Updates on dispersive analysis for $\gamma^*\gamma^*\gamma \to \pi\pi$

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Workshop on Radiative Corrections and Monte Carlo Simulations at Electron-Positron Colliders
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PSI
Universität Zürich UZH

Contents

- Importance of $\gamma^*\gamma^*\gamma \to \pi\pi$
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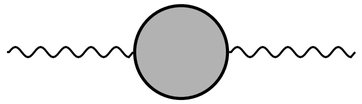
Motivation : Importance of $\gamma^*\gamma^*\gamma \to \pi\pi$

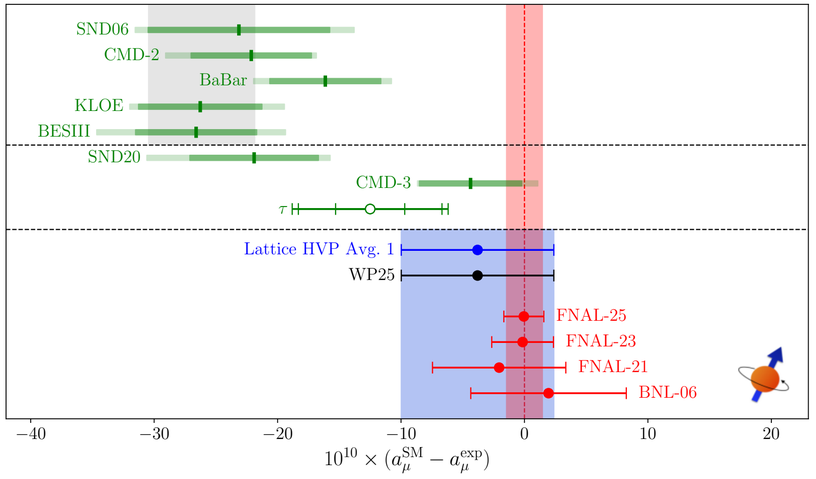
- Data-driven approach for HVP : main error contribution



↪ cross section data $\sigma(e^+e^- \to \text{hadrons}(+\gamma))$

↪ radiative-return experiments :
need of improvement for hadronic modeling for
$e^+e^- \to \pi^+\pi^-\gamma$

↪ inclusion of description of hadronic matrix element
(dispersive)



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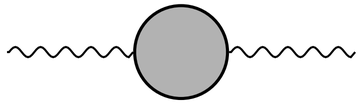

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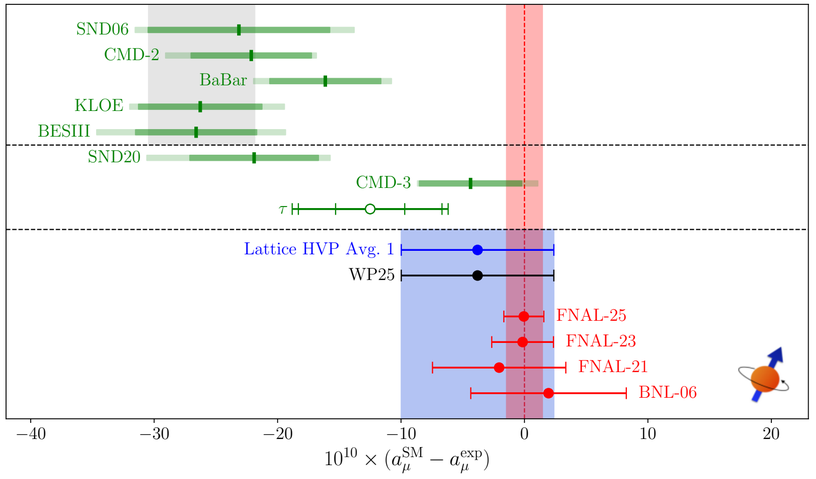

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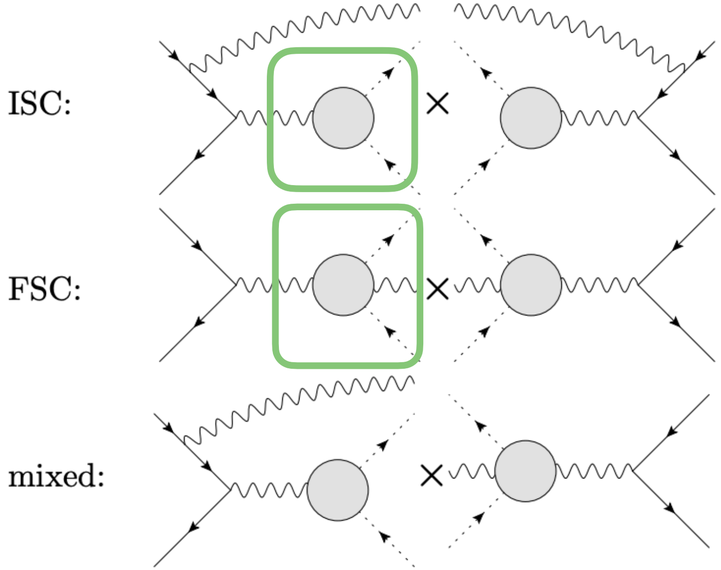
LO terms : well-known dispersively

↪ Pion VFF

↪ $\gamma^* \to \pi^+\pi^-\gamma$: crossed Compton tensor



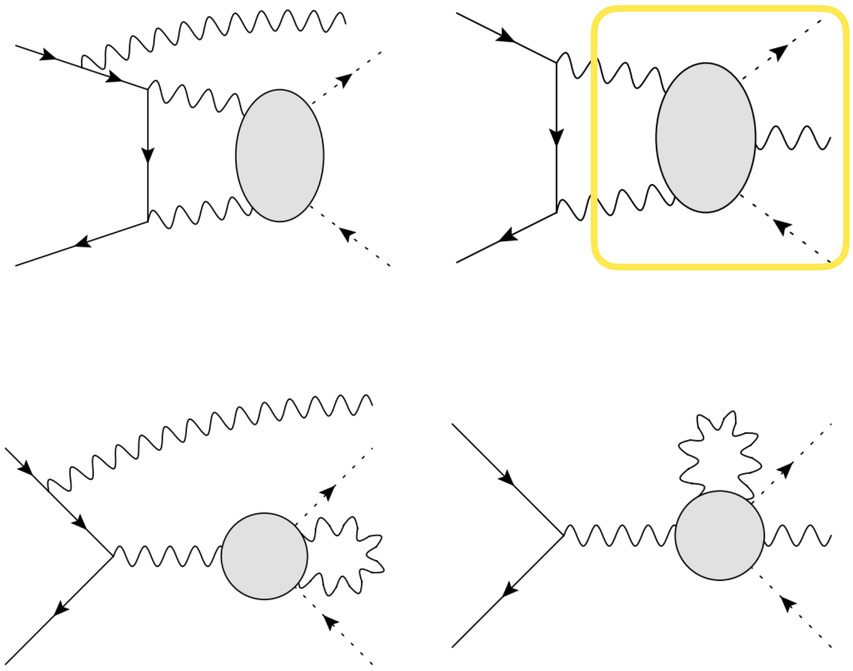
[arXiv:2410.22882]

Motivation : Importance of $\gamma^*\gamma^*\gamma \to \pi\pi$

NLO terms : $\gamma^*\gamma^* \to \pi\pi\gamma$ (among others)

↪ FsQED : not sufficient, need to account for important intermediate states

↪ **no** dispersive description **so far**



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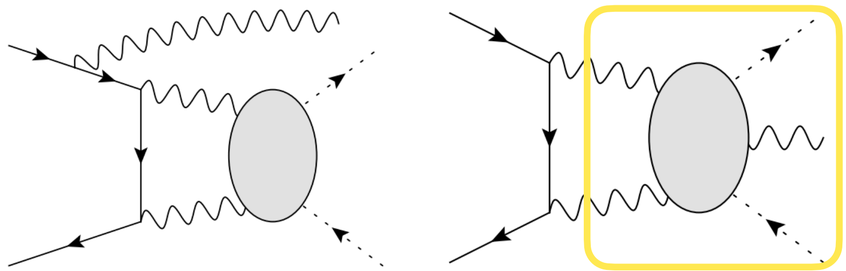
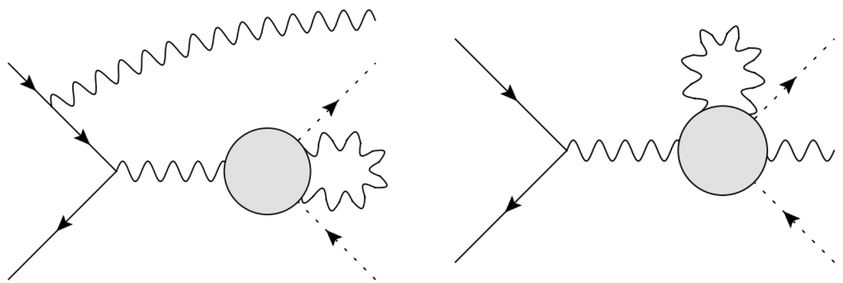


- Start with $\gamma^*\gamma^*\gamma \to \pi\pi$ in HLbL (**soft photon case**)
  - ↪ systematic deviations still present for certain contributions : necessary precision improvement
  - ↪ **then** consider hard photon case : **useful for HVP**



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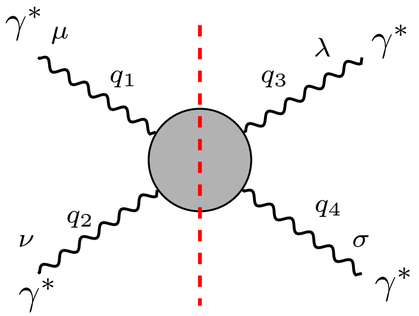
HLbL : 4-point vs triangle kinematics framework

Usual approach → dispersions relations for HLbL tensor in general **four-point kinematics**

- dispersion relations for singly-on-shell limit for basis of scalar functions



[arXiv:1506.01386]
[arXiv:1702.07347]

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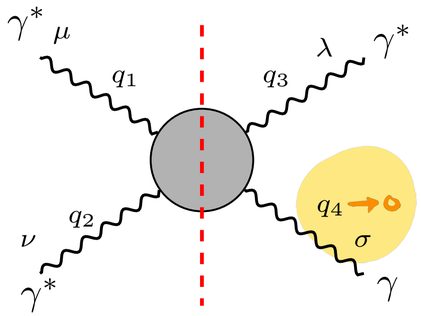

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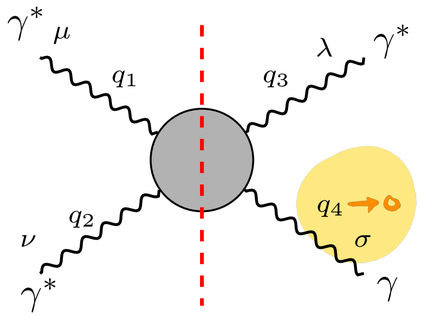
  sum rules not necessarily satisfied



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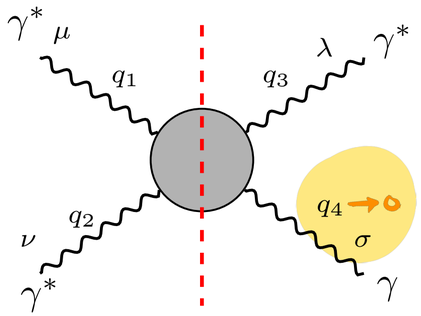

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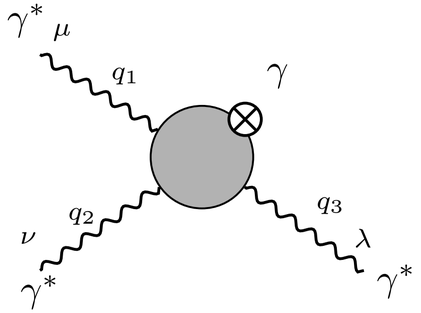

[arXiv:2302.12264]

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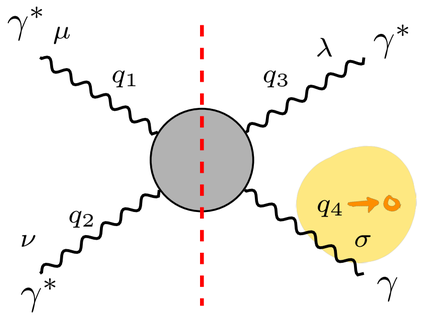

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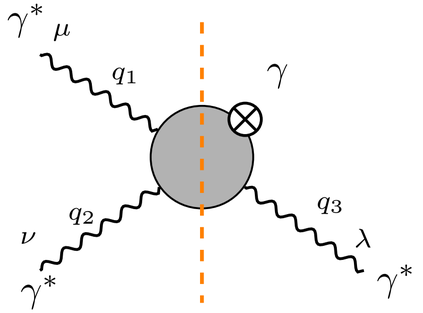
New approach → dispersions relations for HLbL tensor in **three-point kinematics**

- first take $q_4 \to 0$ limit then write dispersion relations

  ↪ need to take both cuts → reconstruct additional hadronic sub-processes

  ↪ no spurious singularities → evaluation of additional contributions



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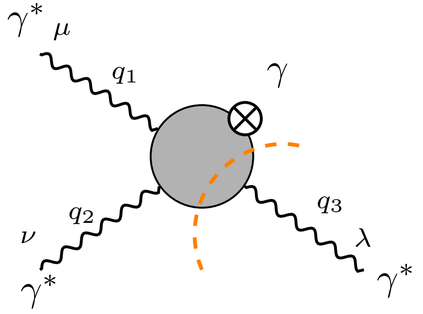
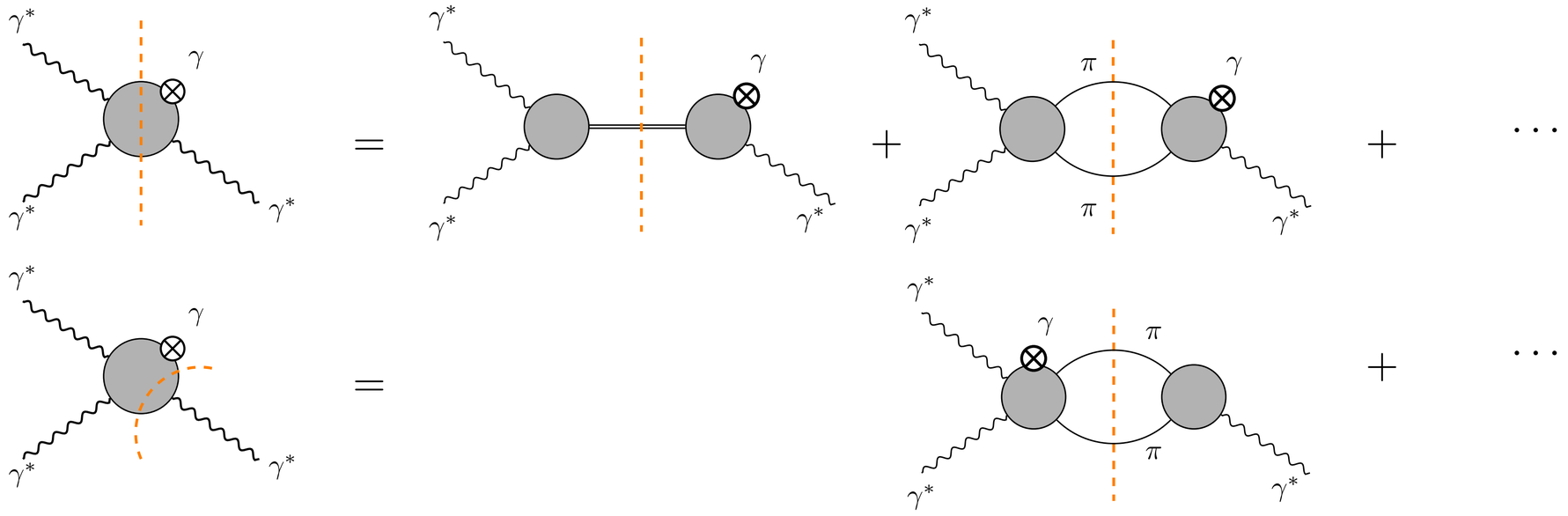


Equivalent unitary cuts in **three-point kinematics**



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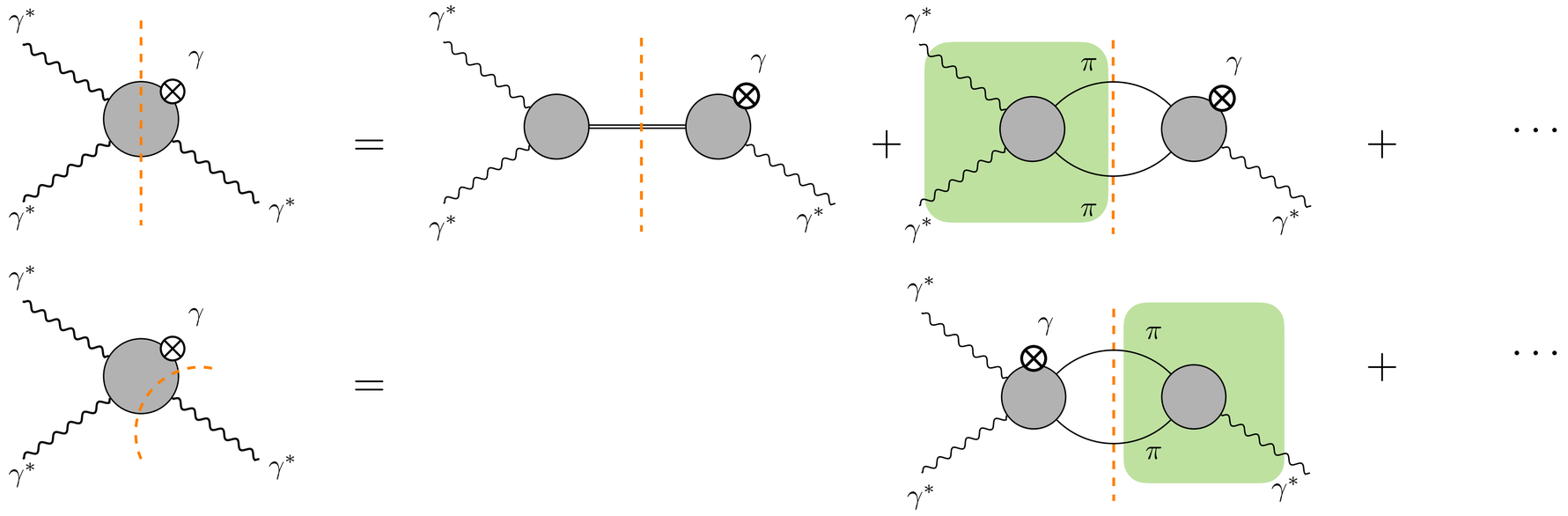

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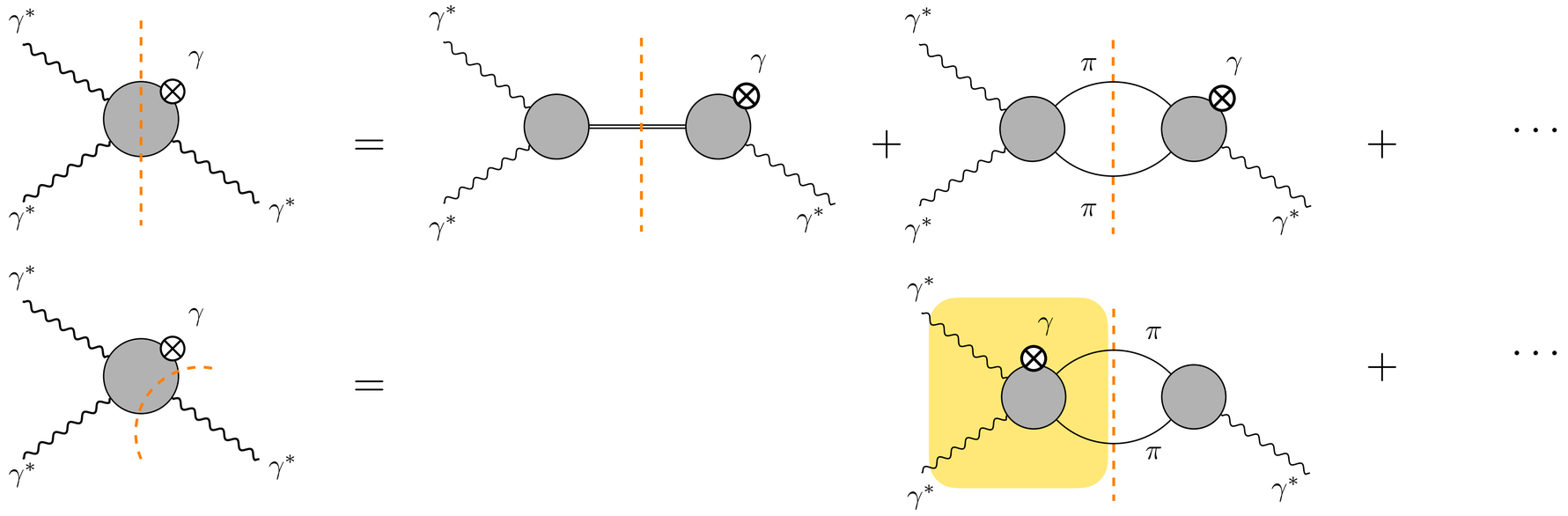

- 5-particle process $\gamma^*(q_1)\gamma^*(q_2)\gamma^*(q_3) \to \pi(p_1)\pi(p_2)$: **not attempt** helicity partial-wave expansion

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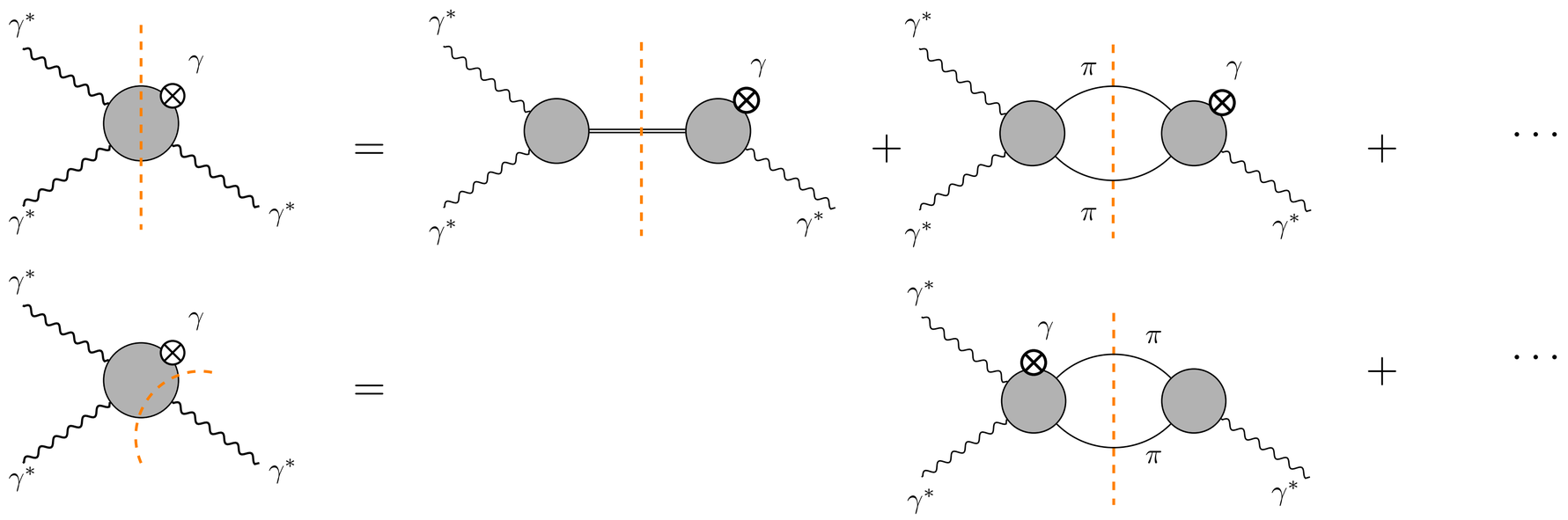
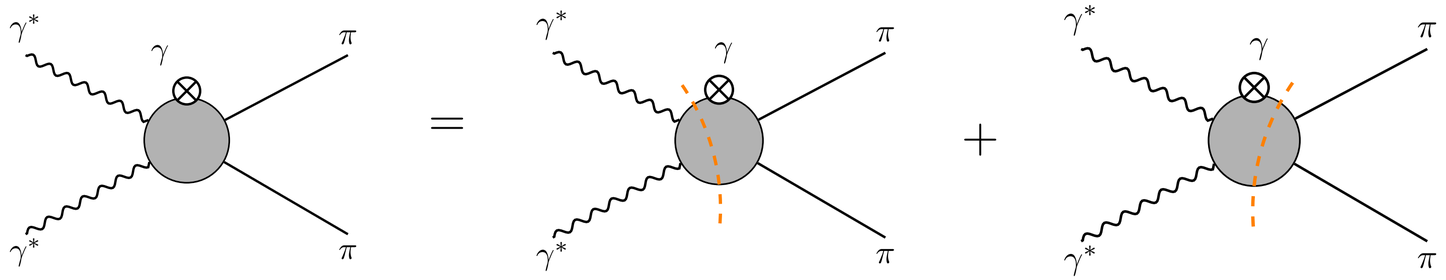


- 5-particle process $\gamma^*(q_1)\gamma^*(q_2)\gamma^*(q_3) \to \pi(p_1)\pi(p_2)$: **not attempt** helicity partial-wave expansion

  ↪ reconstruct dispersively via hadronic sub-processes

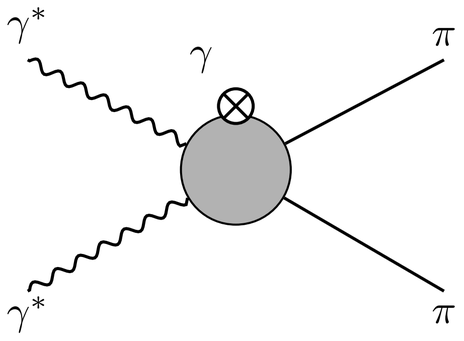


Subprocesses for $\gamma^*\gamma^*\gamma \to \pi\pi$

- Basis for $\gamma^*\gamma^*\gamma \to \pi\pi$ in the soft photon limit : **constructed**

$$\frac{\partial}{\partial q_{3\sigma}} \mathcal{M}^{\mu\nu\lambda}_{\text{reg}}(p_1, p_2, q_1, q_2)\Big|_{q_3=0} = \sum_{i=1}^{27} \mathcal{B}_i^{\mu\nu\lambda;\sigma}(q_1, q_2, q_5)\bar{\mathcal{A}}_i(s, t-u, q_1^2, q_2^2)$$

- Projectors for $\gamma^*\gamma^*\gamma \to \pi\pi$: **constructed**



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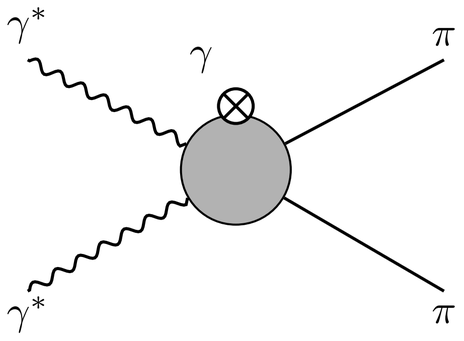

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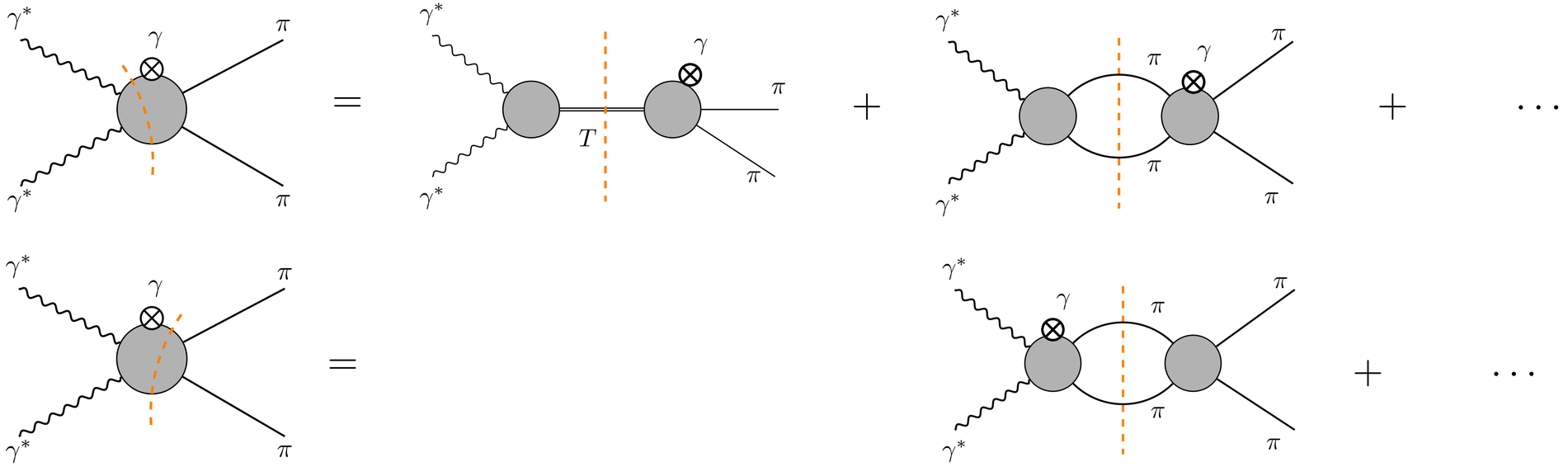
- Projectors for $\gamma^*\gamma^*\gamma \to \pi\pi$: **constructed**

- Contributions :



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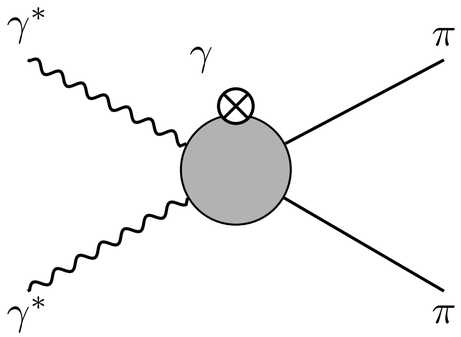

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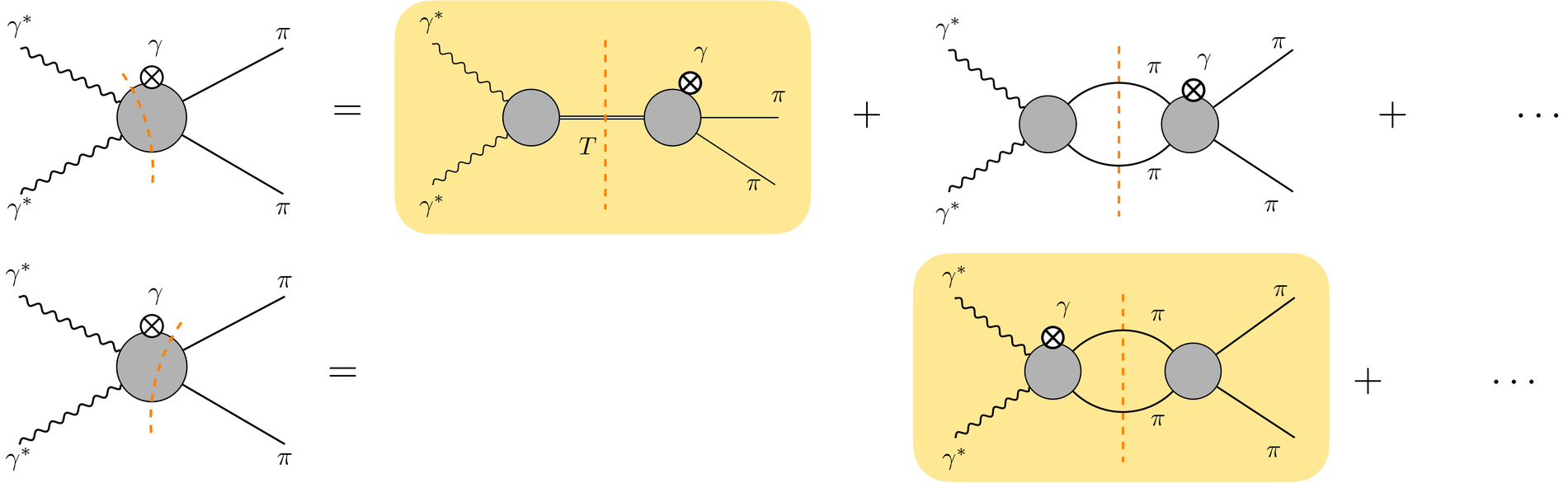
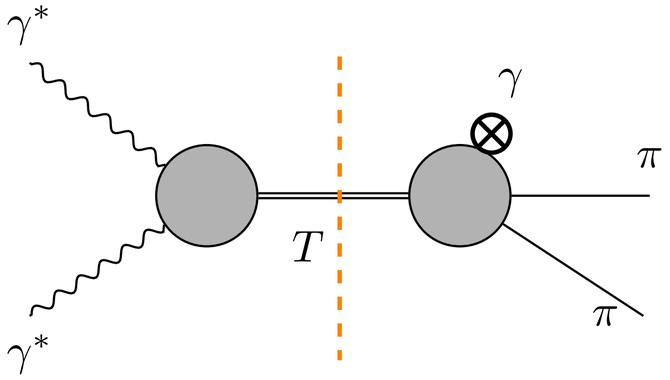
- Projectors for $\gamma^*\gamma^*\gamma \to \pi\pi$: **constructed**

- Contributions : tensor-meson pole & two-pion cut



The tensor-meson cut

Construct bases for sub-processes :



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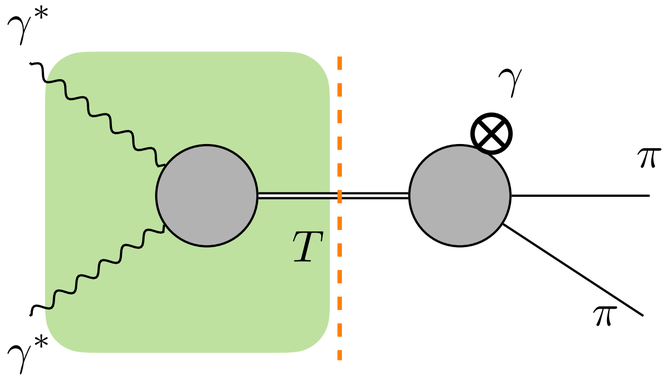
Construct bases for sub-processes :

- tensor meson TFFs [arXiv:2004.06127] : **5 BTT**



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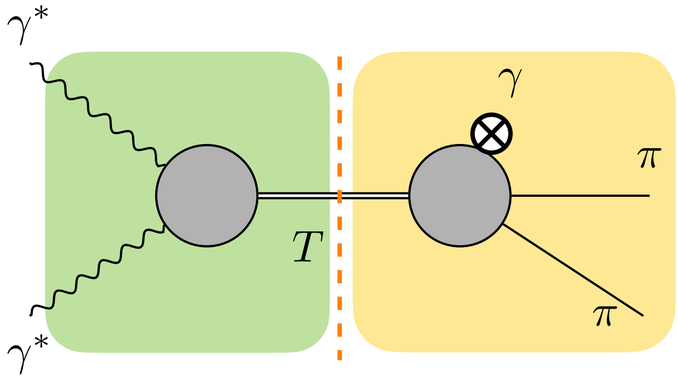
- description of scattering amplitude for $T\gamma^* \to \pi\pi$

  ↪ **BTT** prescription : gauge invariance, remove poles, eliminate redundancies
  8 structures

  ↪ in the soft-photon limit: **only 2** survive

$$M^{\mu\alpha\beta}_{T\gamma^*\pi\pi} = \sum_k^8 T_k^{\mu\alpha\beta}\hat{\mathcal{F}}_k$$



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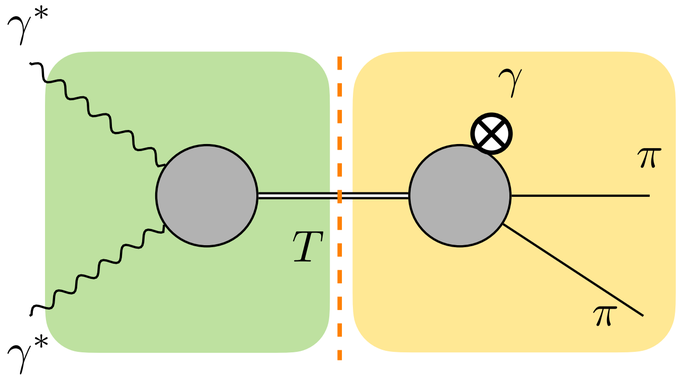

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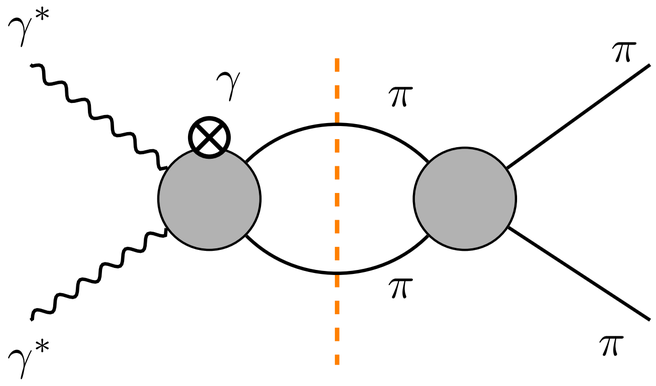
Compute imaginary parts:

$$\mathrm{Im}_s^{\mathrm{T}}\,\bar{\mathcal{A}}_i(s, t-u, q_1^2, q_2^2) = \pi\delta(s - m_T^2)\sum_{j=1}^5\sum_{k=1}^8 \frac{1}{m_{\mathrm{T}}^3} t^i_{j,k}\,\mathcal{F}_j^T(q_1^2, q_2^2)\hat{\mathcal{F}}_k(q_5^2, 0)$$

- dispersion relations for contribution : possibly **reconstruction theorem**

The two-pion cut

Construct bases for sub-processes :

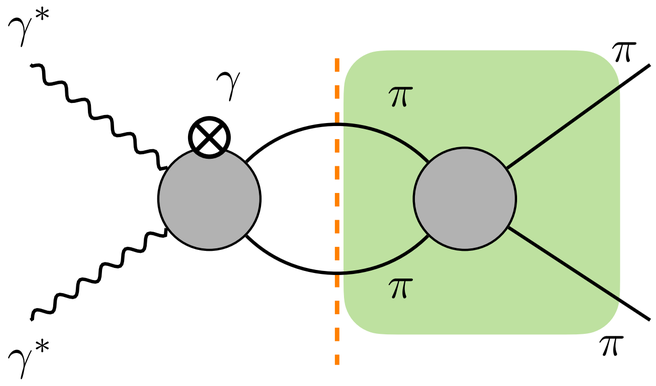


The two-pion cut

Construct bases for sub-processes :

- $\pi\pi-$scattering **:** partial-wave decomposition for fixed isospin I amplitude

  ↪ for $\gamma^*\gamma^*\gamma \to \pi\pi$: only **odd** partial waves



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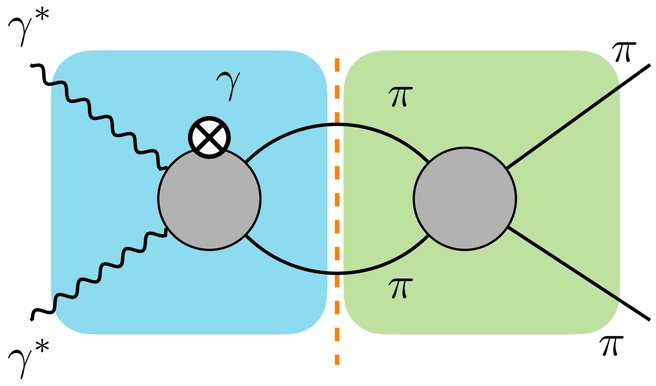
- sub-process $\gamma^*\gamma^*\gamma \to \pi\pi$: **Omnès problem**

  ↪ **not attempt** helicity partial-wave expansion :
  general expansion over $P_j(z)$

$$\bar{\mathcal{A}}_i = \sum_j a_j^i(s) \cdot P_j(z)$$



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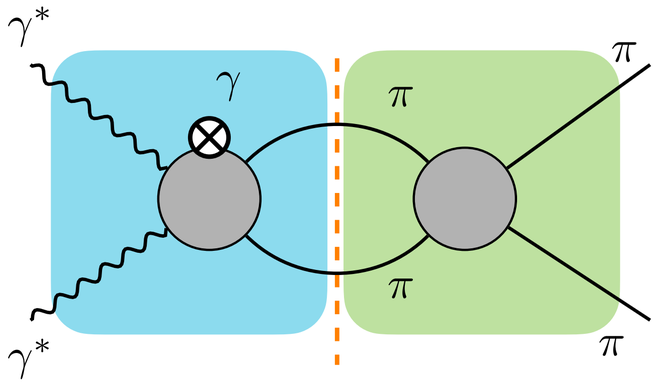
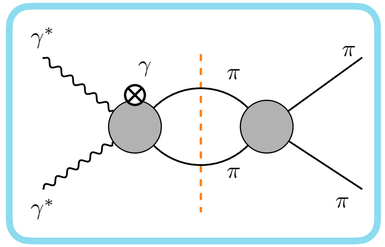
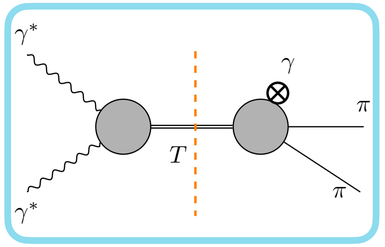
  ↪ inhomogeneous contribution from tensor-pole



$$\bar{A}_{i'}^{\text{full}}(s,z') = \bar{A}_{i'}^{\text{resc.}}(s,z') + \bar{A}_{i}^{\text{tensor}}(s,z')$$





Compute imaginary parts: for only P-wave contributing to $\pi\pi$—scattering

$$\mathrm{Im}_s^{\pi\pi}\bar{\mathcal{A}}_i = \mathrm{Im}_s^{\pi\pi} a_0^{i\,\mathrm{resc.}}(s) = \left[a_0^{i\,\mathrm{resc.}}(s) + \hat{F}_i^{\mathrm{inh.}}(s)\right] \cdot \sin\delta_1^1(s) e^{-i\delta_1^1(s)}$$

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- scalar function contributions : inhomogeneous Omnès solution

$$\bar{\mathcal{A}}_i^{\mathrm{resc.}} = \Omega_1(s)\left[P(s) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_1^1(s')}{|\Omega_1(s')|(s'-s)}\hat{F}_i^{\mathrm{inh.}}(s')\right]$$, with $\hat{F}_i^{\mathrm{inh.}}(s) = \bar{\mathcal{A}}_i^{\mathrm{tensor,P}}$

↪ account for asymptotic behavior

Compute imaginary parts: for only P-wave contributing to $\pi\pi-$scattering

$$\mathrm{Im}_s^{\pi\pi}\bar{\mathcal{A}}_i = \mathrm{Im}_s^{\pi\pi} a_0^{i\,\mathrm{resc.}}(s) = \left[a_0^{i\,\mathrm{resc.}}(s) + \hat{F}_i^{\mathrm{inh.}}(s)\right] \cdot \sin\delta_1^1(s) e^{-i\delta_1^1(s)}$$

- scalar function contributions : inhomogeneous Omnès solution

$$\bar{\mathcal{A}}_i^{\mathrm{resc.}} = \Omega_1(s)\left[P(s) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_1^1(s')}{|\Omega_1(s')|(s'-s)}\hat{F}_i^{\mathrm{inh.}}(s')\right] \quad \text{, with } \hat{F}_i^{\mathrm{inh.}}(s) = \bar{\mathcal{A}}_i^{\mathrm{tensor,P}}$$

↪ account for asymptotic behavior

- tensor-pole contribution: for P-wave case (J=1) : **only 1** scalar function $\hat{\mathcal{F}}_k$ contributes

$$\mathrm{Im}_s^{\mathrm{T}}\,\bar{\mathcal{A}}_i(s, t-u, q_1^2, q_2^2) = \pi\delta(s-m_T^2)\sum_{j=1}^{5}\sum_{k=1}^{8}\frac{1}{m_{\mathrm{T}}^3} t_{j,k}^i\,\mathcal{F}_j^T(q_1^2, q_2^2)\hat{\mathcal{F}}_k(q_5^2, 0)$$

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↪ need TFFs and $\hat{\mathcal{F}}_k$ → Emilis' talk

Conclusions & Outlook

- Dispersive construction of scalar functions for $\gamma^*\gamma^*\gamma \to \pi\pi$: tensor-meson pole & two-pion cut

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✦ Fully study and understand the results of scalar functions $\bar{\mathcal{A}}_i$

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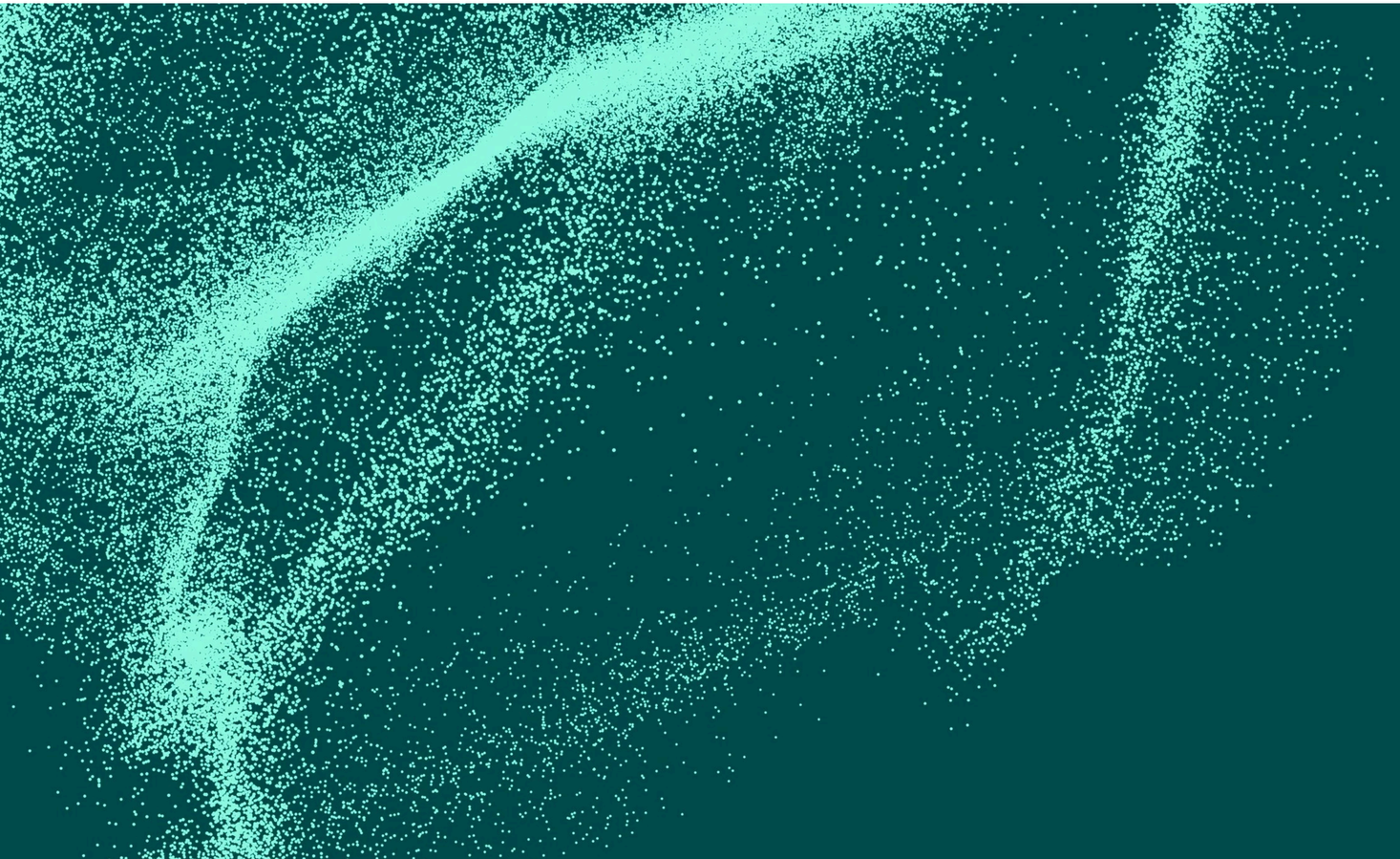
  ↪ compute contribution to a_μ^{HLbL}

✦ Steps towards the hard-photon case : 5-particle process

  ↪ hadronic modeling for radiative corrections

Thank you for your attention!

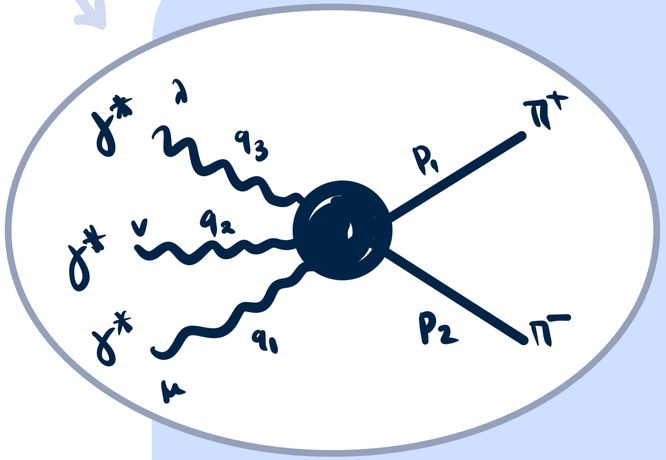


PSI

Universität Zürich[UZH]

B1. Bardeen-Tung-Tarrach

- Bardeen-Tung-Tarrach (BTT) prescription for the scattering amplitude

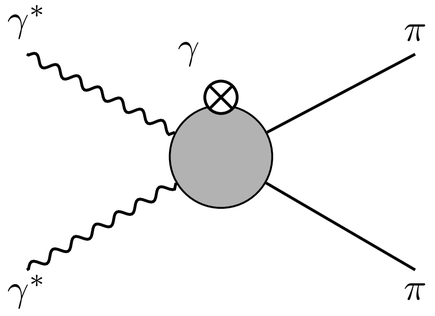


Lorentz decomposition of the amplitude $M^{\mu\nu\lambda} = \sum_i T_i^{\mu\nu\lambda} A_i$

1. consider all possible combinations of Lorentz structures
   ↪ building blocks: external momenta, metric, Levi-Civita
2. ensure gauge invariance (via projectors): $I^{\mu\nu} = g^{\mu\nu} - \frac{v^\mu q_i^\nu}{v \cdot q_i}$
3. remove poles (via linear combinations)
4. consider physical properties of the process
   ↪ photon crossing symmetry, Bose symmetry
5. identify and deal with Tarrach redundancies in the basis

B.2 Construction of $\gamma^*\gamma^*\gamma \to \pi\pi$ basis

Focusing on the process $\gamma^*\gamma^*\gamma \to \pi\pi$ in the soft photon limit



[arXiv:2302.12264]

$$\frac{\partial}{\partial q_{3\sigma}}\mathcal{M}^{\mu\nu\lambda}_{\rm reg}(p_1,p_2,q_1,q_2)\Big|_{q_3=0} = \sum_{i=1}^{28}\tilde{T}_i^{\mu\nu\lambda;\sigma}(q_1,q_2,q_5)\tilde{\mathcal{A}}_i(s,t-u,q_1^2,q_2^2)$$

↪ with one redundancy raining:

$$0 = (t-u)^2\tilde{T}_{22}^{\mu\nu\lambda;\sigma} - (q_1^2-q_2^2-s)(q_1^2-q_2^2+s)\tilde{T}_{27}^{\mu\nu\lambda;\sigma} + (q_1^2+q_2^2-s)\tilde{T}_{28}^{\mu\nu\lambda;\sigma}$$

- Change of basis

$$\boxed{\frac{\partial}{\partial q_{3\sigma}}\mathcal{M}^{\mu\nu\lambda}_{\rm reg}(p_1,p_2,q_1,q_2)\Big|_{q_3=0} = \sum_{i=1}^{27}\mathcal{B}_i^{\mu\nu\lambda;\sigma}(q_1,q_2,q_5)\bar{\mathcal{A}}_i(s,t-u,q_1^2,q_2^2)}$$

↪ with certain kinematic constraints for scalar functions

$$\bar{\mathcal{A}}_{22} = (q_1^2+q_2^2-s)\tilde{\mathcal{A}}_{22} - (t-u)^2\tilde{\mathcal{A}}_{28}$$
$$\bar{\mathcal{A}}_{27} = (q_1^2+q_2^2-s)\tilde{\mathcal{A}}_{27} - (q_1^2-q_2^2-s)(q_1^2-q_2^2+s)\tilde{\mathcal{A}}_{28}$$
$$\bar{\mathcal{A}}_i = \tilde{\mathcal{A}}_i \quad \text{for } i \in \{1,\cdots,21,23,\cdots,26\}$$

B.3 Projectors for $\gamma^*\gamma^*\gamma \to \pi\pi$

Building projectors for $\gamma^*\gamma^*\gamma \to \pi\pi$

$$\frac{\partial}{\partial q_{3\sigma}} \mathcal{M}^{\mu\nu\lambda}_{\text{reg}}(p_1, p_2, q_1, q_2)\Big|_{q_3=0} = \sum_{i=1}^{27} \mathcal{B}_i^{\mu\nu\lambda;\sigma}(q_1, q_2, q_5)\bar{\mathcal{A}}_i(s, t-u, q_1^2, q_2^2)$$

- isolate each scalar function: $\mathcal{P}_{j\,\mu\nu\lambda\sigma}\mathcal{B}_i^{\mu\nu\lambda;\sigma} = \delta_{ij}$

Definition of projectors : $\mathcal{P}_j^{\mu\nu\lambda\sigma} = C_{jk}T_k^{\mu\nu\lambda\sigma}$

↪ contracting & inverting matrix of coefficients : $C_{jk} = I_{27}\,\mathrm{CTM}^{-1}$

↪ preservation of matrix rank: $\mathrm{CTM}_{(27\times 27)} := T_{k\,\mu\nu\lambda\sigma}\mathcal{B}_i^{\mu\nu\lambda;\sigma}$

✅ inversion of matrix via C++ program [arXiv:1904.00009v2]

B.4 Expansion with Legendre polynomials

$$\bar{\mathcal{A}}_i = \sum_j a^i_j(s) \cdot P_j(z) \;\rightarrow\quad a^i_j(s) = \frac{2j+1}{2}\int_{-1}^{+1} dz\; P_j(z)\,\bar{\mathcal{A}}_i(z)$$

- Taking Imaginary part:

$$\begin{aligned}
\mathrm{Im}^{\pi\pi}_s a^i_j(s) &= \frac{2j+1}{2}\int_{-1}^{+1} d\,z\,P_j(z)\,\mathrm{Im}^{\pi\pi}_s \bar{\mathcal{A}}_i(z) \\
&= \frac{2j+1}{2}\int_{-1}^{+1} d\,z\,P_j(z)\,\mathrm{Im}^{\pi\pi}_s \left(\mathcal{P}_{i,\,\mu\nu\lambda;\sigma}\cdot\sum_{k=1}^{27}\mathcal{B}^{\mu\nu\lambda;\sigma}_k\bar{\mathcal{A}}_k\right) \Rightarrow \\
\mathrm{Im}^{\pi\pi}_s a^i_j(s) &= \frac{2j+1}{2}\int_{-1}^{+1} d\,z\,P_j(z)\,\mathcal{P}_{i,\,\mu\nu\lambda;\sigma}\cdot\sum_{k=1}^{27}\mathcal{B}^{\mu\nu\lambda;\sigma}_k\cdot \boxed{\mathrm{Im}^{\pi\pi}_s\bar{\mathcal{A}}_k(s,z)}
\end{aligned}$$

replace the unitarity relation in here

** Macfarlane, A. J. "Relativistic partial wave analysis." *Reviews of Modern Physics* 34.1 (1962): 41.

B.5 Tensor-meson pole cut (extra results)

$$\mathrm{Im}_s^{\mathrm{T}}\,\bar{\mathcal{A}}_i(s,t-u,q_1^2,q_2^2) = \pi\delta(s-m_T^2)\sum_{j=1}^{5}\sum_{k=1}^{8}\frac{1}{m_{\mathrm{T}}^3}t_{j,k}^i\,\mathcal{F}_j^T(q_1^2,q_2^2)\hat{\mathcal{F}}_k(q_5^2,0)$$

with

$$\boxed{\mathrm{Im}_s^{\mathrm{T}}\bar{\mathcal{A}}_i = 0 \quad \text{for } i\in\{11,12,21,22,24,27\}}$$

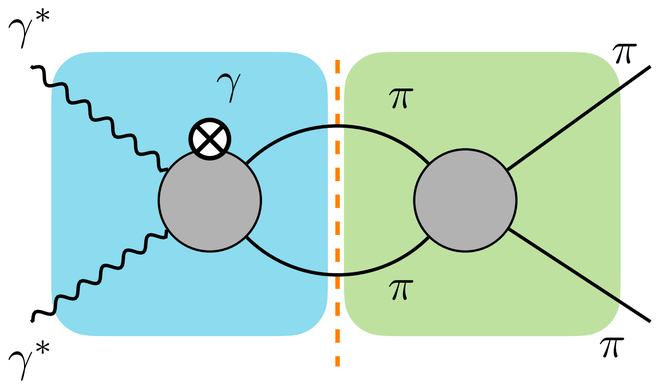
* structures 22 and 27 do not have contributing terms :)

- s,t,u dependance in the coefficients $t_{j,k}^i$

↪ 21 non-zero 5×8 matrices of coefficients, e.g.

$$t_{3,1}^1 = 2m_T^2\,,\quad t_{4,1}^1 = q_1^2 - q_2^2 - m_T^2\,,\quad t_{3,5}^3 = (t-u)^2\,,\quad t_{4,5}^3 = \frac{(t-u)^2(m_T^2-q_1^2+q_2^2)}{2(q_1^2-q_2^2)}$$

B.6 Two-pion cut (extra calculation)



Computation of imaginary parts:

$$\mathrm{Im}_s^{\pi\pi} a_j^{i\,\mathrm{resc.}}(s) = \frac{2j+1}{2}\int_{-1}^{+1} d\,z\, P_j(z)\, \mathcal{P}_{i,\,\mu\nu\lambda;\sigma}(s, t_1 - u_1) \cdot \int d\Pi_2 \left(\mathcal{T}^{(I)}(p_1 p_2 \to k_1 k_2)\right)^* \sum_{i'=1}^{27} \mathcal{B}_{i'}{}^{\mu\nu\lambda}(q_1, q_2, k_5) \cdot \bar{\mathcal{A}}_{i'}(s, t_3 - u_3) \quad \Rightarrow$$

$$\bar{A}_{i'}^{\mathrm{full}}(s, z') = \bar{A}_{i'}^{\mathrm{resc.}}(s, z') + \bar{A}_i^{\mathrm{tensor}}(s, z')$$

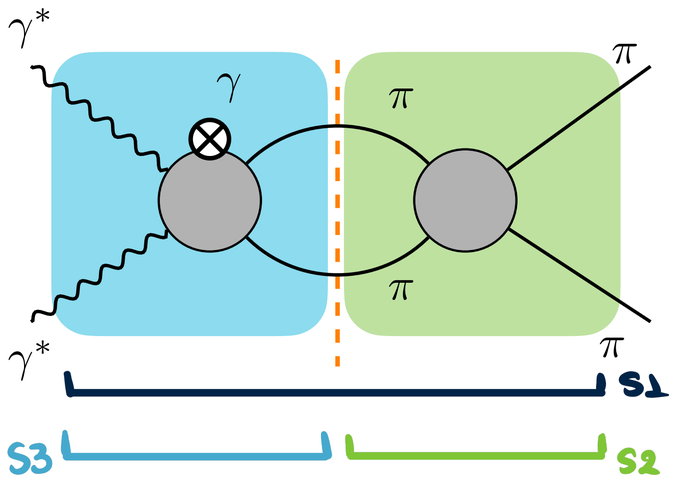
$$\mathrm{Im}_s^{\pi\pi} a_j^{i\,\mathrm{resc.}}(s) = \frac{(2j+1)}{8\pi}\sum_J (2J+1)\sin\delta_J^I(s) e^{-i\delta_J^I(s)} \int_{-1}^{+1} dz \int_{-1}^{+1} dz' \int_0^{2\pi} d\phi'\, P_j(z) P_J(z_0) \sum_{i'=1}^{27} C_{ii'}(s, z, z', z_0) \cdot A_{i'}^{\mathrm{full}}(s, z')$$

$$\bar{\mathcal{A}}_i = \sum_j a_j^i(s) \cdot P_j(z)$$

$$P_J(z_0) = P_J(z') P_J(z) + 2\sum_{m=1}^{J} \frac{(J-m)!}{(J+m)!} P_J^m(z') P_J^m(z) \cos m(\phi - \phi')$$

Kinematics for subprocesses:

$$\gamma^*(q1)\gamma^*(q_2)\gamma(q_3) \to \pi(k_1)\pi(k_2) \to \pi(p_1)\pi(p_2)$$



$$s_1 = (q_1 + q_2)^2 = (p_1 + p_2)^2 \qquad s_2 = (k_1 + k_2)^2 = (p_1 + p_2)^2 \qquad s_3 = (q_1 + q_2)^2 = (k_1 + k_2)^2$$

$$t_1 = (q_1 - p_1)^2 = (q_2 - p_2)^2 \qquad t_2 = (k_1 - p_1)^2 = (k_2 - p_2)^2 \qquad t_3 = (q_1 - k_1)^2 = (q_2 - k_2)^2$$

$$u_1 = (q_1 - p_2)^2 = (q_2 - p_1)^2 \qquad u_2 = (k_1 - p_2)^2 = (k_2 - p_1)^2 \qquad u_3 = (q_1 - k_2)^2 = (q_2 - k_1)^2$$

$$z = \cos\theta = \frac{t_1 - u_1}{\sigma_\pi(s)\lambda^{1/2}(s, q_1^2, q_2^2)}$$

$$z_0 = \cos\theta' \cos\theta + \cos(\phi - \phi') \sin\theta \sin\theta' = \frac{t_2 - u_2}{(s - 4M_\pi^2)}$$

$$z' = \cos\theta' = \frac{t_3 - u_3}{\sigma_\pi(s)\lambda^{1/2}(s, q_1^2, q_2^2)}$$

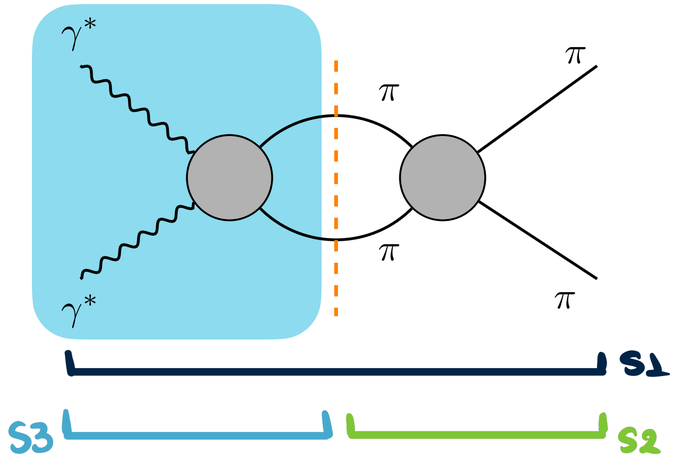
✅ cross-check for $\gamma^*\gamma^* \to \pi\pi$

↪ for $J = 0$:

$$\mathrm{Im}_s^{\pi\pi} a_0^{1\,\mathrm{resc.}}(s) = \left[a_0^{1\,\mathrm{resc.}}(s) + \hat{F}_1^{\mathrm{inh.}}(s)\right] \cdot \sin\delta_0^0(s) e^{-i\delta_0^0(s)}$$

$$\mathrm{Im}_s^{\pi\pi} a_0^{2\,\mathrm{resc.}}(s) = \left[a_0^{2\,\mathrm{resc.}}(s) + \hat{F}_2^{\mathrm{inh.}}(s)\right] \cdot \sin\delta_0^0(s) e^{-i\delta_0^0(s)}$$

$$\mathrm{Im}_s^{\pi\pi} a_0^{3\,\mathrm{resc.}}(s) = \mathrm{Im}_s^{\pi\pi} a_0^{4\,\mathrm{resc.}}(s) = \mathrm{Im}_s^{\pi\pi} a_0^{5\,\mathrm{resc.}}(s) = 0$$



$$A_{i'}^{\mathrm{full}}(s,z') = A_{i'}^{\mathrm{resc.}}(s,z') + F_\pi^V(q_1^2) F_\pi^V(q_2^2)\, A_i^{\mathrm{Born}}(s,z')$$

↪ solution for inhomogeneous Omnès problem :

$$A_1^{\mathrm{resc.}} = a_0^{1\,\mathrm{resc.}}(s) = \frac{\Omega_0(s)}{\pi} \int_{4M_\pi^2}^{\infty} ds' \, \frac{\sin\delta_0^0(s')}{|\Omega_0(s')|} \frac{\hat{F}_1^{\mathrm{inh.}}(s')}{(s'-s)}$$

$$A_2^{\mathrm{resc.}} = a_0^{2\,\mathrm{resc.}}(s) = \frac{\Omega_0(s)}{\pi} \int_{4M_\pi^2}^{\infty} ds' \, \frac{\sin\delta_0^0(s')}{|\Omega_0(s')|} \frac{\hat{F}_2^{\mathrm{inh.}}(s')}{(s'-s)}$$

$$A_3^{\mathrm{resc.}} = A_4^{\mathrm{resc.}} = A_5^{\mathrm{resc.}} = 0$$

same to the results from the helicity partial-wave approach (S-wave contribution)
[arXiv:1905.13198]

Two-pion cut (extra calculation) : $\gamma^*\gamma^* \to \pi\pi$

$$\mathrm{Im}_s^{\pi\pi} a_j^{i\,\mathrm{resc.}}(s) = \frac{(2j+1)}{8\pi}\sum_J (2J+1)\sin\delta_J^I(s)e^{-i\delta_J^I(s)}\int_{-1}^{+1}dz\int_{-1}^{+1}dz'\int_0^{2\pi}d\phi'\,P_j(z)P_J(z_0)\sum_{i'=1}^{27}C_{ii'}(s,z,z',z_0)\cdot A_{i'}^{\mathrm{full}}(s,z')$$

$$A_{i'}^{\mathrm{full}}(s,z') = A_{i'}^{\mathrm{resc.}}(s,z') + F_\pi^V(q_1^2)F_\pi^V(q_2^2)\,A_i^{\mathrm{Born}}(s,z')$$

$$z_0 = \cos\theta'\cos\theta + \cos(\phi-\phi')\sin\theta\sin\theta' = z'\cdot z + \cos(\phi-\phi')\sin\theta\sin\theta'$$

$$\begin{aligned}
I_{\mathrm{Born}}^i =& \frac{(2j+1)}{8\pi}\sum_J (2J+1)\sin\delta_J^I(s)e^{-i\delta_J^I(s)}\int_{-1}^{+1}dz\int d\Omega'\,P_j(z)P_J(z_0)\sum_{i'=1}^{5}C_{ii'}(s,z,z',z_0)\cdot A_{i'}^{\mathrm{Born}}(s,z')\\
=& \frac{(2j+1)}{8\pi}\sum_J (2J+1)\sin\delta_J^I(s)e^{-i\delta_J^I(s)}\int_{-1}^{+1}dz\int_{-1}^{+1}dz'\int_0^{2\pi}d\phi' P_j(z)\\
&\times\left(P_J(z)P_J(z') + 2\sum_{m=1}^{J}\frac{(J-m)!}{(J+m)!}P_J^m(z)P_J^m(z')\cos(m(\phi-\phi'))\right)\sum_{i'=1}^{5}C_{ii'}(s,z,z',\phi')\cdot A_{i'}^{\mathrm{Born}}(s,z')\\
=& \frac{(2j+1)}{8\pi}\sum_J (2J+1)\sin\delta_J^I(s)e^{-i\delta_J^I(s)}\left[I_{\mathrm{Born}}^{(\mathrm{P1})} + I_{\mathrm{Born}}^{(\mathrm{P2})}\right]
\end{aligned}$$

B.7 Tensor TFFs