[\bar\psi,\psi,A] + S_{\rm QCD}[\bar\psi,\psi,A] + S_{\rm QCD}[\bar\psi,\psi,A] + S_{\rm QCD}[\bar\varphi,\varphi,B] + S_{\rm int}[\bar\varphi,\varphi,B;\bar\psi,\psi,A;\bar\psi,\psi,A;\bar\psi,\psi,A]$$

All connected graphs are deduced from

$$e^{iS_{\rm eff}[\bar\psi,\psi,A;\bar\psi,\psi,A;\bar\psi,\psi,A]} = e^{iS_{\rm QCD}[\bar\psi,\psi,A]}\, e^{iS_{\rm QCD}[\bar\psi,\psi,A]}\, e^{iS_{\rm QCD}[\bar\psi,\psi,A]} \times \int \mathcal{D}\bar\varphi\,\mathcal{D}\varphi\,\mathcal{D}B\; e^{iS_{\rm int}[\bar\varphi,\varphi,B;\bar\psi,\psi,A;\bar\psi,\psi,A;\bar\psi,\psi,A]}\, e^{iS_{\rm QCD}[\bar\varphi,\varphi,B]}\,.$$

# BFM+S AND CURRENT OPERATOR




**Step I :** current power expansion

$$J^\mu_{\rm eff}(0) = \left[J^\mu_{\rm eff}(0)\right]^{(2)} + \left[J^\mu_{\rm eff}(0)\right]^{(2.5)} + \left[J^\mu_{\rm eff}(0)\right]^{(3)} + \mathcal{O}(\lambda^{7/2})$$

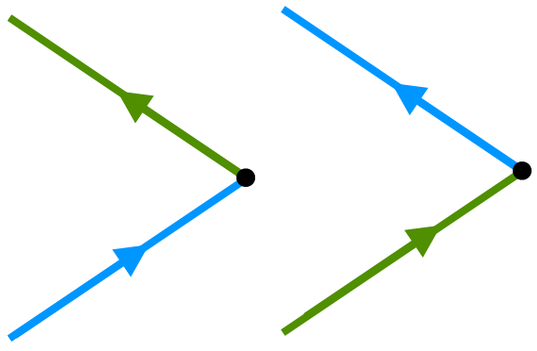
$$\left[J^\mu_{\rm eff}(0)\right]^{(k)} \sim \lambda^k \,.$$

$$\begin{aligned}
J^\mu_{\rm eff}(0) = {}& \bar\psi\gamma^\mu\left[1 - g\frac{1}{{\rm i}\partial^-}A^- + g\frac{1}{{\rm i}\partial^+}A^+ - g\frac{1}{{\rm i}\partial^-}A^- + g\frac{1}{{\rm i}\partial^+}A^+\right]\psi + (n \leftrightarrow \bar n) \\
& + g\Bigg\{-n^\mu\bar\psi\left(\mathcal{A}_T - \frac{{\rm i}\partial\!\!\!/_T}{{\rm i}\partial^-}A^-\right)\frac{1}{{\rm i}\partial^+}\psi - n^\mu\bar\psi\left(\mathcal{A}_T - \frac{{\rm i}\partial\!\!\!/_T}{{\rm i}\partial^-}A^-\right)\frac{1}{{\rm i}\partial^+}\psi \\
& - \bar\psi\gamma_T^\mu\left(\frac{1}{{\rm i}\partial^-}A_T^\rho - \frac{{\rm i}\partial_T^\rho}{({\rm i}\partial^-)^2}A^-\right)\frac{{\rm i}\partial_T^\rho}{{\rm i}\partial^+}\psi - \frac{1}{2}\bar\psi\gamma_T^\mu\left(\frac{{\rm i}\partial\!\!\!/_T}{{\rm i}\partial^-}\mathcal{A}_T - \frac{({\rm i}\partial\!\!\!/_T)^2}{({\rm i}\partial^-)^2}A^-\right)\frac{1}{{\rm i}\partial^+}\psi \\
& - \frac{1}{2}\bar\psi\gamma_T^\mu\gamma^-\left(\mathcal{A}_T - \frac{{\rm i}\partial\!\!\!/_T}{{\rm i}\partial^+}A^+\right)\frac{1}{{\rm i}\partial^-}\psi - \frac{1}{2}\bar\psi\gamma_T^\mu\gamma^-\left(\mathcal{A}_T - \frac{{\rm i}\partial\!\!\!/_T}{{\rm i}\partial^+}A^+\right)\frac{1}{{\rm i}\partial^-}\psi \\
& + (n \leftrightarrow \bar n) + \text{h.c.}\Bigg\} + \mathcal{O}(g^2, \lambda^{7/2})\,.
\end{aligned}$$

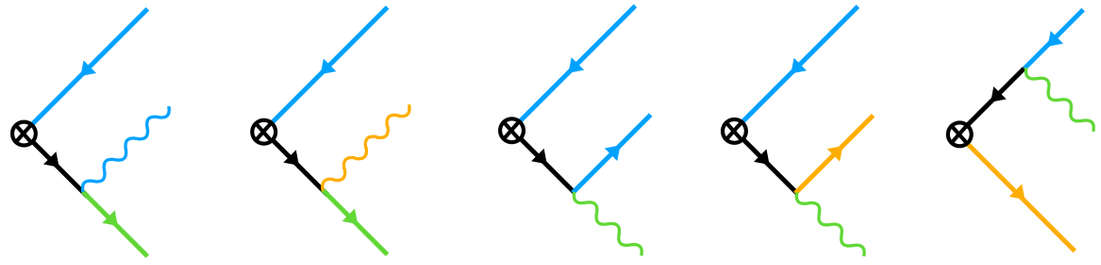
$$\frac{1}{(\partial^-)^k}f(y) = \frac{(-1)^{k-1}}{\Gamma(k)}\int_{\bar L}^0 {\rm d}w^+\,(w^+)^{k-1}\,f(y + w^+\bar n)\,,$$
$$\frac{1}{(\partial^+)^k}g(y) = \frac{(-1)^{k-1}}{\Gamma(k)}\int_{L}^0 {\rm d}w^-\,(w^-)^{k-1}\,f(y + w^-n)\,.$$

# BFM+S AND CURRENT OPERATOR

*LP*



*NLP*



**Step I:** Wilson lines

$$\Psi_n = S_n^\dagger \psi\,, \qquad \mathcal{A}_{T,n}^\rho = S_n^\dagger \big[\mathrm{i}D_T^\rho, S_n\big]\,,$$
$$\Psi_{\bar n} = S_{\bar n}^\dagger \psi\,, \qquad \mathcal{A}_{T,\bar n}^\rho = S_{\bar n}^\dagger \big[\mathrm{i}D_T^\rho, S_{\bar n}\big]\,.$$
$$\chi = \frac{\gamma^+\gamma^-}{2} W^\dagger \psi\,, \qquad \mathcal{A}_T^\rho = W^\dagger \big[\mathrm{i}D_T^\rho, W\big]\,,$$
$$\chi = \frac{\gamma^-\gamma^+}{2} W^\dagger \psi\,, \qquad \mathcal{A}_T^\rho = W^\dagger \big[\mathrm{i}D_T^\rho, W\big]\,.$$

**Step III:** C-,P-,T-symmetries, RPI, current conservation, $i\partial_\mu J^\mu_{\text{eff}} = \mathcal{O}(\lambda^{7/2})$ impose relations among the hard coefficients

*LP*

$$\big[J^\mu_{\text{eff}}(0)\big]^{(2)} = (\gamma_T^\mu)_{\alpha\beta} \int \mathrm{d}y^+\,\mathrm{d}y^-\, C_1(y^+,y^-)\, \bar\chi_\alpha(y^+\bar n)\, S_n^\dagger S_{\bar n}\, \chi_\beta(y^- n) + \text{h.c.}$$

Hard coefficient; Soft WL; Fermion+WL; Fermion+WL

*Hadronic tensor at LP*

# OVERLAPPING OPERATORS

- FORMAL DEFINITION OF OPERATORS IS NOT THE END OF STORY: WE HAVE EXPERIENCE OF TMD FACTORIZATION AND ZERO-BIN SUBTRACTION
- WE WOULD LIKE TO HAVE AN "OPERATOR DEFINITION OF SUBTRACTIONS" TO DEFINE "PURE COLLINEAR STATES"
- WE WOULD LIKE TO HAVE A REGULATOR AT AN OPERATOR LEVEL
- WE NEED TO REGULARIZE RAPIDITY DIVERGENCES

# $\delta$-regulator

- IT IS DEFINED AT AN OPERATOR LEVEL:

$$W(x) = \lim_{\delta^+ \to 0} \mathcal{P}\left\{ \exp\left[ ig \int_L^0 dy^+ \, e^{-s\delta^+ |y^-|} \, A^+(x + y^- n) \right] \right\},$$

$$W(x) = \lim_{\delta^- \to 0} \mathcal{P}\left\{ \exp\left[ ig \int_{\bar{L}}^0 dy^+ \, e^{-\bar{s}\delta^- |y^+|} \, A^-(x + y^+ \bar{n}) \right] \right\},$$

$$S_n(x) = \lim_{\delta^+ \to 0} \mathcal{P}\left\{ \exp\left[ ig \int_L^0 dy^+ \, e^{-s\delta^+ |y^-|} \, A^+(x + y^- n) \right] \right\},$$

$$S_{\bar{n}}(x) = \lim_{\delta^- \to 0} \mathcal{P}\left\{ \exp\left[ ig \int_{\bar{L}}^0 dy^+ \, e^{-\bar{s}\delta^- |y^+|} \, A^-(x + y^+ \bar{n}) \right] \right\},$$

- IT PROVIDES AN EXPONENTIAL DUMPING IN POSITION SPACE. INTEGRALS CAN BE EASIER IN THIS SPACE
- IN MULTILOOP CALCULATIONS WE HAVE FROM WILSON LINE EXPANSION

$$\frac{1}{k_1^+ - i0} \frac{1}{k_2^+ - i0} \frac{1}{k_3^+ - i0} \cdots \to \frac{1}{k_1^+ - i\delta} \frac{1}{k_2^+ - 2i\delta} \frac{1}{k_3^+ - 3i\delta} \cdots$$

*And this is consistent with calculations at 2 loops*

# SUBTRACTION METHOD STEPS

- WE SPLIT THE COLLINEAR BACKGROUND FIELDS UP INTO A PURE COLLINEAR FIELD AND A BACKGROUND FIELD THAT CONTAINS THE OVERLAP WITH THE SOFT REGION.

$$S_{\mathrm{QCD}}[\phi+\phi] = \int \mathrm{d}^d x \left[\bar{\psi}(x)\left(\mathrm{i}\partial\!\!\!/ + g\mathcal{A}\!\!\!/(x) + g\mathcal{A}\!\!\!/(x)\right)\psi(x) + \bar{\psi}(x) g\mathcal{A}\!\!\!/(x)\psi(x) + \bar{\psi}(x) g\mathcal{A}\!\!\!/(x)\psi(x)\right] + \text{field strength} + S_{\mathrm{QCD}}[\phi]$$

- WE REDEFINE THE PURE COLLINEAR FIELDS SUCH THAT THE LEADING-POWER INTERACTIONS WITH THE OVERLAP BACKGROUND FIELD ARE REMOVED (SEE MEHEN, IDILBI, 2005).

$$\psi \to S_n \psi + \psi \qquad A_T^{\mu} \to S_n A_T{}^{\mu} S_n^{\dagger} + A_T{}^{\mu}$$

# SUBTRACTION METHOD STEPS

- THE STRAIGHT MATRIX ELEMENT THAT WE HAVE, IS BY DEFAULT UNSUBTRACTED

$$\langle 0|\chi_{j,\beta}(b_T+b^+\bar{n})|p,X\rangle\langle p,X|\bar{\chi}_{k,\gamma}(0)|0\rangle \equiv \langle 0|\chi_{j,\beta}(b_T+b^+\bar{n})|p,X\rangle\langle p,X|\bar{\chi}_{k,\gamma}(0)|0\rangle_{\text{unsub}}$$

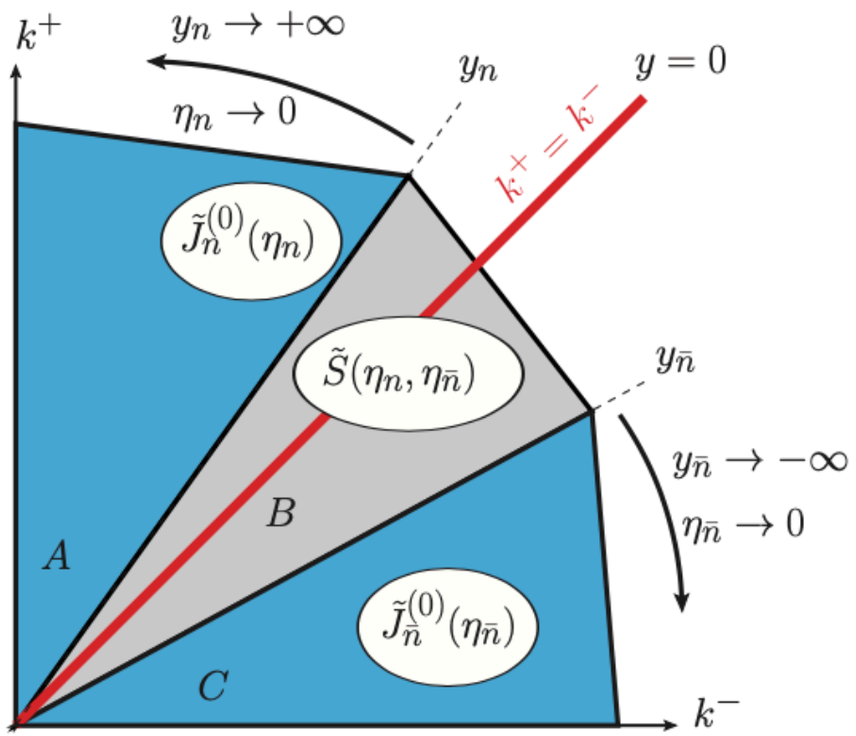
- WE PASS FROM UNSUBTRACTED TO SUBTRACTED WITH ZERO-BIN SUBTRACTIONS

$$\langle 0|\chi_{j,\beta}(b_T+b^+\bar{n})|p,X\rangle\langle p,X|\bar{\chi}_{k,\gamma}(0)|0\rangle_{\text{unsub}}(\delta^+) \longrightarrow \langle 0|\chi_{j,\beta}(b_T+b^+\bar{n})|p,X\rangle\langle p,X|\bar{\chi}_{k,\gamma}(0)|0\rangle_{\text{sub}}(\delta^-)$$

$$\langle 0|\chi_{j,\beta}(b_T+b^+\bar{n})|p,X\rangle\langle p,X|\bar{\chi}_{k,\gamma}(0)|0\rangle_{\text{sub}}(\delta^-) = \frac{\langle 0|\chi_{j,\beta}(b_T+b^+\bar{n})|p,X\rangle\langle p,X|\bar{\chi}_{k,\gamma}(0)|0\rangle(\delta^+)}{\frac{1}{N_c}\text{tr}\left[\langle 0|\left[S_{\bar{n}}^\dagger S_n(b_T)\right]|X\rangle\langle X|\left[S_n^\dagger S_{\bar{n}}(0)\right]|0\rangle(\delta^+\delta^-)\right]}$$

Only the subtracted matrix element depend onthe $\delta$-regulator consistent with its sector ($\delta^\pm$ for $\pm$ sectors)

# LOOKING BACKWARDS




This already discussed more then a decade ago,

Echevarria, Idilbi, I.S. *Phys.Lett.B* 726 (2013) 795,
see also Collins, Rogers *Phys.Rev.D* 87 (2013) 3, 034018

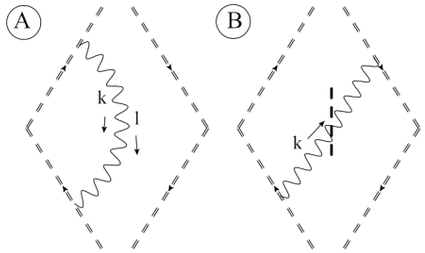
$e^{y_n} \gg 1, \quad e^{y_{\bar{n}}} \ll 1, \quad e^{y} \sim 1,$ **where** $y = \frac{1}{2} \ln \frac{k^+}{k^-}$

In the figure $\eta_{n,\,\bar{n}}$ are rapidity cutoffs, $\eta_{n,\,\bar{n}} \sim \delta^{\pm}$ and pure (or subtracted) TMD elements are drawn $\tilde{J}^{(0)}_{\bar{n},\,n}$

# LOOKING BACKWARDS

## More technical questions: How to distinguish between IR and rapidity divergences?

An example in the soft function, Phys.Rev.D 93 (2016) 5, 054004




$$S^{[1]}_{[A]} = 2\frac{a_s}{(4\pi)^{-\varepsilon}}\mu^{2\varepsilon}\underbrace{(-)(\delta_+\delta_-)^{-\varepsilon}\Gamma^2(\varepsilon)\Gamma(1-\varepsilon)}_{\text{IR}}$$

$$S^{[1]}_{B} = \frac{2a_s}{(4\pi)^{-\varepsilon}}\mu^{2\varepsilon}\left[(\delta_+\delta_-)^{-\varepsilon}\Gamma^2(\varepsilon)\Gamma(1-\varepsilon) - \left(\frac{b^2}{4}\right)^{\varepsilon}\Gamma(-\varepsilon)\left(\ln\frac{b^2|\delta_+\delta_-|}{4e^{-2\gamma_E}} - \psi(-\varepsilon) - \gamma_E\right)\right]$$

The soft function linearly depends only on logs of rapidity divergences at all orders. The same occurs in other TMD matrix elements. All this checked at higher orders.

Ordering the limits, we can see that virtual corrections explicit calculations are not needed

$$\lim_{\delta^\pm\to 0} S^{[1]}_{[A]} = 0$$

$$\lim_{\delta^\pm\to 0} S^{[1]}_{B} = \frac{2a_s}{(4\pi)^{-\varepsilon}}\mu^{2\varepsilon}\left[-\left(\frac{b^2}{4}\right)^{\varepsilon}\Gamma(-\varepsilon)\left(\ln\frac{b^2|\delta_+\delta_-|}{4e^{-2\gamma_E}} - \psi(-\varepsilon) - \gamma_E\right)\right] = S^{[1]}_{A} + S^{[1]}_{B}$$

# TMD DEFINITION WITH $\delta$-REGULATOR

The Collins-Soper kernel K appears inside the soft function

$$S^{\text{bare}}(b,\delta^+\delta^-) = \exp\left[K^{\text{bare}}(b)\ln(\delta^+\delta^-) + s^{\text{bare}}(b)\right]$$

We use the linearity in $\ln(\delta^+\delta^-)$

$$S^{\text{bare}}(b,\delta^+\delta^-) = \sqrt{S^{\text{bare}}\left(b,\zeta(\delta^+/q^+)^2\right)}\ \sqrt{S^{\text{bare}}\left(b,\bar{\zeta}(\delta^-/q^-)^2\right)}$$

And get the TMDPDF definition

$$\left[\mathcal{F}^{\text{bare}}_{q,11}(x,b_T,\zeta)\right]_{\beta\alpha} = \int\frac{db^-}{2\pi}\,e^{+ib^-q^+}\,\langle P|\bar{\chi}_{i,\alpha}(b_T+b^-n)|X\rangle\langle X|\chi_{i,\beta}(0)|P\rangle(\delta^-)_{\text{sub}}\sqrt{S\left(b_T,\zeta(\delta^-/q^+)^2\right)}$$

$$\left[\mathcal{F}^{\text{bare}}_{q,11}(x,b_T,\bar{\zeta})\right]_{\beta\alpha} = \int\frac{db^-}{2\pi}\,e^{+ib^-q^+}\,\langle P|\bar{\chi}_{i,\alpha}(b_T+b^-n)|X\rangle\langle X|\chi_{i,\beta}(0)|P\rangle(\delta^+)_{\text{sub}}\sqrt{S\left(b_T,\bar{\zeta}(\delta^+/q^+)^2\right)}$$

with $\quad\zeta\bar{\zeta} = 2q^+q^- = Q^4 + \mathcal{O}(q_T^2Q^2)$

# REVIEW OF MOST USED REGULATORS

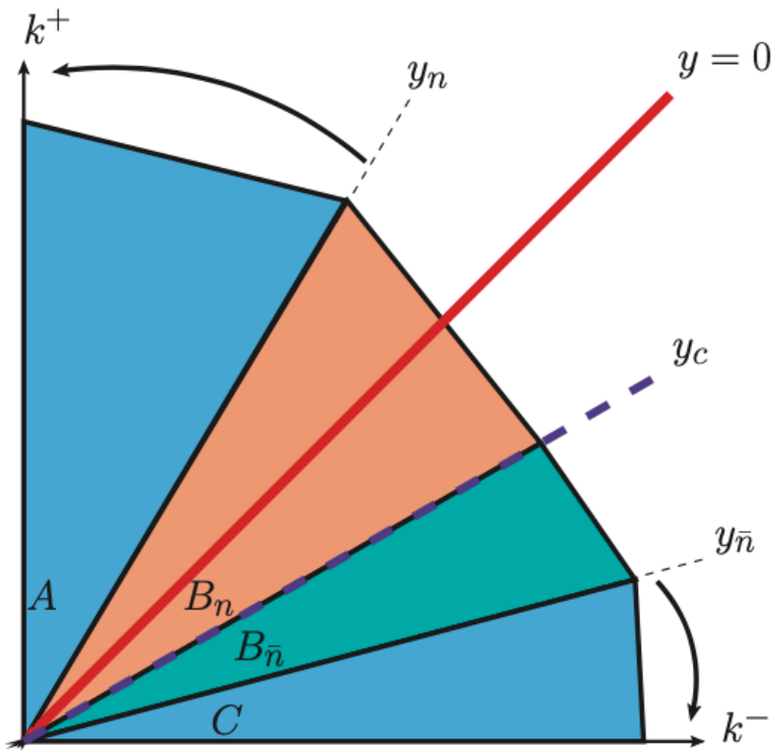
- BECHER, NEUBERT (2010), ***ANALYTIC REGULATOR***, *USEFUL FOR EXTRACTIONS OF EVOLUTION OF THE PRODUCT OF 2 TMD. IMPOSSIBLE TO GET A DEFINITION OF A TMD*

- COLLINS (2011), ***WILSON LINES OFF-THE LIGHT CONE,*** *USED FOR THE DEMONSTRATIONS AND DEFINITION OF TMD. MORE DIFFICULT FOR PERTURBATIVE CALCULATIONS BEYOND ONE LOOP.*

- ECHEVARRIA, IDILBI, SCIMEMI (2012), AND VLADIMIROV (2016), $\delta$-REGULATOR, HAS AN OPERATOR DEFINITIONS, EASY FOR THE DEMONSTRATIONS (ESPECIALLY IN POSITION SPACE), IT IS USED IN THE DEFINITION OF (HIGHER TWIST) TMD AND FOR PERTURBATIVE CALCULATIONS BEYOND ONE LOOP. USED AT NLP

- CHIU, JAIN, NEILL, ROTHSTEIN (2012) AND ROTHSTEIN, STEWART (2016), ***RAPIDITY REGULATOR,*** *EASY FOR THE DEMONSTRATIONS, AND FOR PERTURBATIVE CALCULATIONS BEYOND ONE LOOP. USED AT NLP AND GLAUBER*

# WILSON LINES OFF THE LIGHT-CONE




In Collins (2011) definition and explicit Rapidity cutoff $y_c$ is introduced to split the soft region.

This strategy may lead to the naive definition

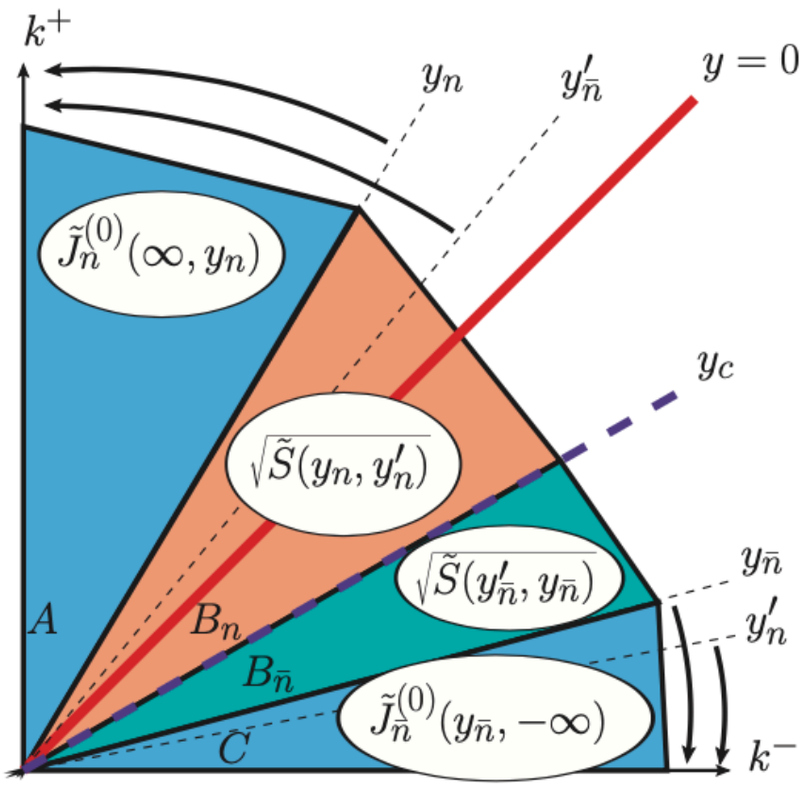
$$\tilde{F}_n^{\text{JCC(naive)}}(x_n, b; \sqrt{\zeta_n}, \mu) = \lim_{y_{\bar{n}} \to -\infty} \frac{\tilde{J}_n(y_{\bar{n}})}{\tilde{S}(y_c, y_{\bar{n}})}$$

Which is spoiled by uncanceled self-energy at $y_c$.
The correct definition reads

$$\tilde{F}_n^{\text{JCC}}(x_n, b; \sqrt{\zeta_n}, \mu) = \lim_{y_n \to +\infty \& y_{\bar{n}} \to -\infty} \tilde{J}_n(y_{\bar{n}}) \sqrt{\frac{\tilde{S}(y_n, y_c)}{\tilde{S}(y_c, y_{\bar{n}})\, \tilde{S}(y_n, y_{\bar{n}})}}$$

And it is independent of rapidity cuts

# EQUIVALENCE OF $\delta$-REGULATOR AND WILSON LINES OFF-THE-LIGHT-CONE




$\delta$-regulator in practice introduces a set of rapidity cuts different from the ones of Collins

$$y'_n = \ln \frac{\nu\delta^-}{\mu} \qquad y'_{\bar{n}} = \ln \frac{\nu\mu}{\delta^+}$$

$$y_c = \lim_{\substack{y_n \to +\infty \\ y'_n \to -\infty}} \frac{1}{2}(y_n + y'_n) = \lim_{\substack{y'_{\bar{n}} \to +\infty \\ y_{\bar{n}} \to -\infty}} \frac{1}{2}(y'_{\bar{n}} + y_{\bar{n}}) = \frac{1}{2}\ln\nu$$

And one recovers Collins'

$$\zeta = (p^+)^2 e^{-2y_c}, \quad \bar{\zeta} = (p^-)^2 e^{2y_c}$$

# RAPIDITY REGULATOR

$$\int d^{4-2\varepsilon}k \to \int d^{4-2\varepsilon}k \left(\frac{2k_z}{\nu}\right)^{\eta}$$

- ZERO-BIN IS ZERO
- THERE IS STILL A SOFT FUNCTION
- IT IS USUALLY ASSUMED JUST ONE REGULATOR PARAMETER FOR COLLINEAR AND ANTI-COLLINEAR SECTORS: IT IS NOT FORBIDDEN TO DISTINGUISH THE TWO BUT PRACTITIONERS DO NOT CARE
- SEVERAL 3-LOOPS CALCULATIONS ARE DONE USING THIS REGULATOR
- IT IS USEFUL FOR THE DISCUSSION OF GLAUBER GLUONS
- THE DEFINITION OF TMD WORKS SIMILARLY TO $\delta$-regulator

# PERTURBATIVE RESULTS

| Name | Leading matching function | Twist of leading matching | Highest perturbative order |
|---|---|---|---|
| Unpolarized, $f_1(x,b)$ | $f_1(x)$ | tw-2 | N$^3$LO ($\alpha_s^3$) |
| Helicity, $g_1(x,b)$ | $g_1(x)$ | tw-2 | N$^3$LO ($\alpha_s^3$) |
| Transversity, $h_1(x,b)$ | $h_1(x)$ | tw-2 | N$^3$LO($\alpha_s^3$) |
| Sivers, $f_{1T}^{\perp}(x,b)$ | $T(-x,0,x)$ | tw-3 | NLO ($\alpha_s^1$) |
| Boer-Mulders, $h_1^{\perp}(x,b)$ | $E(-x,0,x)$ | tw-3 | NLO ($\alpha_s^1$) |
| Worm-gear-T, $g_{1T}^{\perp}(x,b)$ | $g_1$, $T$, $\Delta T$ | tw-2/3 | NLO ($\alpha_s^1$) |
| Worm-gear-L, $h_{1L}^{\perp}(x,b)$ | $h_1$, $H$ | tw-2/3 | NLO ($\alpha_s^1$) |
| Pretzelosity, $h_{1T}^{\perp}(x,b)$ | $H$ | tw-3/4 | LO ($\alpha_s^0$, only tw-3) |

| REGULATORS | OFF-LIGHT-CONE | DELTA | RAPIDITY |
|---|---|---|---|
| tw-2 | X(NLO) | X(NNLO) | X(N3LO) |
| tw-3 | | X(NLO) | |
| Soft factor | X(NLO) | X(NNLO) | X(N3LO) |
| NLP | | X | X |
| Jets | | X | X |
| GTMD | | X | |

# SOFT GLUONS IN TW3: "ENDPOINT" DIVERGENCES (ED)

*Higher twist operators have also "end-point" divergencies (ED)*

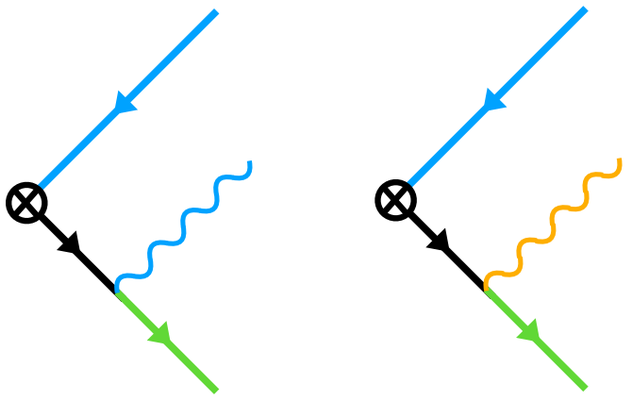
$$\left[F_{21}^{\text{naive}}(x,\xi,b_T)\right]_{\delta\alpha} = \int \frac{\mathrm{d}b_1^-}{2\pi}\frac{\mathrm{d}b_2^-}{2\pi}\, e^{-\mathrm{i}\bar{\xi}xb_1^- P^+}\, e^{-\mathrm{i}\xi x b_2^- P^+} \times \langle P|\left[\bar{\chi}(b_T+b_1^- n)\mathcal{A}_T^\rho(b_T+b_2^- n)\right]_{i,\alpha}|X\rangle\langle X|\,\chi_{i,\delta}(0)\,|P\rangle\,.$$

*ED are due to soft gluon limit of twist-3 matrix elements that overlap with twist-2*

$$\left[F_{21}^{\text{o.s.}}(x,\xi,b_T)\right]_{\delta\alpha} = \mathrm{i}xP^+\int \frac{\mathrm{d}b_1^-}{2\pi}\frac{\mathrm{d}b_2^-}{2\pi}\, e^{-\mathrm{i}\bar{\xi}xb_1^- P^+}\, e^{-\mathrm{i}\xi x b_2^- P^+}\,\langle P|\,\bar{\chi}_\alpha(b_T+b_1^- n)\,|X\rangle\langle X|\,\chi_\delta(0)\,|P\rangle \times \langle 0|\,S_{\bar{n}}^\dagger S_n(b_T)\,\mathcal{A}_{T,n}^\rho(b_T+b_2^- n)\,|X\rangle\,\langle X|\,S_n^\dagger S_{\bar{n}}(0)\,|0\rangle\,.$$

- *Here we use the arguments of Del Rio, Jaarsma, I.S. ,Waalewijn, 2025 which are more general*
- *In Vladimirov, Moos, Scimemi, 2022 the same cancellation is related to derivatives of the CS kernel*

# SOFT GLUON SUBTRACTION AT NLP WITH ED



*ED appear already at LO when the gluons become soft*

$$\left[F_{21}^{\text{naive}}(x,\xi,b_T)\right]^{\text{LO}}_{\delta\alpha} = a_s\,\frac{b_T^\sigma}{b_T^2}\theta(-\xi x)\left[\gamma_T^\sigma\gamma_T^\rho\gamma^- x - 2(1+\xi x)\frac{g_T^{\rho\sigma}}{\xi - \mathrm{i}\delta^+/xP^+}\gamma^-\right]_{\delta\alpha}$$

*ED cancellation need an operator subtraction. At LO*

$$\lim_{\xi\to 0}\xi\left(\left[F_{21}^{\text{naive}}(x,\xi,b_T)\right]^{\text{LO}}_{\delta\alpha} - \left[F_{21}^{\text{o.s.}}(x,\xi,b_T)\right]^{\text{LO}}_{\delta\alpha}\right) = 0\,,$$

*The final contribution to the hadronic tensor is*

$$\int \mathrm{d}\xi\, H_2(\xi,Q^2)\left(F_{21}^{\text{naive}}(x,\xi,b_T) - F_{21}^{\text{o.s.}}(x,\xi,b_T)\right) D_{11}(z,b_T) = \text{finite}\,.$$

# SUBTRACTION METHOD AT NLP: GENUINE PC

$$
\begin{aligned}
&\left[\mathcal{F}^{\text{bare}}_{q,21}(x,\xi,b_T,\zeta)\right]_{\delta\alpha} \\
&= \mathrm{i}q^+ \int \frac{\mathrm{d}b_1^-}{2\pi}\,\frac{\mathrm{d}b_2^-}{2\pi}\, e^{-\mathrm{i}\bar{\xi}b_1^- q^+}\, e^{-\mathrm{i}\xi b_2^- q^+} \left\{ \frac{\langle P|\left[\bar{\chi}(b_T+b_1^- n)\mathcal{A}_T(b_T+b_2^- n)\right]_{i,\alpha}|X\rangle\langle X|\,\chi_{i,\delta}(0)\,|P\rangle}{\sqrt{S(b_T,\zeta(\delta^+/q^+)^2)}} \right. \\
&\qquad - \frac{\langle P|\left[\bar{\chi}(b_T+b_1^- n)\,\gamma_T^\rho\right]_{i,\alpha}|X\rangle\langle X|\,\chi_{i,\delta}(0)\,|P\rangle}{\sqrt{S(b_T,\zeta(\delta^+/q^+)^2)}} \\
&\qquad \left. \times \frac{1}{N_c}\mathrm{tr}\left[\langle 0|\,S_{\bar{n}}^\dagger S_n(b_T)\,\mathcal{A}^\rho_{T,n}(b_T+b_2^- n)\,|X\rangle\langle X|\,S_n^\dagger S_{\bar{n}}(0)\,|0\rangle\right] \right\}
\end{aligned}
$$

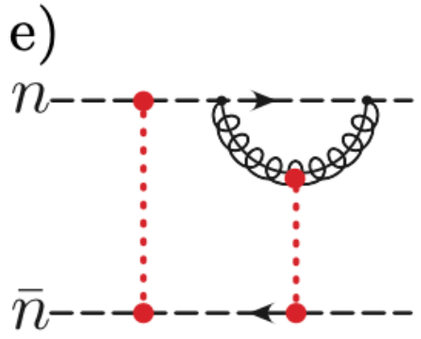
*Tw3 contributions subtractions happen in 2 steps*

*Tw3 contributions to the hadronic tensor*

$$
\begin{aligned}
&W^{\mu\nu}_{\text{gNLP}}(q) = \left[\mathcal{W}^{\mu\nu}_{\text{sub.}}\right]^{(3,2)}_D + \left[\mathcal{W}^{\mu\nu}_{\text{sub.}}\right]^{(2,3)}_D \\
&= \frac{\mathrm{i}}{N_c}\left[\frac{\bar{n}^\mu}{q^-} - \frac{n^\mu}{q^+}\right](\mathbb{1})_{\alpha\beta}(\gamma_T^\nu)_{\gamma\delta}\int \mathrm{d}\xi\, H_2(\xi,q^2)\int \frac{\mathrm{d}^2 b_T}{(2\pi)^2}\, e^{\mathrm{i}b_T\cdot q_T} \\
&\quad \times \sum_q \Big\{ \left[\mathcal{F}_{q,21}\right]_{\delta\alpha}\left[\mathcal{D}_{q,11}\right]_{\beta\gamma} - \left[\mathcal{F}_{\bar{q},21}\right]_{\beta\gamma}\left[\mathcal{D}_{\bar{q},11}\right]_{\delta\alpha} \\
&\qquad + \left[\mathcal{F}_{q,11}\right]_{\delta\alpha}\left[\mathcal{D}_{q,21}\right]_{\beta\gamma} - \left[\mathcal{F}_{\bar{q},11}\right]_{\beta\gamma}\left[\mathcal{D}_{\bar{q},21}\right]_{\delta\alpha}\Big\} \\
&\quad + \frac{\mathrm{i}}{N_c}\left[\frac{\bar{n}^\nu}{q^-} - \frac{n^\nu}{q^+}\right](\gamma_T^\mu)_{\alpha\beta}(\mathbb{1})_{\gamma\delta}\int \mathrm{d}\xi\, H_2^*(\xi,q^2)\int \frac{\mathrm{d}^2 b_T}{(2\pi)^2}\, e^{\mathrm{i}b_T\cdot q_T} \\
&\quad \times \sum_q \Big\{ \left[\mathcal{F}_{q,12}\right]_{\delta\alpha}\left[\mathcal{D}_{q,11}\right]_{\beta\gamma} - \left[\mathcal{F}_{\bar{q},12,}\right]_{\beta\gamma}\left[\mathcal{D}_{\bar{q},11}\right]_{\delta\alpha} \\
&\qquad + \left[\mathcal{F}_{q,11}\right]_{\delta\alpha}\left[\mathcal{D}_{q,12}\right]_{\beta\gamma} - \left[\mathcal{F}_{\bar{q},11}\right]_{\beta\gamma}\left[\mathcal{D}_{\bar{q},12}\right]_{\delta\alpha}\Big\},
\end{aligned}
$$

# GLAUBER GLUONS

- $\delta$-regulator FAILS TO WORK FOR GLAUBER GLUONS WHOSE MOMENTA SCALE AS $(\lambda^2, \lambda^2, \lambda)$. ONE NEEDS TO INTRODUCE A SPECIFIC REGULATOR FOR GLAUBER (NOT ATTEMPTED YET)
- RAPIDITY REGULATOR WORKS FOR GLAUBER BUT WITH SOME CAVEATS: ONE NEEDS A DIFFERENT RAPIDITY REGULATOR FOR RAPIDITY DIVERGENCIES( $\sim 1/\eta$) AND GLAUBER DIVERGENCIES( $\sim 1/\eta'$).THE $\eta'$ ZERO-LIMIT SHOULD BE TAKEN FIRST (MOULT, RAMAN, RIDGWAY, STEWART,2022)




$$\sim \frac{\eta'}{\eta}$$

And consistency of instantaneous Glauber potential requires this contribution to be 0

# CONCLUSIONS

- THE PHYSICAL OBSERVABLES REMAIN REGULATOR-INDEPENDENT, BUT THE $\delta$-REGULATOR OFFERS FORMAL OPERATOR-LEVEL DEFINITIONS PARTICULARLY WELL-SUITED FOR POSITION-SPACE CALCULATIONS AND HIGHER TWISTS/NLP.
- *THE RAPIDITY REGULATOR IS CONNECTED TO ZERO-BIN SUBTRACTION. WE HAVE AN OPERATOR DEFINITION OF ZERO-BIN SUBTRACTION.*
- *FOR PRACTICAL CALCULATIONS IN MOMENTUM SPACE $\delta$-REGULATOR AND RAPIDITY REGULATORS HAVE SIMILAR ADVANTAGES AND DISADVANTAGES IN MULTI-LOOPS CALCULATIONS. OFF-THE-LIGHT-CONE REGULATOR IS NOT USED IN HIGHER LOOPS, BUT IT SHOULD WORK TOO (PERHAPS IT IS JUST A QUESTION OF PRACTICE IN OVERCOMING MATHEMATICAL DIFFICULTIES )*
- FOR FORMAL DEMONSTRATIONS $\delta$-REGULATOR IS THE ONLY ONE THAT HAS A FORMAL OPERATOR DEFINITION WHICH DISTINGUISHES COLLINEAR AND ANTI COLLINEAR SECTORS TOO, UP TO NOW.

# DISCUSSION SESSION

- MONTECARLO AND TMD: THEORETICAL DIFFICULTIES FROM KT SPACE?
- EVENT SHAPES AND JETS WITH TMD: CAN WE EXTRAPOLATE FROM BELLE TO LEP AND FCC? ARE LHC NEW MEASUREMENTS INTERESTING?
- PDF AT N3LO: WHAT TO EXPECT FOR PRECISION MEASUREMENTS?
- REGULATORS AND SMALL-X VS TMD: WHY IR EXTRA DIVERGENCE
- $\alpha_s$ FROM THRUST MEASUREMENTS: WHY DISCREPANCIES AMONG THEORETICAL EXTRACTIONS
- EXTRACTIONS… PION, AZIMUTHAL, GLUON TMD,…