Enhanced coherence-violating logarithms in non-global QCD observables

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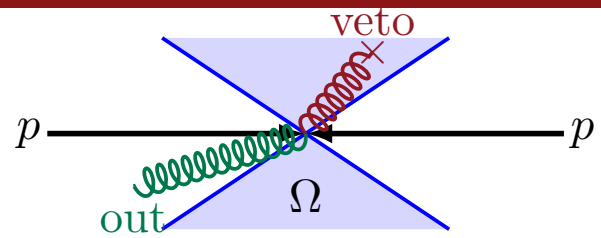
Based on PRL 137 (2026) 111906 [2511.14681] with M. Dasgupta, A. Fraley and P. Monni

Transverse-energy flow into gaps

The standard case: rapidity gaps

Fraction of events that survive a veto E_T on the transverse-energy flow into a region Ω:



$$\Sigma(E_T) = \frac{1}{\sigma_0}\int_0^{E_T} \mathrm{d}E_{T,\Omega}\,\frac{\mathrm{d}\sigma}{\mathrm{d}E_{T,\Omega}}, \qquad E_{T,\Omega} = \sum_{k_i\in\Omega} k_{t,i}, \qquad L = \ln\frac{Q}{E_T} \gg 1.$$

- Standard choice: $\Omega = \{|\eta| < \eta_{\text{cut}}\}$ with $\eta_{\text{cut}} \sim \mathcal{O}(1)$
  $\Longrightarrow$ the veto only probes soft wide-angle radiation: *naively single-logarithmic*.
- Non-global logarithms (NGLs): correlated soft emissions, $\alpha_s^n L^n$
  [Dasgupta, Salam: hep-ph/0104277; Banfi, Marchesini, Smye: hep-ph/0206076]
- Coherence-violating logarithms (CVLs), aka super-leading: $\alpha_s^n L^{2n-3}$, i.e. double logarithms at hadron colliders
  [Forshaw, Kyrieleis, Seymour: hep-ph/0604094, 0808.1269]

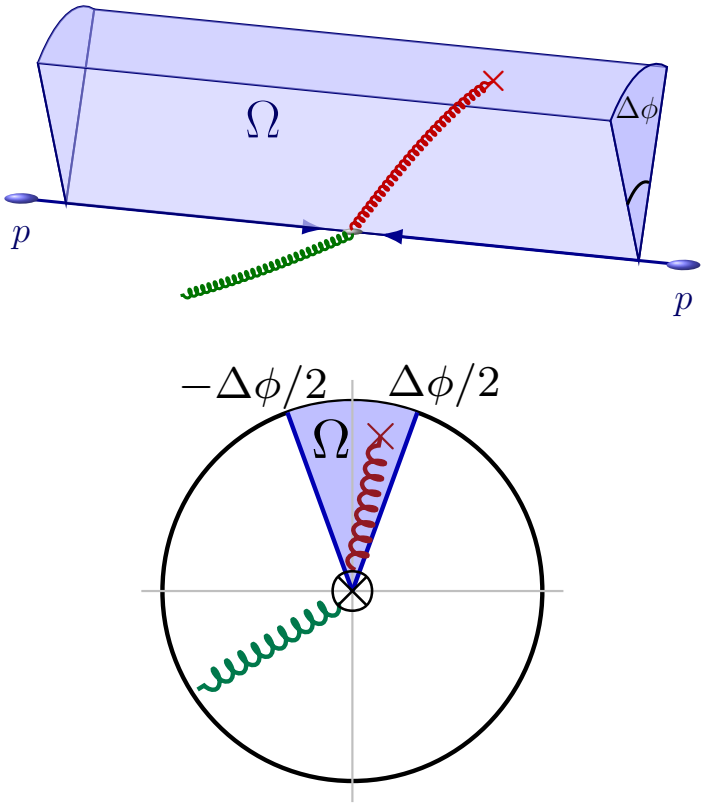
What if the veto region extends to large rapidities, $\eta_{\text{cut}} \gtrsim L$?

A different regime: azimuthal vetoes

- Production of a colour singlet of invariant mass Q in pp collisions (Drell-Yan).
- Veto radiation in an azimuthal region around the beam axis $\hat{n}$:

$$\Omega = \{\, |\phi| < \Delta\phi/2 \,\}, \qquad f \equiv \frac{\Delta\phi}{2\pi}.$$

- No rapidity boundary: only the kinematic bound $|\eta| \leq \ln(Q/k_t)$, i.e. $\eta_{\text{cut}} > L$.
- Vetoed radiation can now be soft *and* collinear to the beam.

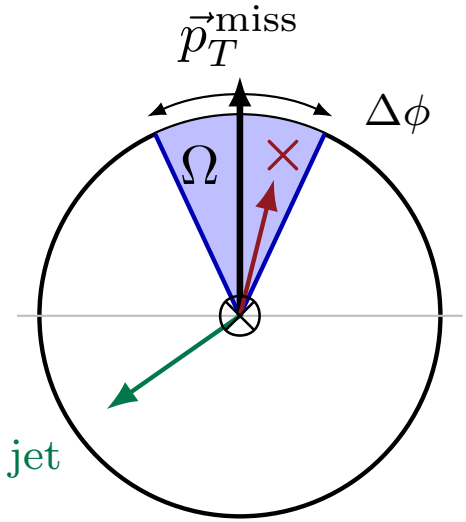


A simple change of geometry exposes new logarithmic structures.

Why this matters

Azimuthal isolation in missing-E_T analyses at the LHC

- In missing-E_T searches, jets within an azimuthal distance of $\vec{p}_T^{\,\text{miss}}$ are vetoed as an isolation criterion.
- The veto comes with a rapidity acceptance η_{cut} for the jets, which is typically large.
- Large logarithm: $L = \ln\left(p_T^{\text{miss}}/p_T^{\text{jet}}\right)$, from L_{min} at the p_T^{miss} threshold upwards.



[GeV]	recoil	p_T^{miss}	$\Delta\phi$ cut on	$\Delta\phi/2$	η_{cut}	L_{min}
ATLAS mono-jet	jet, $p_T > 150$	> 200	≤ 4 jets, $p_T > 30$	0.4–0.6	2.8	1.9
CMS mono-γ	γ, $E_T > 225$	> 200	4 leading jets, $p_T > 30$	0.5	5	1.9

[ATLAS: 2102.10874; CMS: 2511.17310; see also CMS: 2107.13021, ATLAS: 2403.02793]

Whenever $\eta_{\text{cut}} \gtrsim L$, the structures discussed in this talk are present.

Logarithmic counting and headline results

Two ways of organising the perturbative series:

$$\Sigma = 1 + g_{\mathrm{DL}}(\alpha_s L^2) + \frac{1}{L}\, g_{\mathrm{NDL}}(\alpha_s L^2) + \frac{1}{L^2}\, g_{\mathrm{NNDL}}(\alpha_s L^2) + \cdots$$

$$\ln \Sigma = L\, g_{\mathrm{LL}}(\alpha_s L) + g_{\mathrm{NLL}}(\alpha_s L) + \cdots \qquad \text{(if the logarithms exponentiate)}$$

Leading term of each class of logarithms ($\bar{\alpha}_s \equiv \alpha_s/2\pi$):

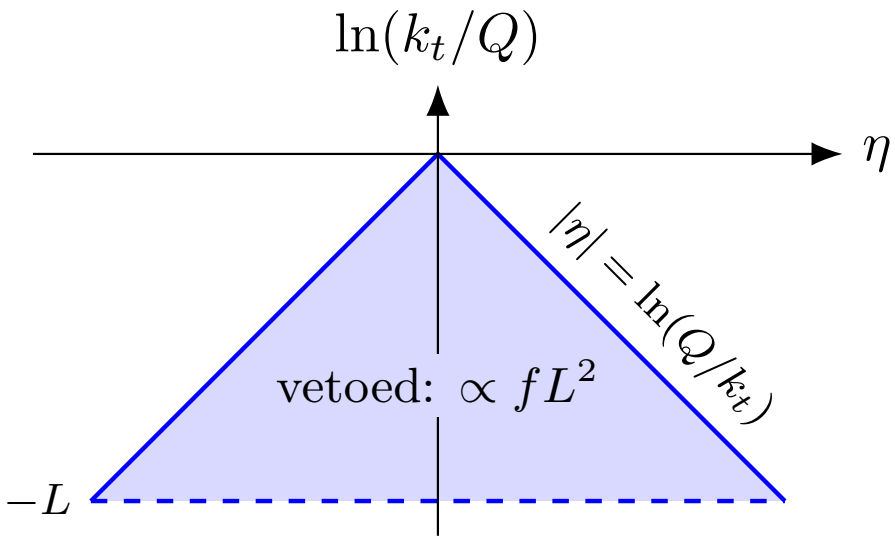
	Rapidity gap ($\eta_{\mathrm{cut}} \ll L$)	**Azimuthal veto** ($\eta_{\mathrm{cut}} > L$)
Primary (global)	$\bar{\alpha}_s L$ (single log)	$\bar{\alpha}_s L^2$ (double log)
NGLs	$\bar{\alpha}_s^2 L^2$	$\bar{\alpha}_s^2 L^3$ (LL in $\ln \Sigma$)
CVLs (Drell-Yan)	$\bar{\alpha}_s^5 L^7$ ($\mathrm{N^3DL}$)	$\bar{\alpha}_s^5 L^8$ (NNDL)

Both NGLs and CVLs acquire one extra power of L:
more singular than in any standard non-global observable.

Global double logarithms
One emission, and naive exponentiation

- At $\mathcal{O}(\alpha_s)$, one gluon is emitted from the incoming $q\bar{q}$ pair.
- The veto constrains its azimuth, not its rapidity.
- Vetoing soft-collinear radiation in Ω gives a double logarithm:

$$\Sigma^{(1)}_{\mathrm{DL}} = -R_{\mathrm{DL}}(L) = -4C_F f\,\bar{\alpha}_s L^2.$$



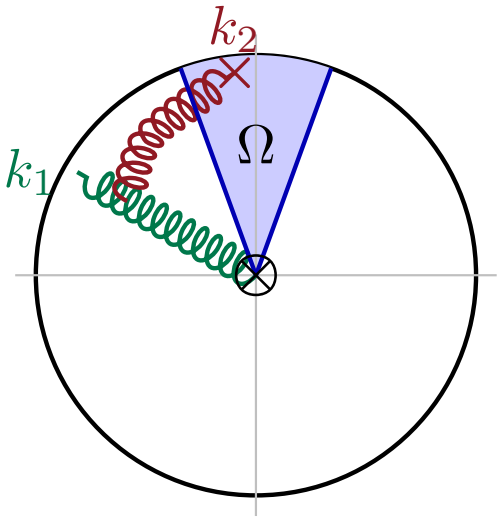
Naive expectation for the LL tower: exponentiate, with a running coupling,

$$\Sigma \overset{?}{\approx} e^{-R(L)}, \qquad R(L) = \frac{4C_F f}{\pi}\int_0^L \mathrm{d}\ell\,\alpha_s(Qe^{-\ell})\,\ell.$$

Here $e^{-R(L)}$ is not the full LL answer:
non-global logarithms enter at the same order.

Non-global logarithms
Enhanced already at $\mathcal{O}(\alpha_s^2)$

- Soft parent k_1 *outside* Ω emits a softer daughter k_2 *into* Ω ($k_{t,1} \gg k_{t,2}$).



- Correlated-emission kernel

$$\propto C_F C_A \, \frac{\cos\phi_{12}}{\cosh\eta_{12} - \cos\phi_{12}} \,,$$

localised at $\eta_{12} \approx 0$.

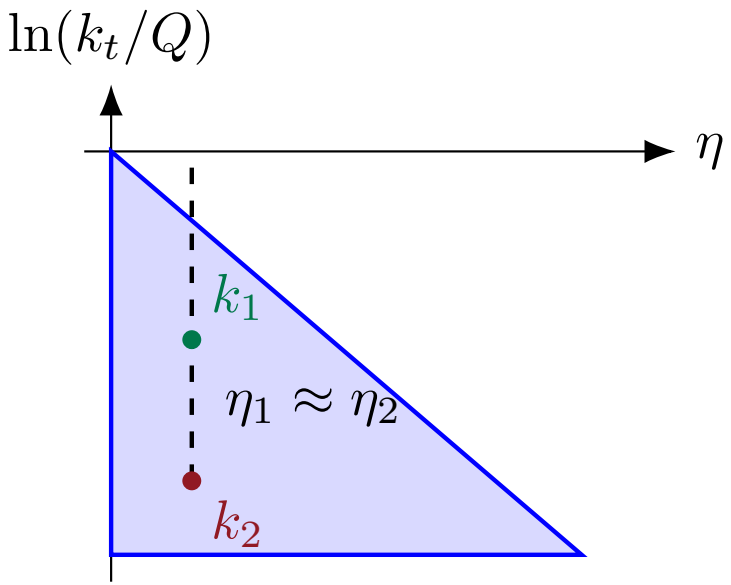
- Real–virtual mismatch: a real k_1 outside Ω changes what radiates k_2 (two dipoles instead of one); a virtual k_1 does not $\implies$ no cancellation.
- Standard gaps: k_1, k_2 at wide angle near the edge of Ω; only the two energy integrals give logs $\implies$ $\Sigma^{(2)}_{\mathrm{NG}} \sim C_F C_A \, \bar{\alpha}_s^2 L^2$.

With an azimuthal veto, does the rapidity integral also give a log?

Why the leading NGL is $\bar{\alpha}_s^2 L^3$

For standard gaps, the correlated pair lives near the boundary of a fixed angular region. Here, with $\eta_1 \approx \eta_2$, the pair can sit anywhere along the boundary of the azimuthal veto region, up to the kinematic bound:

$$\int \frac{\mathrm{d}v_1}{v_1} \int \frac{\mathrm{d}v_2}{v_2} \int \mathrm{d}\eta_1 \mathrm{d}\eta_2 \, K(\eta_1 - \eta_2) \sim L^2 \times L.$$



η_{12} integral: finite ($\to I_{\mathrm{NG}}$); common rapidity: $\sim L$.

$$\boxed{\Sigma_{\mathrm{NG}}^{(2)} = -\frac{8}{3} C_F C_A \, I_{\mathrm{NG}}(f) \, \bar{\alpha}_s^2 L^3}$$

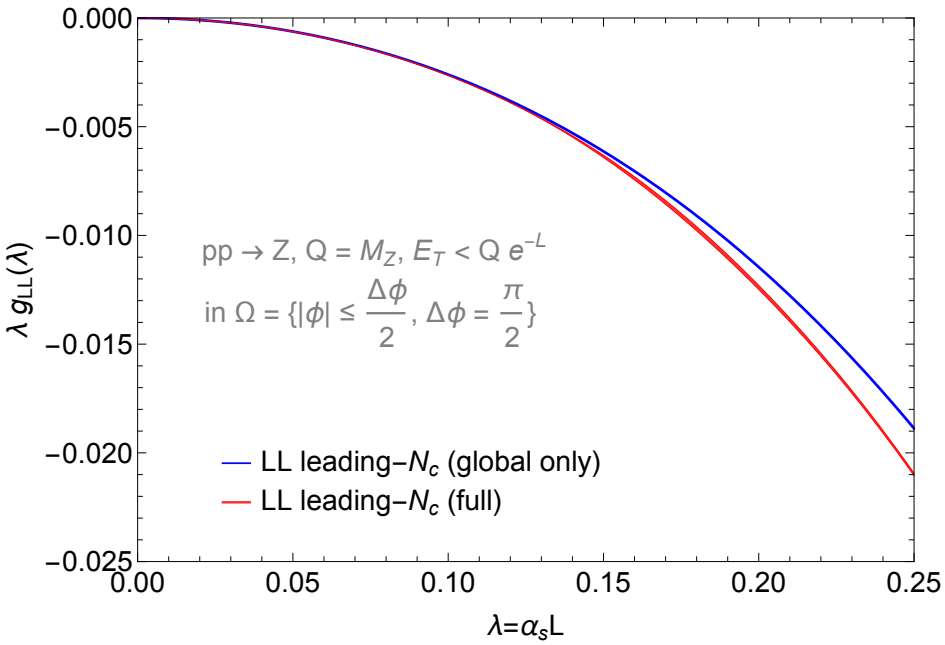
(standard gaps: $\bar{\alpha}_s^2 L^2$)

$$I_{\mathrm{NG}}(f) = \int_{k_1 \notin \Omega} \frac{\mathrm{d}\phi_1}{2\pi} \int_{k_2 \in \Omega} \frac{\mathrm{d}\phi_2}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\eta \, \frac{\cos\phi_{12}}{\cosh\eta - \cos\phi_{12}}.$$

LL resummation at large N_c

Non-global logarithms at leading-logarithmic accuracy

- The enhanced NGLs form a full LL tower, $L\, g_{\mathrm{LL}}(\alpha_s L)$, in $\ln \Sigma$.
- Resummed numerically in the Veneziano limit ($N_c \sim n_f \gg 1$) with a modified `Gnole` algorithm [Banfi, Dreyer, Monni: 2104.06416, 2111.02413]
- Blue: naive global result.
  Red: full LL result.



Key modification: physical rapidity bound for each emission,
$|\eta_i| \leq \ln(Q/k_{t,i})$,
which captures the additional collinear enhancement.

Counting in Σ keeps the full colour structure. At NDL:

$$R(L) = 2C_F f \int_0^L \mathrm{d}\ell \, \frac{\alpha_s(Qe^{-\ell})}{2\pi}(4\ell - 3),$$

$$\Sigma_{\mathrm{NDL}} = e^{-R(L)} \left(1 - \frac{8}{3} C_F C_A I_{\mathrm{NG}}(f) \bar{\alpha}_s^2 L^3\right)$$
$$- 2\bar{\alpha}_s C_F f \, e^{-R(L)} L \left(\frac{(\hat{P}^{(0)} \otimes h)_q(x_1, Q)}{h_q(x_1, Q)} + \frac{(\hat{P}^{(0)} \otimes h)_{\bar{q}}(x_2, Q)}{h_{\bar{q}}(x_2, Q)} \right).$$

- Second line: real hard-collinear initial-state emission. Its factor $f \neq 1$ prevents the usual reabsorption into the evolution of the parton densities $h_i(x, \mu)$.

Beyond NDL: interplay of the veto with hard-collinear dynamics and PDF evolution, and coherence-violating logarithms.

Coherence-violating logarithms

Coherence, and how Coulomb phases violate it

- Coherence: a soft wide-angle gluon sees only the *total* colour charge of a collinear system $\implies$ out-of-gap soft-collinear emissions cancel.
- Hadron colliders: Coulomb/Glauber exchanges between the *incoming* partons give the soft anomalous dimension an imaginary part:

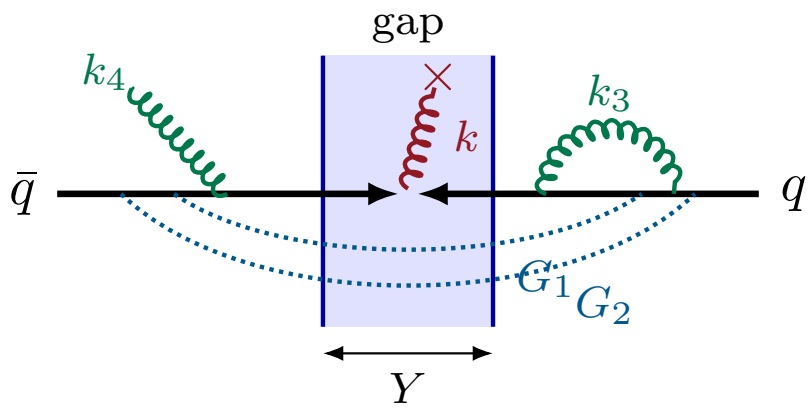
$$\boldsymbol{S}_{\mu_2,\mu_1} = \exp\left(-\frac{2\alpha_s}{\pi}\int_{\mu_1}^{\mu_2}\frac{dk_t}{k_t}\left[\boldsymbol{\Gamma}^{\mathrm{R}}+\boldsymbol{\Gamma}^{\mathrm{I}}\right]\right), \qquad \boldsymbol{\Gamma}^{\mathrm{I}} = \mathrm{i}\pi\,\boldsymbol{T}_1\cdot\boldsymbol{T}_2+\cdots$$

$$\boldsymbol{\Gamma}^{\mathrm{R}} = -\frac{1}{2}\sum_{i<j}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\int_{k\in\Omega}\mathrm{d}\eta\frac{\mathrm{d}\phi}{2\pi}\,\omega_{ij}, \quad (\omega_{ij}\text{: radiation pattern of dipole } ij)$$

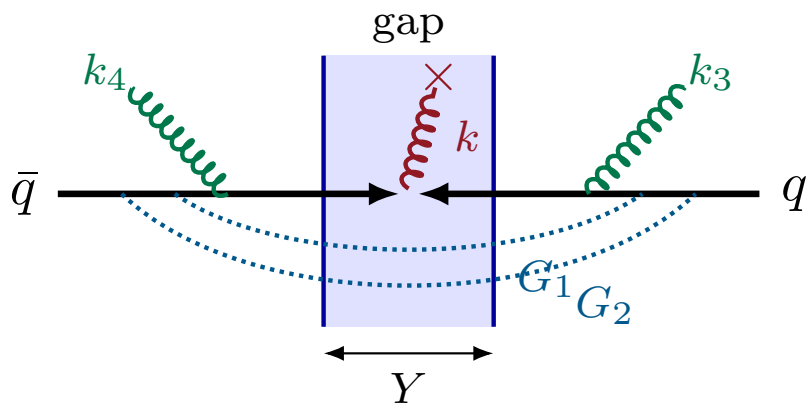
- $\left[\boldsymbol{\Gamma}^{\mathrm{R}},\boldsymbol{\Gamma}^{\mathrm{I}}\right]\neq 0 \implies$ real–virtual miscancellation for initial-state soft-collinear emissions [Forshaw, Kyrieleis, Seymour: hep-ph/0604094, 0808.1269]
- Linked to the breaking of strict collinear factorisation [Catani, de Florian, Rodrigo: 1112.4405; Forshaw, Seymour, Siodmok: 1206.6363]
- Leading tower resummed for standard gaps [Becher, Neubert, Shao: 2107.01212; Becher, Neubert, Shao, Stillger: 2307.06359]

Coherence-violating logarithms
Physical picture for a rapidity gap

- A soft gluon vetoed in the gap is emitted coherently by everything on one side of it: colour factor $\boldsymbol{T}_{\mathrm{R}}^2$, the squared total colour charge on that side.



k_3 virtual: $\boldsymbol{T}_{\mathrm{R}}^2 = C_F$, a number.
Coulomb exchange cannot change it.



k_3 real: $\boldsymbol{T}_{\mathrm{R}}^2 = (\boldsymbol{T}_q + \boldsymbol{T}_{k_3})^2$, an operator.
Coulomb exchange rotates it.

- Real and virtual out-of-gap emissions no longer cancel: $\left[\boldsymbol{\Gamma}^{\mathrm{I}}, \left[\boldsymbol{\Gamma}^{\mathrm{I}}, \boldsymbol{\Gamma}^{\mathrm{R}}\right]\right] \propto Y$.
- Drell-Yan: with one out-of-gap emission the colour structure is trivial ($q\bar{q}g$ has a single colour state) $\Longrightarrow$ two are needed, one on each side: $\Sigma_{\mathrm{CVL}}^{\mathrm{gap}} \sim \bar{\alpha}_s^5 L^7\, Y$ (N^{3}DL, suppressed by $1/N_c^2$).

CVLs with an azimuthal veto

Can they be enhanced too?

For the azimuthal veto in Drell-Yan, the same ingredients as for a rapidity gap:

- two out-of-gap soft-collinear emissions k_3, k_4;
- two Coulomb/Glauber exchanges between the incoming partons;
- one softer emission k into the azimuthal gap, which is vetoed.

To isolate the *leading* CVLs, work in the strongly-ordered limit:

$$Q \gg k_{t,3} \gg k_{t,4} \gg k_t \gg E_T$$

The rapidities of the out-of-gap emissions go up to their kinematic boundaries, $|\eta_i| < \eta_i^{\max} \equiv \ln(Q/k_{t,i})$.

New here: the vetoed in-gap gluon can be soft *and* collinear.
It has access to a large rapidity phase space.

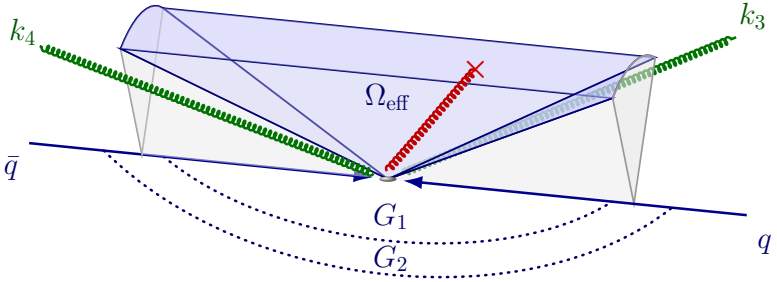
CVLs with an azimuthal veto

Master formula at $\mathcal{O}(\alpha_s^5)$

$$\hat{\sigma}_{\text{CVL}} = \left(-\frac{2\alpha_s}{\pi}\right)^5 (1-f)^2 \int_{E_T}^{Q} \frac{dk_{t,3}}{k_{t,3}} \int_{-\eta_3^{\max}}^{\eta_3^{\max}} d\eta_3 \int_{E_T}^{k_{t,3}} \frac{dk_{t,4}}{k_{t,4}} \int_{-\eta_4^{\max}}^{\eta_4^{\max}} d\eta_4 \int_{E_T}^{k_{t,4}} \frac{dk_t}{k_t}$$
$$\times \frac{1}{3} \ln^2\left(\frac{k_{t,4}}{E_T}\right) \left\langle m_0 \middle| \boldsymbol{D}_{a_3}^{\mu_3\,\dagger} \boldsymbol{D}_{a_4}^{\mu_4\,\dagger} \left[\boldsymbol{\Gamma}^{\text{I}}, \left[\boldsymbol{\Gamma}^{\text{I}}, \boldsymbol{\Gamma}^{\text{R}}\right]\right] \boldsymbol{D}_{a_4}^{\mu_4} \boldsymbol{D}_{a_3}^{\mu_3} \middle| m_0 \right\rangle,$$

- Born amplitude $\left|m_0\right\rangle$, real emissions:

$$\boldsymbol{D}_{a_\ell}^{\mu_\ell} = \sum_{i=1}^{n} \boldsymbol{T}_i^{a_\ell} h_{i,\ell}^{\mu_\ell}, \qquad h_{i,\ell}^{\mu_\ell} = \frac{1}{2} k_{t,\ell} \frac{k_i^{\mu_\ell}}{k_i \cdot k_\ell}$$



- $(1-f)^2$: k_3, k_4 outside Ω.
- $\frac{1}{3}\ln^2$: the two Coulomb exchanges.

Same double commutator as for gaps between jets [Forshaw, Kyrieleis, Seymour: 0808.1269].

The crucial difference: the structure of $\boldsymbol{\Gamma}^{\text{R}}$.

CVLs with an azimuthal veto

The real part of the anomalous dimension

$$\mathbf{\Gamma}^{\mathrm{R}} = -\frac{1}{2}\sum_{i<j} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \int_{k\in\Omega} \mathrm{d}\eta \frac{\mathrm{d}\phi}{2\pi}\, \omega_{ij}\,, \qquad \omega_{ij} \equiv \frac{1}{2} k_t^2 \frac{p_i \cdot p_j}{(k\cdot p_i)(k\cdot p_j)}$$

$$\omega_{ij} = \frac{1}{2} \frac{\cosh(\eta_i - \eta_j) - \cos(\phi_i - \phi_j)}{(\cosh(\eta - \eta_i) - \cos(\phi - \phi_i))\,(\cosh(\eta - \eta_j) - \cos(\phi - \phi_j))}$$

$$\approx \Theta(\eta_i - \eta)\,\Theta(\eta - \eta_j) + \Theta(\eta_j - \eta)\,\Theta(\eta - \eta_i) \qquad \text{(strongly-ordered rapidities)}$$

In the strongly-ordered limit, the vetoed soft gluon is radiated by dipole (ij) only if its rapidity η lies between the rapidities η_i, η_j of the two partons forming the dipole.

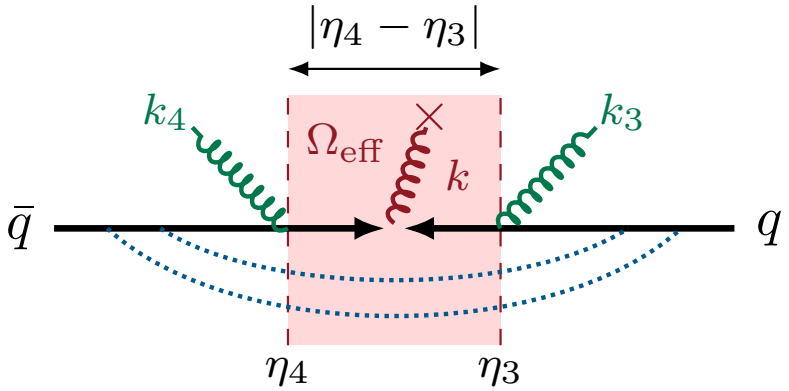
For incoming q (1), $\bar{q}$ (2) and outgoing soft gluons k_3, k_4 (neglecting subleading terms):

$$\mathbf{\Gamma}^{\mathrm{R}} \approx f\left[C_F \ln\left(\frac{Q}{k_t}\right) + \left\{\Theta(\eta_3 - \eta_4)(\eta_3 - \eta_4)\left(\frac{C_A}{2} + \boldsymbol{T}_1 \cdot \boldsymbol{T}_3\right) + (3 \leftrightarrow 4)\right\}\right]$$

CVLs with an azimuthal veto

The effective gap

- k_3, k_4 split Ω into three rapidity regions.
- Between η_4 and η_3: an *effective gap*, with the same coherence violation as for a rapidity gap.
- Collinear regions: colour factor still C_F, the Coulomb exchange has no effect.



The $C_F \ln(Q/k_t)$ and C_A terms of $\mathbf{\Gamma}^{\mathrm{R}}$ drop out of the commutator: the collinear singularity of the vetoed gluon does not contribute to CVLs.

$$\left\langle m_0 \middle| \boldsymbol{D}_{a_3}^{\mu_3\,\dagger} \boldsymbol{D}_{a_4}^{\mu_4\,\dagger} \left[\mathbf{\Gamma}^{\mathrm{I}}, \left[\mathbf{\Gamma}^{\mathrm{I}}, \mathbf{\Gamma}^{\mathrm{R}}\right]\right] \boldsymbol{D}_{a_4}^{\mu_4} \boldsymbol{D}_{a_3}^{\mu_3} \middle| m_0 \right\rangle = \pi^2 C_F \frac{N_c^2}{8} f \, |m_0|^2 \, |\eta_4 - \eta_3|$$

Rapidity gap $\to$ azimuthal veto: $\quad Y \longrightarrow f\,|\eta_4 - \eta_3|$

Its size is set by the kinematics of k_3, k_4, and is integrated over.

Performing the remaining k_t and rapidity integrals:

$$\boxed{\Sigma^{(5)}_{\text{CVL}} = -\pi^2\, C_F N_c^2\, f(1-f)^2\, \frac{8}{45}\, \bar{\alpha}_s^5 L^8}$$

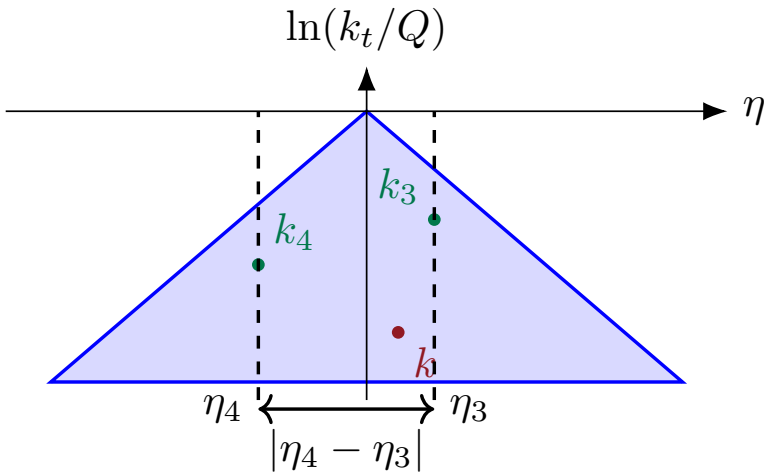
- NNDL: one power of L more than for a rapidity gap (N^3DL).
- The whole tower contributes at this order: $\Sigma_{\text{CVL}} \sim \sum_{n\geq 5} c_n\, \bar{\alpha}_s^n L^{2n-2}$
- π^2: two Coulomb phases. $f(1-f)^2$: k inside Ω; k_3, k_4 outside.
- Usual colour suppression relative to leading-colour terms.
- Starts at $n = 5$ for Drell-Yan: two out-of-gap soft-collinear emissions needed.

For NNDL accuracy, the resummation of CVLs is necessary.

Where does the extra logarithm come from?

Ingredient	Rapidity gap Y	Azimuthal veto
2 soft-collinear out-of-gap gluons k_3, k_4	$(\bar{\alpha}_s L^2)^2$	$(\bar{\alpha}_s L^2)^2$
2 Coulomb exchanges	$\pi^2\,(\bar{\alpha}_s L)^2$	$\pi^2\,(\bar{\alpha}_s L)^2$
1 vetoed in-gap gluon k	$\bar{\alpha}_s L \times Y$	$\bar{\alpha}_s L \times \lvert\eta_4 - \eta_3\rvert \to \bar{\alpha}_s L^2$
Leading CVL	$\bar{\alpha}_s^5 L^7\, Y$	$\bar{\alpha}_s^5 L^8$

- k_3, k_4 lie on a different "leaf" (in azimuth) from the in-gap gluon k.
- k_3, k_4 explore the full kinematic range in k_t *and* η; k lies between η_4 and η_3.
- $|\eta_4 - \eta_3|$ grows with the available rapidity range: integrating over η_3, η_4 gives the extra L.

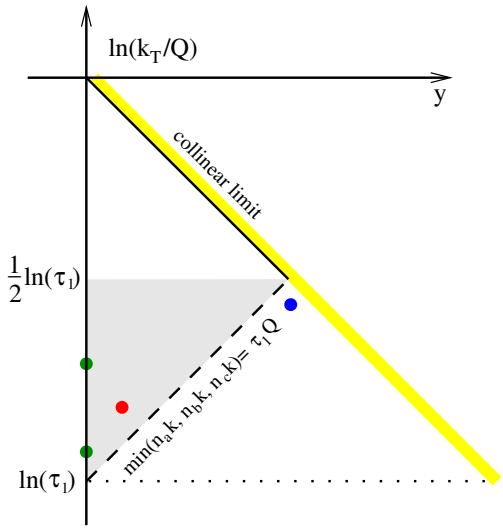


Enhanced CVLs in other observables

N-jettiness

- One-jettiness (*global*) in colour singlet + jet: $\sigma_1^{\mathrm{CVL}} \sim \alpha_s^4 L^6$, $L \equiv \ln(1/\tau_1)$. With a hard jet, *one* soft-collinear emission suffices [Banfi, Forshaw, Holguin: 2511.11799]

- Factorisation theorem for N-jettiness with Glauber modes: CVLs from four loops ($N \geq 1$) or five loops ($N = 0$)
  [Becher, Hager, Neubert, Schwienbacher: 2603.12383]



- Same origin as in Drell-Yan: *phase-space enhancement* of the soft "vetoed" emission. Azimuthal vetoes with a hard jet: also $\bar{\alpha}_s^4 L^6$.

Enhanced CVLs arise whenever the vetoed soft gluon has a double-logarithmic phase space.

Resummation of the NNDL tower $\bar{\alpha}_s^n L^{2n-2}$

- Standard gaps: the in-gap gluon is single-logarithmic, one soft insertion suffices [Becher, Neubert, Shao (+ Stillger): 2107.01212, 2307.06359; Schalch: 2609.34984]
- Here: arbitrary numbers of soft in-gap *and* soft-collinear out-of-gap emissions, with an effective gap set by the out-of-gap rapidities.

Collinear-factorisation breaking

- Initial-state collinear emissions with $E_T < k_t < Q$ do not decouple from the soft colour evolution. Gaps between jets: PDF factorisation restored below the veto scale [Becher et al.: 2408.10308, 2509.07082]; expected here too.
- Standard gaps: $\mathrm{N^3DL}$ and colour suppressed. Here: one logarithm larger, with cuts already used at the LHC.

A new probe of collinear-factorisation breaking at hadron colliders.

Lepton-collider case

No CVLs, but the choice of axis matters

$e^+e^- \to q\bar{q}$: no CVLs, but the logarithmic structure depends on the reference axis $\hat{n}$.

- Thrust axis: due to recoil from soft-collinear emissions, naive exponentiation is broken already at DL accuracy.

$$\Sigma^{(\mathrm{DL})}_{\mathrm{Thrust}} = \frac{\Big(1-2f+\exp\big(\kappa\, f^2 L^2\big)\,\sqrt{\kappa(1-2f)^2}\; fL\sqrt{\pi}\,\Big[\mathrm{erf}\Big(-\sqrt{\kappa}\,(1-f)\,L\Big)+\mathrm{erf}\Big(\sqrt{\kappa}\,f\,L\Big)\Big]\Big)^2}{(1-2f)^2}\,,$$

$\kappa \equiv 2\bar{\alpha}_s C_F/(1-2f),\;\; 0<f<\frac{1}{2}$.

- Winner-Takes-All (WTA) axis: no axis recoil from soft-collinear emissions, which restores exponentiation at DL accuracy. NDL resummation, verified against Event2 at $\mathcal{O}(\alpha_s^2)$, after considering NGLs and axis misalignment from hard-collinear emissions:

$$\Sigma^{(\mathrm{NDL})}_{\mathrm{WTA}} = e^{-\hat{R}(L)}\left(1-\tfrac{8}{3}C_F C_A\, I_{\mathrm{NG}}(f)\bar{\alpha}_s^2 L^3\right)+\frac{1}{2}\,C_F\,\bar{\alpha}_s\,e^{-\hat{R}(L)}\,(1-f)\,(16\ln 2-5)\left(\sqrt{\pi}\,\frac{\mathrm{erf}(\xi L)}{2\xi}-L\right),$$

$\hat{R}(L)$: NDL Sudakov radiator, $\qquad \xi \equiv \sqrt{2\bar{\alpha}_s f\,(C_A-C_F)}$.

Implications for parton showers

k_t-ordered dipole showers with dipole-local recoil

- For many standard observables, k_t-ordered dipole showers with dipole-local recoil reproduce LL terms, but can fail at NLL because recoil is assigned incorrectly in regions with $k_{t,1} \sim k_{t,2}$, $|\eta_1 - \eta_2| \gg 1$. [Dasgupta, Dreyer, Hamilton, Monni, Salam: 1805.09327]
- For the azimuthal veto, the same recoil issue is *enhanced*: the observable is sensitive to large rapidities, $|\eta| \lesssim \ln(Q_{\text{dip}}/k_t)$, so an incorrect recoil can move an emission *into* or *out of* the veto region.
- The resulting shower error starts already at

$$\delta\Sigma^{(2)}_{\text{shower}} \sim \bar{\alpha}_s^2 L^3,$$

i.e. at LL in the usual $\ln \Sigma$ counting.

The azimuthal veto is a sharp diagnostic of shower recoil schemes.

Conclusions and outlook

- Azimuthal vetoes on transverse-energy flow expose *new, more singular* logarithmic structures, already present in current LHC cuts ($\eta_{\text{cut}} \gtrsim L$).
- NGLs and CVLs are both *enhanced* by one power of L:

$$\Sigma^{(2)}_{\text{NG}} \sim \bar{\alpha}_s^2 L^3 \quad \text{(LL)}, \qquad \Sigma^{(5)}_{\text{CVL}} \sim \bar{\alpha}_s^5 L^8 \quad \text{(NNDL)}.$$

- Origin: the vetoed soft radiation reaches large rapidities; for CVLs, an effective gap $|\eta_4 - \eta_3|$.
- Resummation: LL (large N_c) and NDL (full colour) for pp; NDL for e^+e^- with the WTA axis, checked against EVENT2.
- Probe of collinear-factorisation breaking; benchmark for parton showers.

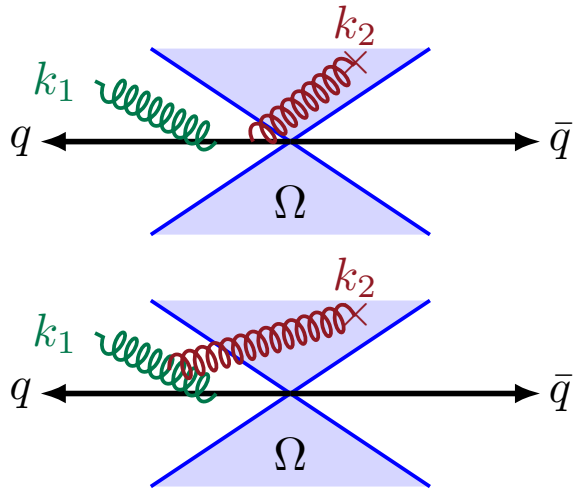
Outlook:

- NNDL resummation of the CVL tower; interplay with NGLs; $\eta_{\text{cut}} \sim L$.
- CVL resummation in Drell-Yan with a rapidity gap $Y \sim L$: a simpler first step towards the azimuthal veto.

Backup
NGL recap: secondary emissions

Two *energy-ordered* soft, large-angle gluons:
$Q \gg k_1 \gg k_2 \gtrsim E_0$.

- k_1: harder, radiated *outside* Ω.
- k_2: much softer, radiated *into* Ω.
- Whether k_1 is real (and lands outside Ω) or virtual, it does *not* change the value of E_Ω directly.



- But it *changes what is radiating* k_2!
- Real (k_1 out) and virtual (k_1 absent) contributions no longer cancel completely: the radiation patterns for k_2 into Ω are different.

A single "primary" Sudakov factor knows nothing about *where* k_1 went. This breakdown of the independent-emission picture is the origin of non-global logarithms.

[Dasgupta, Salam: hep-ph/0104277, hep-ph/0203009]

The leading non-global term comes from the correlated part of the two-gluon soft matrix element

$$|M_{12}|^2_{\text{corr}} \propto C_F C_A \, W_{ab}(1) \left[W_{a1}(2) + W_{1b}(2) - W_{ab}(2) \right],$$

where a, b label the primary $q\bar{q}$ dipole and $W_{ij}(k) = \dfrac{n_i \cdot n_j}{(n_i \cdot n_k)(n_j \cdot n_k)}$.

- C_F^2 term: independent emissions, already in the Sudakov factor Σ_{primary}.
- $C_F C_A$ factor: a genuinely non-Abelian effect.

Integrating over phase space:

$$\Sigma^{(2)}_{\text{NGL}} \sim \int_{E_0}^{Q} \frac{\mathrm{d}\omega_1}{\omega_1} \int_{E_0}^{\omega_1} \frac{\mathrm{d}\omega_2}{\omega_2} \int_{k_1 \notin \Omega} \frac{\mathrm{d}\Omega_1}{4\pi} \int_{k_2 \in \Omega} \frac{\mathrm{d}\Omega_2}{4\pi} \, |M_{12}|^2_{\text{corr}}.$$

Leading non-global correction at $\mathcal{O}(\alpha_s^2)$: soft parent outside Ω, softer daughter inside.

$$\Sigma_{\rm NG}^{(2)} = -\int [\mathrm{d}k_1][\mathrm{d}k_2]\,\Theta(k_{t,1}-k_{t,2})\,\widetilde{\mathcal{M}}^2(k_1,k_2) \times \left(\Theta\left(v_\Omega(k_1,k_2)-e^{-L}\right)-\Theta\left(v_\Omega(k_1+k_2)-e^{-L}\right)\right).$$

- $\widetilde{\mathcal{M}}^2(k_1,k_2) \equiv \mathcal{M}^2(k_1,k_2) - \mathcal{M}^2(k_1)\mathcal{M}^2(k_2)$: correlated part of the squared eikonal amplitude for two soft emissions. $v_\Omega(k_1,\ldots,k_n) \equiv \sum_{k_i\in\Omega} k_{t,i}/Q$.
- At LL: strongly-ordered soft limit, where $\widetilde{\mathcal{M}}^2$ reduces to the standard $C_F C_A$ angle-correlated kernel [Dokshitzer, Marchesini, Oriani '92]

$$\propto \frac{\cos\phi_{12}}{\cosh\eta_{12}-\cos\phi_{12}}$$

- $\Theta\big(v_\Omega(k_1+k_2)-e^{-L}\big) \approx \Theta\big(v_\Omega(k_1)-e^{-L}\big)$: support only for $k_2\in\Omega$, $k_1\notin\Omega$.

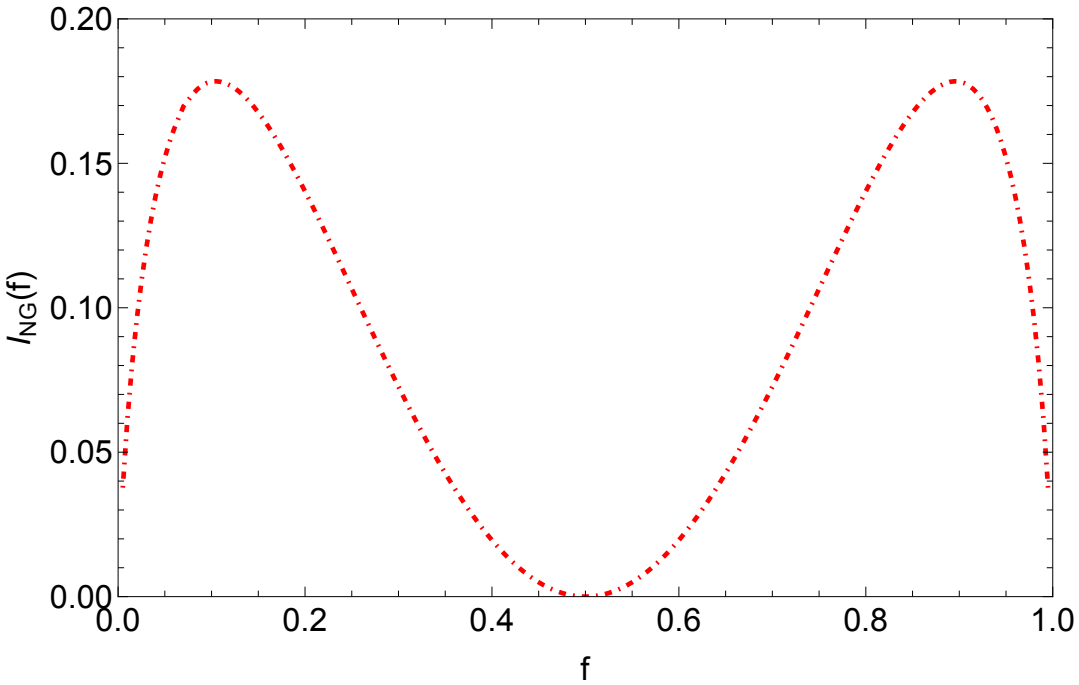
$$\Sigma^{(2)}_{\rm NG} = -16\, C_F C_A\, \bar{\alpha}_s^2 \int_{e^{-L}}^{1} \frac{\mathrm{d}v_1}{v_1} \int_{e^{-L}}^{v_1} \frac{\mathrm{d}v_2}{v_2} \int_0^{\ln \frac{1}{v_1}} \mathrm{d}\eta_1 \int_0^{\ln \frac{1}{v_2}} \mathrm{d}\eta_2 \times \int_{k_1 \notin \Omega} \frac{\mathrm{d}\phi_1}{2\pi} \int_{k_2 \in \Omega} \frac{\mathrm{d}\phi_2}{2\pi} \frac{\cos\phi_{12}}{\cosh\eta_{12} - \cos\phi_{12}}$$

- $\ell_i \equiv \ln(1/v_i)$, with $\ell_1 < \ell_2$ (strong ordering in k_t).
- The kernel falls like $e^{-|\eta_{12}|}$: the η_{12} integral is *finite*, $\int \mathrm{d}\eta_{12}\,(\ldots) = 2(\pi - |\phi_{12}|)\cot|\phi_{12}| \implies$ part of $I_{\rm NG}(f)$.
- The common rapidity of the pair runs up to ℓ_1 (bound of the harder gluon): factor ℓ_1.
- $\displaystyle\int_0^L \mathrm{d}\ell_1 \int_{\ell_1}^L \mathrm{d}\ell_2\, \ell_1 = \frac{L^3}{6} \implies$
  $\Sigma^{(2)}_{\rm NG} = -16 \cdot \frac{1}{6}\, C_F C_A I_{\rm NG}(f)\, \bar{\alpha}_s^2 L^3 = -\frac{8}{3} C_F C_A I_{\rm NG}(f)\, \bar{\alpha}_s^2 L^3.$

L^2 from the two k_t integrals, L from the common rapidity; the relative rapidity η_{12} gives only the $\mathcal{O}(1)$ coefficient.

The geometry dependence is isolated in

$$I_{\rm NG}(f) = \int_{k_1 \notin \Omega} \frac{\mathrm{d}\phi_1}{2\pi} \int_{k_2 \in \Omega} \frac{\mathrm{d}\phi_2}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\eta \, \frac{\cos\phi_{12}}{\cosh\eta - \cos\phi_{12}}.$$



- Non-trivial dependence on the gap opening f.
- Symmetric under $f \leftrightarrow 1 - f$.
- Vanishes at $f = 1/2$ ($\Delta\phi = \pi$) at this order.
- η integral in closed form:

$$2\,(\pi - |\phi_{12}|)\cot|\phi_{12}|,$$

for $|\phi_{12}| \leq \pi$.

- Solve the LL (large-N_c) evolution equation [Banfi, Marchesini, Smye: hep-ph/0206076; Banfi, Dreyer, Monni: 2104.06416, 2111.02413], complemented with the physical rapidity bound in the dipole frame, $|\eta_i| \leq \ln(Q/k_{t,i})$.
- Evolution time:

$$t_i = \int_{k_{t,i}/Q}^{1} \frac{\mathrm{d}x}{x} N_c \frac{\alpha_s(xQ)}{\pi} = -\frac{N_c}{2\pi\beta_0} \ln(1 - 2\beta_0\lambda_i), \qquad \lambda_i = \alpha_s(Q) \ln \frac{Q}{k_{t,i}}.$$

- Infrared cutoff $Q_0 > \Lambda$, with $2\beta_0\alpha_s(Q)\ln(Q/\Lambda) = 1$.
- Veneziano limit, $N_c \sim n_f \gg 1$: set $C_F = C_A/2$.
- The rapidity bound also generates small subleading logarithms. They are removed by taking numerically

$$\lambda\, g_{\mathrm{LL}}(\lambda) = \lim_{\alpha_s \to 0} \alpha_s \ln \Sigma \qquad \text{[as in Dasgupta et al.: 2002.11114]}$$

Backup
LL resummation: the modified `Gnole` algorithm

Generation of one event for $\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$:

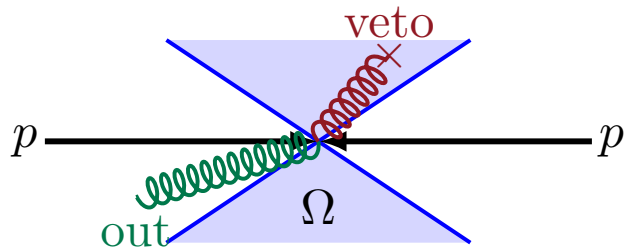
1. Start at $t_0 = 0$ with the Born $q\bar{q}$ dipole. Assign a collinear cutoff η_{M} in the dipole rest frame, which fixes its rapidity span $\Delta\eta_1$ in the event frame.
2. Compute $\Delta\eta_{\mathrm{tot}} = \sum_\ell \Delta\eta_\ell$ over all dipoles. Choose dipole $\mathcal{D}_\ell$ with probability $\Delta\eta_\ell/\Delta\eta_{\mathrm{tot}}$.
3. Increase t by Δt, obtained from $e^{-\Delta\eta_{\mathrm{tot}}\Delta t} = r$ with $r \in [0,1]$ random.
4. Generate $k_{t,i}$ from t_i; generate $\phi_i \in [0, 2\pi]$ and $|\eta_i| \leq \ln(Q/k_{t,i})$ uniformly in the dipole frame. If $|\eta_i| > \eta_{\mathrm{M}}$, discard the emission and generate a new one from t_i.
5. If $k_{t,i} < Q_0$: stop, and generate a new event.
6. Split $\mathcal{D}_\ell$ into two adjacent dipoles, each with its own collinear cutoff. Boost the rapidity bounds to the event frame.
7. If $k_i \in \Omega$: fill the t_i bin with weight $1/N_{\mathrm{events}}$ and start a new event. Otherwise go back to step 2.

- Emissions are generated in the emitting-dipole frame and Lorentz-transformed with *normalised* momenta: no subleading-power corrections. The cutoff must satisfy $L < \eta_{\mathrm{M}}$.

$$\Sigma_{\text{NDL}} = e^{-R(L)} \left(1 - \frac{8}{3} C_F C_A I_{\text{NG}}(f) \bar{\alpha}_s^2 L^3\right) - 2\bar{\alpha}_s C_F f\, e^{-R(L)} L \left(\frac{(\hat{P}^{(0)} \otimes h)_q(x_1, Q)}{h_q(x_1, Q)} + \{1 \leftrightarrow 2\}\right)$$

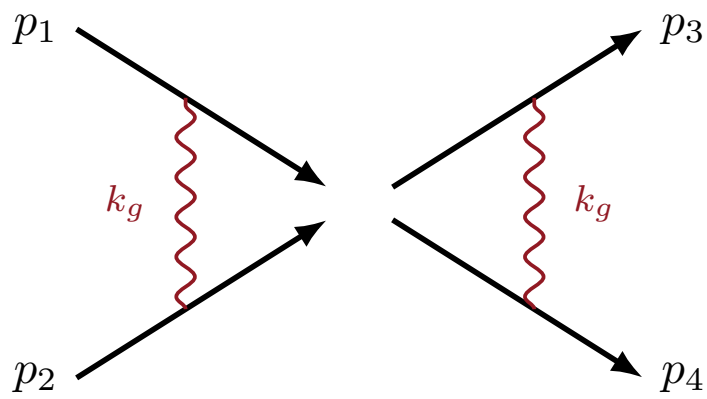
- Radiator promoted to its NLL counterpart, including the hard-collinear contribution (the -3): $R(L) = 2C_F f \int_0^L \mathrm{d}\ell\, \frac{\alpha_s(Qe^{-\ell})}{2\pi}(4\ell - 3)$
- Non-global correction: enters only through $I_{\text{NG}}(f)$, with full N_c dependence.
- Real hard-collinear initial-state emission: $\hat{P}^{(0)}$ regularised LO Altarelli-Parisi splitting function, $h_i(x, \mu)$ parton densities, $x_{1,2}$ momentum fractions.
- In a typical NDL resummation the last term is reabsorbed into the scale evolution of the parton densities. The factor $f \neq 1$ makes this impossible.

- Two *coloured* partons now enter the hard process, not just leave it.
- A gluon can be exchanged *between the incoming legs*, before, during, or after the hard scattering.
- Such an exchange is called a *Coulomb* or *Glauber* gluon: soft, predominantly transverse momentum, t-channel ($k^+k^- \ll |k_\perp^2|$).



- The incoming–incoming exchanges responsible for SLLs are absent in e^+e^- (no coloured initial state) and in DIS (only one coloured incoming leg).
- Resummation of the primary large logarithms, $L \equiv \ln(Q/Q_0)$, with a soft anomalous dimension matrix [Oderda, Sterman: hep-ph/9806530]

- Comes from a pole where one emitting parton goes *on-shell*: the exchanged gluon has $k^+ = k^- = 0$, i.e. purely transverse.
- Its residue is *purely imaginary*: a phase.
- Non-zero *only* if both emitting legs are on the *same side*: both incoming, or both outgoing.



$$\mathbf{\Gamma}^{\mathrm{I}} = \mathrm{i}\pi\, \boldsymbol{T}_1 \cdot \boldsymbol{T}_2 + \frac{\mathrm{i}\pi}{4}\left(\boldsymbol{T}_1^2 + \boldsymbol{T}_2^2 - \sum_{i=3}^{n} \boldsymbol{T}_i^2\right)$$

- The Casimir terms commute with everything: only $\mathrm{i}\pi\, \boldsymbol{T}_1 \cdot \boldsymbol{T}_2$ matters.
- A single phase is imaginary at amplitude level: a real cross section needs two insertions $\implies \left[\mathbf{\Gamma}^{\mathrm{I}}, \left[\mathbf{\Gamma}^{\mathrm{I}}, \mathbf{\Gamma}^{\mathrm{R}}\right]\right]$ and a factor π^2.

Backup

SLL recap: why a single Glauber exchange cancels

Soft evolution $\boldsymbol{S} = \exp\left[-a\int \frac{dk_t}{k_t}(\boldsymbol{\Gamma}^{\mathrm{R}} + \boldsymbol{\Gamma}^{\mathrm{I}})\right]$, $a = \frac{2\alpha_s}{\pi}$: $\boldsymbol{\Gamma}^{\mathrm{R}}$ hermitian, $\boldsymbol{\Gamma}^{\mathrm{I}} = \mathrm{i}\pi\, \boldsymbol{T}_1 \cdot \boldsymbol{T}_2 + \ldots$ anti-hermitian.

- Cross section $\sim \langle m|\boldsymbol{S}^\dagger \cdots \boldsymbol{S}|m\rangle$. At first order $\boldsymbol{\Gamma}^{\mathrm{I}} + \boldsymbol{\Gamma}^{\mathrm{I}\,\dagger} = 0$: the phase cancels between amplitude and conjugate amplitude.
- With $\boldsymbol{\Gamma}^{\mathrm{R}}$ present, $\boldsymbol{\Gamma}^{\mathrm{I}}$ enters only through nested commutators ($t = \ln\frac{\mu_2}{\mu_1}$):

$$\boldsymbol{S}^\dagger \boldsymbol{S} = \exp\left[-2at\,\boldsymbol{\Gamma}^{\mathrm{R}} + a^2t^2\left[\boldsymbol{\Gamma}^{\mathrm{R}}, \boldsymbol{\Gamma}^{\mathrm{I}}\right] - \frac{a^3t^3}{3}\left[\boldsymbol{\Gamma}^{\mathrm{I}}, \left[\boldsymbol{\Gamma}^{\mathrm{I}}, \boldsymbol{\Gamma}^{\mathrm{R}}\right]\right] + \cdots\right]$$

- Colour trace: in a real orthonormal colour basis (e.g. built from the trace basis), all $\boldsymbol{T}_i \cdot \boldsymbol{T}_j$ are real symmetric matrices and the colour state is real. $\left[\boldsymbol{\Gamma}^{\mathrm{R}}, \boldsymbol{T}_1 \cdot \boldsymbol{T}_2\right]$ is then antisymmetric, and $\langle m|A|m\rangle = 0$ for any antisymmetric A.
- Odd number of Glauber phases $\Rightarrow$ antisymmetric $\Rightarrow$ zero. Two phases: $\left[\boldsymbol{\Gamma}^{\mathrm{I}}, \left[\boldsymbol{\Gamma}^{\mathrm{I}}, \boldsymbol{\Gamma}^{\mathrm{R}}\right]\right] = -\pi^2\left[\boldsymbol{T}_{12}, \left[\boldsymbol{T}_{12}, \boldsymbol{\Gamma}^{\mathrm{R}}\right]\right]$ is symmetric and survives.

A real cross section needs an even number of Glauber phases: the leading CVLs carry π^2 from two exchanges.

Amplitudes are vectors in colour space. Each external parton carries a colour-charge operator $\boldsymbol{T}_i^a$ [Catani, Seymour: hep-ph/9605323]:

$$\boldsymbol{T}_i \cdot \boldsymbol{T}_i = C_i \qquad \text{(Casimir)},$$

$$\sum_i \boldsymbol{T}_i = 0 \qquad \text{(colour conservation)},$$

$$\left[\boldsymbol{T}_i^a, \boldsymbol{T}_j^b\right] = 0, \quad i \neq j; \qquad \left[\boldsymbol{T}_i^a, \boldsymbol{T}_i^b\right] = \mathrm{i} f^{abc} \boldsymbol{T}_i^c \qquad \text{(algebra)}.$$

A soft gluon of momentum k, colour a and Lorentz index μ is emitted by

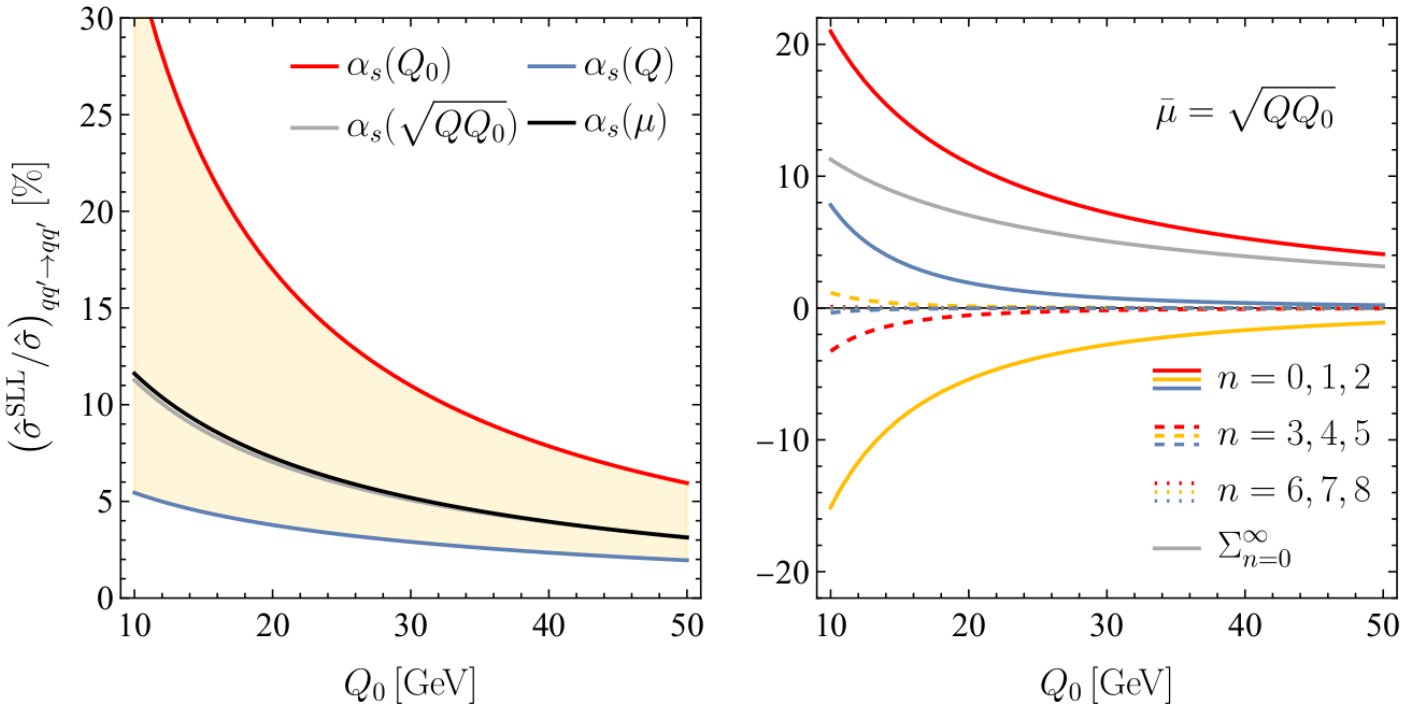
$$\boldsymbol{D}^{a,\mu}(k) = \sum_i \boldsymbol{T}_i^a \, h_i^\mu(k), \qquad h_i^\mu(k) = \frac{1}{2} k_t \frac{p_i^\mu}{p_i \cdot k},$$

$$\omega_{ij}(k) \equiv 2 h_i \cdot h_j = \frac{1}{2} k_t^2 \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)}.$$

- Gluon exchange between legs i and j: "dipole" operator $\boldsymbol{T}_i \cdot \boldsymbol{T}_j \equiv \sum_a \boldsymbol{T}_i^a \boldsymbol{T}_j^a$.
- After a real out-of-gap gluon, the soft anomalous dimension acts in a larger colour space: multiplicity matters.

Backup
SLL recap: size of the resummed effect for standard gaps

[Becher, Neubert, Shao, Stillger: 2307.06359]



- SLL contribution relative to the Born cross section for $qq' \to qq'$: can be phenomenologically important.
- How NGLs (and the single-logarithmic Sudakov factor) sit on top of the resummed SLLs is still open. First Monte Carlo resummation of SLLs: [Schalch: 2609.34984]

Two real out-of-gap emissions, ordered in transverse momentum:

$$\sigma_2 = \frac{1}{2}\left(-\frac{2\alpha_s}{\pi}\right)^2 \left\{ \int_{Q_0}^{Q} \frac{dk_{T3}}{k_{T3}} \int_{k_3 \notin \Omega} \frac{dy_3\, d\phi_3}{2\pi} \int_{Q_0}^{k_{T3}} \frac{dk_{T4}}{k_{T4}} \int_{k_4 \notin \Omega} \frac{dy_4\, d\phi_4}{2\pi} \right.$$
$$\left[\langle m_0 | \boldsymbol{S}^{\dagger}_{Q,k_{T3}} \boldsymbol{D}^{\mu_3\,\dagger}_{a_3} \boldsymbol{S}^{\dagger}_{k_{T3},k_{T4}} \boldsymbol{D}^{\mu_4\,\dagger}_{a_4} \boldsymbol{S}^{\dagger}_{k_{T4},Q_0} \boldsymbol{S}_{k_{T4},Q_0} \boldsymbol{D}^{\mu_4}_{a_4} \boldsymbol{S}_{k_{T3},k_{T4}} \boldsymbol{D}^{\mu_3}_{a_3} \boldsymbol{S}_{Q,k_{T3}} | m_0 \rangle \right]$$
$$\left. + (3 \leftrightarrow 4) \right\}$$

The evolution matrix ($Q_0 \equiv E_T$) accounts for uncancelled virtual emissions into Ω:

$$\boldsymbol{S}_{\mu_2,\mu_1} = \exp\left(-\frac{2\alpha_s}{\pi} \int_{\mu_1}^{\mu_2} \frac{dk_T}{k_T} \left[\boldsymbol{\Gamma}^{\mathrm{R}} + \boldsymbol{\Gamma}^{\mathrm{I}} \right] \right)$$

- No contribution to the CVLs when either or both of the out-of-gap gluons are virtual.
- *Leading* CVLs: expand $\boldsymbol{S}^{\dagger}_{k_{T4},Q_0} \boldsymbol{S}_{k_{T4},Q_0}$ to $\mathcal{O}(\alpha_s^3)$ and keep the term with two $\boldsymbol{\Gamma}^{\mathrm{I}}$. All other $\boldsymbol{S}_{\mu_2,\mu_1}$ can be set to the identity.

With $a \equiv 2\alpha_s/\pi$, $t \equiv \ln(k_{t,4}/E_T)$, and $\mathbf{\Gamma}^{\mathrm{R}}$ hermitian, $\mathbf{\Gamma}^{\mathrm{I}}$ anti-hermitian:

$$\boldsymbol{S}^\dagger \boldsymbol{S} = e^X e^Y, \qquad X = -a\,t\,(\mathbf{\Gamma}^{\mathrm{R}} - \mathbf{\Gamma}^{\mathrm{I}}), \qquad Y = -a\,t\,(\mathbf{\Gamma}^{\mathrm{R}} + \mathbf{\Gamma}^{\mathrm{I}}).$$

Baker-Campbell-Hausdorff:

$$\ln\left(e^X e^Y\right) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X-Y,[X,Y]] + \cdots$$

$$X + Y = -2a\,t\,\mathbf{\Gamma}^{\mathrm{R}}, \qquad [X,Y] = 2a^2t^2\left[\mathbf{\Gamma}^{\mathrm{R}},\mathbf{\Gamma}^{\mathrm{I}}\right], \qquad X - Y = 2a\,t\,\mathbf{\Gamma}^{\mathrm{I}}.$$

The only term with two $\mathbf{\Gamma}^{\mathrm{I}}$ at $\mathcal{O}(\alpha_s^3)$ is therefore

$$\boxed{\boldsymbol{S}^\dagger \boldsymbol{S} \supset \frac{(-a\,t)^3}{3}\left[\mathbf{\Gamma}^{\mathrm{I}},\left[\mathbf{\Gamma}^{\mathrm{I}},\mathbf{\Gamma}^{\mathrm{R}}\right]\right]}$$

- $(-a)^3$ combines with $(-a)^2$ from the two real emissions: $\left(-\frac{2\alpha_s}{\pi}\right)^5$.
- $\frac{t^3}{3} = \int_{E_T}^{k_{t,4}} \frac{dk_t}{k_t}\,\frac{1}{3}\ln^2\frac{k_{t,4}}{E_T}$, as in the master formula.
- The colour-trivial, k_t-dependent part of $\mathbf{\Gamma}^{\mathrm{R}}$ drops out of the commutator.

- No emission ($q\bar{q}$): one-dimensional colour space, $\boldsymbol{T}_1 \cdot \boldsymbol{T}_2 = -C_F$.
- One emission ($q\bar{q}g$): still one-dimensional. Colour conservation fixes all dipoles:

$$\boldsymbol{T}_1 \cdot \boldsymbol{T}_2 = \frac{C_A}{2} - C_F, \qquad \boldsymbol{T}_1 \cdot \boldsymbol{T}_3 = \boldsymbol{T}_2 \cdot \boldsymbol{T}_3 = -\frac{C_A}{2}.$$

  $\boldsymbol{\Gamma}^{\mathrm{R}}$ and $\boldsymbol{\Gamma}^{\mathrm{I}}$ are numbers, so $\left[\boldsymbol{\Gamma}^{\mathrm{I}}, \boldsymbol{\Gamma}^{\mathrm{R}}\right] = 0$: no CVLs. Consistently,

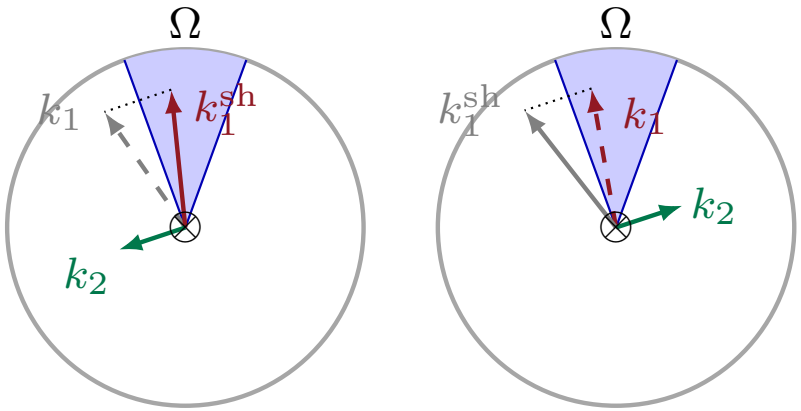
$$\boldsymbol{\Gamma}^{\mathrm{R}} = f\left[C_F \ln\frac{Q}{k_t} + \eta_3\left(\frac{C_A}{2} + \boldsymbol{T}_1 \cdot \boldsymbol{T}_3\right)\right] = f\, C_F \ln\frac{Q}{k_t}.$$

- Two emissions ($q\bar{q}gg$): multi-dimensional colour space, $\boldsymbol{T}_1 \cdot \boldsymbol{T}_3$ is a genuine operator and $[\boldsymbol{T}_1 \cdot \boldsymbol{T}_2, \boldsymbol{T}_1 \cdot \boldsymbol{T}_3] \neq 0$.

Drell-Yan: CVLs start at $\bar{\alpha}_s^5$.
With a hard jet in the Born process: one emission suffices, $\bar{\alpha}_s^4$.

- Two soft emissions with commensurate k_t but $|\eta_1 - \eta_2| \gg 1$.
- A k_t-ordered dipole shower with dipole-local recoil assigns the recoil incorrectly:

$$\boldsymbol{k}_{t,1} = \boldsymbol{k}_{t,1}^{\text{shower}} + \boldsymbol{k}_{t,2}$$

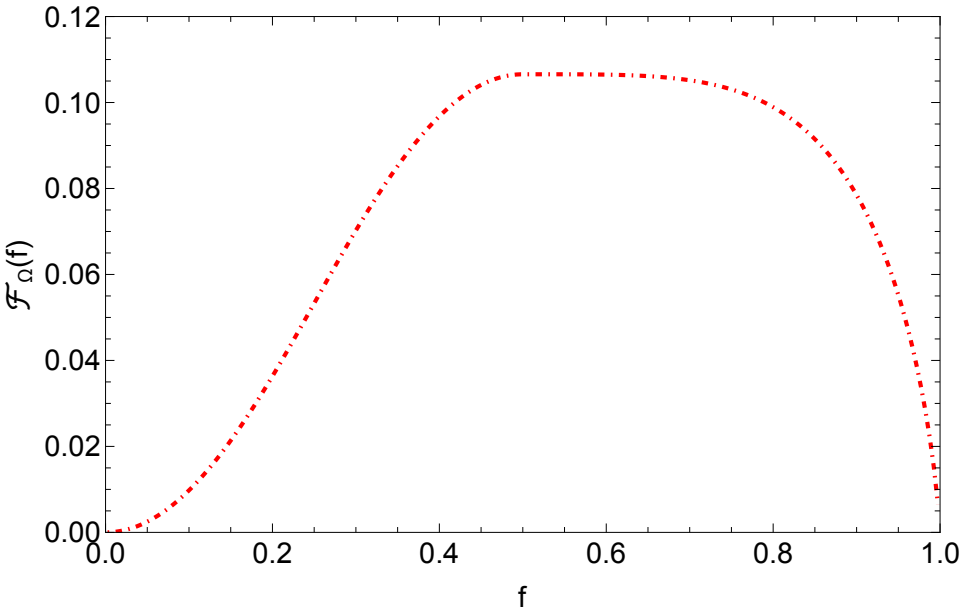


- This can knock k_1 *into* or *out of* Ω.

Spurious contribution to the cumulative cross section:

$$\delta\Sigma_{\text{shower}} = \left(\frac{2C_F\alpha_s}{\pi}\right)^2 \frac{2}{3} L^3 \mathcal{F}_\Omega(f), \qquad \mathcal{F}_\Omega(f) = \mathcal{F}_\Omega^{\text{in}}(f) + \mathcal{F}_\Omega^{\text{out}}(f)$$

- An $\mathcal{O}(\alpha_s^2 L^3)$ error: *LL* in the usual $\ln \Sigma$ counting.
- Standard non-global observables only lose accuracy at NLL.
- Reinforces the case for higher-logarithmic-accuracy showers.



- $x = k_{t,2}/k_{t,1}$, and $k_2 \notin \Omega$ throughout.
- Recoiled azimuth:

$$\phi_1^{\text{shower}} = \tan^{-1} \frac{\sin\phi_1 - x\sin\phi_2}{\cos\phi_1 - x\cos\phi_2}$$

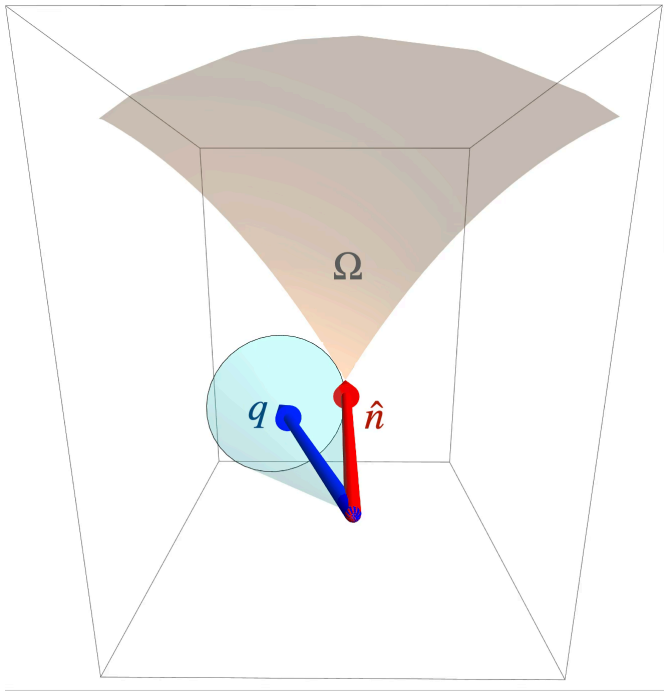
- Evaluated numerically.

$$\mathcal{F}_\Omega^{\text{in}} = \int \frac{dx}{x}\frac{d\phi_1}{2\pi}\frac{d\phi_2}{2\pi}\,\Theta_{k_1^{\text{shower}}\in\Omega}\,\Theta_{k_1\notin\Omega}\,\Theta_{k_2\notin\Omega},$$

$$\mathcal{F}_\Omega^{\text{out}} = -\int \frac{dx}{x}\frac{d\phi_1}{2\pi}\frac{d\phi_2}{2\pi}\,\Theta_{k_1^{\text{shower}}\notin\Omega}\,\Theta_{k_1\in\Omega}\,\Theta_{k_2\notin\Omega}.$$

- A soft-collinear emission at angle θ to its emitter can enter Ω only if θ exceeds the emitter–axis misalignment $\theta_{\hat{n},\mathrm{em}}$.
- For $\theta < \theta_{\hat{n},\mathrm{em}}$ the emission lies inside the emitter's collinear cone (blue) and cannot reach Ω.
- To DL accuracy:

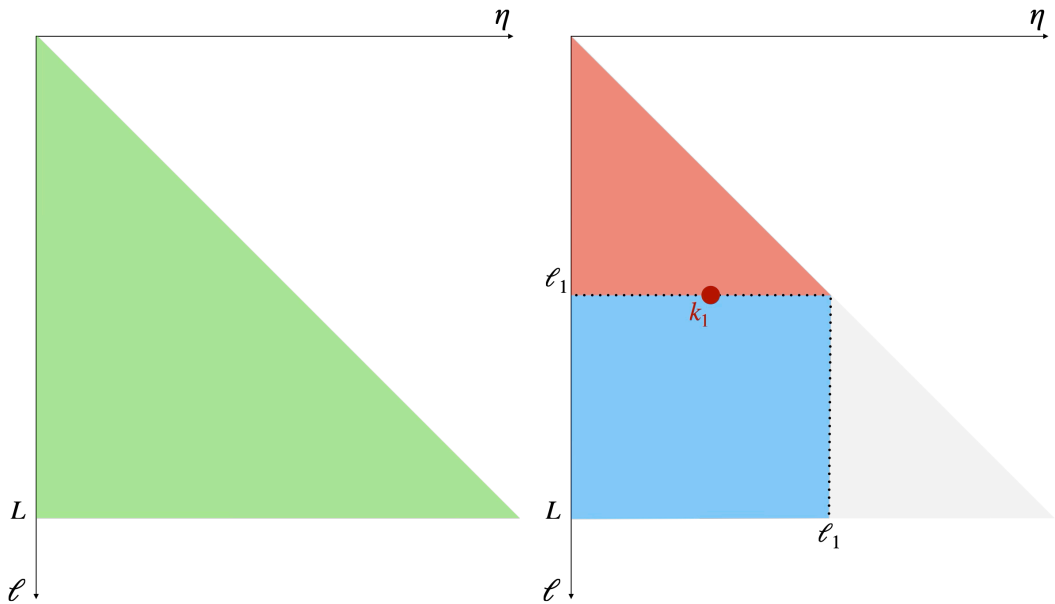
$$\Theta(k \in \Omega) = \Theta(\eta < -\ln \theta_{\hat{n},\mathrm{em}})\,\Theta(|\phi| < \pi f)$$



- Thrust axis: the recoil of the hardest soft-collinear emission k_1 sets $\theta_{\hat{n},q} \approx k_{t,1}/Q \implies$ tilt acts already at DL.
- WTA axis: soft-collinear emissions induce no axis recoil. The misalignment requires a hard-collinear branching $\implies$ an NDL configuration.

Naive hypothesis in the strongly-ordered limit ($\ell \equiv \ln(Q/k_t)$, $v_\Omega \equiv E_{T,\Omega}/Q$):

$$\Theta(e^{-L} - v_\Omega) \overset{\text{naive}}{\approx} \prod_i \Theta(e^{-L} - v_\Omega(k_i)) \quad \Rightarrow \quad \Sigma^{(\text{DL})}_{\text{Thrust,naive}} = e^{-R_{\text{naive}}(L)}$$

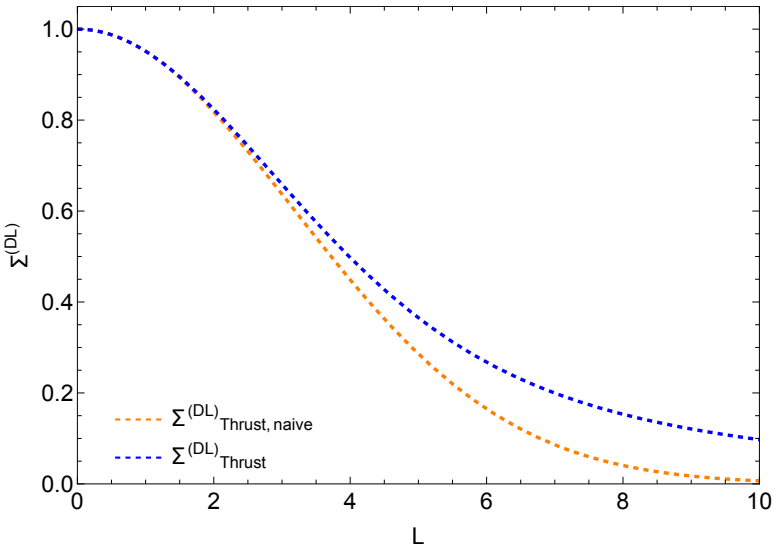


Left: naive veto. Right: correct veto, k_1 lowest ℓ.
Green: azimuthal factor $2f$; blue: f; red: 1.

- One emission: vetoed from Ω *and* from the opposite region Ω_c, where it would recoil the quark into Ω.
- Only the hardest emission k_1 recoils the quark: the others are not vetoed from Ω_c.
- Tilt induced by k_1: no veto on emissions with $\eta > \ell_1$.

Hemisphere factorisation at DL: $\Sigma^{(\mathrm{DL})}_{\mathrm{Thrust}} = \left(\Sigma^{(\mathrm{DL})}_{\mathrm{Thrust,hemi}}\right)^2$, with

$$\Sigma^{(\mathrm{DL})}_{\mathrm{Thrust,hemi}} = e^{-R^{\mathrm{T}}_{\mathrm{hemi}}(0,L)} + 4\bar{\alpha}_s C_F (1-2f) \int_0^L d\ell_1 \int_0^{\ell_1} d\eta_1\, e^{-R^{\mathrm{T}}_{\mathrm{global}}(0,\ell_1)} \times \left(e^{-R^{\mathrm{T}}_{\mathrm{true}}(\ell_1,L)} - e^{-R^{\mathrm{T}}_{\mathrm{hemi}}(\ell_1,L)}\right)$$



$f = 1/4$, $\bar{\alpha}_s = 0.118/(2\pi)$

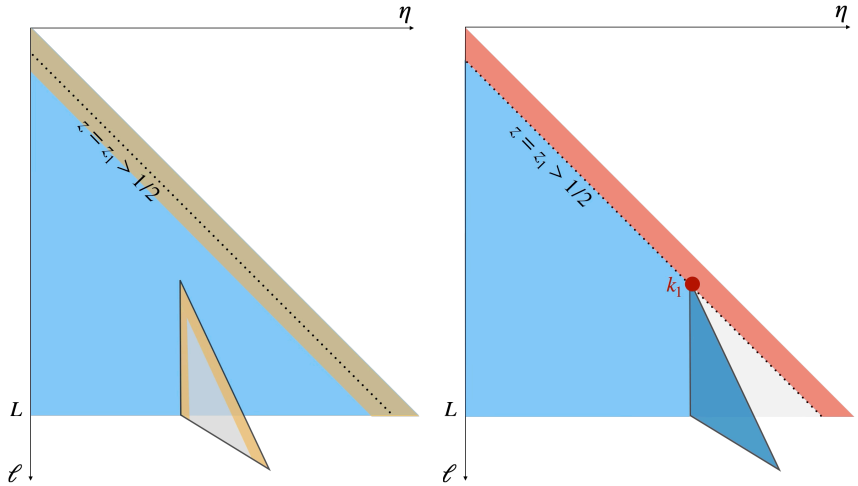
$$R^{\mathrm{T}}_{\mathrm{hemi}}(L_1,L_2) = 4C_F\,(2f) \int_{L_1}^{L_2} d\ell\, \bar{\alpha}_s \int_0^{\ell} d\eta$$

$$R^{\mathrm{T}}_{\mathrm{global}}(L_1,L_2) = 4C_F \int_{L_1}^{L_2} d\ell\, \bar{\alpha}_s \int_0^{\ell} d\eta$$

$$R^{\mathrm{T}}_{\mathrm{true}}(L_1,L_2) = 4C_F\, f \int_{L_1}^{L_2} d\ell\, \bar{\alpha}_s \int_0^{L_1} d\eta$$

- $(1-2f)$: k_1 outside both Ω and Ω_c.
- R_{global}: k_1 is the lowest-ℓ emission.
  R_{true}: recoil-modified veto.

Backup
Lepton collider, WTA axis: NDL structure



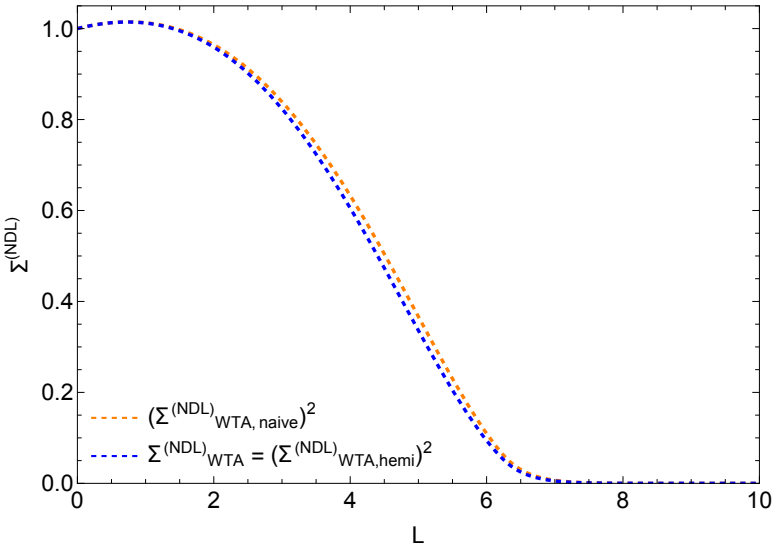
Left: naive NDL veto. Right: correct veto when k_1 triggers an axis swap. Blue: factor f; red: 1; gold: NDL.

- WTA axis aligns with the hardest parton: no recoil from soft-collinear emissions, $\Sigma^{(\mathrm{DL})}_{\mathrm{WTA}} = e^{-4C_F f \bar{\alpha}_s L^2}$.
- At NDL: axis swap onto a hard-collinear gluon for $z > 1/2$.
- $(1-f)$ below: k_1 must keep the quark out of Ω.

$$\Sigma^{(\mathrm{NDL})}_{\mathrm{WTA,hemi}} = \mathcal{S}\, e^{-R^{\mathrm{W}}_{\mathrm{hemi}}(0,L)} + \Sigma^{(\mathrm{NDL})}_{\mathrm{WTA,correc}}, \qquad \mathcal{S} = 1 - \tfrac{8}{3} C_F C_A I_{\mathrm{NG}}(f)\, \bar{\alpha}_s^2 L^3$$

$$R^{\mathrm{W}}_{\mathrm{hemi}}(L_1, L_2) = 2C_F\, f \int_{L_1}^{L_2} d\ell\, \frac{\alpha_s(Qe^{-\ell})}{2\pi} \int_0^{\ell} d\eta\, z\, P_{gq}(z)$$

$$\Sigma^{(\mathrm{NDL})}_{\mathrm{WTA,correc}} = 2C_F(1-f) \int_0^{L} d\ell_1\, \bar{\alpha}_s \int_{\ell_1 - \ln 2}^{\ell_1} d\eta_1\; z_1 P_{gq}(z_1)$$
$$\times \left[e^{-R^{\mathrm{W}}_{\mathrm{true},q}(\ell_1,0,L)}\, e^{-R^{\mathrm{W}}_{\mathrm{true},g}(\ell_1,\ell_1,L)} - e^{-R^{\mathrm{W}}_{\mathrm{hemi}}(0,L)} \right]$$



$f = 1/4$, $Q = M_Z$

Radiators in the presence of an axis swap:

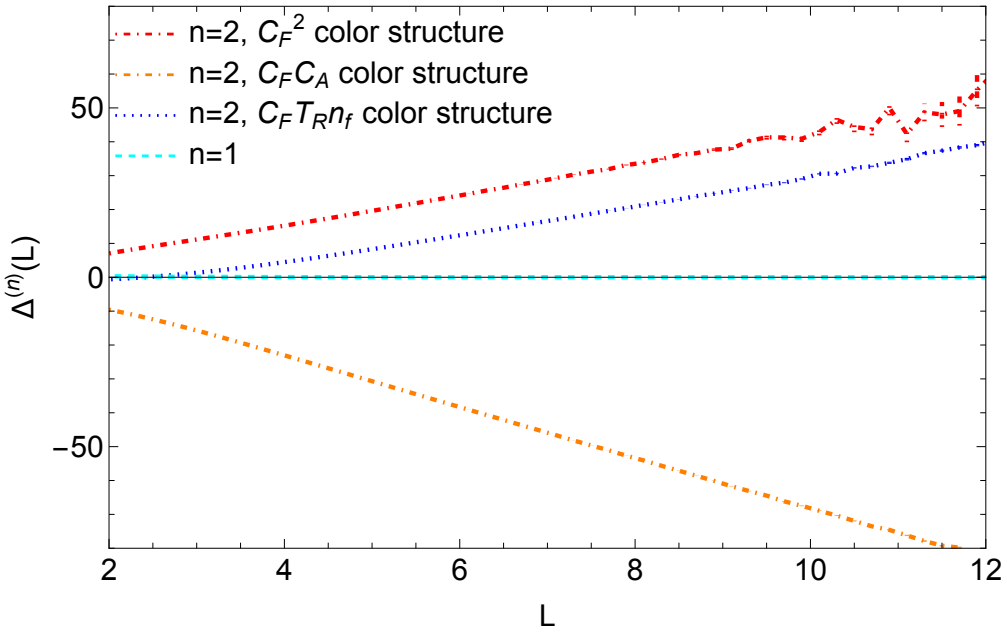
$$R^{\mathrm{W}}_{\mathrm{true},q}(\tilde{\eta}, L_1, L_2) = 4C_F\, f \int_{L_1}^{L_2} d\ell\, \bar{\alpha}_s \int_0^{\min(\ell,\tilde{\eta})} d\eta$$

$$R^{\mathrm{W}}_{\mathrm{true},g}(\tilde{\eta}, L_1, L_2) = 4C_A\, f \int_{L_1}^{L_2} d\ell\, \bar{\alpha}_s \int_{\tilde{\eta}}^{\ell} d\eta$$

- Soft-collinear radiation off the hard gluon must also be vetoed (C_A).

$$\Sigma^{(\mathrm{NDL})}_{\mathrm{WTA}} = e^{-\hat{R}(L)} \left(1 - \tfrac{8}{3} C_F C_A I_{\mathrm{NG}}(f) \bar{\alpha}_s^2 L^3\right) + \frac{1}{2}\, C_F\, \bar{\alpha}_s\, e^{-\hat{R}(L)}\, (1-f)\, (16 \ln 2 - 5) \left(\sqrt{\pi} \frac{\mathrm{erf}(\xi L)}{2\xi} - L\right)$$

with $\xi \equiv \sqrt{2\bar{\alpha}_s f (C_A - C_F)}$ and $\hat{R}(L)$ the NDL radiator of the pp case.



- Expand the NDL result to $\mathcal{O}(\alpha_s^2)$ and compare with `Event2` [Catani, Seymour: hep-ph/9605323] at $f = 1/4$.
- C_F^2, $C_F C_A$ and $C_F T_R n_f$ colour structures checked separately.

$$\Delta^{(n)}(L) \equiv \frac{d\Sigma^{(\mathrm{NDL})\,(n)}_{\mathrm{WTA}}}{dL} - \frac{d\Sigma^{(n)}_{\texttt{Event2}}}{dL}, \qquad \frac{d\Sigma}{dL} = \sum_{m=1}^{2} \frac{\alpha_s^m}{(2\pi)^m} \frac{d\Sigma^{(m)}}{dL}$$

At NDL accuracy, for large L one expects $\Delta^{(1)}(L) \to 0$ and $\Delta^{(2)}(L) \propto L$: confirmed numerically.