CSS evolution in light-cone gauge: studying the impact of residual gauge

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Basics of TMD factorization

Transverse momentum dependent (TMD) factorization for some semi-inclusive processes like Drell-Yan or SIDIS, with measured transverse momentum $\mathbf{q}$ small compared to the hard scale Q

Ex: for Drell-Yan:

$$\frac{d\sigma^{\mathrm{DY}}}{dQdYd^2\mathbf{q}} = \sum_{\text{flavors } f} H(Q^2,\mu^2) \int d^2\mathbf{b}\, e^{i\mathbf{b}\cdot\mathbf{q}}\, q_{f/H_a}(\mathrm{x}_a,\mathbf{b};\mu^2,\zeta_a)\, q_{\bar{f}/H_b}(\mathrm{x}_b,\mathbf{b};\mu^2,\zeta_b) + O\left(\frac{\mathbf{q}^2}{Q^2}\right) + O\left(\frac{\Lambda^2_{QCD}}{Q^2}\right)$$

with $\zeta_a\zeta_b = Q^4$
Valid to all orders in α_s! (Collins (2011))

$$\text{TMD distribution:} \quad q_{f/H}(\mathrm{x},\mathbf{k};\mu^2,\zeta) = \int \frac{d^2\mathbf{b}}{(2\pi)^2}\, e^{i\mathbf{b}\cdot\mathbf{k}}\, q_{f/H}(\mathrm{x},\mathbf{b};\mu^2,\zeta)$$

$\rightarrow$ distribution of quarks of flavor f, momentum fraction x and transverse momentum $\mathbf{k}$ in the hadron H, with "resolutions" μ and ζ

Collins-Soper-Sterman (CSS) evolution equations:

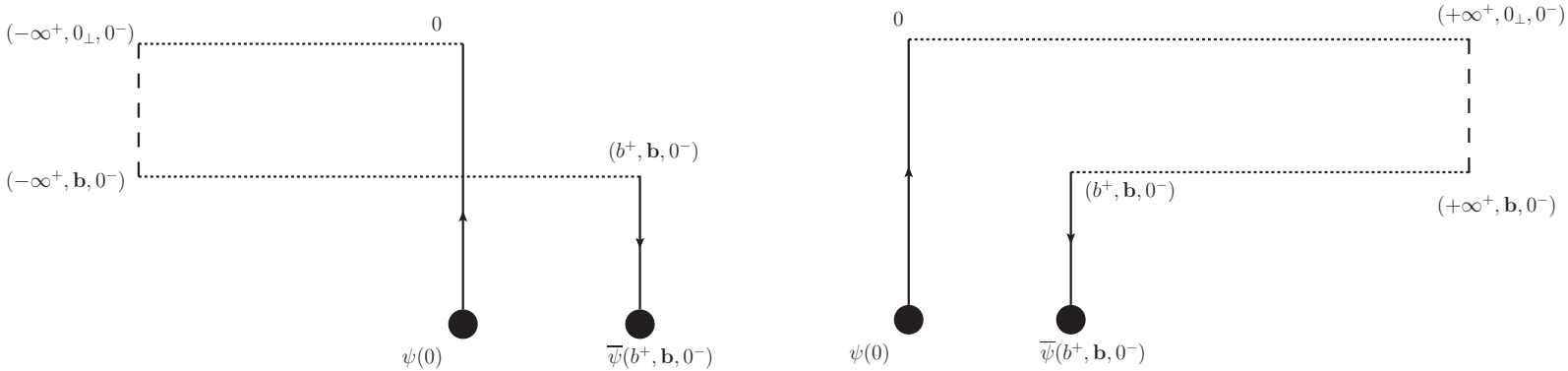
$$\mu^2\frac{d}{d\mu^2} q_{f/H}(\mathrm{x},\mathbf{b};\mu^2,\zeta) = \frac{1}{2}\gamma_q[\alpha_s(\mu^2);\zeta/\mu^2]\;\; q_{f/H}(\mathrm{x},\mathbf{b};\mu^2,\zeta)$$
$$\zeta\frac{d}{d\zeta} q_{f/H}(\mathrm{x},\mathbf{b};\mu^2,\zeta) = \frac{1}{2}\tilde{K}(\mathbf{b};\mu^2)\;\; q_{f/H}(\mathrm{x},\mathbf{b};\mu^2,\zeta)$$

Basics of TMD factorization

Basic operator definition of unpolarized quark TMD, before rapidity and UV renormalization, in a left-moving hadron, with large P^- momentum component :

$$q(\mathrm{x},\mathbf{b}) = \int \frac{db^+}{2\pi}\, e^{-i\mathrm{x}P^- b^+} \left\langle P\middle|\mathcal{T}\,\overline{\Psi}(b^+,\mathbf{b},0^-)\,\frac{\gamma^-}{2}\,U(\sigma\infty^+,\mathbf{b},0^-;b^+,\mathbf{b},0^-)^\dagger \right.$$
$$\left. \times\; U(\sigma\infty^+,\mathbf{b},0^-;\sigma\infty^+,0_\perp,0^-)\,U(\sigma\infty^+,0_\perp,0^-;0)\Psi(0)\middle|P\right\rangle_c$$

$\rightarrow$ Staple shaped gauge link going either to the infinite past ($\sigma = -1$) or to the infinite future ($\sigma = +1$)



Shape of the link determined for each process for TMD factorization to work. For example:

- Drell-Yan: past staple ($\sigma = -1$)
- SIDIS: future staple ($\sigma = +1$)

In the following: focus on the future staple case for simplicity.

CSS evolution equations follow from the renormalization of UV and rapidity divergences in the operator definition of TMDs.

Light-cone gauge and prescriptions for residual gauge

Light-cone gauge (LCG): $n_\mu A^\mu = 0$ with n_μ light-like ($n^2 = 0$)

Free gluon Feynman propagator in LCG:

$$\tilde{G}^{\mu\nu}_{0,F}(k) = \frac{i}{(k^2 + i0)} \left\{ -g^{\mu\nu} + \frac{(k^\mu n^\nu + n^\mu k^\nu)}{[n\cdot k]} \right\}$$

However, need extra prescription to handle $n\cdot k$ denominator in the vicinity of $n\cdot k = 0$.
$\rightarrow$ Issue related to residual gauge freedom after imposing LCG condition.

- Boundary conditions on A^μ can be imposed to fix residual gauge, which implies a prescription for the $n\cdot k$ denominator (Belitsky, Ji, Yuan (2003), Chirilli, Kovchegov, Wertepny (2015))
  - With retarded boundary condition:

$$\tilde{G}^{\mu\nu}_{0,F}(k) = \frac{i}{(k^2 + i0)} \left\{ -g^{\mu\nu} + \frac{k^\mu n^\nu}{(n\cdot k + i0)} + \frac{n^\mu k^\nu}{(n\cdot k - i0)} \right\}$$

  - With advanced boundary condition:

$$\tilde{G}^{\mu\nu}_{0,F}(k) = \frac{i}{(k^2 + i0)} \left\{ -g^{\mu\nu} + \frac{k^\mu n^\nu}{(n\cdot k - i0)} + \frac{n^\mu k^\nu}{(n\cdot k + i0)} \right\}$$

  - With anti-periodic boundary condition:

$$\tilde{G}^{\mu\nu}_{0,F}(k) = \frac{i}{(k^2 + i0)} \left\{ -g^{\mu\nu} + (k^\mu n^\nu + n^\mu k^\nu)\mathrm{P.V.}\left(\frac{1}{n\cdot k}\right) \right\}$$

Light-cone gauge and prescriptions for residual gauge

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  Interpolation: infinite family of prescriptions (and boundary conditions) with a real parameter β

$$\begin{aligned}\tilde{G}^{\mu\nu}_{0,F}(k) &= \frac{i}{(k^2 + i0)} \left\{ -g^{\mu\nu} + \frac{\big(\beta k^\mu n^\nu + (1-\beta) n^\mu k^\nu\big)}{(n\cdot k + i0)} + \frac{\big((1-\beta) k^\mu n^\nu + \beta n^\mu k^\nu\big)}{(n\cdot k - i0)} \right\} \\ &= \frac{i}{(k^2 + i0)} \left\{ -g^{\mu\nu} + \big(k^\mu n^\nu + n^\mu k^\nu\big) \mathrm{P.V.}\left(\frac{1}{n\cdot k}\right) + i\pi(1-2\beta)\big(k^\mu n^\nu - n^\mu k^\nu\big)\delta(n\cdot k) \right\}\end{aligned}$$

- Other possibility: Mandelstam (1983), Leibbrandt (1984) (ML)

$$\frac{1}{[n\cdot k]_{ML}} \equiv \frac{(\bar{n}\cdot k)}{(n\cdot k)(\bar{n}\cdot k) + i0} = \frac{1}{(n\cdot k) + i0(\bar{n}\cdot k)} = \frac{\theta(\bar{n}\cdot k)}{((n\cdot k) + i0)} + \frac{\theta(-\bar{n}\cdot k)}{((n\cdot k) - i0)}$$

  introducing a second light-like vector $\bar{n}^\mu$

Pure rapidity regularization

Dimensional regularization is the standard one for UV divergences.

Numerous regularization schemes have been proposed for rapidity divergences.

Choice for this work: Pure rapidity regulator (Ebert, Moult, Stewart, Tackmann, Vita, Zhu (2019))

$\rightarrow$ Introduce in the integration measure for gluons the factor

$$w^2 \left[\frac{|k^-|}{|k^+|} \frac{\nu^+}{\nu^-} \right]^{\eta/2}$$

- η: dimensionless regulator analog to ϵ in dim reg
- $\log(\nu^+/\nu^-)/2$: arbitrary reference point in rapidity
- w^2: bookkeeping parameter, dimensionless but chosen to make the whole factor independent of ν^+/ν^-, and such that $w^2 \to 1$ for $\eta \to 0$.

Important note: expansion at $\eta \to 0$ should be performed at fixed finite ϵ, before expanding at $\epsilon \to 0$.

With this regulator, the ζ variable in TMDs for left-moving (resp. right-moving) hadron is defined as

$$\zeta \equiv \frac{2(\mathrm{x}P^{\mp})^2 \nu^{\pm}}{\nu^{\mp}}$$

Remark: Soft factor becomes trivial with the pure rapidity regularization

Background field method

The background field method (DeWitt (1967), Abbott (1981), Balitsky, Braun (1989)) is well suited to study TMD factorization (Scimemi, Tarasov, Vladimirov (2019), Moos, Scimemi, Vladimirov (2022)).

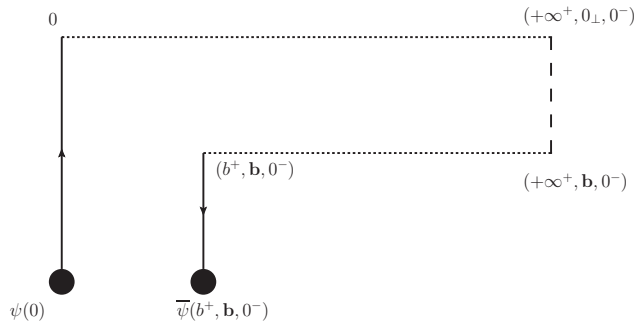
Here: split the fields into background and fluctuations in the TMD operator definition as

$$\Psi(x) = \psi(x) + \delta\Psi(x) \quad \text{and} \quad A_a^\mu(x) = \mathcal{A}_a^\mu(x) + \delta A_a^\mu(x)$$

with the background fields representing the modes collinear to the hadron target (with large P^-), and the fluctuation fields perturbative contributions of any kinematics.

Quadratic terms in the fluctuations provide the $O(\alpha_s)$ corrections in the TMD operator definition.

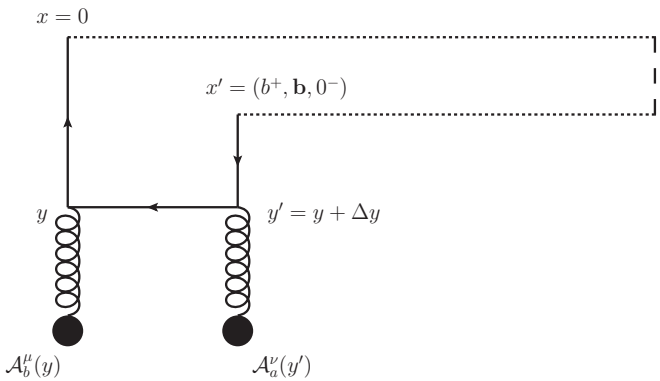
Background modes collinear to the target must have a sizable $-$ momentum
$\Rightarrow$ Should be localized around $x^+ = 0$
$\Rightarrow$ Only fluctuation field can contribute to transverse gauge link a infinity in x^+

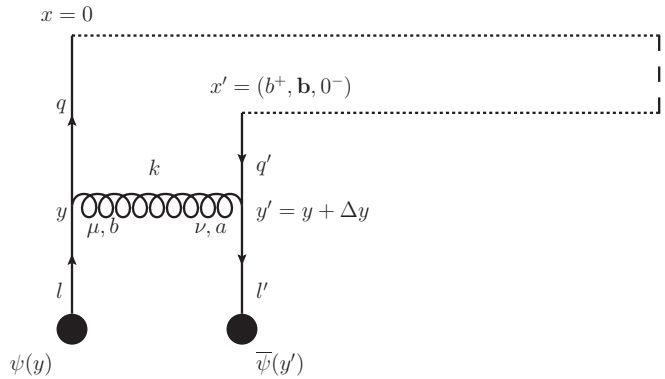


We consider target light-cone gauge, $\delta A_a^-(x) = \mathcal{A}_a^-(x) = 0$ for a target with large P^-, so that gauge links along x^+ are trivial.

For simplicity we focus on the dilute regime, and keep only quadratic terms in the background fields as well.

Ladder diagrams: ML prescription case





For $\mathbf{b} \neq 0$ and $\mathrm{x} \neq 0$:

- gluon-to-quark ladder diagram: finite, in any gauge
- quark-to-quark ladder diagram: finite in LCG with ML prescription
  $\rightarrow$ potential soft or rapidity divergence at $k^- \rightarrow 0$ removed by the ML prescription

In both cases, phase factor $e^{i\mathbf{k}\cdot\mathbf{b}}$ prevents UV divergence for $|\mathbf{k}| \rightarrow \infty$

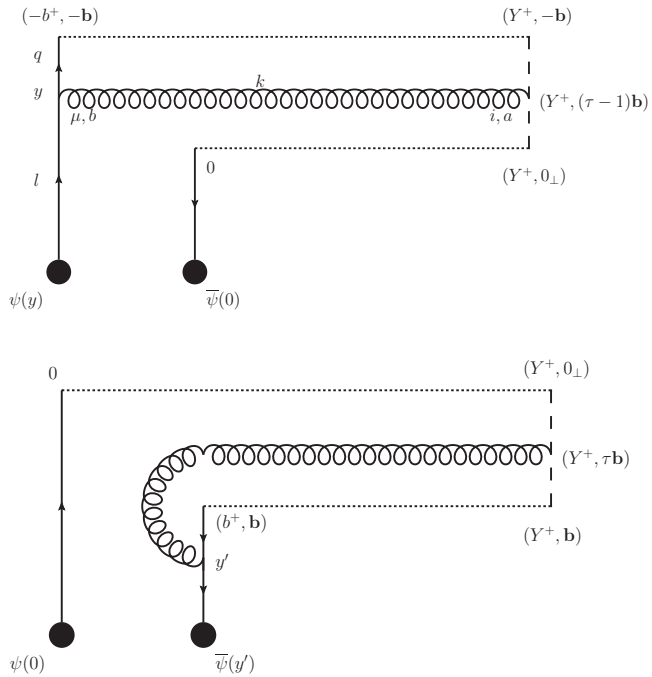
Diagrams with gluon radiated to the transverse gauge link

With ML prescription, these diagrams contain k^- integrations

$$\int \frac{dk^-}{2\pi} \frac{i}{(k^2+i0)} e^{-ik^- Y^+} = \frac{\theta(k^+)}{2k^+} e^{-i\frac{\mathbf{k}^2}{2k^+}Y^+}$$

$$\int \frac{dk^-}{2\pi} \frac{i}{(k^2+i0)} \frac{e^{-ik^- Y^+}}{[k^-]_{ML}} = \frac{\theta(k^+)}{\mathbf{k}^2} \left[e^{-i\frac{\mathbf{k}^2}{2k^+}Y^+} - 1 \right]$$

Only k^- zero-mode contribution -1 survives $Y^+ \to +\infty$ limit



Final result for the contribution of these diagrams:

$$\frac{\alpha_s C_F}{\pi} \left[\pi\mu^2\mathbf{b}^2\right]^\epsilon \frac{w^2}{\eta} \left[\frac{\zeta\,\mathbf{b}^2}{4}\right]^{\frac{\eta}{2}} \frac{\Gamma\left(-\epsilon-\frac{\eta}{2}\right)}{\Gamma\left(1+\frac{\eta}{2}\right)} q_{\mathrm{Bckgd}}(\mathrm{x},\mathbf{b};\mu^2) + \text{finite NLO} + O(\eta)$$

$$= -\frac{\alpha_s C_F}{2\pi} \frac{\Gamma(1-\epsilon)}{\epsilon} \left[\pi\mu^2\mathbf{b}^2\right]^\epsilon \left[\frac{2w^2}{\eta} + \log\left(\frac{\zeta\,\mathbf{b}^2}{c_0^2}\right) - \frac{1}{\epsilon} - \Psi(1-\epsilon) + \Psi(1)\right] q_{\mathrm{Bckgd}}(\mathrm{x},\mathbf{b};\mu^2) + \text{finite NLO} + O(\eta)$$

Both rapidity pole $1/\eta$ and UV pole $1/\epsilon$. Note: double pole $1/\epsilon^2$ also produced by interplay of regulators.

UV Renormalization in LCG with ML

All order renormalization of QCD understood in LCG with ML (Bassetto, Dalbosco, Soldati (1987)):

- Both additive and multiplicative renormalization for $\bar{n}_\mu A^\mu$
- Same renormalization constant for coupling g and transverse components A^i:

$$A_i^{(0)a} = {Z_3}^{\frac{1}{2}}\, A_i^a \quad \text{and} \quad g_0 = \mu^\epsilon\, {Z_3}^{-\frac{1}{2}}\, g(\mu^2) \quad \text{so that} \quad g_0\, A_i^{(0)a} = \mu^\epsilon\, g\, A_i^a$$

- Using n^μ and $\bar{n}^\mu$, Dirac projectors on *good* and *bad* components of quarks can be built (or collinear to target and collinear to projectile)
  $\Rightarrow$ Different renormalization constants for both projections:

$$\left(\frac{\not{\bar{n}}\not{n}}{2\bar{n}\cdot n}\Psi^{(0)}\right) = \frac{\sqrt{Z_2}}{\sqrt{\tilde{Z}_2}}\left(\frac{\not{\bar{n}}\not{n}}{2\bar{n}\cdot n}\Psi\right) \quad \text{and} \quad \left(\frac{\not{n}\not{\bar{n}}}{2\bar{n}\cdot n}\Psi^{(0)}\right) = \sqrt{Z_2\,\tilde{Z}_2}\left(\frac{\not{n}\not{\bar{n}}}{2\bar{n}\cdot n}\Psi\right)$$

$$\text{with} \quad \frac{Z_2}{\tilde{Z}_2} = 1 + \frac{3\alpha_s C_F}{4\pi}\,\frac{S_\epsilon}{\epsilon} + O(\alpha_s^2) \quad \text{and} \quad Z_2\tilde{Z}_2 = 1 - \frac{\alpha_s C_F}{4\pi}\,\frac{S_\epsilon}{\epsilon} + O(\alpha_s^2) \quad \text{where} \quad S_\epsilon = (4\pi e^{-\gamma_E})^\epsilon$$

In our calculation, the TMD contains $\overline{\Psi}(\cdots)\gamma^-[\cdots]\Psi(\cdots) = \overline{\Psi}(\cdots)\not{n}[\cdots]\Psi(\cdots)$
$\Rightarrow$ quark TMD obtained in bare perturbation theory or in renormalized perturbation theory related as

$$q^{(0)}_{\text{unsub.}}(\mathrm{x}, \mathbf{b}) = \frac{Z_2}{\tilde{Z}_2}\, q^{\text{n.r.}}_{\text{unsub.}}(\mathrm{x}, \mathbf{b}; \mu^2)$$

Rapidity and UV renormalization and CSS evolution

So far: fully bare quark TMD at one loop:

$$
\begin{aligned}
q^{(0)}_{\text{unsub.}}(\mathrm{x},\mathbf{b}) &= \frac{Z_2}{\tilde{Z}_2}\, q^{\text{n.r.}}_{\text{unsub.}}(\mathrm{x},\mathbf{b};\mu^2) \\
&= \left\{1 + \frac{3\alpha_s C_F}{4\pi}\,\frac{S_\epsilon}{\epsilon} - \frac{\alpha_s C_F}{2\pi}\,\frac{\Gamma\left(1-\epsilon\right)}{\epsilon}\left[\pi\mu^2\mathbf{b}^2\right]^\epsilon\left[\frac{2w^2}{\eta} + \log\left(\frac{\zeta\,\mathbf{b}^2}{c_0^2}\right) - \frac{1}{\epsilon} - \Psi(1-\epsilon) + \Psi(1)\right]\right\} \\
&\quad \times\, q_{\text{Bckgd}}(\mathrm{x},\mathbf{b};\mu^2) + \text{finite NLO} + O(\alpha_s^2)
\end{aligned}
$$

UV and rapidity renormalization/subtraction of TMD in $\overline{MS}$ scheme:

$$
q(\mathrm{x},\mathbf{b};\mu^2,\zeta) = Z_{UV}\; q^{(0)}_{\text{sub.}}(\mathrm{x},\mathbf{b};\zeta) = Z_{UV}\; Z_{\text{rap.}}\; q^{(0)}_{\text{unsub.}}(\mathrm{x},\mathbf{b})
$$

Divergences removed with

$$
\begin{aligned}
Z_{\text{rap.}} &= 1 + \frac{\alpha_s C_F}{\pi}\,\frac{\Gamma(1-\epsilon)}{\epsilon}\left[\pi\mu^2\mathbf{b}^2\right]^\epsilon\frac{w^2}{\eta} + O(\alpha_s^2) = 1 + \frac{\alpha_s^{(0)} C_F}{\pi}\,\frac{\Gamma(1-\epsilon)}{\epsilon}\left[\pi\mathbf{b}^2\right]^\epsilon\frac{w^2}{\eta} + O(\alpha_s^{(0)2}) \\
Z_{UV} &= 1 - \frac{\alpha_s C_F}{2\pi}\left[\frac{S_\epsilon}{\epsilon^2} + \left(\log\left(\frac{\mu^2}{\zeta}\right) + \frac{3}{2}\right)\frac{S_\epsilon}{\epsilon}\right] + O(\alpha_s^2)
\end{aligned}
$$

$\Rightarrow$ CSS equations: OK!

$$
\begin{aligned}
\mu^2\frac{d}{d\mu^2}\log\Big(q(\mathrm{x},\mathbf{b};\mu^2,\zeta)\Big) &= \mu^2\frac{d}{d\mu^2}\log Z_{UV} = \frac{\alpha_s C_F}{2\pi}\left[\log\left(\frac{\mu^2}{\zeta}\right) + \frac{3}{2} + O(\epsilon)\right] + O(\alpha_s^2) \\
\zeta\frac{d}{d\zeta}\log\Big(q(\mathrm{x},\mathbf{b};\mu^2,\zeta)\Big) &= \zeta\frac{d}{d\zeta}\log Z_{UV} + \zeta\frac{d}{d\zeta}\log Z_{\text{rap.}} = -\frac{\alpha_s C_F}{2\pi}\left[\log\left(\frac{\mu^2\mathbf{b}^2}{c_0^2}\right) + O(\epsilon)\right] + O(\alpha_s^2)
\end{aligned}
$$

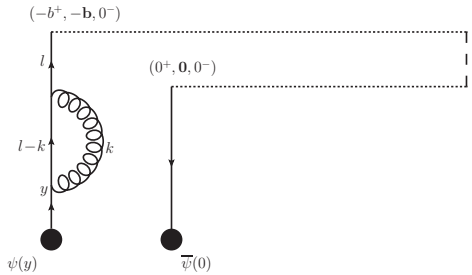
UV Renorm. in LCG with single n^μ vector prescriptions

$$\tilde{G}^{\mu\nu}_{0,F}(k) = \frac{i}{(k^2+i0)}\left\{-g^{\mu\nu} + \left(k^\mu n^\nu + n^\mu k^\nu\right)\text{P.V.}\left(\frac{1}{n\cdot k}\right) + i\pi(1-2\beta)\left(k^\mu n^\nu - n^\mu k^\nu\right)\delta(n\cdot k)\right\}$$

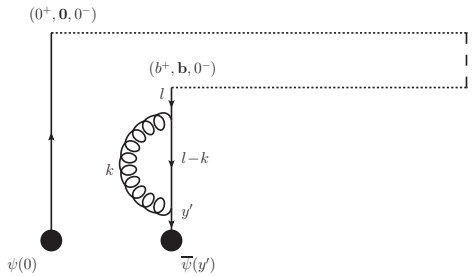
All order renormalizability not clear in that case, but results at one loop:

- Both multiplicative and additive renormalization for quark field and for gluon field

Moreover, IR divergences within renormalization constants for fields!



Ex: Bare one-loop quark self-energy (independent of sub-gauge parameter β!):



$$(-i)\,\Sigma^{1\text{L}}(l) = (-i)\frac{\alpha_s^{(0)}C_F}{4\pi}\,\frac{\Gamma(1+\epsilon)}{\epsilon_{UV}}\,\frac{(\Gamma(1-\epsilon))^2}{\Gamma(1-2\epsilon)}\left[\frac{-l^2-i0}{4\pi}\right]^{-\epsilon}$$
$$\times\left\{-\frac{(1-\delta_s\epsilon)}{(1-2\epsilon)}\,\not{l} + 2\left[\frac{1}{\epsilon_{IR}} + \frac{1}{(1-2\epsilon)}\right] l^2\text{P.V.}\left(\frac{1}{n\cdot l}\right)\not{n}\right\}$$

Contribution of quark self-energy diagram to bare quark TMD:

$$q^{(0)}_{\text{unsub.}}(\text{x},\mathbf{b})\Big|_{q\text{ self-energies}} = \int\frac{d^{4-2\epsilon}\Delta y}{4\pi}\left\langle P\middle|\mathcal{T}\,\overline{\psi}^{(0)}(\Delta y)\,\gamma^-\,\psi^{(0)}(0)\middle|P\right\rangle_c \int\frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}\,2\pi\delta(l^- - \text{x}P^-)\,e^{-il\cdot\Delta y}\,e^{-i\mathbf{l}\cdot\mathbf{b}}$$
$$\times\;\frac{\alpha_s^{(0)}C_F}{\pi}\,\Gamma(1+\epsilon)\left[\frac{-l^2-i0}{4\pi}\right]^{-\epsilon}\left\{\frac{1}{\epsilon_{UV}\epsilon_{IR}} + \frac{3}{4\,\epsilon_{UV}} + O(\epsilon^0)\right\}$$

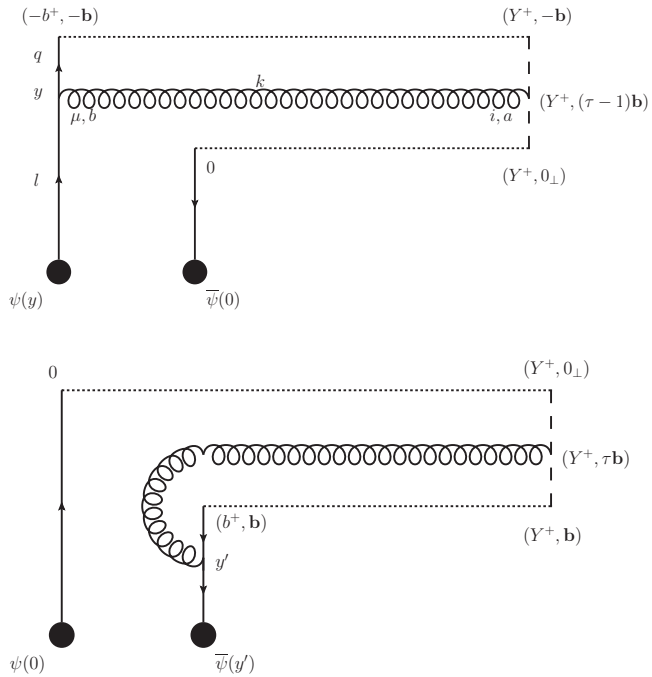
Diagrams with gluon radiated to the transverse gauge link

For prescriptions with single n^μ vector, extra gauge denominator $1/[k^-]$ independent of k^+

$\Rightarrow$ can integrate over k^+ first, as

$$\int \frac{dk^+}{2\pi} \frac{i}{(k^2+i0)} \frac{1}{[2\mathrm{x}P^-(l^+-k^+)-(\mathbf{l}-\mathbf{k})^2+i0]}$$
$$= \frac{\theta(k^-)}{2k^-} \frac{1}{\left[2\mathrm{x}P^- l^+ - \frac{\mathrm{x}P^-}{k^-}\mathbf{k}^2-(\mathbf{l}-\mathbf{k})^2+i0\right]}$$

$k^- > 0 \Rightarrow$ sub-gauge parameter β now irrelevant.



Then, $Y^+ \to +\infty$ forces $k^- \to 0$. With pure rapidity regulator (with $\eta > 0$), diagrams proportional to

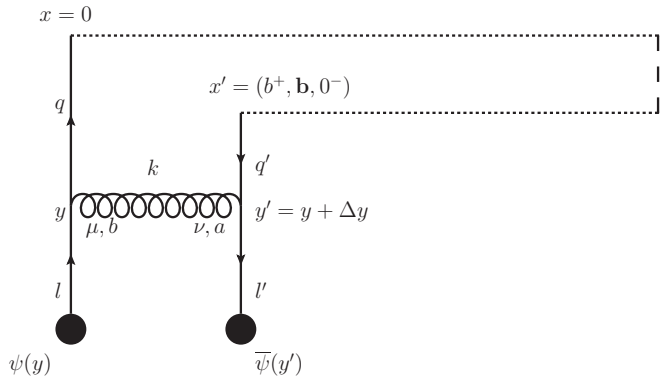
$$\int_0^{+\infty} \frac{dk^-}{k^-} e^{-ik^- Y^+} w^2 \left[\frac{2}{\mathbf{k}^2}\frac{\nu^+}{\nu^-}\right]^{\eta/2} (k^-)^\eta \left[1+O\left(k^-\right)\right] = O\left(\frac{1}{(Y^+)^\eta}\right) \to 0$$

No contribution from these diagrams with single n^μ vector prescriptions, when transverse gauge link pushed to infinity!

Quark ladder diagram with single n^μ prescriptions

β dependent $\delta(k^-)$ contribution leads to trivial pole integration in k^+ (pure double pole)

$\Rightarrow$ Still no dependence on sub-gauge parameter β.



Eventually, by adding and subtracting the $\mathbf{b}=0$ limit of the diagram, the leftover (with PV prescription) can be split into a rapidity and UV divergent piece, and a UV and IR divergent piece as

$$q^{(0)}_{\text{unsub.}}(\mathrm{x},\mathbf{b})\Big|_{q\text{ ladder}} = \int \frac{d^{4-2\epsilon}\Delta y}{4\pi} \langle P|\mathcal{T}\,\overline{\psi}^{(0)}(\Delta y)\,\gamma^-\,\psi^{(0)}(0)|P\rangle_c \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}\, 2\pi\delta(l^- - \mathrm{x}P^-)\, e^{-il\cdot\Delta y}\, e^{-i\mathbf{l}\cdot\mathbf{b}}$$

$$\times\ \frac{\alpha_s^{(0)}C_F}{\pi}\left\{\frac{w^2}{\eta}\left[\frac{\zeta\mathbf{b}^2}{4}\right]^{\frac{\eta}{2}}\frac{\Gamma\left(-\epsilon-\frac{\eta}{2}\right)}{\Gamma\left(1+\frac{\eta}{2}\right)}\left[\pi\mathbf{b}^2\right]^{\epsilon} + \Gamma(1+\epsilon)\left[\frac{-l^2-i0}{4\pi}\right]^{-\epsilon}\left(-\frac{1}{\epsilon_{UV}\,\epsilon_{IR}}+\frac{1}{\epsilon_{IR}}\right)+O(\epsilon^0)\right\}$$

Final results for prescriptions with single n^μ vector

- Dependence of sub-gauge parameter β (or equivalently boundary conditions) found to drop for each diagram
- Unexpected UV times IR divergences are found to cancel between ladder and self-energies: OK!
- Rapidity times UV divergences leading to CSS double logs comes from ladder in this case
- UV divergence leading to CSS single log still comes from field renormalization / self-energy diagrams
- Unexpected leftover IR divergence from the ladder diagram

$$q^{(0)}_{\text{unsub.}}(\mathrm{x},\mathbf{b}) = \left\{1 - \frac{\alpha_s C_F}{2\pi}\,\frac{\Gamma(1-\epsilon)}{\epsilon_{UV}}\left[\pi\mu^2\mathbf{b}^2\right]^\epsilon\left[\frac{2w^2}{\eta} + \log\left(\frac{\zeta\,\mathbf{b}^2}{c_0^2}\right) - \frac{1}{\epsilon} - \Psi(1-\epsilon) + \Psi(1)\right]\right.$$
$$\left. + \frac{3\alpha_s C_F}{4\pi}\,\frac{S_\epsilon}{\epsilon_{UV}} + \frac{\alpha_s C_F}{\pi}\,\frac{S_\epsilon}{\epsilon_{IR}}\right\} q^{(0)}_{\text{Bckgd}}(\mathrm{x},\mathbf{b}) + \text{finite NLO} + O(\alpha_s^2)$$

Same rapidity and UV divergences
$\Rightarrow$ same $Z_{\text{rap.}}$ and Z_{UV} to perform rapidity and UV renormalization as with ML prescription
$\Rightarrow$ Standard CSS equations at one loop recovered: OK!

Unclear significance of the leftover IR divergence:

- Still present at $\mathbf{b} = 0$ in the case of collinear PDFs.
- Does not affect derivation of CSS evolution for TMDs, DGLAP evolution of PDFs, or matching coefficients between TMDs and PDFs

Does it signal a fundamental issue with these prescriptions with single n^μ vector for residual gauge fixing in LCG?

Or merely some issue in their implementation with BFM and/or pure rapidity regularization?