The University of Manchester
DEPT. OF PHYSICS & ASTRONOMY

Jet fragmentation functions for general cut variables

A DGLAP boundary prescription & its use as a parton-shower benchmark

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Fragmentation in $e^+e^- \to$ hadrons

INTRODUCTION

The cross section factorises into a hard coefficient and a fragmentation function

– Single-inclusive hadron spectrum ($z = E_h/E$, $E = Q/2$):

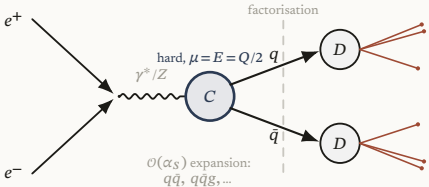
$$\frac{1}{\sigma_0}\frac{d\sigma^h}{dz}(z,Q) = \sum_i C_i(z,Q,\mu) \otimes D_i^h(z,\mu)$$

– C_i – hard coefficient, *perturbative*: knows Q only via $\ln Q^2/\mu^2 \Rightarrow$ no large log at $\mu = E = Q/2$.

– D_i^h – fragmentation function: *non-perturbative*, universal (no Q).

– Collinear singularities of D_i^h are absorbed into an $\overline{\text{MS}}$ renormalisation $\Rightarrow$ the FF obeys (timelike) DGLAP:

$$\frac{d\,D^h(z,\mu)}{d\ln\mu^2} = P \otimes D^h.$$



each leg dressed independently – D blind to the hard process.

Two ingredients: a *calculable* C and a *renormalised, DGLAP-evolving* FF D.

Gribov, Lipatov, Sov. J. Nucl. Phys. 15 (1972) 675; Dokshitzer, Sov. Phys. JETP 46 (1977) 641; Altarelli, Parisi, NPB 126 (1977) 298; Mitov, Moch, Vogt, PLB 638 (2006) 61 [hep-ph/0604053]; Almasy, Moch, Vogt, NPB 854 (2012) 133 [1107.2263]

From a hadron to a jet

JETS

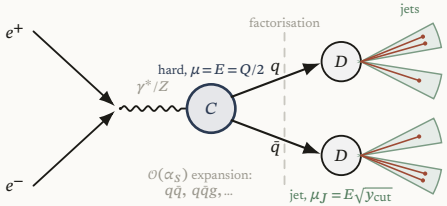
y_{cut} replaces the collinear pole with a resummable logarithm

- Replace each hadron by a **jet**: cluster the collinear particles, *inclusive* over emissions below a resolution y_{cut} (C/A: $y_{\text{cut}} = R^2$, an angular radius).
- y_{cut} regulates the collinear region — collinear safe, **no** $1/\epsilon$ pole. The pole becomes a **large log** $\ln \frac{1}{y_{\text{cut}}}$ (SL $\alpha_s^n \ln^n$, NSL $\alpha_s^n \ln^{n-1}$).
- Jet FF *factorises* $\Rightarrow \ln \frac{1}{y_{\text{cut}}}$ resummed by a *DGLAP-like* evolution, $\mu_J = E\sqrt{y_{\text{cut}}} \to \mu = E$, effective kernel $\hat{P}$:

$$\frac{1}{\sigma_0}\frac{d\sigma^{\text{jet}}}{dz} = \sum_i C_i(z, Q, \mu) \otimes D_i^{\text{jet}}(z, \mu, \mu_J)$$

$$\frac{d\, D^{\text{jet}}(z, \mu, \mu_J)}{d \ln \mu^2} = \hat{P} \otimes D^{\text{jet}}$$

$z = E_{\text{jet}}/E$; same C_i as for hadrons.



cluster to resolution y_{cut}: the collinear pair → one jet, a wider emission → its own.

The question

THE QUESTION

Does the jet FF evolve with the $\overline{\text{MS}}$ splitting function at NSL?

The $\ln \frac{1}{y_{\text{cut}}}$ resummation is *DGLAP-like*, from μ_J up to $\mu = E$, with an effective kernel $\hat{P}$; expand it order by order:

$$\frac{d\, D^{\text{jet}}}{d \ln \mu^2} = \hat{P} \otimes D^{\text{jet}}, \qquad \hat{P} = \hat{P}^{(0)} + \hat{P}^{(1)} + \cdots$$

SL ($\alpha_s^n \ln^n$) is fixed by $\hat{P}^{(0)}$; **NSL** ($\alpha_s^n \ln^{n-1}$) needs $\hat{P}^{(1)}$ – is it the textbook $\overline{\text{MS}}$ splitting function?

- $\hat{P}^{(0)}$ – ordinary (timelike) $\overline{\text{MS}}$ DGLAP.
- An initial **SCET** analysis suggested $\hat{P}^{(1)}$ is $\overline{\text{MS}}$ too: NSL still ordinary DGLAP, up to a recomputed coefficient function.
- **Two-loop small-R jets** (+ updated SCET): otherwise – an **angular (R) cut** gives the **spacelike** $\hat{P}^{(1)}$.
- But a k_T **cut** gives the **timelike** $\hat{P}^{(1)}$ (the hadron-FF case).

The *clustering cut* chooses timelike vs. spacelike evolution – **what about the general case?**

Dasgupta, Dreyer, Salam, Soyez, JHEP 04 (2015) 039 [1411.5182]; Kang, Ringer, Vitev, JHEP 10 (2016) 125 [1606.06732]; van Beekveld *et al.*, JHEP 07 (2024) 239 [2402.05170]

A general cut variable

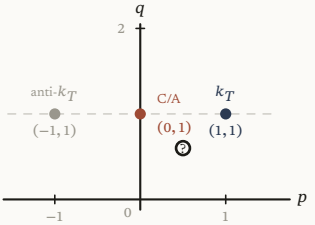
THE GENERAL CASE

One cut family – from timelike to spacelike, and between

Cluster the closest pair i, j iff $v_{ij} = \min\left(\frac{E_i}{Q}, \frac{E_j}{Q}\right)^{2p} \theta_{ij}^{2q} < y_{\text{cut}}$ $(E = Q/2)$, else resolve.

- One family (p, q): C/A (p=0) angular/**spacelike**; k_T (p=q=1) **timelike** – a general cut = ?
- Exclusive e^+e^- gen-k_T with a y_{cut} (like k_T) – **not** the *inclusive* gen-k_T (beam distance) at pp.
- The angular exponent q is our extension – connects to **shower ordering variables**.

Aim: the kernel $\hat{P}$ to $\mathcal{O}(\alpha_s^2)$ – the NSL input for *any* cut in the family.



? = a general cut – which evolution does it pick?

Salam, EPJC 67 (2010) 637 [0906.1833]

Why this family

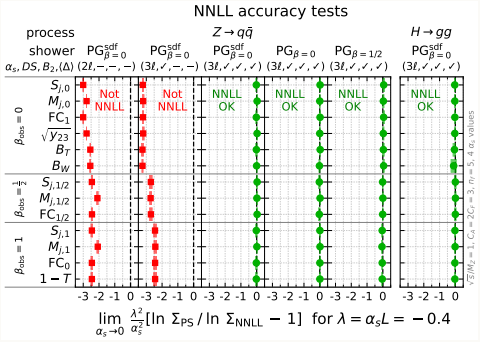
MOTIVATION

Two reasons: the structure of the logs, and a target for showers

(1) **Theory.** What structure do the $\ln(1/y_{\text{cut}})$ logs have, and how is it related to the ordinary $\overline{\text{MS}}$ FF? One function with one DGLAP-like equation? Which kernel: timelike, spacelike, or in between?

(2) **Shower targets.** Showers are climbing in logarithmic accuracy, into the *triple-collinear* region. The jet FF is a family of **genuinely triple-collinear-sensitive** observables: NSL needs the $1 \to 3$ splitting, and (p, q) is a continuous dial. Any shower, clustered with the same cut, must reproduce it.

- *Analogue:* skip the clustering; evolve to a cutoff in the shower's own ordering variable. PanScales $v = k_t\, e^{-\beta_{\text{PS}}|\eta|} \leftrightarrow p = 1$, $q = 1 + \beta_{\text{PS}}$, so $\kappa = \beta_{\text{PS}}$. Argued to share the NSL structure (backup); the $\mathcal{O}(\alpha_s)$ constant depends on the shower's exact v.



PanScales event shapes: deviation from NNLL $\to 0$ as $\alpha_s \to 0$ once the NNLL ingredients are in (green). We apply the same test later.

Dasgupta *et al.*, PRL 125 (2020) 052002 [2002.11114]; van Beekveld *et al.*, PRL 134 (2025) 011901 [2406.02661]; van Beekveld *et al.*, JHEP 03 (2025) 209 [2409.08316]

Outline

ROADMAP

1. **The NLO calculation** – the effect of clustering, organised as a difference to the inclusive parton FF, checked against Event2; and what $\kappa = q - p$ means: the cut's boost charge (timelike ↔ spacelike).

2. **The all-orders prescription** – the coupling scale, and the shifted anomalous dimension as a change of *evolution scale*, read three ways, across every cut; and a target for triple-collinear showers.

3. **Summary.**



1 · The NLO calculation

the clustering effect, checked against Event2

The clustering effect at NLO

§1 · NLO CALCULATION

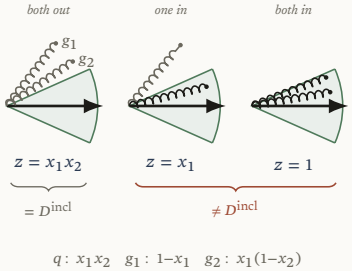
Organise as a difference to the inclusive ($\overline{\text{MS}}$) parton FF

– Jet FF and inclusive *parton* FF ($\overline{\text{MS}}$): the *same* matrix elements, differing only by the clustering.

– Compute the difference $D^{\text{jet}} - D^{\text{incl}}$: D^{incl} keeps an IR collinear pole, the **collinear-safe** jet a finite $\ln \frac{1}{y_{\text{cut}}}$.

– The leading logs $\alpha_s^n \ln^n \frac{1}{y_{\text{cut}}}$ are fixed by the LO splitting function $P^{(0)}$ – read off $1 \to 2$ branchings, and **identical** to the inclusive (and hadron) FF.

– The next tower $\alpha_s^n \ln^{n-1} \frac{1}{y_{\text{cut}}}$ needs the NLO splitting function $P^{(1)}$ – **here** the jet and inclusive FF first differ.



Argued in the quark (non-singlet) channel; the same reasoning carries over to *all* colour channels.

Ritzmann, Waalewijn, PRD 90 (2014) 054029 [1407.3272]; van Beekveld *et al.*, JHEP 07 (2024) 239 [2402.05170]

The four pieces

§1 · NLO CALCULATION

$q \to qgg$ configurations — two give no log, two survive

NO LOG in the difference

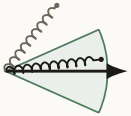
SURVIVE → $\delta P^{(1)}$



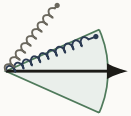
1 triple-collinear
commensurate, at the edge



2 virtual
loop on a single emission



3 piece I $(+2q)$
angle part of the cut
vs. MS's k_t



4 piece II $(-2p)$
energy part of the cut
depleted parent energy

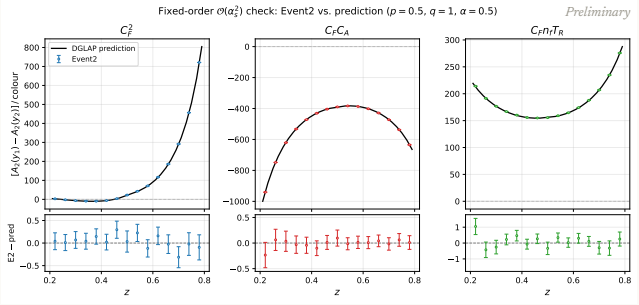
$$\delta P^{(1)} = 2\kappa\, P^{(0)} \otimes (\ln z\, P^{(0)}), \quad \kappa = q - p$$

Fixed-order check against Event2

§1 · EVENT2

Boost-non-invariant $p = \frac{1}{2}, q = 1 - \delta P^{(1)}$ *is active*

- Boost-non-invariant point $p = \frac{1}{2}, q = 1$, so $\kappa = q - p = \frac{1}{2} \neq 0$ and the clustering correction $\delta P^{(1)}$ is *active*.
- Difference across two cut scales $y_{\text{cut}} = 10^{-5},\ 2.5 \times 10^{-6}$ of the $\mathcal{O}(\alpha_s^2)$ coefficient – cancels the uncomputed NNSL (L^0) constant.
- Prediction: $\delta P^{(1)}$, plus the $\mathcal{O}(\alpha_s)$ boundary condition $D_{\text{BC}}(z; p, q)$ and hard matching C, each iterated with $P^{(0)}$ – these survive the difference too, at the same order.



Catani, Seymour, NPB 485 (1997) 291 [hep-ph/9605323]

What κ means: the cut's boost charge

§1 · INTERPRETATION

$\kappa = q - p$ is the cut's charge under boosts — and it dials timelike ↔ spacelike

– Boost along the jet axis: $\omega \to e^{\eta}\omega$, $\theta \to e^{-\eta}\theta$, $k_t = \omega\theta$ fixed. The cut has a definite charge:

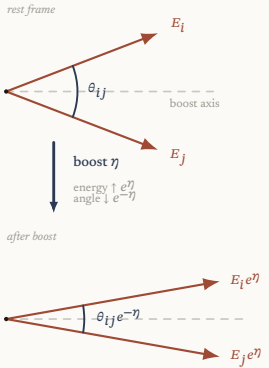
$$v = \frac{k_t^{2q}}{Q^{2p}}\,\omega^{-2\kappa} \longrightarrow e^{-2\kappa\eta}\,v, \qquad \kappa = q - p$$

– Boost-invariant $\iff$ $p = q$; and κ dials the effective NLO kernel:

at $\mathcal{O}(\alpha_s^2)$: $\quad P_{\text{eff}}^{(1)} = (1-\kappa)\,P_T^{(1)} + \kappa\,P_S^{(1)}$

$\kappa = 0$: timelike (k_T, hadron FF) $\quad \kappa = 1$: spacelike (C/A)

Beyond $\mathcal{O}(\alpha_s^2)$ the κ-piece rides $\tilde{\alpha}_s(E^2 y_{\text{cut}})$ — next.



energy factor $\times e^{2p\eta}$, angle factor $\times e^{-2q\eta}$:

they cancel $\iff$ $p = q$.



2 · The all-orders prescription

the coupling scale; one shift, three equivalent forms

The scale of the coupling

§2 · COUPLING SCALE

The clustering piece runs at the jet scale, not the hard scale

– $\delta P^{(1)}$ is generated at the **jet boundary** $\mu_J = E\sqrt{y_{\text{cut}}}$, where the second emission sits at the edge of the cut – so it rides $\bar{\alpha}_s(E^2 y_{\text{cut}})$.

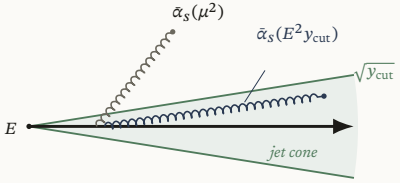
– The full all-orders (NSL) prescription, $\mu_J \to \mu = E$:

$$\frac{dD^{\text{jet}}}{d\ln\mu^2} = \hat{P} \otimes D^{\text{jet}}$$

$$\hat{P} = P_{\text{AP}} - \bar{\alpha}_s(\mu^2)\,\bar{\alpha}_s(E^2 y_{\text{cut}})\,\delta P^{(1)}$$

$$P_{\text{AP}} = \bar{\alpha}_s(\mu^2)P^{(0)} + \bar{\alpha}_s^2(\mu^2)P^{(1)}_{\overline{\text{MS}}}, \quad \delta P^{(1)} = 2\kappa P^{(0)} \otimes (\ln z\, P^{(0)})$$

– Boundary condition at μ_J: $\mathcal{O}(\alpha_s)\, D_{\text{BC}}(z; p, q)$, **computed** (backup); $P^{(n)} \equiv P^{(n)}_{\text{AP}}/q$, $\kappa = q - p$.



Still a timelike–spacelike mix: using $\delta P^{(1)} = \kappa\big(P_T^{(1)} - P_S^{(1)}\big)$,

$$\hat{P} = \bar{\alpha}_s(\mu^2)\Big\{P^{(0)} + \kappa\,\bar{\alpha}_s(E^2 y_{\text{cut}})\,P_S^{(1)} + \big[\bar{\alpha}_s(\mu^2) - \kappa\,\bar{\alpha}_s(E^2 y_{\text{cut}})\big]P_T^{(1)}\Big\}$$

The spacelike share rides the jet-scale coupling: a fraction κ at $\mu = \mu_J$, growing as $\mu \to E$.

van Beekveld *et al.*, JHEP 07 (2024) 239 [2402.05170]

Shift the kernel, or shift the boundary

§2 · ALL ORDERS

Forms (A) and (B): modify the kernel, or keep $\overline{\text{MS}}$ and start from the jet's own scale $\mu_J z^\kappa$

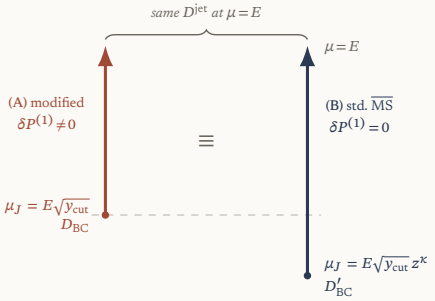
(A) Modified kernel – fixed boundary $\mu_J = E\sqrt{y_{\text{cut}}}$, boundary condition D_{BC}:

$$\frac{dD^{\text{jet}}(z,\mu,\mu_J)}{d\ln\mu^2} = \left(P_{\text{AP}} - \bar{\alpha}_s(\mu^2)\,\bar{\alpha}_s(E^2 y_{\text{cut}})\,\delta P^{(1)}\right) \otimes D^{\text{jet}}$$

(B) Shifted boundary – plain $\overline{\text{MS}}$ evolution from $\mu_J = E\sqrt{y_{\text{cut}}}\, z^\kappa$, boundary condition re-expressed at that scale:

$$\frac{dD^{\text{jet}}(z,\mu,\mu_J z^\kappa)}{d\ln\mu^2} = P_{\text{AP}} \otimes D^{\text{jet}}$$

with $D'_{\text{BC}}(y) = D_{\text{BC}}(y) + 2\kappa \ln y\, P^{(0)}_{qq}(y)$.



Same $\bar{\alpha}_s(\mu^2)\,\bar{\alpha}_s(E^2 y_{\text{cut}})\,\delta P^{(1)}$ term: identical at NSL, differ at NNSL (α_s^2 at fixed $\alpha_s L$)

The modified convolution, and what it generalises

§2 · ALL ORDERS

Form (C): each daughter evolves from its own jet scale — at κ=1, the SCET small-R factorisation

$$\frac{1}{\sigma_0}\frac{d\sigma^{\text{jet}}}{dz} = \sum_i \int_z^1 \frac{dx}{x}\, C_i(x, Q, \mu)\, D_i^{\text{jet}}\Big(\frac{z}{x}, \mu, \mu_J\, x^\kappa\Big)$$

$$\frac{dD^{\text{jet}}(z,\mu,\mu_J)}{d\ln\mu^2} = \int_z^1 \frac{dy}{y}\, P_{\text{AP}}(y)\, D^{\text{jet}}\Big(\frac{z}{y}, \mu, \mu_J\, y^\kappa\Big), \qquad \text{boundary condition } D_{\text{BC}} \text{ (as in (A))}$$

- The jet scale follows the parton that starts the jet, outside and in the evolution: with standard DGLAP for C this is exactly μ-independent.
- The extra evolution sits at the *jet* end, so it carries $\tilde{\alpha}_s(E^2 y_{\text{cut}})$: (C) ≡ (A) ≡ (B) at NSL. (A), (B) keep the plain $C \otimes D^{\text{jet}}$; the outer box changes only NNSL.
- **SCET small-R jets**: at κ=1, form (C) *is* the factorisation and jet-function RGE of Lee, Moult and Zhang. Their names: $H \leftrightarrow C$, $J \leftrightarrow D^{\text{jet}}$, $J^{(1,0)} \leftrightarrow D_{\text{BC}}$, $P_T \leftrightarrow P_{\text{AP}}$, $\frac{QR}{2} \leftrightarrow \mu_J$. Our cut generalises it to any $\kappa = q - p$.

The shift sits on the splitting fraction y (the kernel), not on the observed z: the other choice is an NSL error.

Lee, Moult, Zhang, JHEP 05 (2025) 129 [2409.19045]; 2410.01902

The shower target

§2 · SHOWER TARGET

$p = \frac{1}{2}$, $q = 1$, quark non-singlet: forms (A), (B), (C) and a toy triple-collinear shower converge as $\alpha_s \to 0$

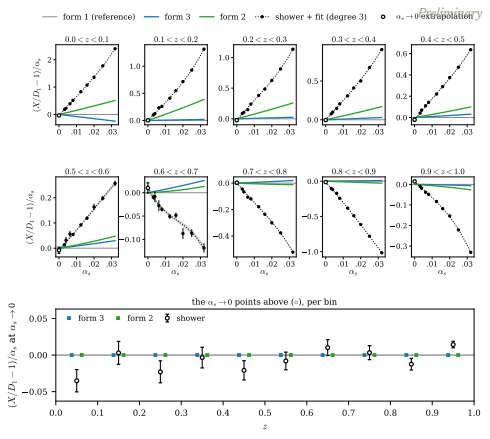
A shower target: an NSL shower, clustered with the same jet definition, must reproduce D^{jet}. Here: the toy triple-collinear (3C) non-singlet shower of van Beekveld *et al.*

Plotted, in each of ten z bins, against α_s at fixed $\alpha_s \ln R = -0.4$:

$$\frac{1}{\alpha_s}\left(\frac{X}{D_{(\mathrm{A})}} - 1\right)$$

with X = form (B), form (C) (legend: forms 2, 3) and the toy 3C shower. NSL agreement $\Leftrightarrow \to 0$ as $\alpha_s \to 0$.

Beyond NSL the three forms differ, and the shower carries its own subleading terms: the slopes are NNSL. Only the intercept at $\alpha_s \to 0$ tests NSL.



van Beekveld *et al.*, JHEP 03 (2025) 209 [2409.08316]



3 · Summary

one shift, two uses, a structural bridge

Summary & outlook

§3 · SUMMARY

One shift, two uses, a structural bridge

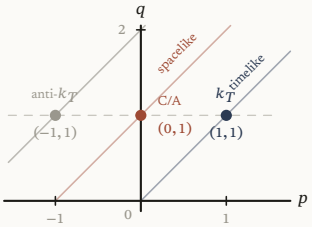
Result. Small-R jet FF at NSL = standard DGLAP with $\mu_J = E\sqrt{y_{\text{cut}}}\, z^{\kappa},\ \kappa = q - p.$

(i) **Shower benchmark** for *any* ordering variable – incl. fractional β_{PS}.

(ii) **Jet-clustering** NSL correction for non-boost-invariant definitions.

Outlook.

- Confirm *triple-collinear showers* hit this target (PanScales).
- Pursue the **shower ↔ $\overline{\text{MS}}$ connection** – collinear evolution, *non-perturbative* $\overline{\text{MS}}$ FFs.
- *Measurable* – an e^+e^- y_{cut}-scan over distance measures (LEP / FCC-ee), and a pp Lund-plane line in standard jets.



Cuts on one diagonal share their NSL logs: $\kappa = q - p$ alone sets them ($\kappa = 0$ timelike, $\kappa = 1$ spacelike); p/q moves only the $\mathcal{O}(\alpha_s)$ constant.

ATLAS, PRL 124 (2020) 222002 [2004.03540]

BACKUP

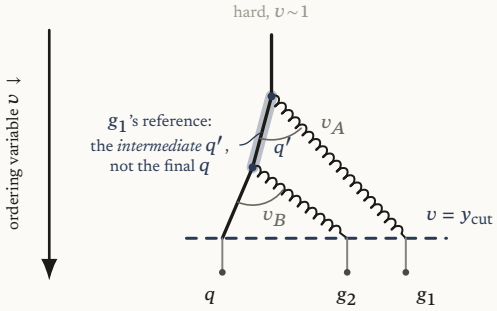
Backup

Backup: the parton-shower variant

BACKUP

Run a shower to $v = y_{\text{cut}}$: what sets the final-state z distribution?

- Both **coarse-grain the same radiation** to $v = y_{\text{cut}}$; they differ only in the *reference* each emission's v is measured against.
- Run a **triple-collinear-accurate** shower, ordered on *its* v, to $v = y_{\text{cut}}$; histogram the final-parton z. The jet *clusters* the *final* partons; the shower tests each emission against the other parton in its splitting – often an **intermediate** line.
- A single emission ⇒ identical. The two-emission difference is confined to the doubly-pinned corner $v_1 \approx v_2 \approx y_{\text{cut}}$ – no scale hierarchy, **no log**.



⇒ shower z-distribution ≡ jet FF at NSL (difference is NNSL).

van Beekveld *et al.*, JHEP 03 (2025) 209 [2409.08316]; Jadach, Skrzypek, Acta Phys. Polon. B 40 (2009) 2071 [0905.1399]

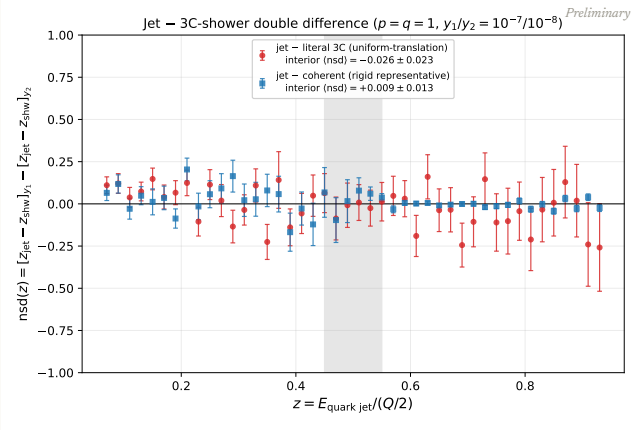
Backup: jet vs. unwound shower at NLO

BACKUP

$p = q = 1$; the double difference isolates any clustering log

What Event2 computes.

- nsd(z): the correlated jet − unwound-shower difference, itself differenced across two cut scales $y_{\text{cut}} = 10^{-7}$ and 10^{-8}.
- A clustering log would *grow* with the cut-scale ratio; a flat-at-0 result signals its absence.
- Faithful unwindings (literal 3C, coherent): consistent with 0 — **no clustering log.**
- Up to 1.5×10^{11} events / variant.



Backup: all-orders closed form

BACKUP

Stirling numbers fix the mixed coupling scale

– SL solution of the DGLAP recursion, unsigned Stirling numbers of the first kind:

$$A_k(z) = \frac{1}{(k+1)!} \sum_{j=0}^{k} |c(k+1, j+1)|\, b_0^{k-j}\, P_0^{\otimes(j+1)}.$$

– Substituting into the NSL recursion, the residual piece is accounted for only if $\delta P^{(1)}$ is evaluated at the *mixed* scale:

$$\bar{\alpha}_s^2(\mu^2) \to \bar{\alpha}_s^2(\mu^2) + \bar{\alpha}_s(\mu^2)\big(\bar{\alpha}_s(E^2 y_{\text{cut}}) - \bar{\alpha}_s(\mu^2)\big).$$

Backup: boundary conditions at general (p, q)

BACKUP

Distinct from κ: depends on the ratio p/q

$$w(z; p, q) = \ln\big(z(1-z)\big) + \frac{p}{q}\,\ln\frac{2}{\min(z, 1-z)}$$

$$D^{(0)}_{q,\mathrm{BC}}(z; p, q) = -2\big[\hat{P}_{qq}(z)\,w\big]_+ - 2\hat{P}_{gq}(z)\,w - C_F + \frac{C_F}{2}\,\delta(1-z)$$

$$D^{(0)}_{g,\mathrm{BC}}(z; p, q) = -2\big[\hat{P}_{gg}(z)\,w\big]_+ - 4n_f\hat{P}_{qg}(z)\,w - 4n_f T_R\, z(1-z) + c_g\,\delta(1-z)$$

Unregularised $\hat{P}_{ij}$. Quark: plus on the whole bracket, δ fixed by quark number. Gluon: plus on the $z \to 1$ pole, c_g fixed by the momentum sum rule.

- Same w for both: the angular limit is flavour-blind; only the splitting functions change.
- Affine in p/q: any (p, q) interpolates between $p = 0$ (C/A) and $p = q$ (the k_T-type cut).
- $\kappa = q - p$ sets the *NSL log*; p/q sets the $\mathcal{O}(\alpha_s)$ matching constant.