# Local infrared finiteness at two loops

$q\bar{q} \to g + mH$ with a massive-fermion loop

An exact graph-by-graph rearrangement using the two sided $G$–$K$ decomposition

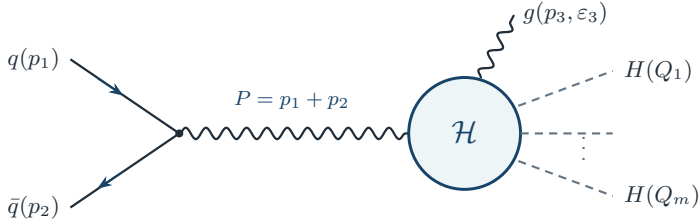
**Ani Venkata**
See also: Sterman and Anaatasiou

REF 2026

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**Goal:** Utilize region independent “factorization" to separate finite feynman diagrams from IR divergent ones and carry out higher order calculations.

# The process and kinematics




$$q(p_1) + \bar{q}(p_2) \longrightarrow g(p_3, \varepsilon_3) + H(Q_1) + \cdots + H(Q_m)$$

## External momenta

$p_1, p_2$: incoming, massless; $p_3$: *outgoing physical gluon*.

$p_1^2 = p_2^2 = p_3^2 = 0, \quad Q_i^2 = m_H^2.$

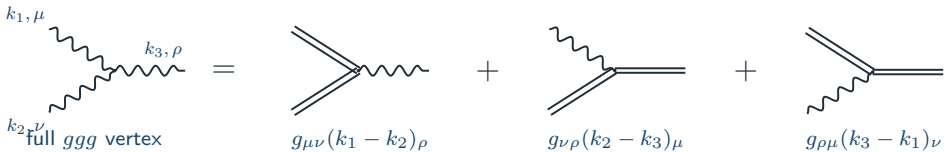
$Q = \sum_{i=1}^{m} Q_i, \quad P = p_1 + p_2 = p_3 + Q.$

## Conventions

$\varepsilon_3 \equiv \varepsilon(p_3; p_1)$: $\quad \varepsilon_3 \cdot p_3 = \varepsilon_3 \cdot p_1 = 0.$

$\mathcal{H}_{\mu_1 \cdots \mu_r}$: summed massive-fermion-loop gluon attachment.

# Tool 1: exact scalar–scalar–gluon (SSG) decomposition




Three terms from the three different Lorentz contractions.

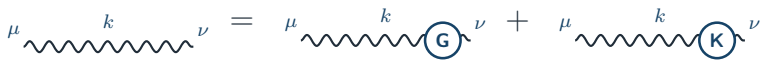
$$\boxed{V^{abc}_{\mu\nu\rho}(k_1, k_2, k_3) = -g_s f^{abc}\left[g_{\mu\nu}(k_1 - k_2)_\rho + g_{\nu\rho}(k_2 - k_3)_\mu + g_{\rho\mu}(k_3 - k_1)_\nu\right]}$$

All three momenta are taken incoming: $k_1 + k_2 + k_3 = 0$.

- **notation:** each *double line* represents a scalar Lorentz contraction in the exact three-gluon vertex.

# Tool 2: one-sided and two-sided $G$–$K$

**One-sided** insert $G$ or $K$ at one end of the propagator




$$D_{\mu\nu}(k) = \frac{-i}{k^2 + i0}\big[G_{\mu\nu}(k;q) + K_{\mu\nu}(k;q)\big], \qquad K_{\mu\nu} = \frac{q_\mu k_\nu}{q\cdot k + i0}, \quad G_{\mu\nu} = g_{\mu\nu} - K_{\mu\nu}.$$

- **Use:** one-sided $G$–$K$ for the families with *transient* infrared singularities.

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**Two-sided** a $G/K$ insertion at each end of the *same* propagator: introduced in 2504.20786



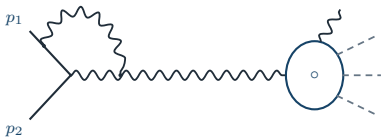

$$D_{\mu\nu}(k) = \frac{-i}{k^2 + i0}\,(G_L + K_L)_\mu{}^\rho\; g_{\rho\sigma}\;(G_R + K_R)^\sigma{}_\nu, \qquad G_{L,R} = \mathbf{1} - K_{L,R}.$$

$(K_L)_\mu{}^\rho = -\frac{k_\mu q^\rho}{k\cdot q + i0}$, $\quad (K_R)^\sigma{}_\nu = -\frac{q^\sigma k_{1\nu}}{k_1\cdot q + i0}$, $\quad q^2>0$ (e.g. $q=2Q+k$, $Q^2>0$).

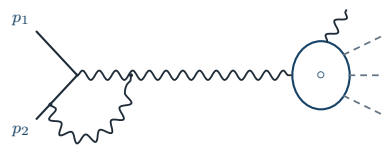
# Graph List I: topological factorization

$$\mathcal{M}^{(2)} = \mathcal{M}^{(2)}_{\mathrm{IR}} + \mathcal{M}^{(2)}_{\overline{\mathrm{IR}}} + \mathcal{M}^{(2)}_{\mathrm{fact}}$$

$\mathcal{M}_{\mathrm{fact}}$: graphs that are *topologically factorized*, with a lower-order factor attached to the hard process. Examples from the quark/antiquark legs:




$(\beta)$: upper leg

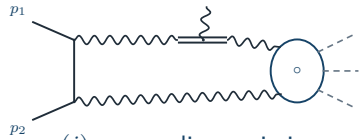



$(\gamma)$: lower leg

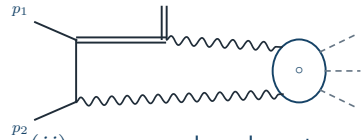
$\mathcal{M}_{\mathrm{IR}}$ and $\mathcal{M}_{\overline{\mathrm{IR}}}$ are the two nonfactorized families; the next two slides distinguish them.

# Graph List II: $\mathcal{M}_{\overline{\mathrm{IR}}}$

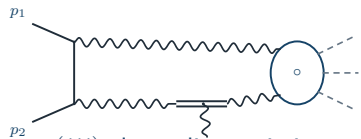
**After SSG decomposition: no IR poles after integration, but some graphs are pointwise singular.**



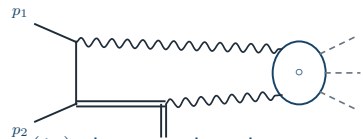

$(i)$: upper-line emission




$(ii)$: upper scalar–gluon term




$(iii)$: lower-line emission

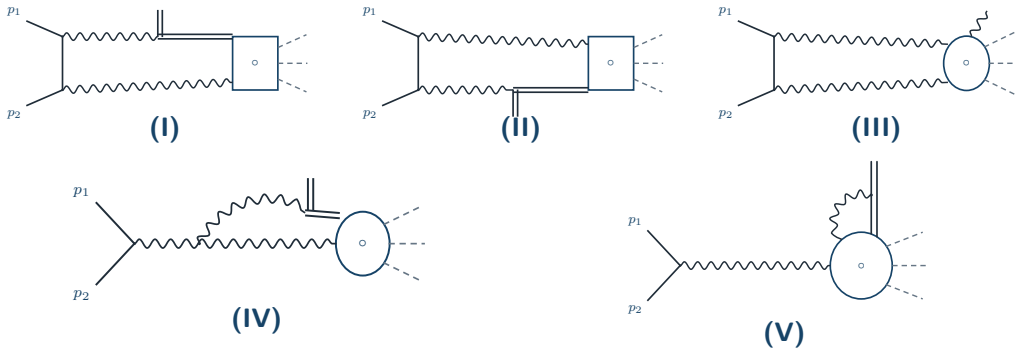



$(iv)$: lower scalar–gluon term

These are **representative SSG graphs**, not the complete $\overline{\mathrm{IR}}$ list. Some contributions only become locally finite after a one-sided $G$–$K$ Ward rearrangement.

# Graph List III: genuinely IR-singular graphs

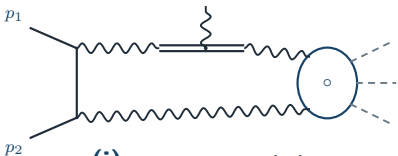
**Five topologies remain after the SSG decomposition (distinguishable hard-loop attachments).**



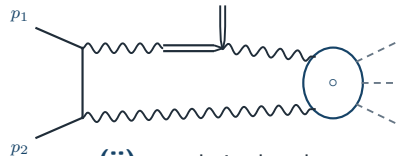

**This is the complete genuinely IR-singular catalogue in the stated sector.** Double lines are scalar Lorentz contractions; squares and ovals are the different hard attachment structures.

# Two transient topologies: (i) and (ii)

**These belong to $\mathcal{M}_{\overline{\text{IR}}}$: integrated IR finite, but not manifestly finite at every pinch.**




**(i)** wavy $p_3$ emission

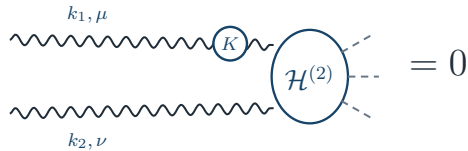



**(ii)** scalarized $p_3$ leg

The upper-line SSG term differs; each graph has a second gluon connecting the $p_2$ line to the massive fermion loop.

**Plan:** decompose the gluons entering the loop with one-sided $G + K$; use the Ward identity to remove the $K$ pieces; then check the soft and collinear pinches of the $G$ representatives.

# The $K$ part entering the fermion loop vanishes




$$k_1^\mu \, \mathcal{H}^{(2)ab}_{\mu\nu}[k_1, k_2; \{Q_i\}] = 0 \qquad (\text{also } k_2^\nu \mathcal{H}^{(2)ab}_{\mu\nu} = 0).$$

**Why?** A longitudinal insertion on a fermion line telescopes:

$$S(\ell) \not{k} \, S(\ell + k) = S(\ell) - S(\ell + k).$$

Summing *all* gluon attachments around the closed massive-fermion loop cancels the endpoint terms.

$K_{\alpha\mu}(k; q) = q_\alpha k_\mu / (q \cdot k + i0)$ inserts $k_\mu$ *toward* $\mathcal{H}^{(2)}$. Thus the $K$ piece vanishes for the summed hard subgraph, **at fixed** $k$—not by an integrated IR cancellation.

## Why a $G$ insertion suppresses a collinear pinch

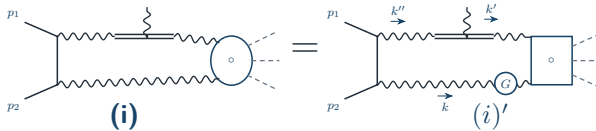



The numerator of a jet collinear to $k$ is longitudinal: $J^\mu \sim p_i^\mu$.

$$p_i^\mu\, G_{\mu\nu}(k;q) = p_{i\nu} - \frac{p_i\cdot q}{k\cdot q}\, k_\nu \xrightarrow[k^\mu \to \alpha p_i^\mu]{} 0$$

**Consequence:** the leading collinear jet numerator cancels between $g$ and $K$. The remaining $G$ contribution starts at a suppressed power near the pinch (for an admissible reference $q\cdot p_i \neq 0$).

# Topology (i): the numerator kills the remaining pinches




The hard-loop Ward identity drops the lower $K$. The exact surviving SSG numerator is

$$N_{\mathrm{SSG}}(k', k'') = \varepsilon_{3\mu}\,(k' + k'')^{\mu}, \qquad \varepsilon_3 \cdot p_3 = \varepsilon_3 \cdot p_1 = 0.$$

**$k'' \to 0$ (soft)**

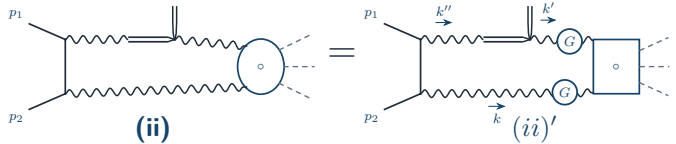
As $k'' \to 0$, the adjacent momentum approaches $k' \to -p_3$. Thus $N_{\mathrm{SSG}} = \mathcal{O}(|k''|)$, rather than a constant soft numerator.

**$k'' \parallel p_1$**

$k'' = \alpha p_1 + k''_\perp + O(k''^2_\perp)$; $k'$ differs by $p_3$. Then $N_{\mathrm{SSG}} = \mathcal{O}(|k''_\perp|)$ by $\varepsilon_3 \cdot p_{1,3} = 0$.

The soft and collinear numerator powers are sufficient in the pinch-region counting. The independent $k \parallel p_2$ limit is suppressed by the lower $G$ insertion.

# Topology (ii): two $G$ projectors remove the pinches




Apply the same hard-loop WI to both gluons: lower $G(k; q_2)$ with $q_2 = 2p_2 + k$, upper $G(k'; q_3)$ with $q_3 = 2p_3 + k'$.

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**Two collinear regions**

$k \parallel p_2$: lower $G$ suppresses the pinch.

$k' \parallel p_3$: upper $G$ suppresses it.

**Soft $k'' \to 0$**

The pinched subgraph eikonalizes on $p_1$; its leading numerator is proportional to $\varepsilon_3 \cdot p_1 = 0$.

The one-sided $G$ representatives $(i)'$ and $(ii)'$ are locally finite in all their pinched soft and collinear limits. No polarization change is needed for this pair.

## The genuinely singular family needs two-sided $G$–$K$




$$D_{\mu\nu}(r) = \frac{-i}{r^2 + i0}(G_L + K_L)_\mu{}^\alpha \, g_{\alpha\beta} \, (G_R + K_R)^\beta{}_\nu .$$

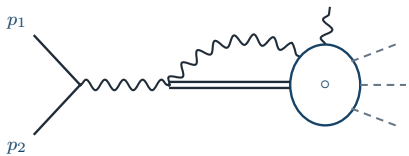
In the two-gluon graphs (I)–(II), decompose **each** exchanged gluon, with $r = k$ and $r = P - k$. The $GG$ sector has **two** $G$ **projectors on each propagator**, including the line entering the fermion loop directly.

$$D(k)D(P-k) = \sum_{X,Y,U,V \in \{G,K\}} D^{XY}(k)\, D^{UV}(P-k),$$
including $D^{GG}(k)D^{GG}(P-k)$.

**First examine $GG$ on both lines. The mixed and longitudinal sectors come afterward.**

# Complete the Ward identities with unpinched classes

**Two additional graph classes close the longitudinal-attachment sums.**




**extra insertion along the hard connector**

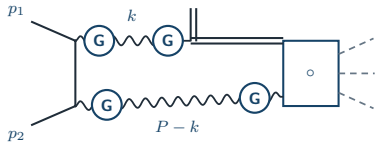



**additional insertion on the massive loop**

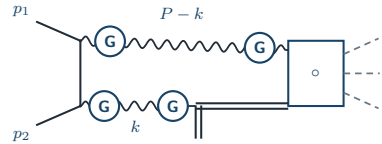
$$\mathcal{M}_{\mathrm{IR}} \longmapsto [\mathcal{M}_{\mathrm{IR}} + \mathcal{U}] - \mathcal{U}, \qquad \mathcal{U} = \mathcal{U}_{\mathrm{connector}} + \mathcal{U}_{\mathrm{loop}}.$$

The SSG descendants supply the missing longitudinal attachments. Once the Ward identities are completed, **all internal gluons in each resulting graph** are decomposed; the added classes do not change the amplitude.

## The $GG$ sector means $GG$ on both exchanged gluons




**(I):** $GG$ **on** $k$ ***and*** $P-k$




**(II):** $GG$ **on** $k$ ***and*** $P-k$

$$\mathcal{D}^{GG}_{\mu\nu}(r;q_r) \equiv \frac{-i}{r^2+i0}\,(G_L)_\mu{}^\alpha(r;q_r)g_{\alpha\beta}(G_R)^\beta{}_\nu(r;q_r), \quad r\in\{k,P-k\}.$$

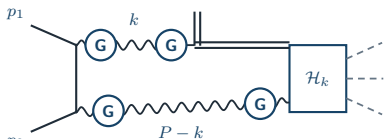
**Pinch-by-pinch result for the completed $GG, GG$ sector:**

**Collinear:** the two-sided $G$ numerators suppress the pinch regions, giving locally integrable limits.

**Soft $k\to 0$:** a leading term can remain. Subtract its hard-factorized approximation.

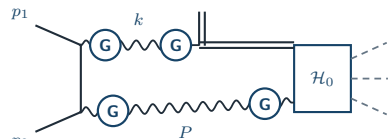
# The $GG, GG$ soft subtraction freezes the hard line too

**Example (I). Both exchanged propagators have $G$ bubbles at both ends.**




full graph: $D^{GG}(k)\, D^{GG}(P-k)$

$\Longrightarrow$




soft CT: $D^{GG}(k)\, D^{GG}(P)$

$k \to 0$ in the nonpinched hard factors :

$$D^{GG}(P-k; q_{P-k}) \longrightarrow D^{GG}(P; q_P),$$
$$\mathcal{H}(P-k, k-p_3) \longrightarrow \mathcal{H}(P, -p_3).$$

Keep the $k$-dependent soft propagator $D^{GG}(k)$, its exterior quark factors, the $p_3$ SSG splitting, and their pinched denominators. The same construction applies to (II), with the two lines exchanged.

## Soft counterterms: both lines remain $GG$

$J_a$: both quark–gluon couplings; $V_a$: $p_3$ SSG splitting and pinched scalar denominator; $\mathcal{H}_a$: the massive fermion-loop block. $P = p_1 + p_2$, $a = I, II$.

$$I^{(a)}_{GG,GG}(k) = J^{\mu\lambda}_a(k)\, D^{GG}_{\mu\nu}(k;q_k)\, V^{\nu\beta}_a(k,p_3) \times D^{GG}_{\lambda\rho}(P-k;q_{P-k})\, \mathcal{H}_{a\,\beta}{}^{\rho}(P-k,k-p_3).$$

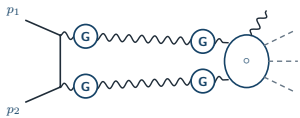
$$\boxed{C^{(a)}_{GG,GG}(k) = J^{\mu\lambda}_a(k)\, D^{GG}_{\mu\nu}(k;q_k)\, V^{\nu\beta}_a(k,p_3) \times D^{GG}_{\lambda\rho}(P;q_P)\, \mathcal{H}_{a\,\beta}{}^{\rho}(P,-p_3).}$$

**Soft approximation:** $k \to 0$ in the hard $D^{GG}(P-k)$ *and* $\mathcal{H}_a$, not in $D^{GG}(k)$, $J_a$, or $V_a$.

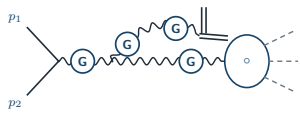
$$\boxed{I^{(a),\mathrm{sub}}_{GG,GG}(k) = I^{(a)}_{GG,GG}(k) - C^{(a)}_{GG,GG}(k).}$$

# The remaining $\mathcal{M}_{\mathrm{IR}}$ graphs need no soft counterterm

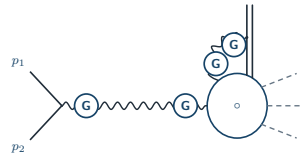
## After the Ward completion, the other genuinely singular topologies have no soft pinch.




**(III)**




**(IV)**



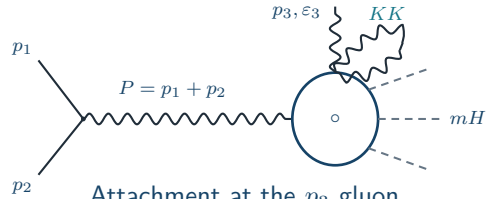

**(V)**

$$p_i^{\mu} G_{\mu\nu}(r;q) = p_{i\nu} - \frac{p_i \cdot q}{r \cdot q}\, r_{\nu} \xrightarrow{r \| p_i} 0, \qquad i = 1, 2.$$

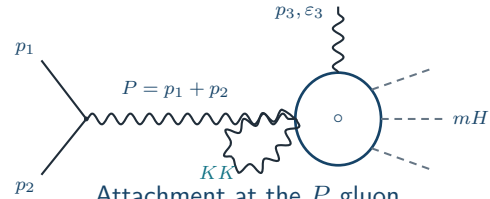
Since there is **no soft pinch**, the only candidate IR limits are collinear. The $G$ numerators suppress these limits, so the $GG$ **contributions from (III)–(V) are locally integrable** without an additional soft subtraction.

# What survives in the $KK$ sector?

**After the boundary-term cancellations, only two tadpole-like placements remain.**




Attachment at the $p_3$ gluon




Attachment at the $P$ gluon

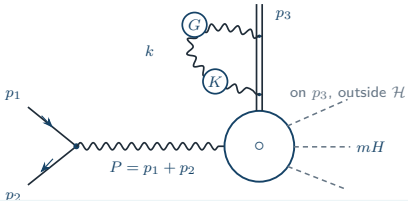
$$\mathcal{M}^{(2)}_{KK} = 2\,\mathcal{M}^{(1)}\,\mathcal{T}_{KK}, \qquad \mathcal{T}_{KK} \equiv -\frac{1}{2}\int_k \frac{q(k)^2}{(k^2+i0)\,[k\cdot q(k)+i0]^2}$$

**One explicit reference choice:** $q(k) = 2P + k,\ \ P = p_1 + p_2,\ s = P^2.$

$$\mathcal{T}_{KK} = -\frac{1}{2}\int_k \frac{4s + 4P\cdot k + k^2}{(k^2+i0)\,[k^2+2P\cdot k+i0]^2}.$$

# Mixed $GK/KG$: a factorized $p_3$ self-energy

**The $K$ end is moved off the hard loop and onto the $p_3$ leg.**




**After the Ward rearrangement:**

$$\boxed{\mathcal{M}_{GK}^{(p_3)} = \mathcal{M}^{(1)}\,\mathcal{T}_{GK}(p_3)}$$

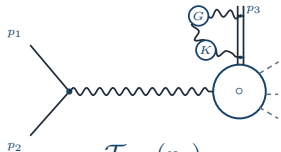
**Ward sum:** the longitudinal attachment at the hard loop telescopes into a $K$ insertion on the external $p_3$ line.

$$\begin{aligned}\mathcal{T}_{GK}(p_3) &= \int_k \frac{a_3^{\mu} G_{\mu}{}^{\nu}(k;q)\, q_{\nu}}{(k^2+i0)(A_3+i0)(U+i0)} \\ &= \int_k \left[\frac{a_3\cdot q}{(k^2+i0)(A_3+i0)(U+i0)} - \frac{q^2}{(k^2+i0)(U+i0)^2}\right].\end{aligned}$$

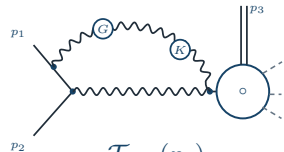
$a_3 = 2p_3 + k$, $A_3 = (p_3+k)^2$, $U = k\cdot q$. Coupling/color factors suppressed.

# Mixed $GK/KG$: surviving graph families

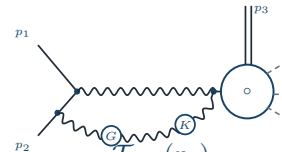
## The three soft/collinear kernels




$\mathcal{T}_{GK}(p_3)$

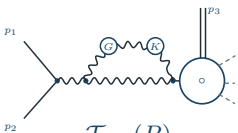



$\mathcal{T}_{GK}(p_1)$

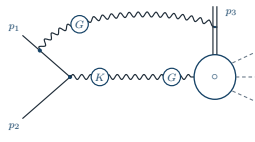



$\mathcal{T}_{GK}(p_2)$

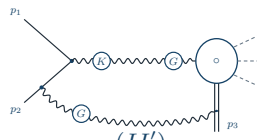
## Other surviving graphs needed for completeness




$\mathcal{T}_{GK}(P)$




$(I')$




$(II')$

The factorized kernels multiply $\mathcal{M}^{(1)}$; $I'$ and $II'$ are locally integrable.

# Conclusion: local infrared finiteness

$$\boxed{\mathcal{I}^{(2)}(k) = \mathcal{I}^{(2)}_{\text{fact}}(k) + \mathcal{I}^{GG}_{\text{soft CT}}(k) + \mathcal{I}^{(2)}_{\text{rem}}(k)}$$

$\mathcal{I}^{(2)}_{\text{rem}}$ is locally integrable at every IR pinch surface.

- **Exact identities:** SSG, $G$–$K$, and Ward rearrangements do not require an infrared approximation.
- **Factorized terms:** $KK$ and mixed $GK/KG$ kernels; the $GG$ soft counterterm is added and subtracted.
- **Proof of finiteness:** independent pinch-by-pinch power counting, not just agreement of integrated poles.

Only the counterterm construction evaluates the hard subgraph at $k = 0$; adding and subtracting it is exact.