Coherent Soft-Photon Resummation in Phokhara

RMCL2 - Università di Torino

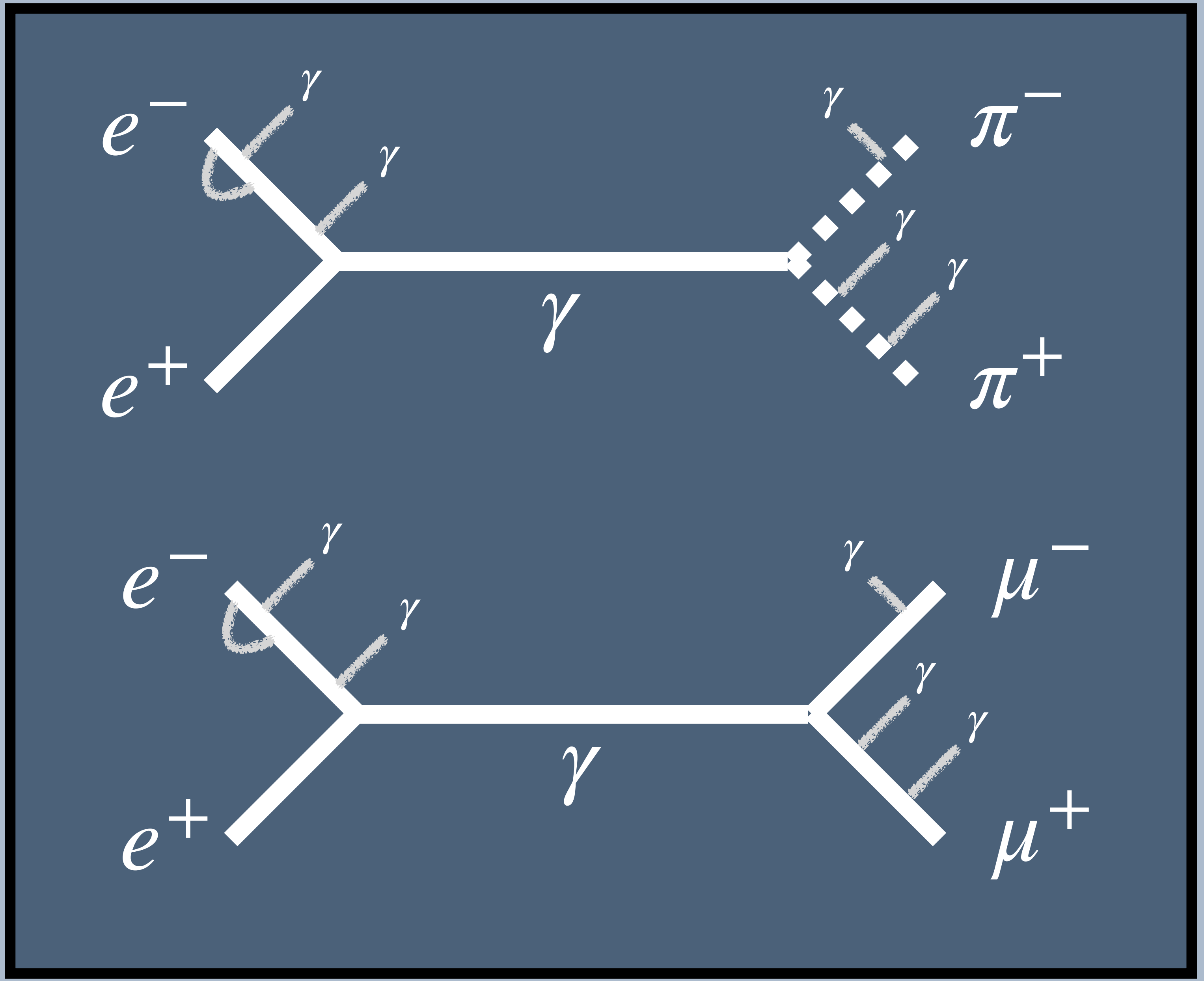
Jérémy Paltrinieri

Why bother?

Fixed-order calculations beyond one-loop are difficult: for radiative return experiments, this requires the calculation of a **5-point amplitude at two-loops with two masses**: many scales!

William, Tom and Mattia's talks

By approximating the amplitude, can probe the dominating effects coming from **radiative corrections**.



tree-level process corrected with soft photons

When two-loop amplitudes become available, these will provide checks on **soft phase-space regions**.

Recently observed tensions in fixed-order calculations: could be resolved with an independent calculation.

E. Budassi, C.M.C. Calame, M. Ghilardi et al. [2601.19530]

Route chosen: **YFS exponentiation**.

Yennie, Frauschi, Suura [Annals Phys. 13 (1961) 379-452]

S. Jadach, B.F.L. Ward, Z. Was [0006359]

Frank Krauss, Alan Price, Marek Schönherr [2203.10948]

This talk: **pions** only, with **FxsQED** (same analyses can be done for muons)

GVMD: Pau's talk

Master Formula

Controls the **IR divergences** in QED at all orders.

Assumption: all fermions must be **massive**: no collinear singularities (but collinear logs).

YFS FF (contains IR divergences)

Regularised amplitudes

$$d\sigma = \sum_{n_\gamma=0}^{\infty} \frac{\exp Y(\Omega)}{n_\gamma!} d\Phi_Q \frac{1}{4} \sum_h \left| \sum_\phi \left[\prod_{i=1}^{n_\gamma} d\phi_i^\gamma S(k_i) \Theta(k_i, \Omega) \right] \left(\beta_0 + \sum_{j=1}^{n_\gamma} \frac{\beta_1(k_j)}{S(k_j)} + \sum_{1=j<k}^{n_\gamma} \frac{\beta_2(k_j, k_k)}{S(k_j)S(k_k)} + \ldots \right) \right|^2$$

Partitions (ISR/FSR) Eikonal factors Resolved domain

No approximations here!

Accuracy at fixed-order directly related to the accuracy of the β's.

Note: need **momenta mappings** to map n-photon events to i-photon events for β_i.

YFS Form Factor

$$d\sigma = \sum_{n_\gamma=0}^{\infty} \frac{\exp Y(\Omega)}{n_\gamma!} d\Phi_Q \sum_{\phi} \left[\prod_{i=1}^{n_\gamma} d\phi_i^\gamma S(k_i)\Theta(k_i,\Omega) \right] \left| \left(\beta_0 + \sum_{j=1}^{n_\gamma} \frac{\beta_1(k_j)}{S(k_j)} + \sum_{1=j<k}^{n_\gamma} \frac{\beta_2(k_j,k_k)}{S(k_j)S(k_k)} + \ldots \right) \right|^2$$

with the **YFS Form Factor** $Y(\Omega) = 2\alpha \sum_{i<j} \mathscr{R}\left(B(p_i,p_j) + \tilde{B}(p_i,p_j,\Omega)\right)$ fermions and hadrons dipoles

Virtual

Real

$$B(p_i,p_j) \propto \int \frac{d^4k}{k^2}\left(\frac{2p_i - k}{k^2 - 2k\cdot p_i} - \frac{2p_j + k}{k^2 + 2k\cdot p_j}\right)^2$$

$$\tilde{B}(p_i,p_j,\Omega) \propto \int_\Omega d^4k\,\delta(k^2)\left(\frac{p_i}{p_i\cdot k} - \frac{p_j}{p_j\cdot k}\right)^2$$

YFS Form Factor

Schematically:
$$\sigma = \sum_{n_\gamma} \int_{\Omega} (dLips)_{n_\gamma+2} \frac{e^{\alpha Y_{\mathrm{FF}}(\tilde{\Omega})}}{n_\gamma!} \sum_{\mathrm{hel}} |\mathcal{M}_{reg}|^2$$

The real photons are emitted in the **resolved** domain Ω

The form factor is calculated in the **unresolved** domain $Y_{\mathrm{FF}}(\tilde{\Omega}) = Y_R(\tilde{\Omega}) + Y_V$

After integration, the cross-section is independent of the definition of $\tilde{\Omega}$

Sherpa, KKMC use a non Lorentz-Invariant parametrisation of the domain.

Results for the integrals are analytical, but weights have to be corrected.

S. Jadach, B.F.L. Ward, Z. Was [0006359]

We compute it in dim-reg in a **Lorentz-Invariant way**: no further corrections needed.
The integral is evaluated numerically (quadrature).

J. Williams [Quadpack github]

YFS Form Factor

$$Y(\Omega) = 2\alpha \sum_{i<j} \mathscr{R}\left(B(p_i, p_j) + \tilde{B}(p_i, p_j, \Omega)\right)$$

Sum over dipoles (electrons and pions)

$$B(p_i, p_j) \propto \int \frac{d^4k}{k^2}\left(\frac{2p_i - k}{k^2 - 2k\cdot p_i} - \frac{2p_j + k}{k^2 + 2k\cdot p_j}\right)^2$$

Virtual

$$\tilde{B}(p_i, p_j, \tilde{\Omega}) \propto \int_{\tilde{\Omega}} d^4k\,\delta(k^2)\left(\frac{p_i}{p_i\cdot k} - \frac{p_j}{p_j\cdot k}\right)^2$$

Real

Define: $\tilde{\Omega} = \left\{K\cdot k < \sqrt{s}\omega\right\}$ with: $K = p(e^-) + p(e^+)$ In the lab frame: $\tilde{\Omega} = \left\{E_\gamma < \omega\right\}$

In the (p_i, p_j) rest frame, rotated, we have $\tilde{\Omega} = \{\,|k|(p^0 - \vec{p}\cdot\vec{k})\}$ with:

$$p^0 = E_{ij}^{(1)}$$
$$p^x = \sqrt{(E_{ij}^{(1)})^2 - s_{ij}}\sin\Theta$$
$$p^y = 0$$
$$p^z = \sqrt{(E_{ij}^{(1)})^2 - s_{ij}}\cos\Theta$$

By Lorentz Invariance, we calculate the real integral in this domain.

YFS Form Factor

Schematically:

$$\sigma = \sum_{n_\gamma} \int_{\Omega} (dLips)_{n_\gamma+2} \frac{e^{\alpha Y_{\mathrm{FF}}(\tilde{\Omega})}}{n_\gamma!} \sum_{\mathrm{hel}} |\mathcal{M}_{reg}|^2$$

Define: $\tilde{\Omega} = \left\{ K \cdot k < \sqrt{s}\omega \right\}$ with: $K = p(e^-) + p(e^+)$ In the lab frame: $\tilde{\Omega} = \left\{ E_\gamma < \omega \right\}$

$$Y_{FF}(p_i, p_j, \tilde{\Omega}) = \frac{\alpha}{4\pi} \left(p_{ij} \log \frac{\omega^2}{s} + q_{ij} + v_{ij} \right)$$

The remainder after cancellation of the IR poles: universal

Finite part from real: 1D integral

Finite part from virtual: analytical: logs and dilogs

Extensive list of checks possible, from small photon mass result or direct dim-reg result.

K. Hamilton, P. Richardson [0603034] G. Passarino [0108255]

Form Factor Decomposition / Tensor Decomposition

$$\sigma_{LO}^{res.} = \sum_{n_\gamma} \int_\Omega (dLips)_{n_\gamma+2} \frac{e^{\alpha Y_{\rm FF}}}{n_\gamma!} \sum_{\rm hel} \left| \sum_{\phi \in [I,F]^{n_\gamma}} \left(\prod_{i=1}^{n_\gamma} s_i^{\phi_i} \right) \beta^0(P) \right|^2$$

At LO, we have $\beta^0 = \mathscr{M}^0$

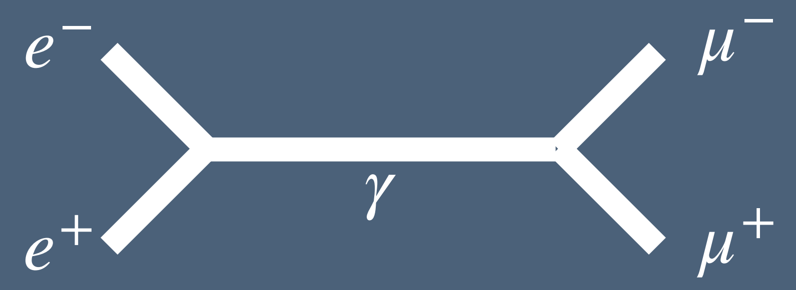
Form Factor Decomposition at l loops:

$$\mathscr{M}^l \left(ee \to f\bar{f} \right) = \sum_{i=1}^{n} \mathscr{F}_i^l \times \mathscr{T}^i$$

Depends on **kinematics** (not helicities)

Depends on **helicities** (Spin tensors)

Example:



$$\mathscr{M}^0 \left(e^+e^- \to \mu^+\mu^- \right) = \left(\bar{v}\gamma^\nu u \; \bar{u}\gamma_\nu v \right) \times \left(\frac{1}{s} \right) = \mathscr{F}^0 \times \mathscr{T}^0$$

Form Factor decomposition of the amplitude

Structure of the amplitude

$$\sigma_{LO}^{res.} = \sum_{n_\gamma} \int_\Omega (dLips)_{n_\gamma+2} \frac{e^{\alpha Y_{FF}}}{n_\gamma!} \sum_{hel} \left| \sum_{\phi\in[I,F]^{n_\gamma}} \left(\prod_{i=1}^{n_\gamma} s_i^{\phi_i} \right) \beta^0(P) \right|^2$$

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Depends on **kinematics** (not helicities)

Depends on **helicities** (Spin tensors)

Notes

- Works for massless or massive particles
- The number of tensors / FFs does not depend on the number of loops
- The coefficients in the FFs can be huge beyond LO, optimisation required (ongoing work)

Eikonal Approximation

In the **soft limit** (all photons soft):

$$\mathcal{M}(\pi\pi\gamma(k_1)\dots\gamma(k_n)) \to \mathcal{M}(\pi\pi) \times s(k_1)\dots s(k_n)$$

In particular, for radiative return processes at LO+resummation, we need for each helicity:

$$\mathcal{M}(\pi\pi\gamma(k)) \to \mathcal{M}(\pi\pi) \times s(k)$$

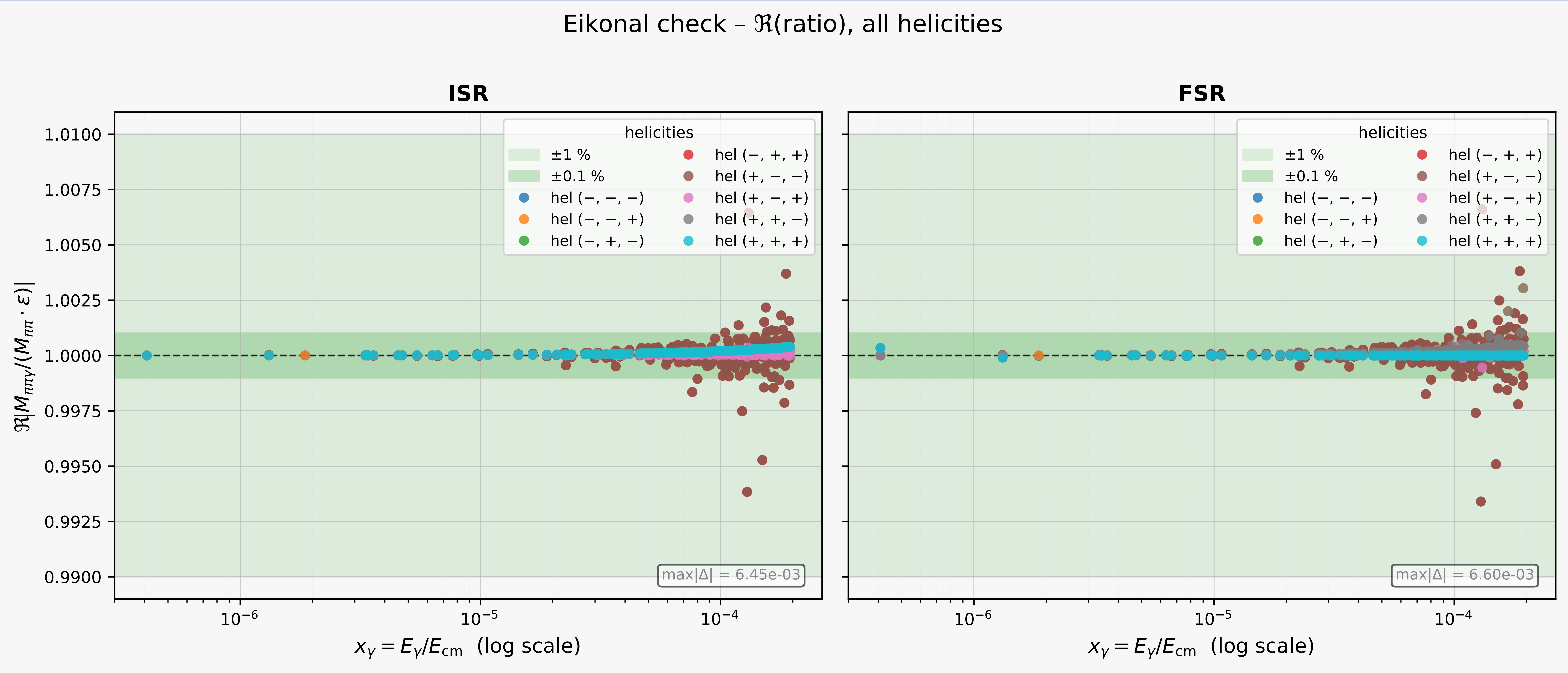
To ensure that the **phases are right**, rewrite all spinors for π π γ and π π in a consistent way:

$$p_i^b = p_i - x_i p_\gamma \qquad x_i = \frac{p_i^2}{2p_i \cdot p_\gamma}, \qquad i = e^+, e^-, \pi^+, \pi^-$$

$$p_\gamma^b = p_\gamma$$

Note: phases must be consistent between ISR and FSR.

Form Factor decomposition of the amplitude



Ratio of amplitudes $\frac{\mathscr{M}(\pi\pi\gamma(k))}{\mathscr{M}(\pi\pi) \times s(k)}$ for ISR and FSR for each helicity configurations (~200 events)

Momenta mappings

To evaluate the resummed amplitudes, we need momenta mappings:

$$\Phi_{n\to i} : \left(e^+e^- \to \pi^+\pi^-\gamma_1\ldots\gamma_n\right) \to \left(e^+e^- \to \pi^+\pi^-\gamma_1\ldots\gamma_i\right)$$

which map n-photon events to i-photon events (i<n).

Scan mode:

- LO resummed: $\Phi_{n\to 0}$
- NLO resummed: $\Phi_{n\to 1}$

Radiative return:

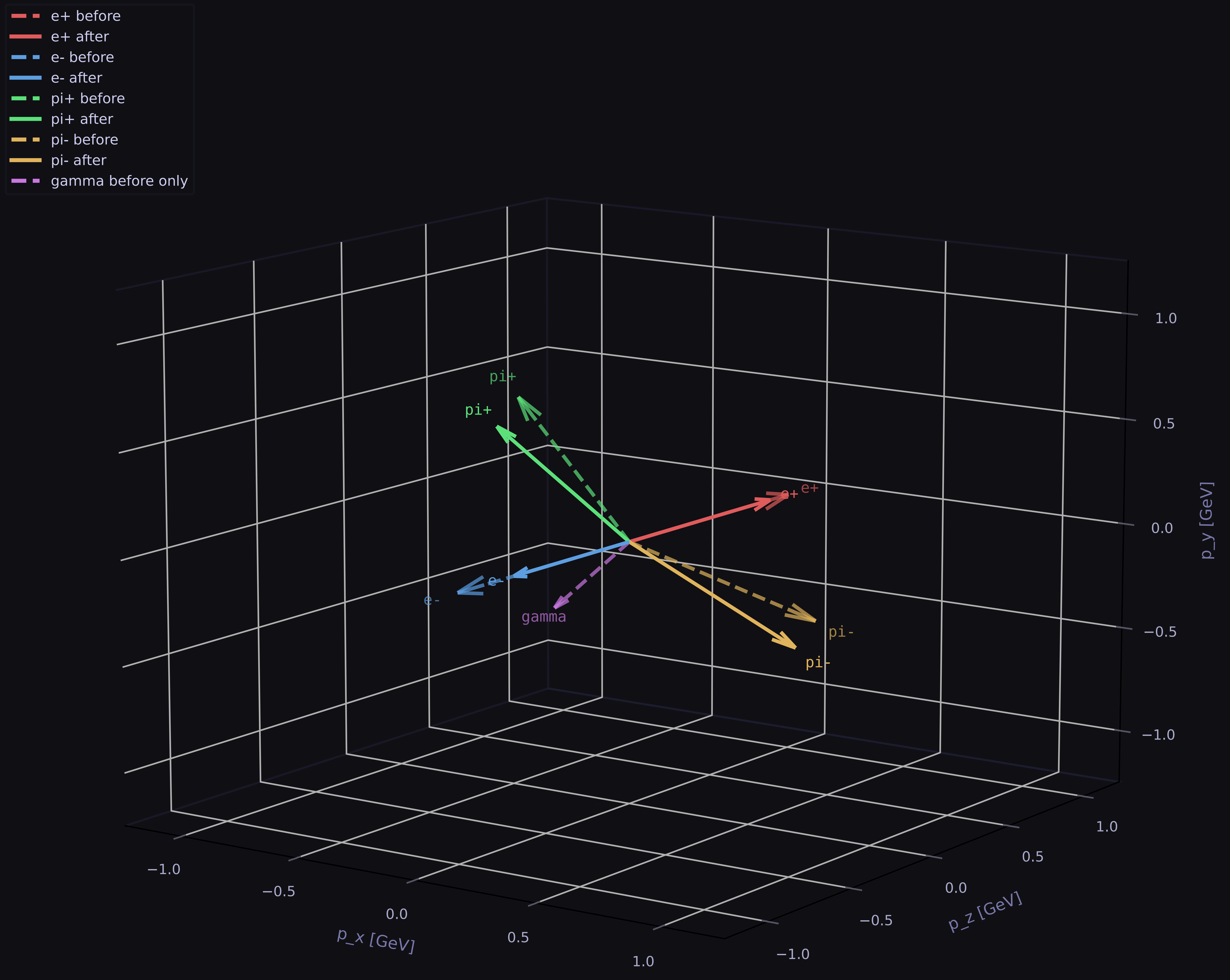
- LO resummed: $\Phi_{n\to 0}, \Phi_{n\to 1}$
- NLO resummed: $\Phi_{n\to 2}$

In particular, the eikonal limit should be written as:

$$\mathscr{M}(\pi\pi\gamma(k)) \to \mathscr{M}(\Phi_{1\to 0}(\pi\pi\gamma)) \times s(k)$$

Momentum Mapping: 1-photon → 0-photon

dashed = before | solid = after | axes: x, z (horiz), y (vert)



Invariant masses [GeV]

particle	m before	m after
e+	0.00051	0.00051
e-	0.00051	0.00051
pi+	0.13957	0.13957
pi-	0.13957	0.13957
gamma	0.00000	–

Momentum conservation

P = P(e⁺)+P(e⁻) − P(π⁺) − P(π⁻) − P(γ)

BEFORE P = (+0.0000, -0.0000, -0.0000, -0.0000)

‖P‖ = 1.161099e-16 GeV

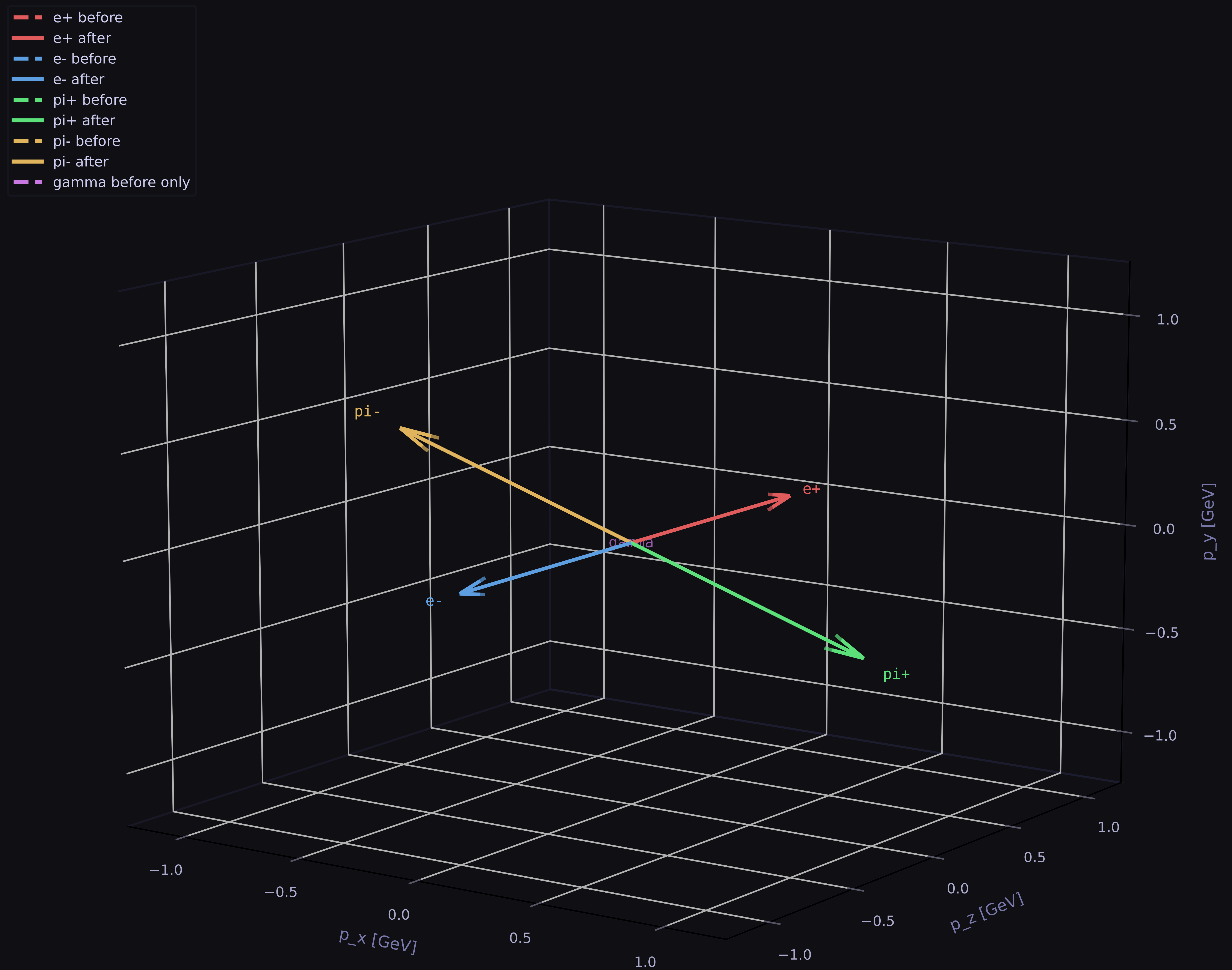
AFTER P = (+0.0000, +0.0000, +0.0000, +0.0000)

‖P‖ = 1.494683e-16 GeV

Non-soft event

Momentum Mapping: 1-photon → 0-photon

dashed = before | solid = after | axes: x, z (horiz), y (vert)



Invariant masses [GeV]

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e+	0.00051	0.00051
e-	0.00051	0.00051
pi+	0.13957	0.13957
pi-	0.13957	0.13957
gamma	0.00000	–

Momentum conservation

$P = P(e^+)+P(e^-) - P(\pi^+) - P(\pi^-) - P(\gamma)$

BEFORE P = (-0.0000, -0.0000, -0.0000, -0.0000)

‖P‖ = 1.191825e-16 GeV

AFTER P = (+0.0000, +0.0000, +0.0000, +0.0000)

‖P‖ = 5.551115e-17 GeV

Soft event

Structure of the code

Scan experiment processes, LO+ Resummation

Master Formula:

$$\sigma_{LO}^{res.} = \sum_{n_\gamma} \int_\Omega (dLips)_{n_\gamma+2} \frac{e^{\alpha Y_{\rm FF}}}{n_\gamma !} \sum_{\rm hel} \left| \sum_{\phi \in [I,F]^{n_\gamma}} \left(\prod_{i=1}^{n_\gamma} s_i^{\phi_i} \right) \beta^0(P) \right|^2$$

For each photon multiplicity

Generate event

G. P. Lepage [VEGAS]

Phase-space algorithm, VEGAS for integral optimisation

G. P. Lepage [2009.05112]

Evaluate amplitude

T. Dave, P. Petit Rosàs, W. Torres Bobadilla, J.P. [2604.16251]

Form Factor Decomposition, polarised amplitudes, Collier for numerical evaluation of one-loop integrals

A. Denner, S. Dittmaier, L. Hofer [1407.0087]

Evaluate YFS form factor

Calculated in Dimensional Regularisation without approximations

G. Passarino [0108255]

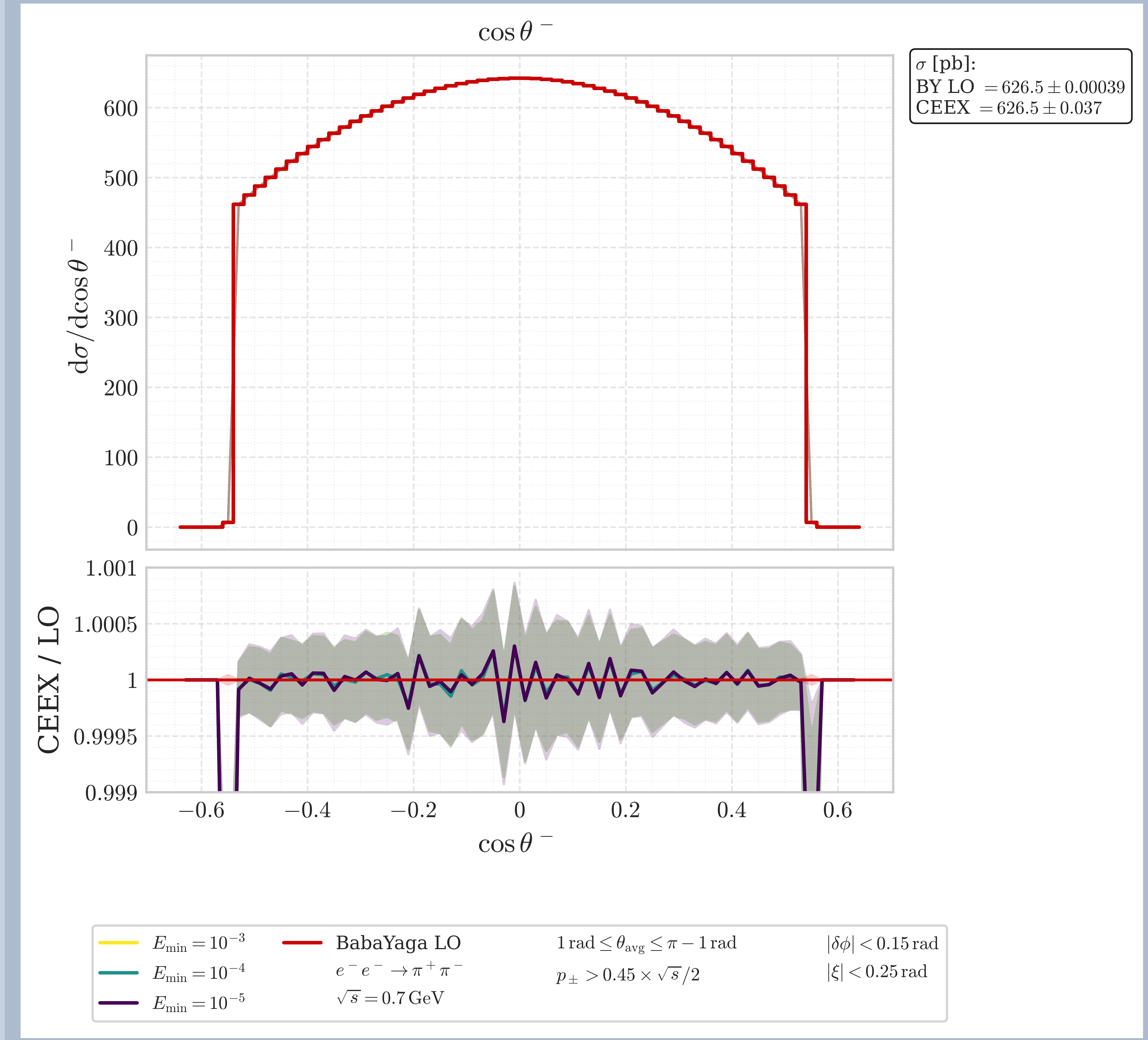
Fill histograms

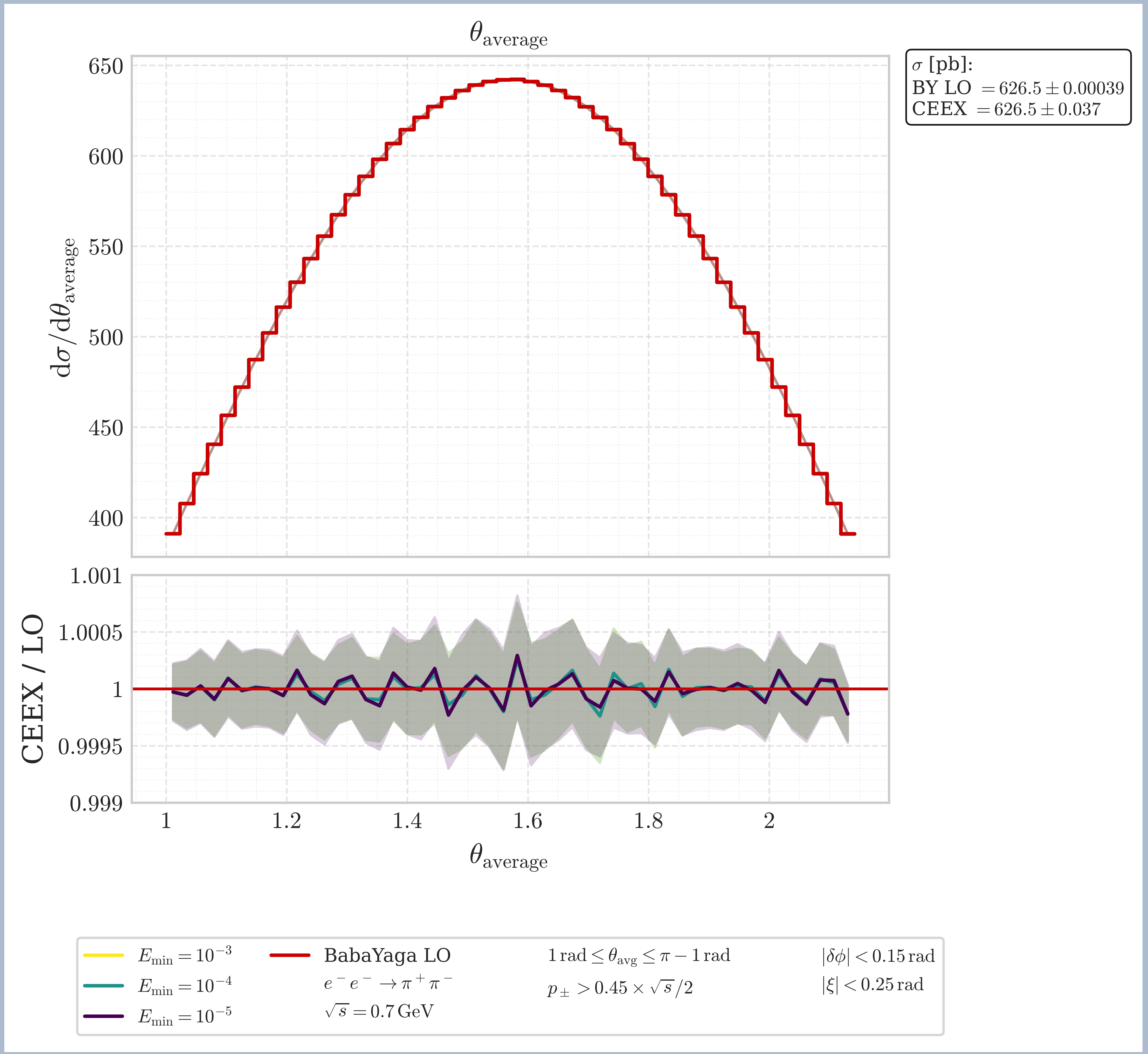
With user-friendly interface and pre-coded scenarios according to RMCL2

SCAN MODE

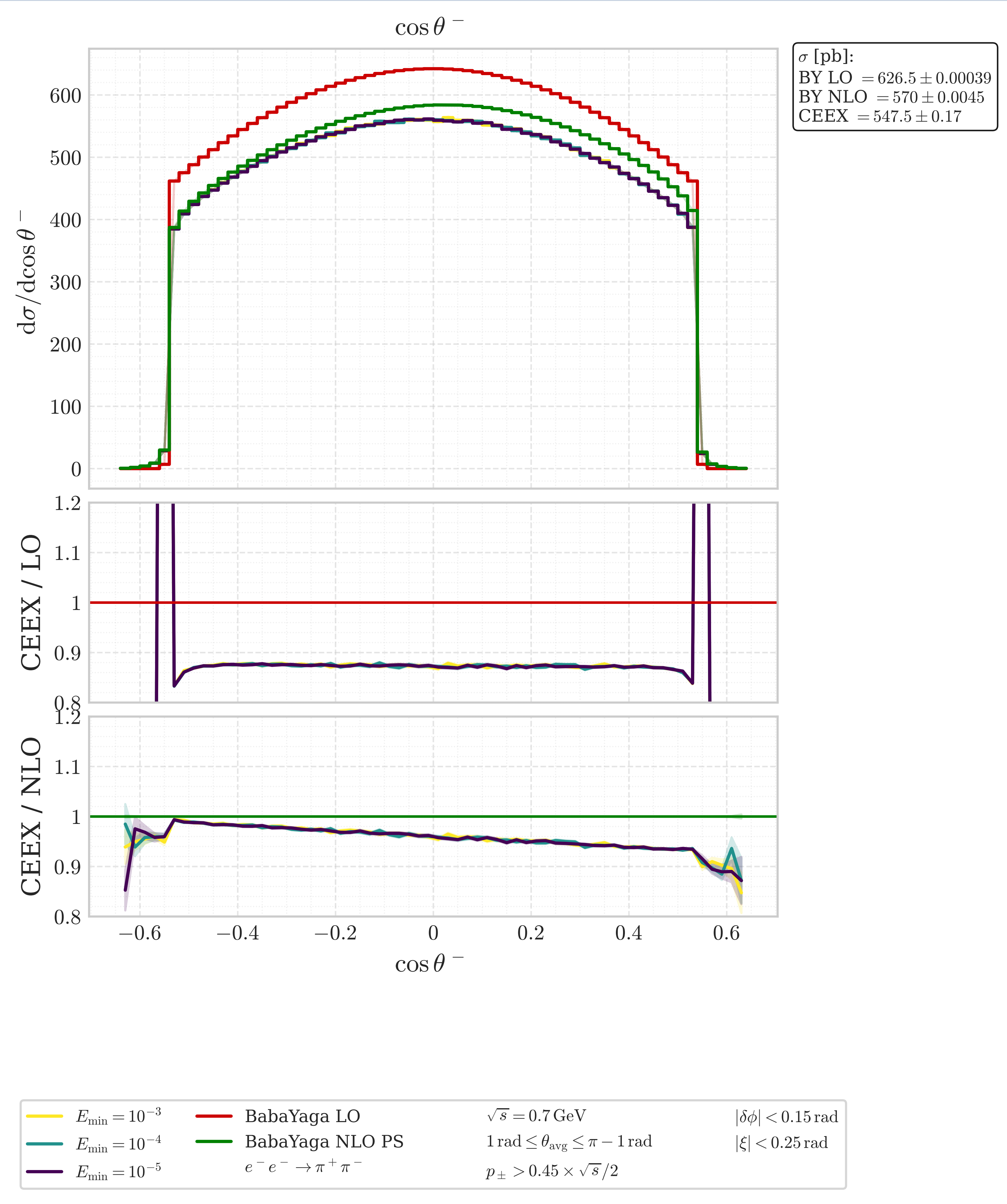
Scan mode

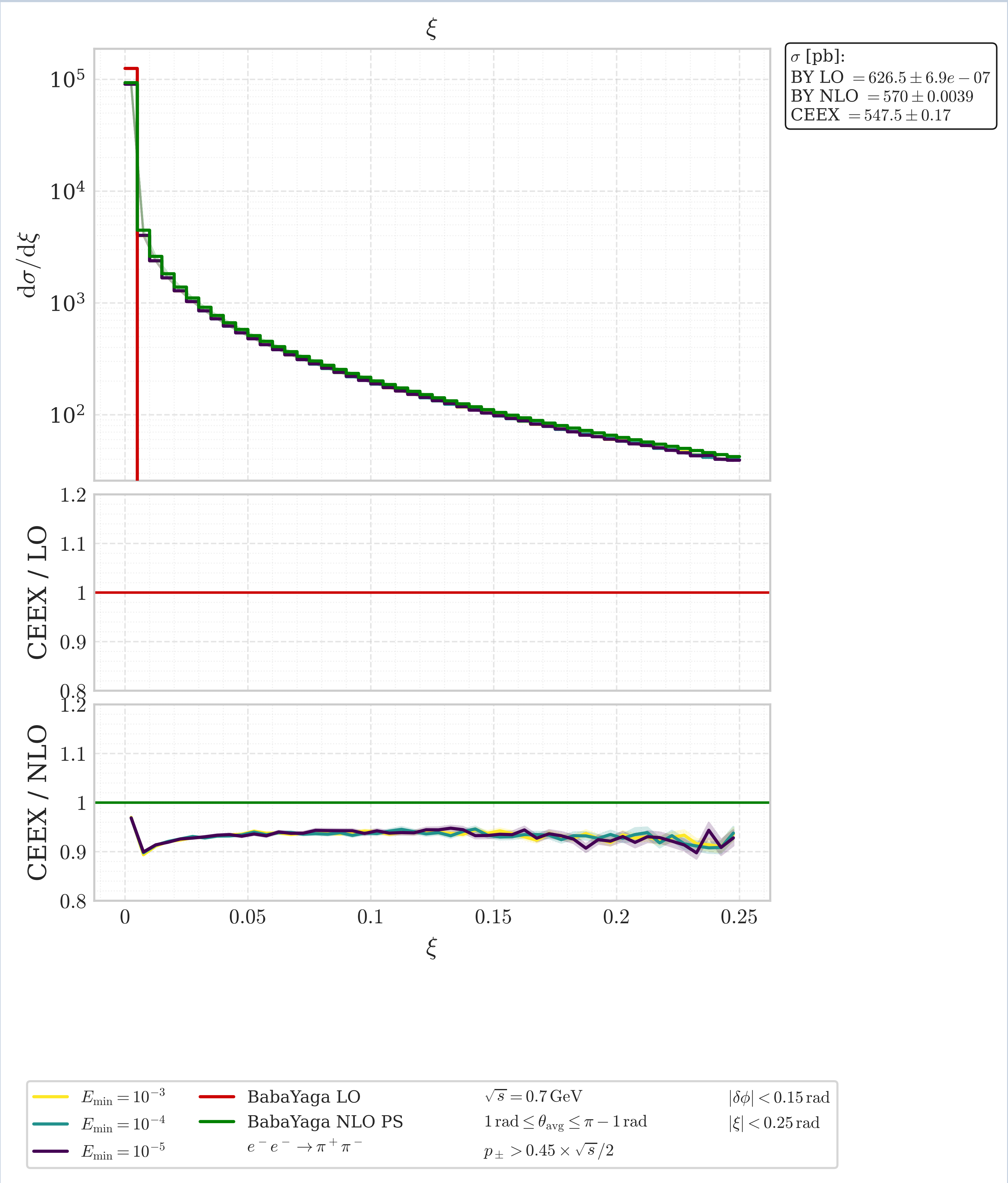
LO Test

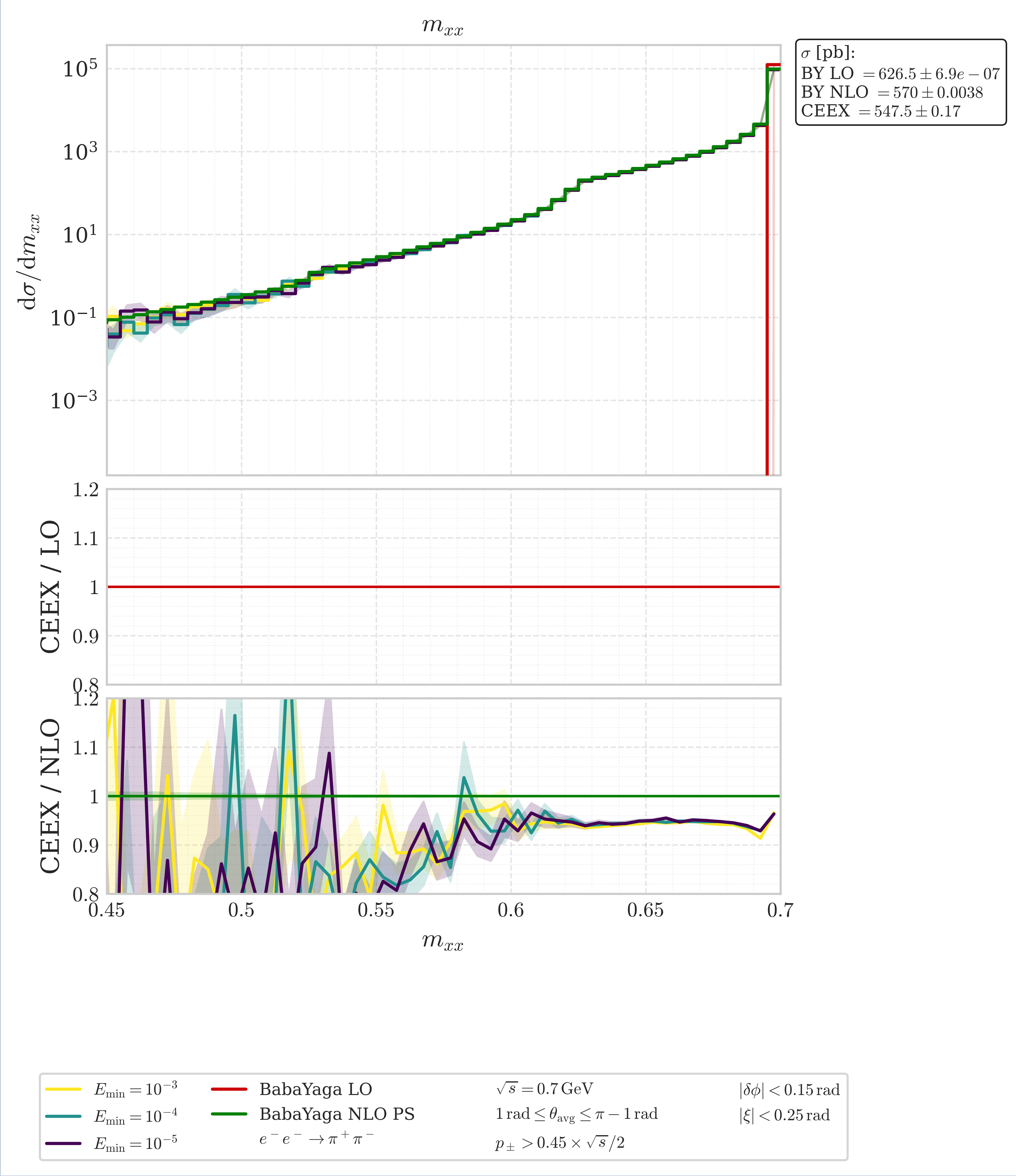




Results from LO + resummation, CMD scenario







Resummed results **do not depend on Emin**: strong test of the **soft limit** handling in the code

The K-factor between LO and LO-resummed is similar to LO-NLOPS K-factor

Next-to-Leading Order truncation

Master Formula:

$$\sigma_{NLO}^{res.} = \sum_{n_\gamma} \int_\Omega (dLips)_{n_\gamma+2} \frac{e^{\alpha Y_{\mathrm{FF}}}}{n_\gamma!} \sum_{\mathrm{hel}} \left| \sum_{\phi\in[I,F]^{n_\gamma}} \left(\prod_{i=1}^{n_\gamma} s_i^{\phi_i} \right) \left(\beta_0^{(1)} + \sum_{j=1}^{n_\gamma} \frac{\beta_1^{(1)}(P,k_j)}{s_j^{\phi_j}} \right) \right|^2$$

Next-to-Leading Order truncation

Master Formula:

$$\sigma_{NLO}^{res.} = \sum_{n_\gamma} \int_\Omega (dLips)_{n_\gamma+2} \frac{e^{\alpha Y_{\mathrm{FF}}}}{n_\gamma!} \sum_{\mathrm{hel}} \left| \sum_{\phi \in [I,F]^{n_\gamma}} \left(\prod_{i=1}^{n_\gamma} s_i^{\phi_i} \right) \left(\beta_0^{(1)} + \sum_{j=1}^{n_\gamma} \frac{\beta_1^{(1)}(P, k_j)}{s_j^{\phi_j}} \right) \right|^2$$

Truncate this (YFS FF, eikonal products, betas):

$$\sigma_{NLO}^{trunc.} = \int (dPS)_2 \left(|\mathcal{M}_{LO}|^2 (1 + \alpha Y_V + \alpha Y_R) + 2Re \left(\mathcal{M}_{LO}^\dagger \left(\mathcal{M}_{1L}^{ren} - \frac{1}{2} \alpha Y_V \mathcal{M}_{LO} \right) \right) \right) + \int_\Omega (dPS)_3 |\mathcal{M}_R|^2$$

Finite Finite Finite

Next-to-Leading Order truncation

Master Formula:

$$\sigma_{NLO}^{res.} = \sum_{n_\gamma} \int_\Omega (dLips)_{n_\gamma+2} \frac{e^{\alpha Y_{FF}}}{n_\gamma!} \sum_{hel} \left| \sum_{\phi \in [I,F]^{n_\gamma}} \left(\prod_{i=1}^{n_\gamma} s_i^{\phi_i} \right) \left(\beta_0^{(1)} + \sum_{j=1}^{n_\gamma} \frac{\beta_1^{(1)}(P, k_j)}{s_j^{\phi_j}} \right) \right|^2$$

Truncate this (YFS FF, eikonal products, betas):

$$\sigma_{NLO}^{trunc.} = \int (dPS)_2 \left(|\mathcal{M}_{LO}|^2 (1 + \alpha Y_V + \alpha Y_R) + 2Re \left(\mathcal{M}_{LO}^\dagger \left(\mathcal{M}_{1L}^{ren} - \frac{1}{2} \alpha Y_V \mathcal{M}_{LO} \right) \right) \right) + \int_\Omega (dPS)_3 |\mathcal{M}_R|^2$$

Finite Finite Finite

Exactly equivalent to a subtraction scheme:

$$\mathcal{S} = \int S \equiv \alpha Y_R |M_{LO}|^2$$

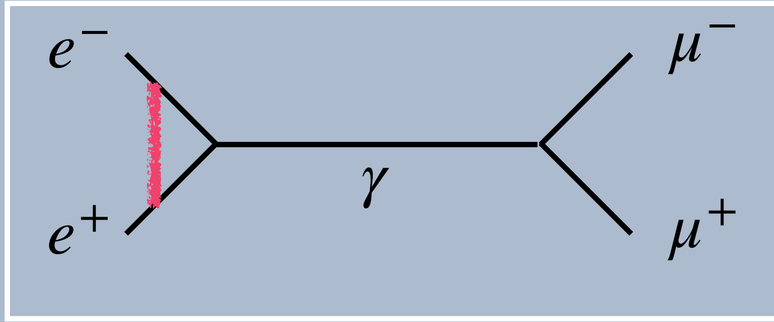
$$\sigma_{NLO}^{sub.sch.} = \int (dPS)_2 \left(\mathcal{B} + \mathcal{V} + \mathcal{S} \right) + \int (dPS)_3 \left(\mathcal{R} - S \right)$$

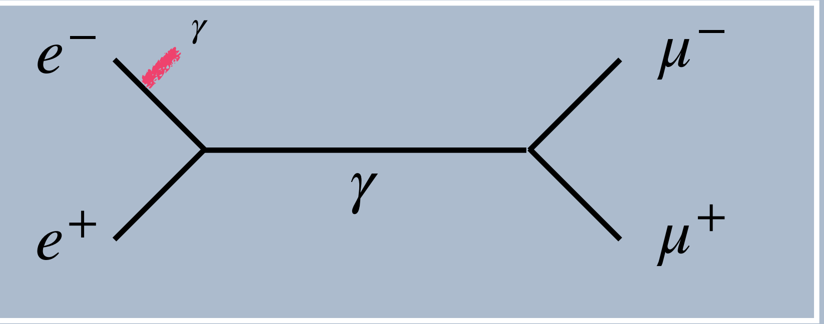
$$\int (dPS)_3 \left(\mathcal{R} - S \right) = \int_\Omega (dPS)_3 |R|^2$$

Form Factor decomposition of the amplitude

Next-to-Leading Order

$$\sigma_{NLO}^{res.} = \sum_{n_\gamma} \int_\Omega (dLips)_{n_\gamma+2} \frac{e^{\alpha Y_{FF}}}{n_\gamma!} \sum_{hel} \left| \sum_{\phi\in[I,F]^{n_\gamma}} \left(\prod_{i=1}^{n_\gamma} s_i^{\phi_i} \right) \left(\beta_0^{(1)} + \sum_{j=1}^{n_\gamma} \frac{\beta_1^{(1)}(P,k_j)}{s_j^{\phi_j}} \right) \right|^2$$





One loop: $$\beta_0^{(1)} = a_0 + \frac{a_{-1}^{IR}}{\epsilon_{IR}} + \frac{a_{-1}^{UV}}{\epsilon_{UV}} - \alpha \sum_{i<j} YFS_{ij}^{virtual} + \frac{ct}{\epsilon_{UV}}$$

IR pole cancels exactly with the virtual YFS form factor pole and has expected structure from known IR operators

UV pole cancels with renormalisation

S. Catani, S. Dittmaier, Z. Trocsanyi [0011222]

Real emission: For $\pi\pi\gamma$, $\beta_1^{(1)}$ is known up to $\mathcal{O}(\epsilon^2)$ (needed for higher-order calculations)

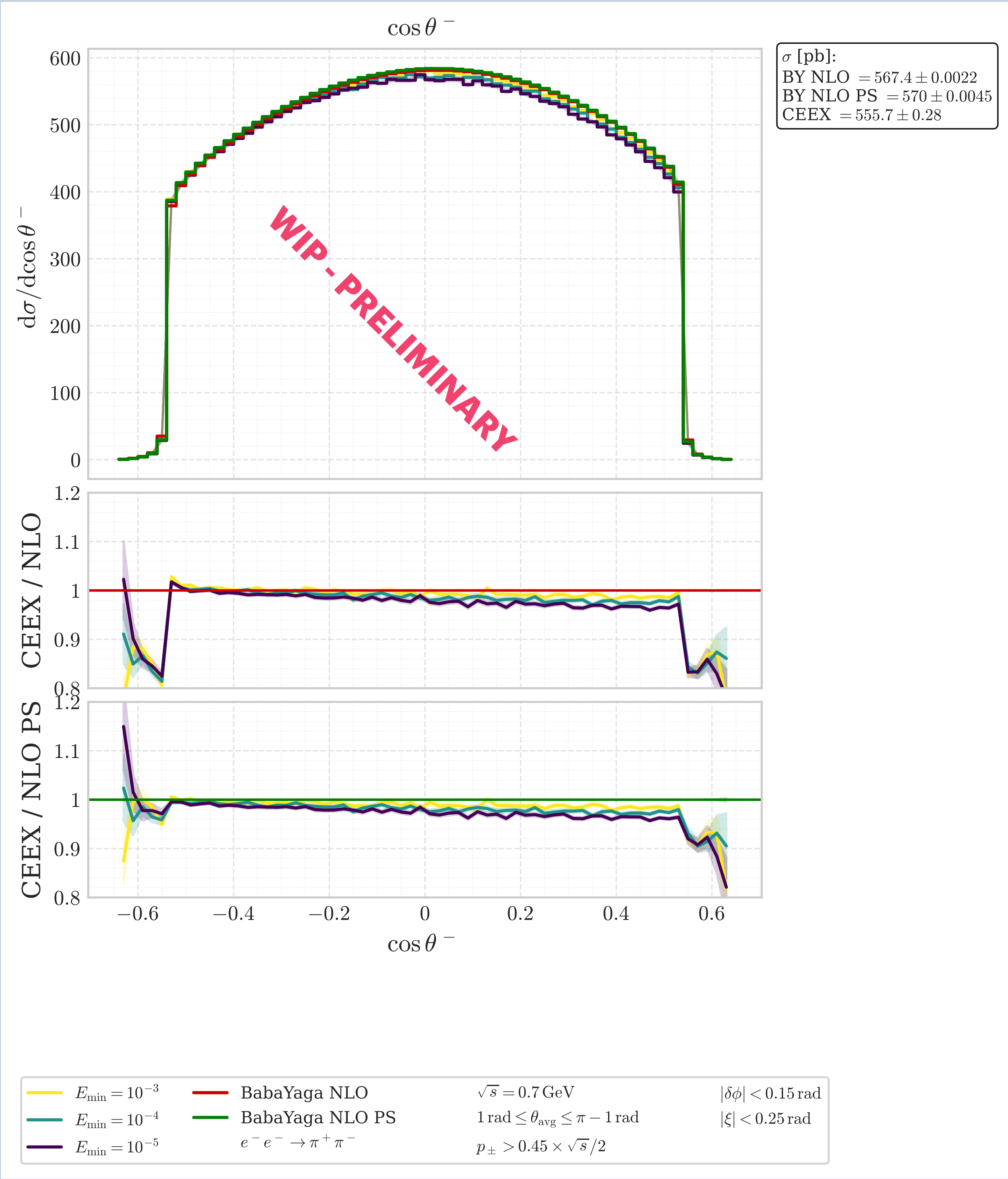
T. Dave, P. Petit Rosàs, W. Torres Bobadilla, J.P. [2604.16251]

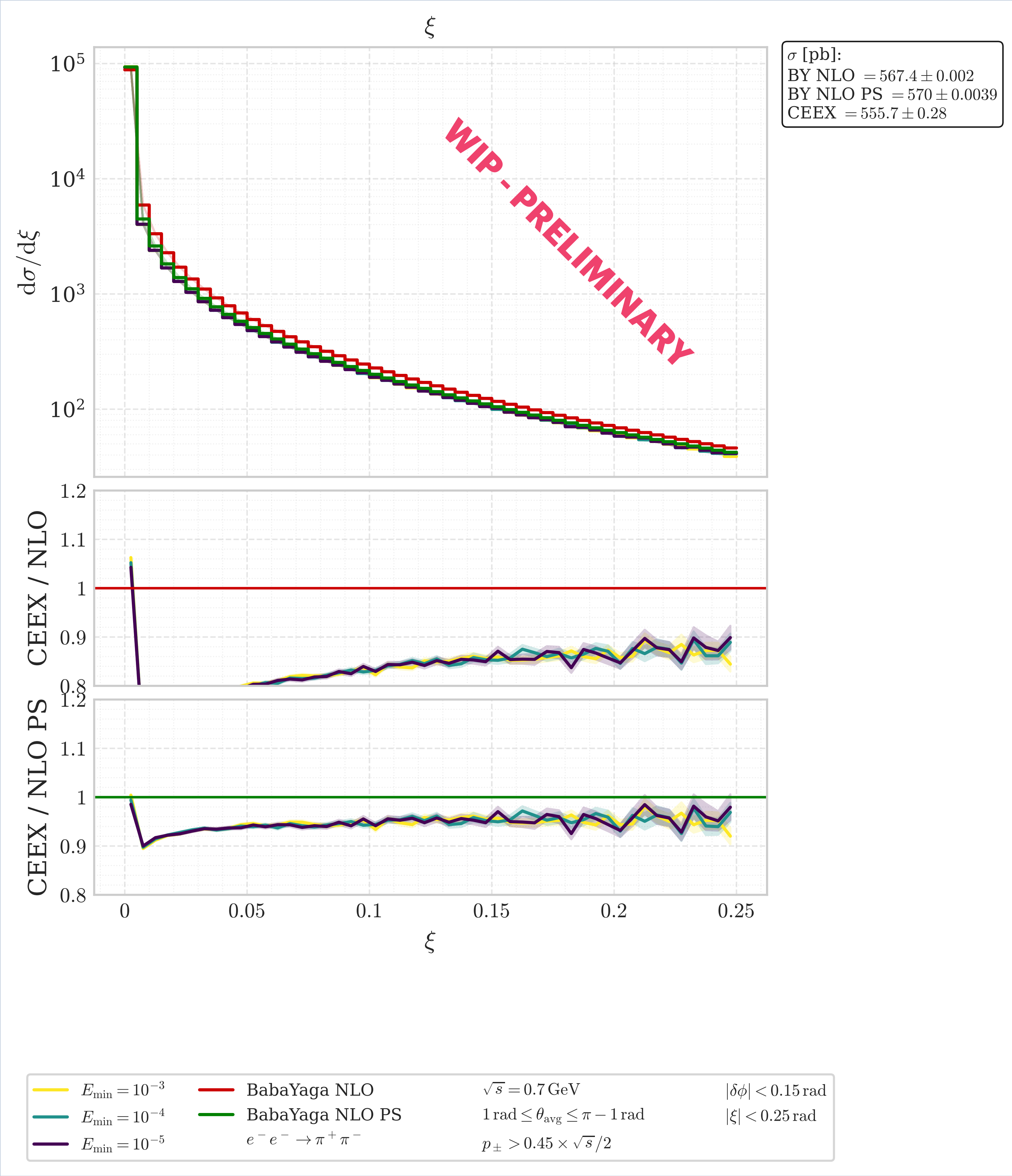
NLO, CMD scenario

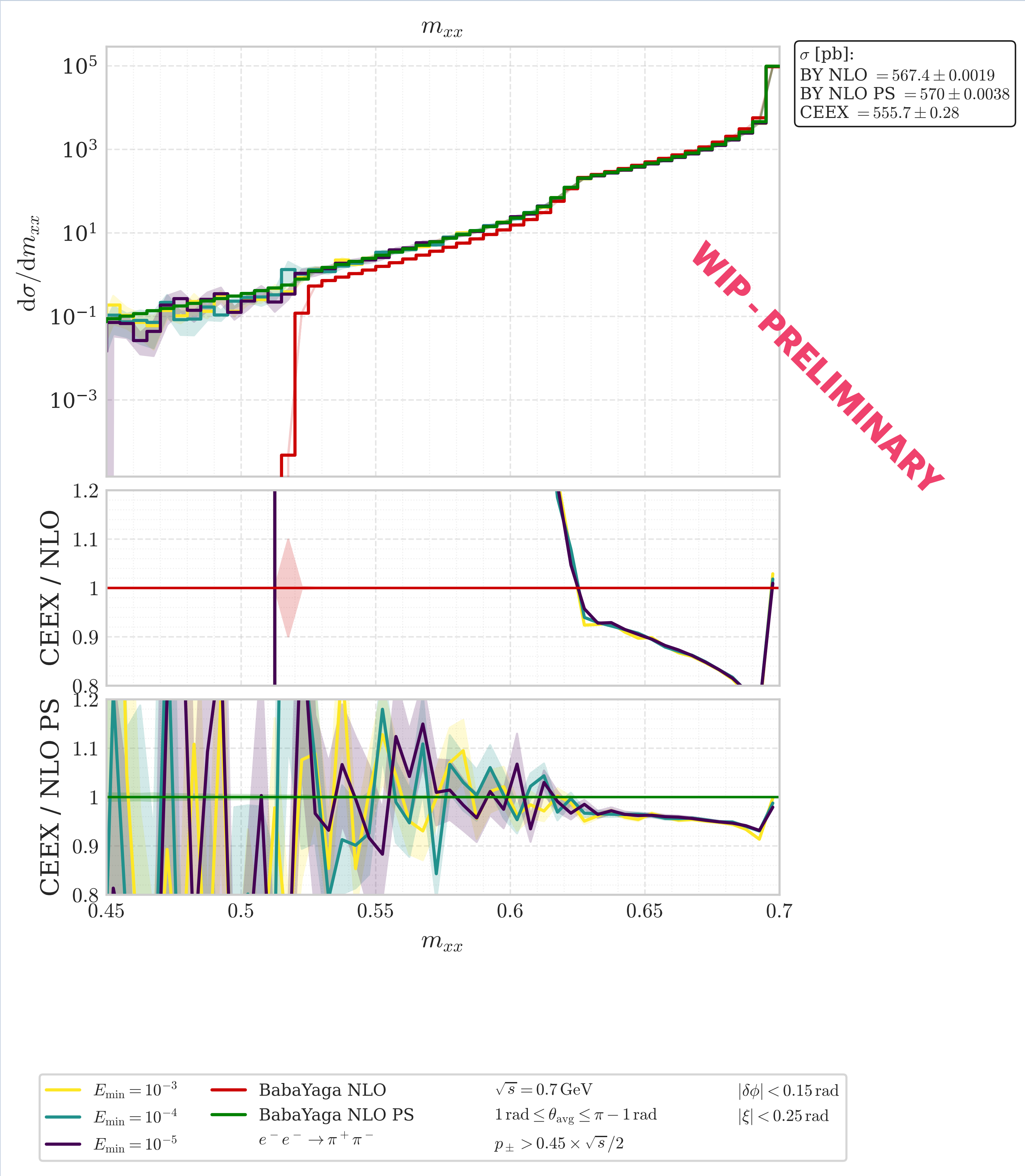
As Emin goes to 0, the predictions converge to NLO: perfect agreement with BabaYaga

This is the result of the **truncation of the resummed results** (up to 1 photon).

NLO resummed, CMD scenario







These are **not right** yet: Emin dependency which hints at a problem in the soft limit.

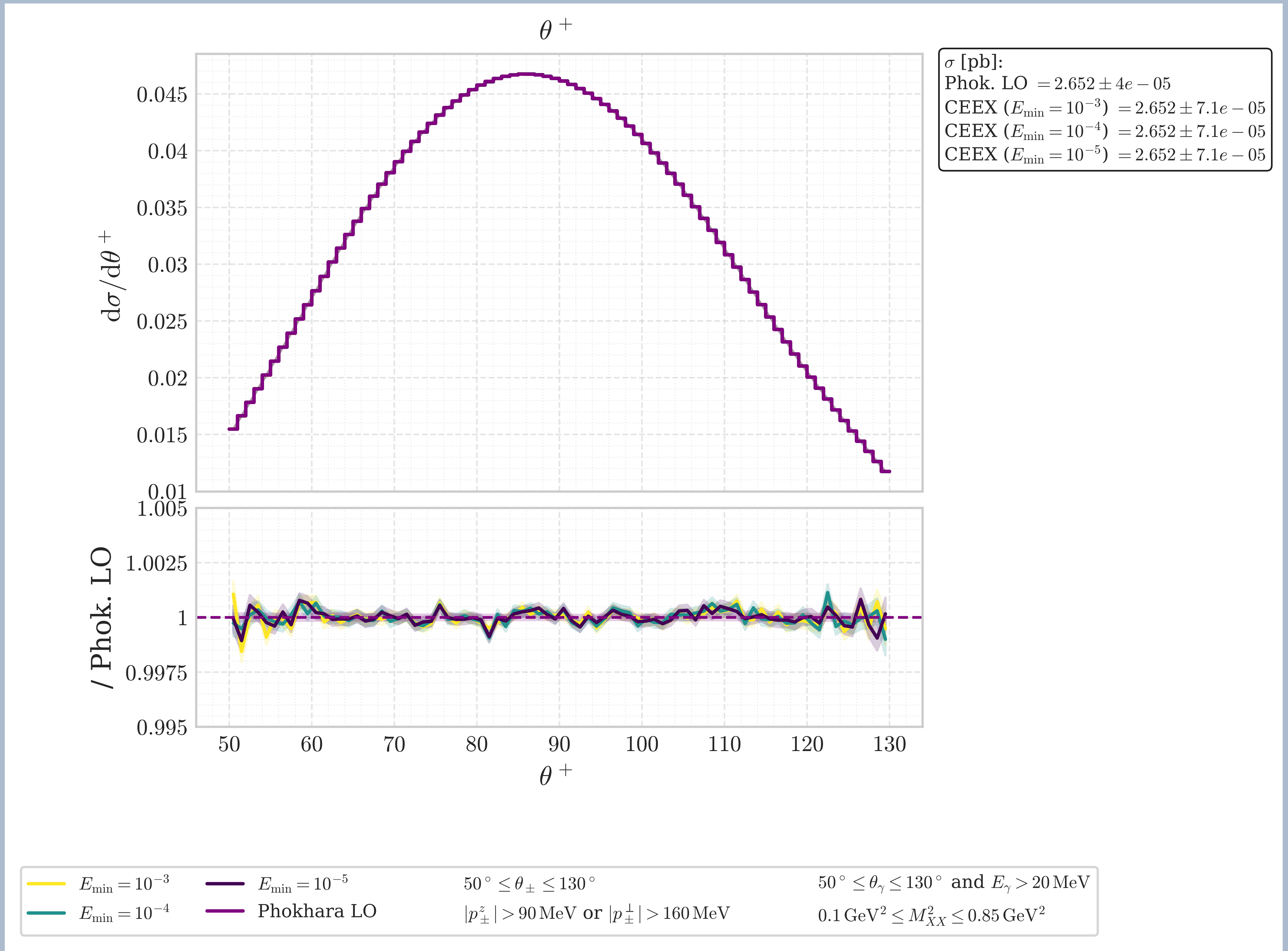
Few percents discrepancy between 2 orders of magnitude for Emin.

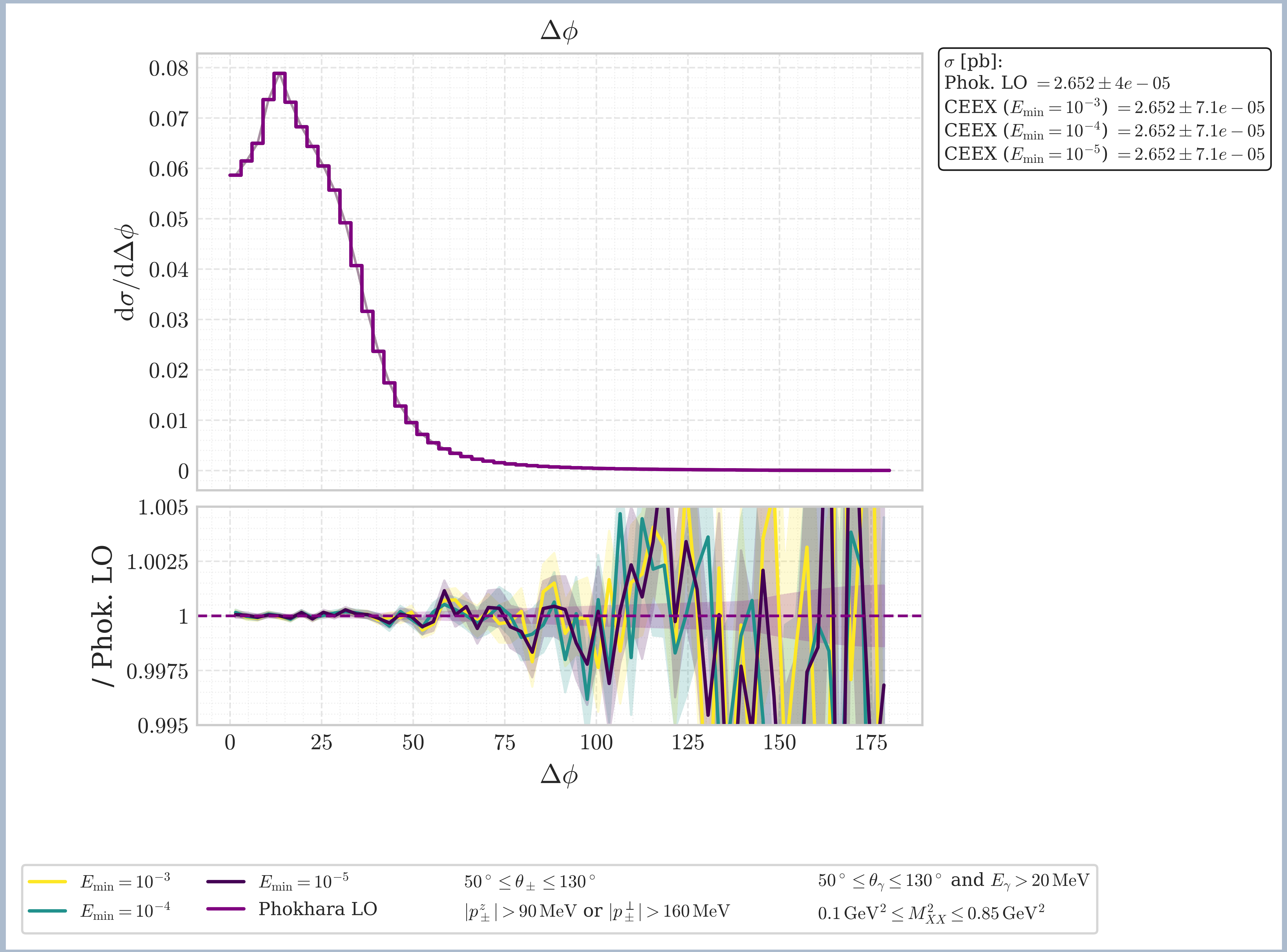
Radiative Return

Radiative Return

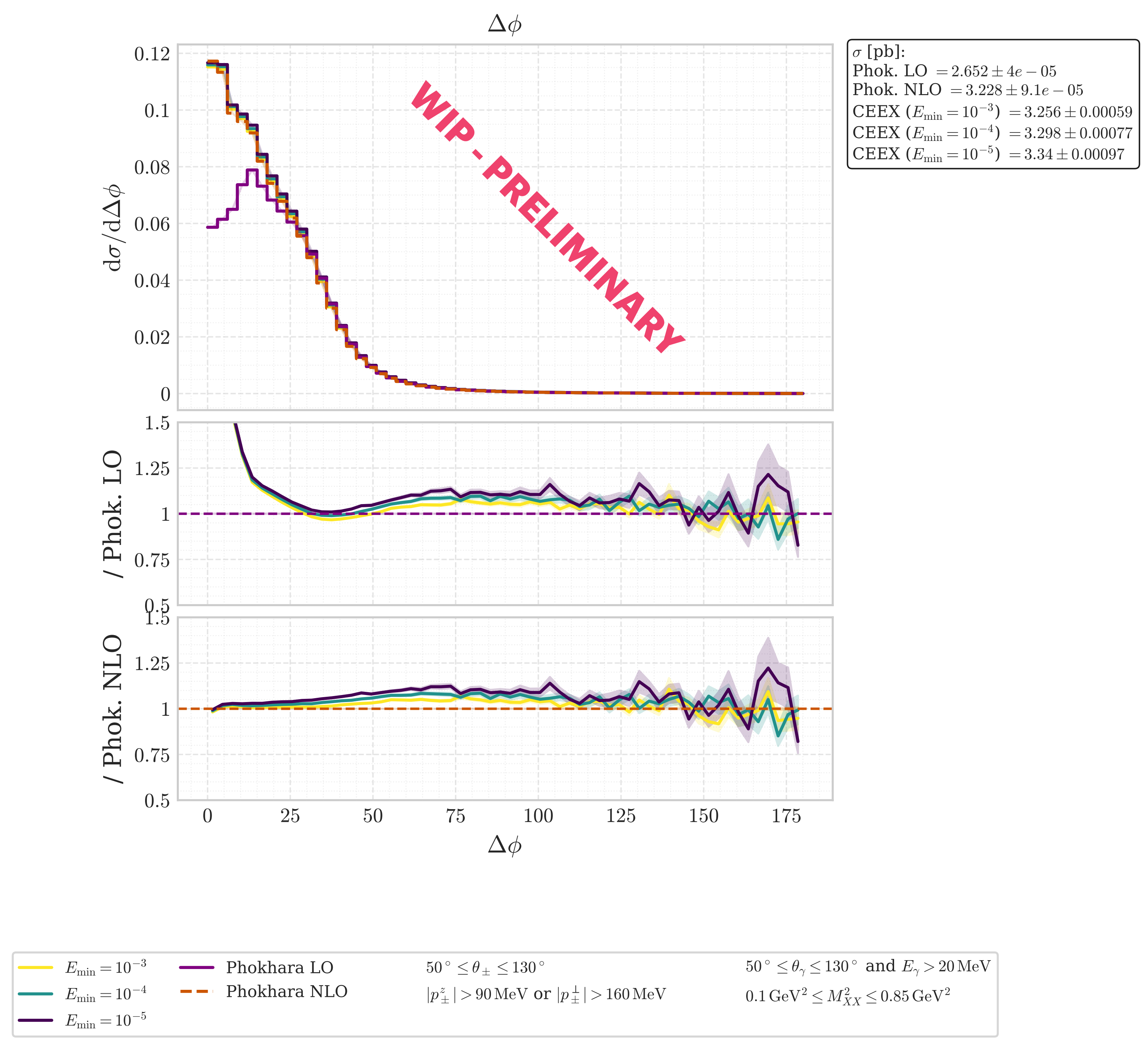
KLOE-LA

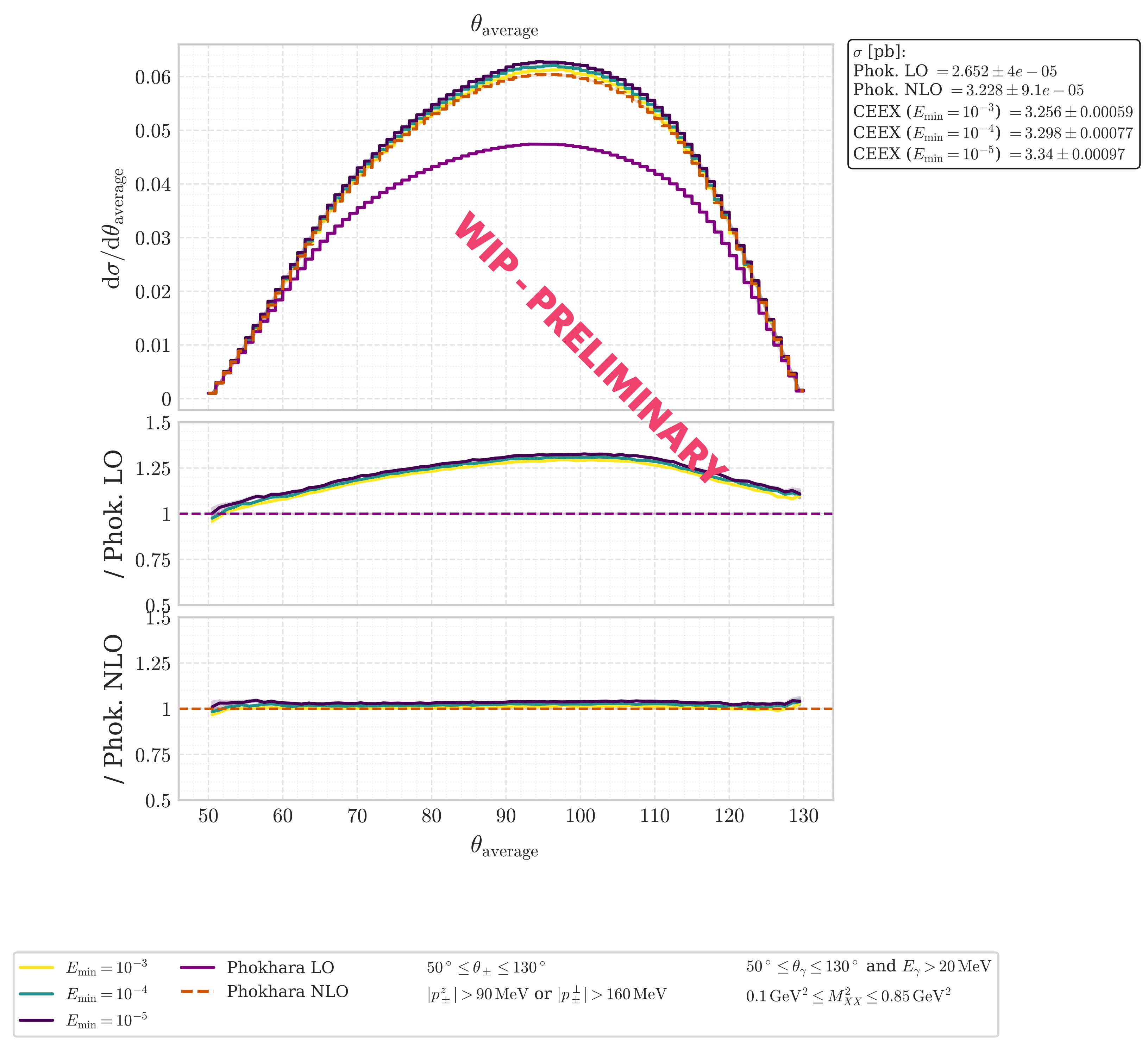
LO Test - KLOE Large Angle

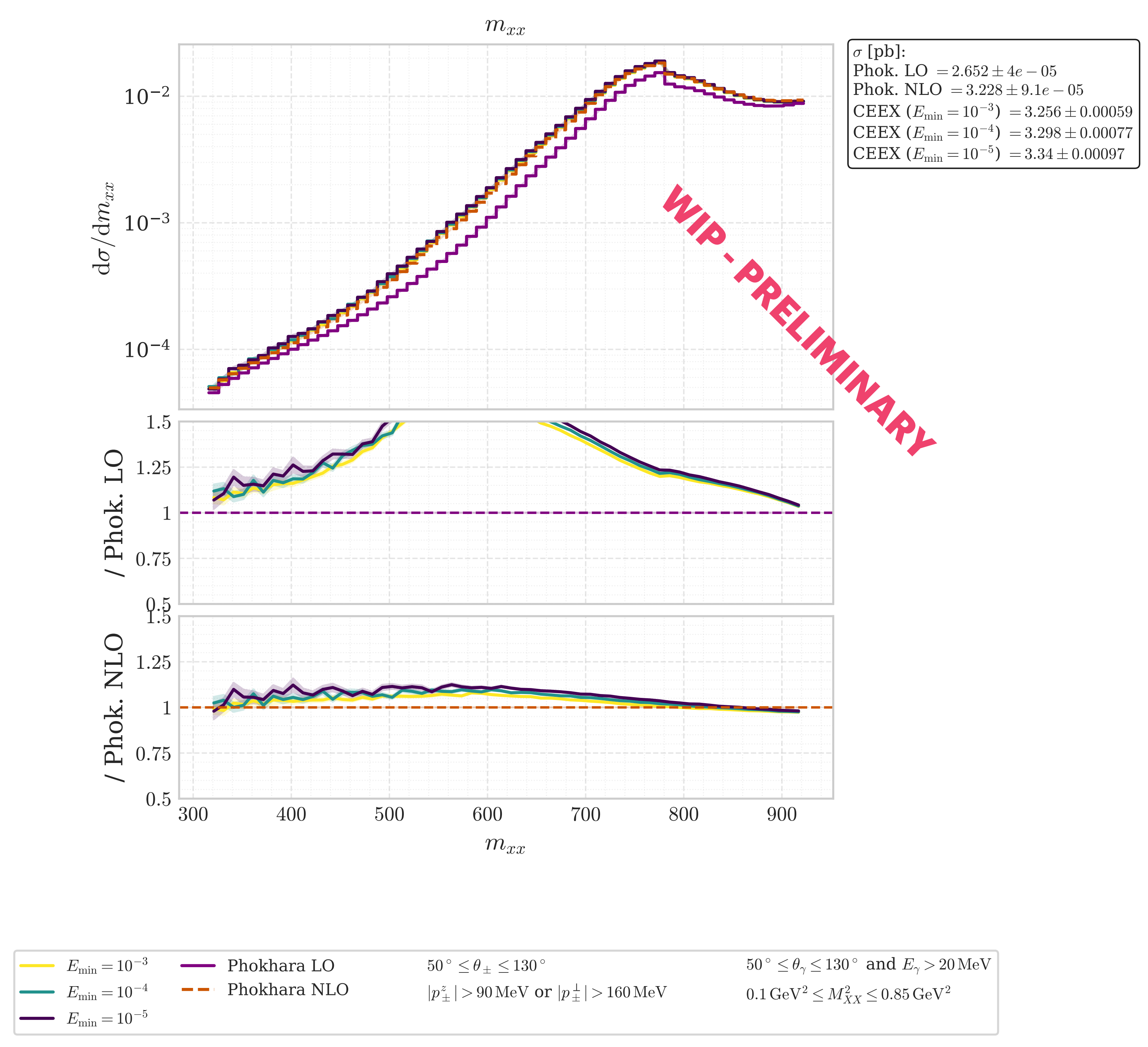




LO Resummation Test - KLOE Large Angle







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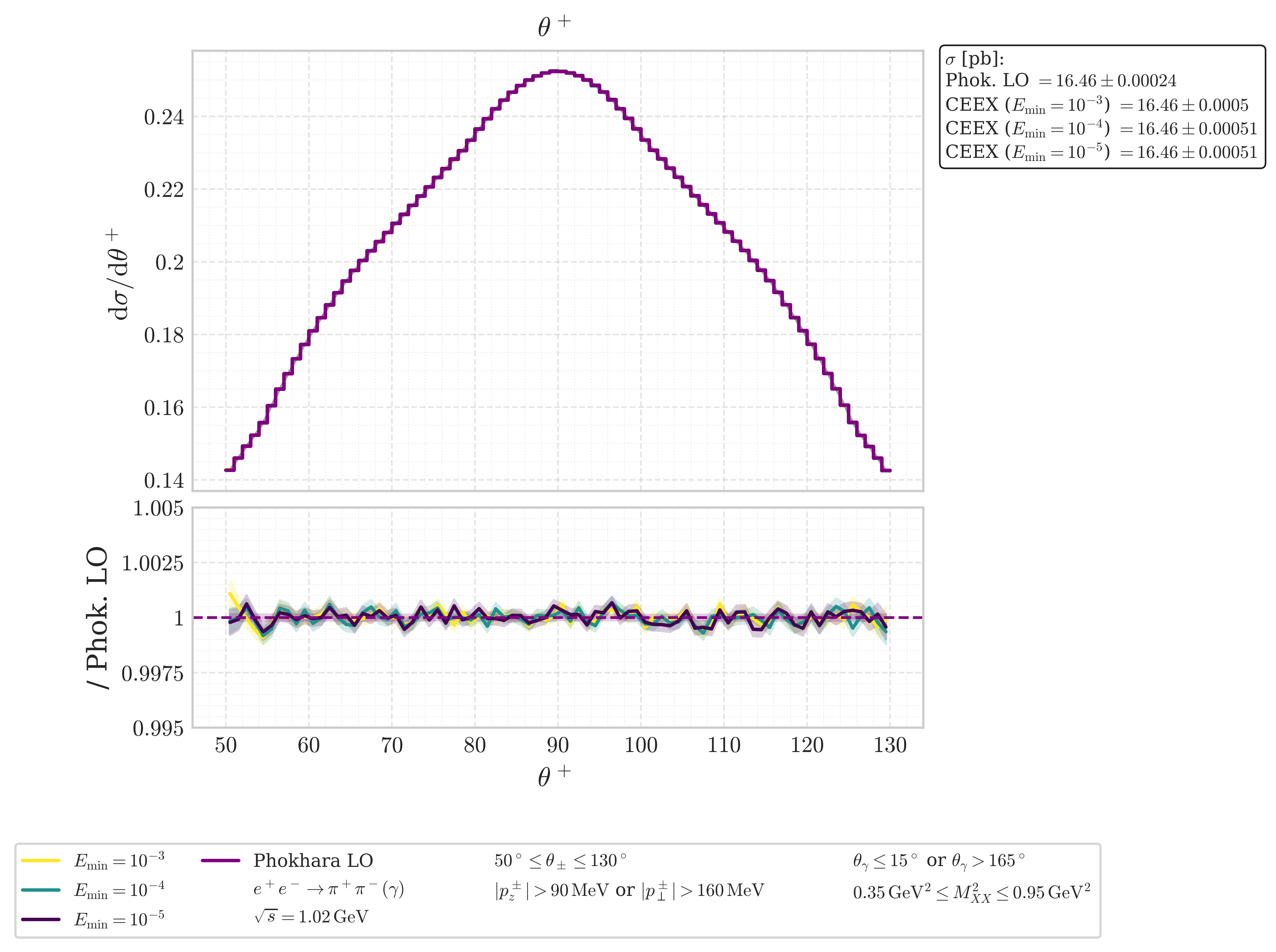
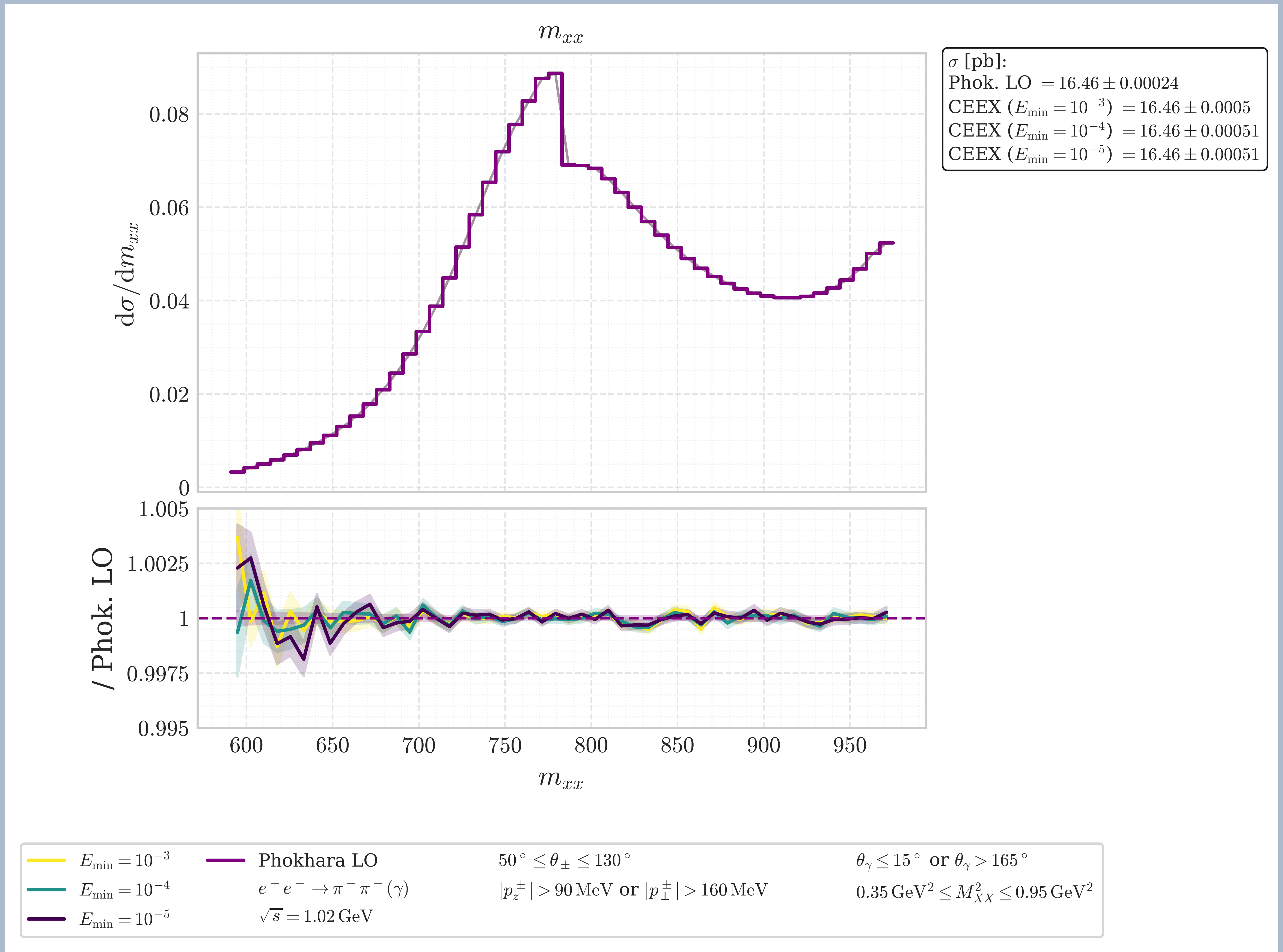
Few percents discrepancy between 2 orders of magnitude for Emin.

Radiative Return

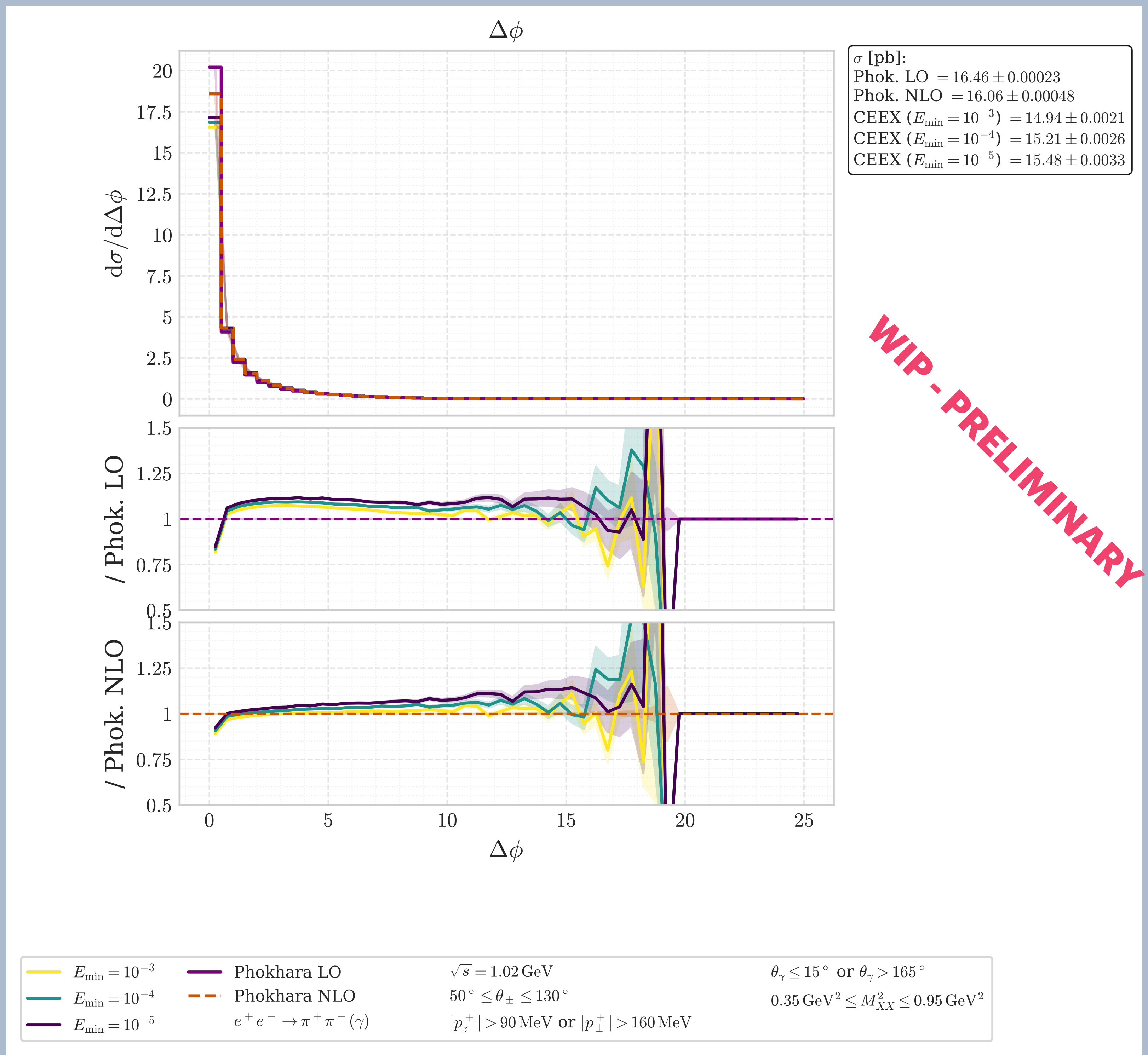
KLOE-SA

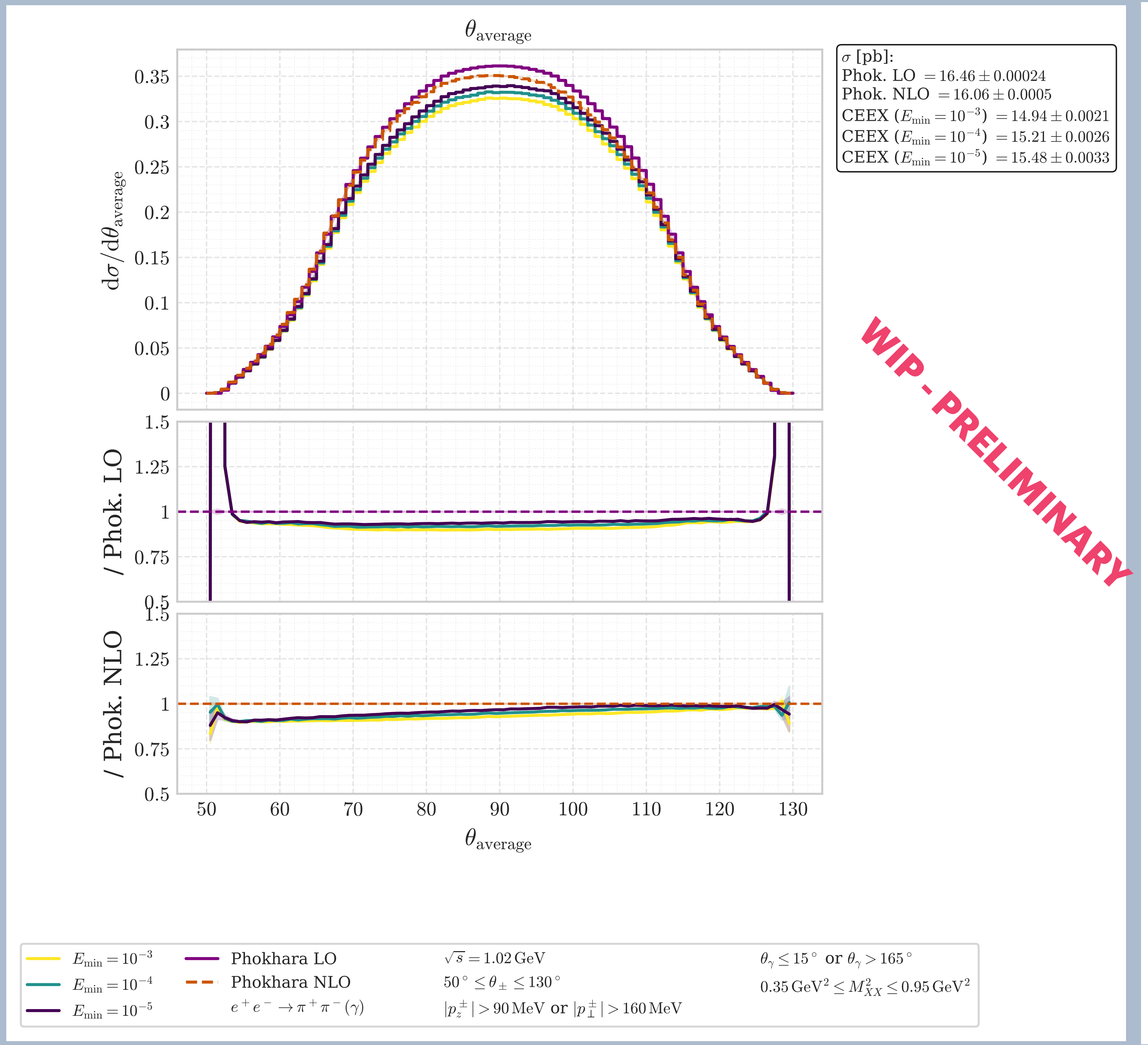
LO Test - KLOE Small Angle

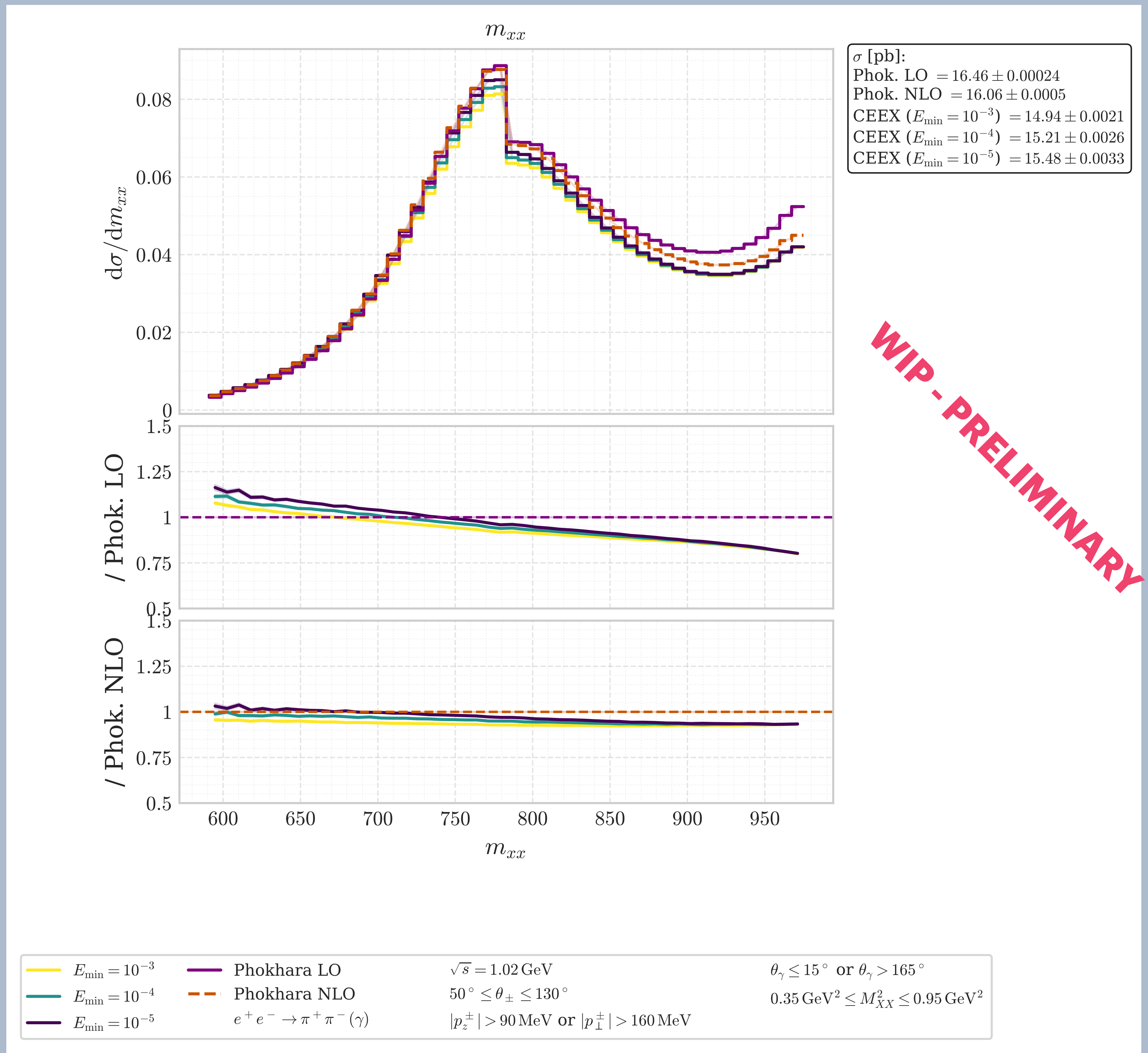




LO Resummation Test - KLOE Small Angle







These are **not right** yet: E_{min} dependency which hints at a problem in the soft limit.

Few percents discrepancy between 2 orders of magnitude for Emin.

Conclusion

Conclusion

Summary

- YFS Form Factor calculation is **understood**: checked against known results in the literature.
- Eikonal approximation works at the level of the **amplitude**, for multi-photon events.
- Truncation of resummation to **LO** (radiative return) and **NLO** (scan mode) **accuracy** is working.
- Preliminary resummed results are encouraging.

Outlook

- Convergence with respect to E_{min}: needs to be understood for NLO scan mode, LO radiative return: need to perform more tests on the soft limit.
- NLO exact (truncation) for radiative return, using YFS as a **subtraction scheme**.
- NLO resummed for radiative return, using **helicity amplitudes already obtained**.
- MC comparisons with BabaYaga@NLOPS, Sherpa YFS, KKMC to **test the resummation**.

Thank you!